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SIMILARITY SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS INVARIANT TO A FAMILY OF AFFINE GROUPS*

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Introductory Overview**

Problems of technological interest can very often be described by partial differential equations (PDEs) with one dependent and two independent variables (call them c, z, and t, respectively). Table 1 shows a few such nonlinear equations and some of the contexts in which they arise.

Table 1. Some typical nonlinear PDEs and some of the contexts in which they arise.

$c_t = (c^n)_{zz}$	Plasma physics; groundwater hydrology; gas flow in porous media; current distribution in type II superconductors
$cc_t = c_{zz}$	Thermal expulsion of fluid from a long, slender, heated pipe; heat conduction in metals at cryogenic temperatures
$c_t = (c_z^{1/3})_z$	Heat transport in superfluid He-II
$c_{tt} = \frac{1}{2} c_{zz} \int_0^1 c_z^2 dz$	Motion of a shock-loaded elastic membrane

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Many such PDEs (including all of those in Table 1) are invariant to one-parameter families of one-parameter affine groups of the form

$$c' = \lambda^{\alpha} c$$

$$t' = \lambda^{\beta} t \qquad 0 < \lambda < \infty \tag{1}$$

$$z' = \lambda z$$

where λ is the group parameter that labels the individual affine transformations and α and β are parameters that label groups of the family. The parameters α and β are connected by a linear relation

$$M\alpha + N\beta = L \tag{2}$$

where M, N, and L are numbers determined by the structure of the PDE. Because of the relation (2), only one of the two parameters α and β may be chosen freely.

Similarity solutions are solutions of the PDE that are invariant to one group of the family – say, that for which $\alpha = \alpha_o$ and $\beta = \beta_o$. Such solutions most generally have the form

$$c = t^{\alpha_o/\beta_o} y(z/t^{1/\beta_o}) \tag{3}$$

where y is a function of the single variable $x = z/t^{1/\beta_o}$. When substituted into the PDE, (3) yields an ordinary differential equation (ODE) for the function of one variable, y(x). I call this ODE the principal ODE.

The great utility of similarity solutions is that they may be calculated by solving an ODE rather than a PDE and are thus much more easily accessible than other solutions. The form of the principal ODE depends, of course, on the form of the PDE, but it can be proved quite generally that the principal ODE is itself invariant to the one-parameter affine group

$$y' = \mu^{L/M} y \qquad 0 < \mu < \infty \qquad (4)$$
$$x' = \mu x$$

I call this group the associated group.

If the PDE is of the second order, as is often the case, so, too, is the principal ODE. According to a theorem of Lie, if we use an invariant $u = yx^{-L/M}$ and a first differential invariant $v = \dot{y}x^{-L/M+1}$ ($\dot{y} \equiv dy/dx$) of the associated group as new independent and dependent variables, we reduce the second-order principal ODE in y and x to a first-order ODE in v and u. I call this first-order ODE the associated ODE. So the computational task for problems of this kind reduces to the solution of a first-order ODE.

Sometimes the principal ODE is soluble in terms of simple functions. When it is not, the associated first-order ODE can be studied graphically by means of its direction field. Two substantial benefits arise from such a study. First, the singular points sometimes provide the asymptotic behavior of similarity solutions without extensive calculation. Second, when the solution y(x) of the principal ODE is determined by two-point boundary conditions, study of the direction field in the (u,v)-plane may enable us to determine both the stable direction of numerical integration and the missing boundary condition.

Because of the invariance of the principal ODE to the associated group, the dependence on the boundary and initial conditions of certain special values of the function y(x), e.g., $y(0)$, $y(\infty)$, $\dot{y}(0)$, $\int_0^\infty y\,dx$, etc., may be predicted a priori without solving the principal ODE.

The nonlinear PDE of heat transport in superfluid He-II (Table 1, line 3) is used as an illustration of these ideas in this review.

The Associated Group

Invariance of any differential equation to a group of transformations means that the image of a solution is also a solution. A solution of the PDE, $c = g(z,t)$, can be viewed as a surface in three-dimensional (c,z,t)-space. Each point (c,z,t) of the surface transforms into an image (c',z',t') under the transformation of a group (1) (λ varies, α and β are fixed). The surface $c = g(z,t)$ then transforms into an image surface $c' = g'(z',t')$. Invariance of the PDE to the groups (1) means that if we substitute from (1) for the unprimed variables in the PDE we recover the same PDE in the primed variables. Thus the image surface $c' = g'(z',t')$ is also a solution of the PDE.

If a solution is its own image, i.e., if it is invariant to transformation by a group of the family, then g' and g are the same function. Thus $c' = g(z',t')$ or, from (1), $\lambda^{\alpha_o} c = g(\lambda z, \lambda^{\beta_o} t)$. Now since $c = g(z,t)$, we have $\lambda^{\alpha_o} g(z,t) = g(\lambda z, \lambda^{\beta_o} t)$. If we differentiate this last equation with respect to λ and then set $\lambda = 1$, we get the first-order linear PDE $\alpha_o g = z g_z + \beta_o t g_t$, the most general solution of which is (3).

If we transform (3) by the group (1) belonging to the values α_o and β_o of α and β we recover (3) again. What happens if we transform it by a group for which $\alpha \neq \alpha_o$ and $\beta \neq \beta_o$? We obtain another solution $c'(z',t')$ of the PDE related to that of (3) as follows:

$$c'(z',t') = \lambda^{\alpha} c(z,t) = \lambda^{\alpha} c(z'/\lambda, t'/\lambda^{1/\beta}) \tag{5a}$$

$$= \lambda^{(\alpha\beta_o - \alpha_o\beta)/\beta_o} (t')^{\alpha_o/\beta_o} y(\lambda^{(\beta-\beta_o)/\beta_o} z'/(t')^{1/\beta_o}) \tag{5b}$$

where the passage from (5a) to (5b) has been achieved by substituting from (3) for c in the last term of (5a). Now because $M\alpha_o + N\beta_o = L$ and $M\alpha + N\beta = L$, it follows that $\alpha\beta_o - \alpha_o\beta = (L/M)(\beta_o - \beta)$. If we introduce $\mu = \lambda^{(\beta-\beta_o)/\beta_o}$, (5b) becomes

$$c'(z',t') = (t')^{\alpha_o/\beta_o} \mu^{-L/M} y(\mu x') \tag{6}$$

The meaning of (6) is that if the function $y(x)$ makes c given by (3) a solution of the PDE, then its transform $\mu^{-L/M}y(\mu x)$ used in place of it will also make c given by (3) a solution of the PDE. Or, more succinctly, if $y(x)$ is a solution of the principal ODE, so is its transform $\mu^{-L/M}y(\mu x)$.

The family of transforms $\mu^{-L/M}y(\mu x)$, $0 < \mu < \infty$, of $y(x)$ is the same as the family of images of $y(x)$ under the associated group (4). Now $y'(x') = \mu^{L/M}y(x) = \mu^{L/M}y(x'/\mu)$, which means that the image of $y(x)$ under the transformation (4) with parameter μ is the transform of $y(x)$ with the parameter $1/\mu$. So there is a one-to-one correspondence between the images under (4) and the transforms $\mu^{-L/M}y(\mu x)$. But this means that if $y(x)$ is a solution of the principal ODE, so is any image of it under the associated group (4). Therefore, the principal ODE itself must be invariant to the associated group (4).

Illustrative Example

At very low temperatures (>2.17 K), helium has a second liquid phase (called He-II or the superfluid phase) with some unusual properties. The one that interests us here is this: when the heat flux is large ($\gtrsim 0.1$ W/cm^2), it is proportional to the cube root of the temperature gradient, rather than to the temperature gradient, as in Fourier's law. A heat balance then leads to the nonlinear PDE $c_t = (c_z^{1/3})_z$ rather than the ordinary diffusion equation. This PDE is invariant to the family of affine groups (1) with $M = 2$, $N = -3$, and $L = -4$, as the reader can easily verify.

If we substitute (3) into the PDE, we find the following principal ODE for $y(x)$:

$$\beta \frac{d}{dx}\left(\frac{dy}{dx}\right)^{1/3} + x\frac{dy}{dx} - \alpha y = 0 \qquad (7)$$

which is invariant to the associated group $y' = \mu^{-2}y$, $x' = \mu x$, as expected. If we introduce the invariant $u = xy^{1/2}$ and the first differential invariant $v = x\dot{y}^{1/3}$ into (7), it becomes the first-order associated ODE

$$\frac{dv}{du} = \frac{2u(\beta v - v^3 + \alpha u^2)}{2\beta u^2 + \beta v^3} \qquad (8)$$

Different choices α_o, β_o of α and β correspond to different physical problems. For example, $\alpha_o = 0$, $\beta_o = 4/3$ gives similarity solutions of the form $c = y(z/t^{3/4})$. Such similarity solutions can be used to solve the PDE under the boundary and initial conditions (BIC) $c(0,t) = c_o$, $t > 0$; $c(z,0) = 0$, $z > 0$; $c(\infty,t) = 0$, $t > 0$. When written in terms of y, these BIC become $y(0) = c_o$, $y(\infty) = 0$. The physical interpretation of these BIC is a half-space initially at zero temperature whose front face is suddenly clamped at temperature c_o at time $t = 0$.

When $\alpha_o = 1$ and $\beta_o = 2$, $c = t^{1/2}y(z/t^{1/2})$, which can be used with the BIC $c_z(0,t) = -q^3$, $t > 0$; $c(z,0) = 0$, $z > 0$; $c(\infty,t) = 0$, $t > 0$ (or $\dot{y}(0) = -q^3$, $y(\infty) = 0$). The physical interpretation of these BIC is a half-space initially at zero temperature, the heat flux through the front face of which is suddenly clamped at the value q at time $t = 0$. When $\alpha_o = -1$ and $\beta_o = 2/3$, $c = t^{-3/2}y(z/t^{3/2})$, which can be used with the BIC $\int_{-\infty}^{\infty} c\,dz = Q$, $t > 0$; $c(z,0) = 0$, $z > 0$; $c(\pm\infty,t) = 0$, $t > 0$ (or $\int_{-\infty}^{\infty} y\,dx = Q$, $y(\infty) = 0$). The physical interpretation of these BIC is a full space initially at zero temperature subjected to a sudden heat pulse Q per unit area in the plane $z = 0$.

In the clamped temperature problem ($\alpha_o = 0$, $\beta_o = 4/3$) and in the pulsed source problem ($\alpha_o = -1$, $\beta_o = 2/3$), the principal ODE is directly integrable in terms of simple functions, and analytic solutions to these problems are known. More interesting from the point of view of this review is the clamped flux problem ($\alpha_o = 1$, $\beta_o = 2$). For it, no simple analytic solution to (7) is known, so we must solve (7) numerically, using the two-point boundary conditions $\dot{y}(0) = -q^3$ and $y(\infty) = 0$.

The simplest method of solving two-point boundary value problems is the so-called shooting method, in which we guess the missing boundary condition at one point, integrate across the interval to the second point, check to see how well the boundary condition at the second point is fulfilled, correct our guess of the missing boundary condition at the first point, and so proceed until sufficient accuracy is obtained. In the case at hand, we would guess $y(0)$ and together with $\dot{y}(0) = -q^3$ we would integrate to large x, testing to see if $y \to 0$ as x becomes very large. Numerical experience shows that if we guess $y(0)$ too large, $y \to \infty$ as $x \to \infty$, whereas if we guess $y(0)$ too small, $y \to -\infty$ as $x \to \infty$. This makes it difficult to decide if a given value of $y(0)$ is the correct value, for as we try to advance numerically along a given integral curve, roundoff and truncation errors will throw us off to one side or the other, and ultimately y will approach either $+\infty$ or $-\infty$. In spite of this, the forward shooting method can be made to work, but it is extremely time-consuming and laborious to achieve high accuracy with it.

Direction Field of the Associated ODE

We shall find a way out of this difficulty when we study the direction field of the associated ODE, Eq. (8). We shall need only the fourth quadrant of it because we expect $y > 0$ and $\dot{y} < 0$ for the solution we are looking for. Since $x > 0$, this means $u = xy^{1/2} > 0$ and $v = x\dot{y}^{1/3} < 0$. Figure 1 shows this direction field when $\alpha = 1$ and $\beta = 2$. Its general features can be understood by first considering the curves C_1 and C_2, on which the slope dv/du is 0 and ∞, respectively. (The curves C_1 and C_2 are described by the respective equations $u^2 = v^3 - 2v$ and $2u^2 + v^3 = 0$, obtained by equating to zero the numerator and the denominator on the right-hand side of Eq. (8).) The special significance of curves C_1 and C_2, as well as the curve $u = 0$, is that they divide the direction field into regions in each of which the slope dv/du has one sign only. The intersections of the curves C_1, C_2, and $u = 0$ are the singular points of the ODE (8), and they occur at the origin O and the point P: $(2/3^{3/4}, -2/\sqrt{3})$.

The solution of (7) that we seek must have finite y and $\dot{y}$ at $x = 0$, so that the curve in the (u,v)-plane that corresponds to it must pass through the origin of the (u,v)-plane. Of the family of curves that do so, some eventually intersect C_1, others eventually intersect C_2, and one, the separatrix S between these two sub-families, passes through the singular point P. It is the separatrix S that we want, for, as we shall see in a moment, as we move along S from 0 to P, $x \to \infty$. In the neighborhood of P, where x is large and changing rapidly, u is nearly constant at the value u_P. Thus $xy^{1/2} \sim u_P = 2/3^{3/4}$ or $y \sim u_P^2/x^2 = 4\sqrt{3}/9x^2$ for $x >> 1$. This behavior is sufficient to satisfy the boundary condition $y(\infty) = 0$, so the separatrix S provides us with a solution satisfying the BIC. As a by-product of this analysis we obtain the asymptotic behavior of the solution y(x) without extensive calculation.

(It remains to be seen that as we approach P along S from 0, $x \to \infty$. We can obtain the slope m of the separatrix at P by application of l'Hospital's rule: $m = -(3^{1/4}/2)(1 + \sqrt{17}/3) = -1.562422$. From the definition $u = xy^{1/2}$ we obtain by differentiation $dx/x = 2u\, du/(2u^2 + v^3)$. The denominator of this last expression vanishes at P; along S in the vicinity of P, $2u^2 + v^3$ can be written to first order in $u - u_P$ as $(4u_P + 3v_P^2 m)(u - u_P)$. So near P on S, $dx/x = -2\, du/(\sqrt{17} - 1)(u - u_P)$, which means $x \sim (u_P - u)^{-2/(\sqrt{17} - 1)}$ and so must become infinite as $u \to u_P$ from below.)

It is now easy to see why forward integration in x is unstable, for it corresponds to forward integration along the separatrix from 0 to P. Since the integral curves of (8) separate at the saddle singularity P, a small error (such as a roundoff or finite-difference truncation error) will throw us off the separatrix and we will eventually diverge to one side or the other. On the other hand, if we integrate numerically from P to 0, the integration will be stable because the integral curves converge. It is easy to get starting values for this integration by stepping slightly away from the singularity P using the slope m: $u = u_P - \varepsilon$, $v = v_P - m\varepsilon$. The rest of the separatrix can then be found by a single, stable, backward integration from P to 0.

The Separatrix and the Missing Boundary Condition

Near the origin of the (u,v)-plane $(u, |v| << 1)$, the integral curves shown in Fig. 1 behave linearly. A quick way to see this is to note that near the origin at a fixed value of u, $|v_{C_2}| >> |v_S| >> |v_{C_1}|$. Using the equations for C_1 and C_2 given in the previous section, we find that this is equivalent to $|v|^3 << u << |v|$ on S. Then, to leading order (8) becomes simply $dv/du = v/u$, for which the general solution is $u = -Av$. Now if we substitute into this last equation the definitions of u and v, we get $y^{1/2}(0) = -A\dot{y}^{1/3}(0) = Aq$ (since the origin in the (u,v)-plane corresponds to $x = 0$). This is the missing boundary condition at the origin $x = 0$!

Numerical integration of (9) from P to 0 gives $A = 0.912582$. With this value of A it is possible to integrate (7) forward. Even though the forward direction of integration is unstable, it turns out that the integration behaves sufficiently well that we can reach values of x for which y is very close to its asymptotic limit $4\sqrt{3}/9x^2$. Figure 2 shows y(x) determined in this way for the case in which $q = 1$.

To find y(x) for other values of q, it is not necessary to repeat the numerical integrations. Instead, we merely transform the solution of Fig. 2 with the transformations of the associated group. The image $y'(x')$, where $y' = \mu^{-2}y$, $x' = \mu x$, is also a solution of the principal ODE (8) and furthermore satisfies the BIC $\dot{y}'(0) = \mu^{-3}\dot{y}(0) = -\mu^{-3}$ and $y'(\infty) = \mu^{-2}y(\infty) = 0$. So if we choose $\mu = q^{-1}$, the image $y'(x')$ of the solution y(x) shown in Fig. 2 corresponds to the value of q of the flux. Thus all the solutions are transforms of one another.

This last fact can be used to formulate a method of integrating (7) backward from large x to $x = 0$, i.e., in the stable direction. We proceed by (i) choosing a point (u,v) on S close to P, (ii) guessing a value of x, say x_1, (iii) calculating $y(x_1)$ and $\dot{y}(x_1)$ from the chosen values of u and v, and (iv) using them as starting values for a backward integration from x_1 to 0. This procedure works for the following reason. Any image point of x_1, y_1, $\dot{y}_1$, say $x' = \mu x_1$, $y' = \mu^{-2}y_1$, $\dot{y}' = \mu^{-3}\dot{y}$, has the same values of u and v as the point x_1, y_1, $\dot{y}_1$

itself, because u and v are invariants of the associated group. Thus any value of x can be made to correspond to any u and v on the separatrix. In general, the backward integration will not give the curve for which $\dot{y}(0)$ has some specified value. But once the curve $y(x)$ has been calculated, it can be scaled with the associated group to a curve with any desired $\dot{y}(0)$.

Scaling With the Associated Group

Since all the curves $y(x)$ corresponding to different values of q are images of one another under the associated group (4), all have the same value $A = 0.912582$ of $-y^{1/2}(0)/\dot{y}^{1/3}(0)$ because this quantity is invariant to the transformations of (4). (Note that $y'(0) = \mu^{-2}y(0)$ and $\dot{y}'(0) = \mu^{-3}\dot{y}(0)$ because the point $x = 0$ transforms into the point $x' = 0$.) Thus, $y(0) = A^2\dot{y}^{2/3}(0) = A^2q^2$, from which it follows that $c(0,t) = A^2q^2t^{1/2}$. This formula gives the dependence of the temperature on the front face in the clamped flux problem on the time and on the clamped flux, which are the only two parameters in the problem on which it can depend. To obtain this formula we only need to know of the existence of the associated group. With that information alone we can obtain a formula for $c(0,t)$ correct up to a single undetermined constant. To find the value of the constant we must perform further calculations, in this case the numerical integration of the associated ODE.

A similar result can be obtained in the clamped temperature case. Suppose we ask what is the the dependence of the flux q through the front face on the time t and the clamped temperature c_o. Now $c = y(z/t^{3/4})$, $y(0) = c(0,t) = c_o$, and $q = -c_z^{1/3}(0,t) = -t^{-1/4}\dot{y}^{1/3}(0)$. Again all the curves $y(x)$ corresponding to different values of c_o are images of one another, and again the ratio $-y^{1/2}(0)/\dot{y}^{1/3}(0) = B$ is a group invariant. (It is important to realize that its value B is different in the clamped temperature problem from its value A in the clamped flux problem because α_o and β_o are different in the two problems and therefore so are the forms of the ODEs (7) and (8).) Thus $q = t^{-1/4}c_o^{1/2}/B$.

Concluding Remarks

The method outlined above does not depend on the PDE being linear. On the other hand, it does depend on the PDE being invariant to a one-parameter family of one-parameter affine groups. This is a high degree of algebraic symmetry that is found only in the simplest equations. However, such equations arise in a great variety of technological problems, as Table 1 attempts to show. So the method presented here should be of widespread use; indeed, in my book I expressed the hope that it would become a practical workhorse for dealing with nonlinear partial differential equations.

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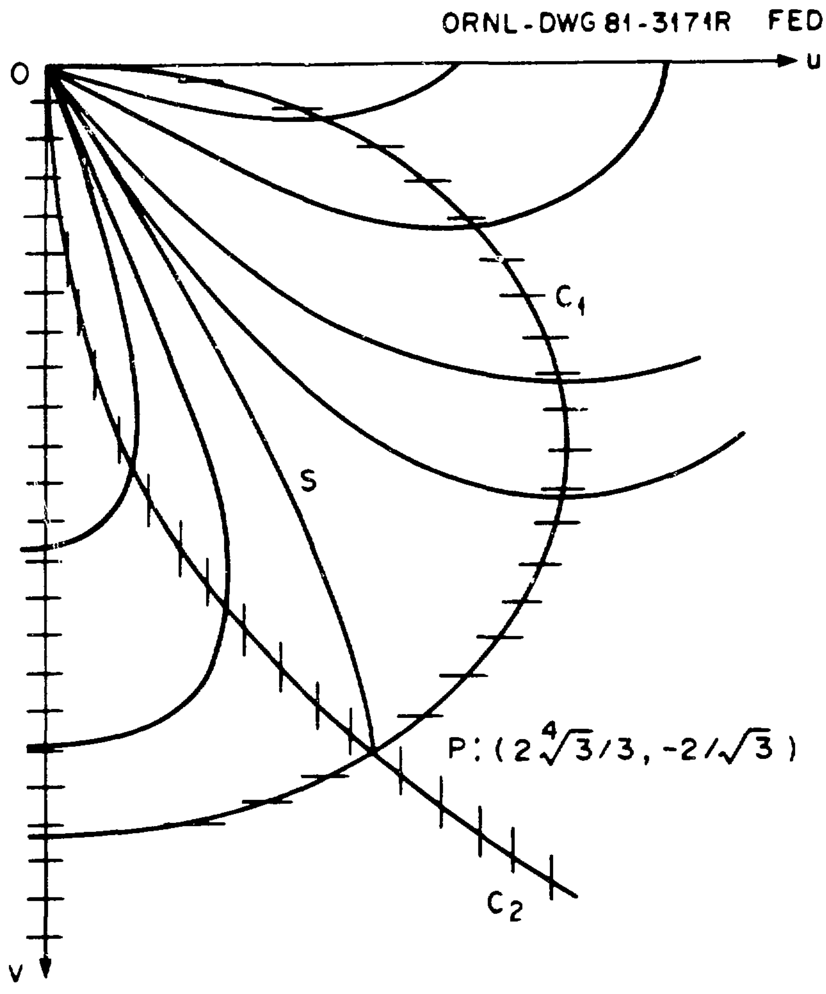

Fig. 1 A sketch of the direction field of (8) when $\alpha = 1$ and $\beta = 2$.

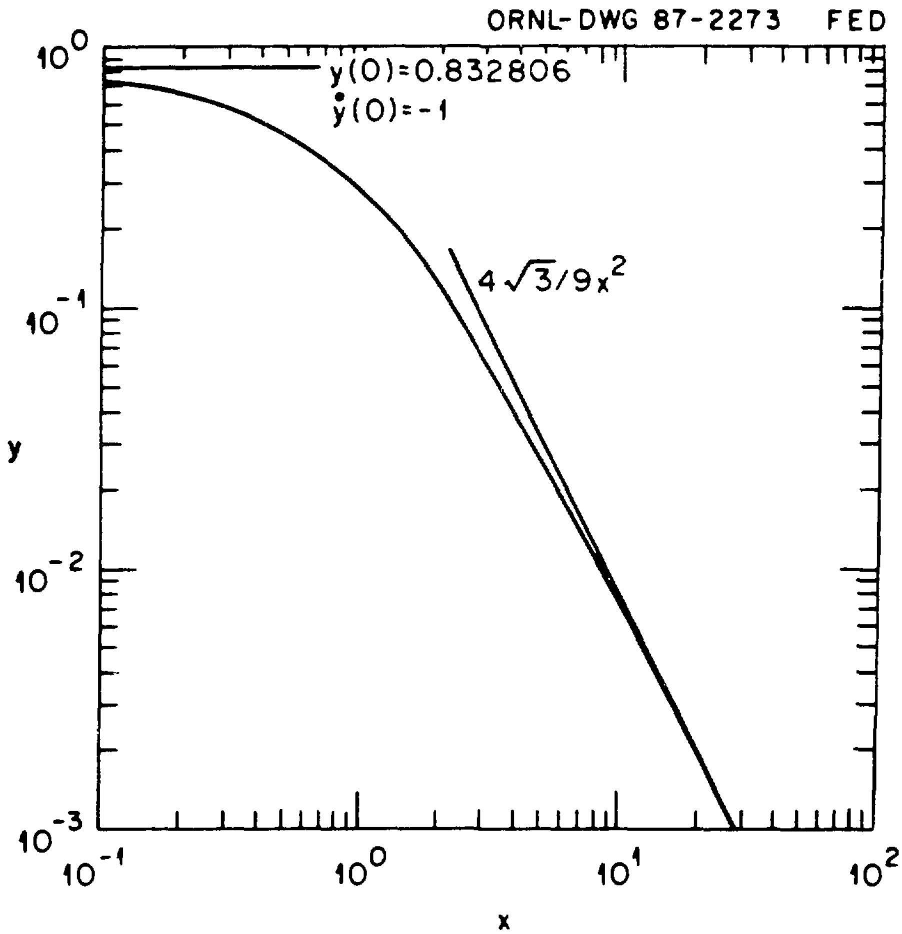

Fig. 2 The solution y(x) of (7) for which $\dot{y}(0) = -1$ and $y(\infty) = 0$, when $\alpha = 1$ and $\beta = 2$.