CONF-800736--1

MASTER

# SOME PROPERTIES OF THE LOG-LAPLACE DISTRIBUTION*

V. R. R. Uppuluri
Mathematics and Statistics Research Department
Computer Sciences Division
Union Carbide Corporation, Nuclear Division
Oak Ridge, Tennessee 37830





*Research sponsored by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Department of Energy, under contract W-7405-eng-26 with the Union Carbide Corporation.

# SOME PROPERTIES OF THE LOG-LAPLACE DISTRIBUTION

V. R. R. Uppuluri

ABSTRACT

A random variable Y is said to have the Laplace distribution or the double exponential distribution whenever its probability density function is given by $\lambda \exp(-\lambda|y|)$, where $-\infty < y < \infty$ and $\lambda > 0$. The random variable $X = \exp(Y)$ is said to have the log-Laplace distribution. In this paper, motivated by the problem of extrapolation to low doses in dose response curves, we obtain an axiomatic characterization of the log-Laplace distribution.

KEYWORDS: Laplace distribution; distribution of the sum of Laplace variates; log-Laplace distribution; an axiomatic characterization; extrapolation to low doses

## 1. INTRODUCTION

In statistical applications the normal distribution and its ramifications play a central role. At times, when the observed variable is nonnegative, it is assumed that the logarithm of the variable has a normal distribution and the theory of lognormal distributions (see Aitchison and Brown, 1969) is applied. The normal theory seems to be more appropriate to phenomena where the first order behavior is well understood (and perhaps controlled), and the second order behavior needs to be understood. For instance, the electrical engineers seem to utilize this theory very aptly.

In problems of epidemiologic nature, or some problems in ecology or biology, it seems to be appropriate to treat them as first order phenomena and use the tools related to the exponential distribution. Though the one-sided exponential distribution has been used a lot, the double exponential distribution, also known as Laplace distribution, is simple. The log-Laplace distribution, which will be studied in this paper, seems to be quite an appropriate model in the study of first order phenomena such as the behavior of dose response curves at low doses.

## 2. LOG-LAPLACE DISTRIBUTION

In statistical literature, the double exponential distribution is referred to as the First Law of Laplace, just as the normal distribution is referred to as the Second Law of Laplace (see Johnson, 1955, p. 283). Johnson (1955) touched on the moments of the log-Laplace distribution while considering the problems of interest to him. In this section we will introduce the log-Laplace distribution in parallel to the lognormal distribution. In the next section we will give an axiomatic derivation of this distribution.

A lognormal distribution may be defined starting with a normal distribution. Let V be a normal variable with probability density function given by

$$(1/\sigma\sqrt{2\pi})\ \exp(-(\nu-\mu)^2/2\sigma^2),\ -\infty < \nu < \infty,\ -\infty < \mu < \infty,\ \sigma > 0. \qquad (1)$$

Let $U = \exp(V)$. Then $0 < U$ is said to be a lognormal variable whose probability density function is given by

$$(1/u\sigma\sqrt{2\pi})\ \exp(-(\ell n\ u-\mu)^2/2\sigma^2)\ . \qquad (2)$$

We shall define a log-Laplace distribution in an anologous way. A random variable Y is said to have a double exponential or a Laplace distribution if its probability density function is given by

$$(\lambda/2)\exp(-\lambda|y|), \quad -\infty < y < \infty, \ \lambda > 0. \tag{3}$$

Let X = exp(Y). Then 0 < X is said to have a log-Laplace distribution whose probability density function is given by

$$f_\lambda(x) = \begin{cases} (\lambda/2)\, x^{\lambda-1}, & \text{for } 0 \leqslant x \leqslant 1 \\ (\lambda/2x^\lambda), & \text{for } 1 \leqslant x . \end{cases} \tag{4}$$

The cumulative distribution function $F_\lambda(x)$ of X is given by

$$F_\lambda(x) = \begin{cases} (1/2)\, x^\lambda, & \text{for } 0 \leqslant x \leqslant 1 \\ 1 - (1/2x^\lambda), & \text{for } 1 \leqslant x \end{cases} \tag{5}$$

It may be noted that the reciprocal of a log-Laplace random variable also has the same distribution. This can be seen from the probability statements:

$$\begin{aligned} &\text{Prob.}[Z \equiv (1/X) \leqslant z] \\ &= \text{Prob.}\ [X \geqslant (1/z)] \\ &= 1 - \text{Prob.}\ [X \leqslant (1/z)] . \end{aligned} \tag{6}$$

The likelihood ratio criterion of a simple hypothesis versus a simple alternative about the parameter λ depends on the product of independent identically distributed log-Laplace random variables. The distribution of this product can be deduced from the distribution of the sum of independent identically distributed Laplace random variables. This result is stated in the following:

Proposition 2.1: The probability density function of the sum Y of n independent identically distributed Laplace ($\lambda$) variates is given by

$$\sum_{k=0}^{n-1} \binom{n+k-1}{k} \frac{1}{2^{n+k}} \frac{\lambda^{n-k}}{(n-k-1)!} e^{-\lambda|y|} |y|^{n-k-1} \quad . \tag{7}$$

Proof: We shall give an outline of the proof. The characteristic function of Y is equal to

$$\phi_n(t) = \{ 1/[1+(t/\lambda)^2]\}^n \quad . \tag{8}$$

This can be expressed as

$$\phi_n(t) = \sum_{k=0}^{n-1} \binom{n+k-1}{k} \frac{1}{2^{k-1+n}} \left[ \left(\frac{i}{i+t/\lambda}\right)^{n-k} + \left(\frac{i}{i-t/\lambda}\right)^{n-k} \right] \tag{9}$$

Next, we use the relations between the characteristic function and the probability density function given by

$$\psi(t) = \int_{-\infty}^{\infty} f(x) \exp(itx)\, dx$$

$$\tag{10}$$

$$\Leftrightarrow \quad f(x) = (1/2\pi) \int_{-\infty}^{\infty} \psi(t) \exp(-itx)\, dt \ .$$

We also have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \left(\frac{i}{i+t/\lambda}\right)^{n-k} dt = \frac{\lambda^{n-k} y^{n-k-1} e^{-\lambda y}}{\Gamma(n-k)} \tag{11}$$

and the proposition follows.

Remarks: (i) This proposition shows that the probability density function of the sum of n independent Laplace variates is equal to the weighted sum of double gamma probability density functions.

(ii) Special cases of this result for $n = 2,3$, and 4 were posed in a problem by Feller (1966, p. 64).

(iii) In the special case $\lambda = 1$, Feller (1966, p. 559) also shows that $\sum_{k=1}^{\infty} (Y_k/k)$ converges to a random variable, Z, with characteristic function

$$E[\exp(itZ)] = \pi t/\sinh(\pi t) \quad . \tag{12}$$

The associated probability density function of Z is given by

$$1/[2 + \exp(z) + \exp(-z)] = 1/4[\cosh(z/2)]^2 \quad . \tag{13}$$

## 3. A CHARACTERIZATION OF THE LOG-LAPLACE DISTRIBUTION

One of the problems of current interest (see Brown, 1976, and Lewis, 1980) is the problem of linearity versus nonlinearity of dose response for radiation carcinogenesis.

Since animal experiments can only be performed at reasonable doses, the problem of extrapolation to low doses becomes an awkward problem unless there are acceptable mathematical models. Several authors believe that the problem of linearity versus quadratic hypothesis cannot be resolved in the present day context (see Lewis, 1980) and Alvin M. Weinberg refers to this as a "trans-scientific problem." In the past, this problem was considered in literature using the lognormal and special cases of the Weibull distribution to get an insight into the behavior at low doses.

We will now assume a set of properties about the dose-response curve and derive a mathematical function that possesses these properties.

(1) At small doses, the percent increase in the cumulative proportion of deaths is proportional to the percent increase in the dose,

(2) at larger doses, the percent increase in the cumulative proportion of survivors is proportional to the percent decrease in the dose and

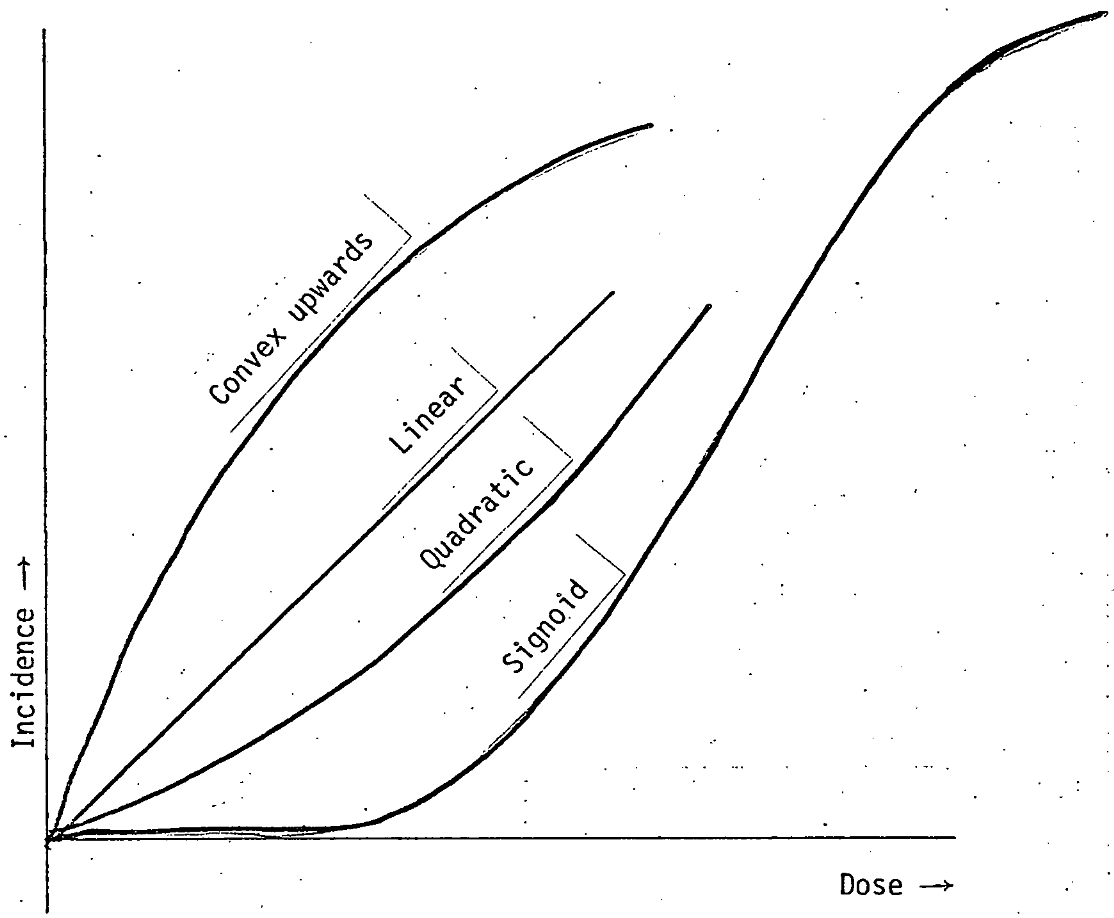


Figure 1. Four simple dose-response curves for radiation carcinogenesis.

(3) at zero dose, no deaths, and when the dose is infinite, no survivors, and the cumulative proportion of deaths F(x) is a monotonic, nondecreasing function of the dose x.

We shall now establish the following:

Proposition 3.1. Under (1), (2), and (3) we have

$$F(x) = \begin{cases} F(1)\, x^{\mu} , & 0 \leqslant x \leqslant 1 \\ 1 - \dfrac{[1-F(1)]}{x^{\lambda}} , & 1 \leqslant x . \end{cases} \qquad (14)$$

Proof: From assumption (1),

$$\frac{F(x + \Delta x) - F(x)}{F(x)} = \mu \frac{(x + \Delta x) - x}{x} \qquad (15)$$

or

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \mu \frac{F(x)}{x} \qquad (16)$$

Taking the limit as $\Delta x \to 0$, and dividing by F(x) and integrating we obtain

$$F(x) = F(1)\, x^{\mu} . \qquad (17)$$

If we let $G(x) = 1 - F(x)$, from assumption (2) we have

$$\frac{G(x + \Delta x) - G(x)}{G(x)} = - \lambda \frac{(x + \Delta x) - x}{x} \qquad (18)$$

or $\qquad \ell n\, G(x) = - \lambda\, \ell$

and $\qquad \text{Const} = \ell n\, G(1) . \qquad (19)$

$$\therefore \quad G(x) = G(1)/x^{\lambda} \qquad (20)$$

and the proposition is proved.

Remarks: (i) For the special case $\lambda = \mu$ and $F(1) = 1/2$, we have

$$F(x) = \begin{cases} (1/2)\, x^{\lambda} , & 0 \leqslant x \leqslant 1 \\ 1 - (1/2x^{\lambda}) , & 1 \leqslant x \end{cases}$$

which was referred to as the log-Laplace distribution in the previous section.

(ii) The cumulative distribution function obtained in the above proposition may be considered as a more general form of the log-Laplace distribution.

(iii) For $\mu = 1$, we have a linear behavior of $F(x)$ at the origin and for $\mu = 2$, we have a quadratic behavior at the origin. Thus if we have adequate data, one can perform the test of a simple hypothesis versus a simple alternative.

(iv) Furthermore, $x = 1$ corresponds to the cusp in the probability density of the log-Laplace distribution or the point of discontinuity of the cumulative distribution function. By proper normalization, one may make this correspond to the threshold dose and if need be, can easily be incorporated into the model.

ACKNOWLEDGMENT

The author would like to express his thanks to Ms. Glennis Abrams and Mr. S. R. Gaddie for their contributions to section 2 of this paper.

# REFERENCES


Aitchison, J. and Brown, J. A. C. (1969). The Lognormal Distribution. Cambridge University Press, New York.

Brown, J. M. (1976). Linearity versus non-linearity of dose response for radiation carcinogenesis. Health Physics, 31, 231-245.

Feller, W. (1966). An Introduction to Probability Theory and its Applications. John Wiley & Sons, New York.

Johnson, N. L. (1954). Systems of frequency curves derived from the first law of Laplace. Trabajos de Estadistica, 5, 283-291.

Johnson, N. L. and Kotz, S. (1970). Continuous Univariate Distributions-2. Houghton Miflin Company, Boston.

Lewis, H. W. (1980). The safety of fission reactors. Scientific American, 242, 53-65.