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Fixed Points by Ishikawa Iterations

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FIXED POINTS BY ISHIKAWA ITERATIONS

JEN-CHIH YAO

ABSTRACT. In this paper, we introduce a class of mappings called generalized quasi-nonexpansive mappings in a Hilbert space. It is shown that a certain Ishikawa iterative process generated by a continuous generalized quasi-nonexpansive and monotone mapping on a compact and convex subset of a Hilbert space always converges strongly to a fixed point of the mapping without any precondition.

1. INTRODUCTION

In [1], Ishikawa has shown that a certain mean value sequence generated by a Lipschitzian and pseudo-contractive mapping on a compact and convex subset of a Hilbert space with arbitrary chosen initial point converges strongly to a fixed point of the mapping. In this paper, we introduce a class of mappings called generalized quasi-nonexpansive mappings on a Hilbert space and show that a certain Ishikawa iterative process generated by a continuous generalized quasi-nonexpansive and monotone mapping on a compact and convex subset of a Hilbert space always converges strongly to a fixed point of the mapping without any precondition.

2. PRELIMINARIES

Let K be a nonempty subset of a Hilbert space X . A mapping T from K into itself is said to be *generalized quasi-nonexpansive* on K if T has a fixed point in K and for any fixed point p of T in K , we have

$$\|T(x) - p\|^2 \leq \alpha\|x - p\|^2 + \beta\|T(x) - x\|^2$$

for all $x \in K$, where $\alpha, \beta \geq 0$ with $\alpha + 2\beta \leq 1$. We note that there exists a generalized quasi-nonexpansive mapping which is not pseudo-contractive. For example, let T be a mapping from $[0, 2/3]$ into itself defined by $T(x) = x^2$ for all $x \in [0, 2/3]$. Then $x = 0$ is the only fixed point of T and it is easy to see that T is generalized quasi-nonexpansive on $[0, 2/3]$. But T is not pseudo-contractive because if we let $x = 2/3$ and $y = 1/2$, then

$$\|T(x) - T(y)\|^2 > \|x - y\|^2 + \|(I - T)(x) - (I - T)(y)\|^2.$$

Given $x_1 \in K$ and sequences of real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ with $0 \leq \alpha_n, \beta_n \leq 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) = \infty$, let $I(x_1, \alpha_n, \beta_n, T)$ be a sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[\beta_n T(x_n) + (1 - \beta_n)x_n].$$

Ishikawa [1] introduced this iteration scheme by imposing the following different restrictions on α_n and β_n : $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n , $\lim_{n \rightarrow \infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

3. THE MAIN RESULT

We now state and prove the main result of this paper.

THEOREM 1. *Let K be a nonempty compact and convex subset of a Hilbert space X and T be a continuous generalized quasi-nonexpansive and monotone mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $I(x_1, \alpha_n, \beta_n, T)$ converges strongly to a fixed point of T .*

PROOF. Since T is monotone, $\langle T(x) - T(y), x - y \rangle \geq 0$ for all $x, y \in K$. Also since T is generalized quasi-nonexpansive, for any fixed point $p \in K$, we have for any $x \in K$,

$$\begin{aligned} \|T(x) - p\|^2 &= \|T(x) - T(p)\|^2 \\ &\leq \alpha \|x - p\|^2 + \beta \|T(x) - x\|^2 \\ &= \alpha \|x - p\|^2 + \beta (\|T(x) - p\|^2 + \|x - p\|^2 - 2\langle T(x) - p, x - p \rangle) \\ &\leq (\alpha + \beta) \|x - p\|^2 + \beta \|T(x) - p\|^2. \end{aligned}$$

Hence for all $x \in K$,

$$\|T(x) - p\| \leq [(\alpha + \beta)/(1 - \beta)]^{1/2} \|x - p\| \leq \|x - p\|. \quad (1)$$

Since X is a Hilbert space, it can be shown that for all $x, y, z \in X$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2)$$

If $x_1 = p$, then we are done. So we may assume that $x_1 \neq p$. Then by (1) and (2), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|T[\beta_n T(x_n) + (1 - \beta_n)x_n] - p\|^2 - \\ &\quad \alpha_n(1 - \alpha_n)\|T[\beta_n T(x_n) + (1 - \beta_n)x_n] - x_n\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\beta_n T(x_n) + (1 - \beta_n)x_n - p\|^2 \\ &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\beta_n\|T(x_n) - p\|^2 + \alpha_n(1 - \beta_n)\|x_n - p\|^2 - \\ &\quad \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 \\ &\leq (1 - \alpha_n\beta_n)\|x_n - p\|^2 + \alpha_n\beta_n\|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2. \end{aligned}$$

Hence for every positive integer m , $\|x_{m+1} - p\| \leq \|x_m - p\|$ and

$$\|x_{m+1} - p\|^2 \leq \|x_1 - p\|^2 - \sum_{n=1}^m \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2$$

from which it follows that $\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 < \infty$. Therefore, from the hypothesis $\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n) = \infty$, we have

$$\lim_{n \rightarrow \infty} \|T(x_n) - x_n\| = 0. \quad (3)$$

Since K is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in K$. From (3) it follows that q is a fixed point of T . Now, for any $\epsilon > 0$, there exists an integer n_k so that $\|x_{n_k} - q\| \leq \epsilon$. Thus for all $n \geq n_k$, we have $\|x_n - q\| \leq \|x_{n_k} - q\| \leq \epsilon$. Consequently, the entire sequence $\{x_n\}$ also converges to q , and the result follows. \square

In the case that the Hilbert space X in Theorem 1 is finite-dimensional, the compactness of the set K can be eliminated.

THEOREM 2. *Let K be a nonempty closed and convex subset of a finite-dimensional Hilbert space X and T be a continuous generalized quasi-nonexpansive and monotone mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $I(x_1, \alpha_n, \beta_n, T)$ converges to a fixed point of T .*

PROOF. Let p be any fixed point of T and let B be the closed ball of X with center p and radius $\|x_1 - p\|$. Then $x_n \in B$ for all n . Since B is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in B$. By the same argument as that in the proof of Theorem 1, it can be shown that q is a fixed point of T , and hence the result follows. \square

We note that the result of Theorem 2 may not be true if the Hilbert space X is infinite-dimensional. This is because that in this case, the set B in the proof of Theorem 2 is no longer compact.

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1. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.

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