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ABSTRACT

A code has been written to use the algebraic computer system MACSYMA to generate systematically the infinitesimal similarity groups corresponding to systems of quasi-linear partial differential equations. The infinitesimal similarity groups can be used to find exact solutions of the partial differential equations. In an example from fluid mechanics the similarity method using the computer code reproduces immediately a solution obtained from dimensional analysis.

1. Introduction

In recent years Lie infinitesimal contact transformation group theory has become a widely used tool for systematic explorations of similarity solutions of partial differential equations [1]. Though the implementation of this method involves mostly standard manipulations (algebra and chain derivatives), in practice these become extremely involved for all but the simplest of equations.

Here we introduce the use of the algebraic computing system MACSYMA to facilitate these calculations. Specifically, MACSYMA is used to calculate systematically the generators of the infinitesimal group under which the considered equations are invariant. As far as the operational side of the similarity method is concerned, this is the most involved and tedious step. Once the general similarity group is found, a (hopefully nontrivial) subgroup is found which leaves invariant boundary curves and boundary conditions. Finally, once this subgroup is found, its invariants and consequently the similarity forms of the solutions of the partial differential equations can be found.

A primary advantage of the similarity method is that it is one of the few general techniques for obtaining exact (nonlinear) solutions of partial differential equations. A primary disadvantage of the similarity method is that the solution found may satisfy only a very restricted set of initial and boundary conditions.

In general, the existence of a one parameter group of transformations leads to a reduction by one in the number of independent variables. Thus, in the most encountered case of two independent variables, the reduction leaves us with an ordinary differential equation.

We remark here that the similarity method contains as a special case those solutions which can be obtained via dimensional analysis. That is, if an equation has a solution which can be obtained from dimensional analysis, then the present method will reproduce immediately that special similarity solution, and possibly more general similarity solutions as well.

2. Sketch of the Similarity Method

In this section we present a brief summary of the similarity method. The similarity transformation solver (STS) uses MACSYMA to construct the similarity transformations which are admitted by a set of partial differential equations of the form (sums over j,k, and l)

$$a_{ij} \frac{\partial^2}{\partial t^2} u_j + b_{ij} \frac{\partial}{\partial t} u_j + c_{ijk} \frac{\partial}{\partial x_k} u_j + d_{ijkl} \frac{\partial^2}{\partial x_l \partial x_k} u_j = e_i \ , \tag{1}$$

where $u_j = u_j(\vec{x},t)$ are solutions, and a,b,c,d, and e are arbitrary, specified functions of $(\vec{x},t,\vec{u})$. The one parameter infinitesimal transformations considered are

$$\vec{x}' = \vec{x} + \epsilon\vec{X}(\vec{x},t,\vec{u}) \tag{2a}$$

$$t' = t + \epsilon T(\vec{x},t,\vec{u}) \tag{2b}$$

$$\vec{u}' = \vec{u} + \epsilon\vec{U}(\vec{x},t,\vec{u}) \quad . \tag{2c}$$

MACSYMA is used to determine the differential equations, called the determining equations, that are satisfied by the infinitesimal groups $\vec{X},T,\vec{U}$. Technically, this is achieved by requiring that the partial differential equations (1) be invariant to order ϵ^2 in terms of the new variables $(\vec{x}',t',\vec{u}')$ of Eqs. (2). Thus the annuling of $0(\epsilon)$ terms becomes an invariance condition.

Let $\vec{M}[\vec{u}] = 0$ be a system of partial differential equations in the old variables, and let $\vec{M}'[\vec{u}'] = 0$ be the same system in the new variables, where $\vec{M}'[\vec{u}']$ is obtained from $\vec{M}[\vec{u}]$ by replacing everywhere $(\vec{x},t,\vec{u}) \rightarrow (\vec{x}',t',\vec{u}')$. The application of the transformations (2) to $\vec{M}'[\vec{u}']$ shows that $\vec{M}'[\vec{u}']$ is of the form $M'[u'] = M[u] + \epsilon H + 0(\epsilon^2)$, where $\vec{H} = \vec{H}(\vec{x},t,\vec{u},\vec{X},T,\vec{U})$. We make the system invariant to the infinitesimal transformations (2) by requiring that $\vec{H} \equiv 0$. Suppose we denote by $\{S_m\}$, for an appropriate set of values m, all possible first or second derivatives of u_j with respect to x_k and/or t . Then the expression $\vec{H}$ is a sum of terms involving the S_m's, with coefficients functions of $(\vec{X},T,\vec{U})$ and their derivatives with respect to (x_k,t,u_j) . After the original system of partial differential equations is used to guarantee that all the S_m's that appear in $\vec{H}$ are independent, $\vec{H}$ is made to vanish by requiring that all the coefficients of the S_m's vanish. This, as explained in [1], is a sufficient but not a necessary condition. The coefficients of the S_m's are differential equations satisfied by the infinitesimal generators $(\vec{X},T,\vec{U})$; these are the determining equations mentioned earlier. In practice the determining equations usually can be solved easily, as they are an overdetermined set of linear equations. A knowledge of the infinitesimal transformations enables one to construct a continuous transformation or in turn to find the invariants of the group that serve as the new dependent and independent variable. It is in these new variables that Eqs. (1) exhibit their symmetry and result in a reduction by one in the number of independent variables.

To illustrate the desirability of using MACSYMA to calculate the determining equations, suppose a partial differential equation contains a derivative $\frac{\partial u}{\partial x}$ (assume scalar u and x). In terms of the new variables, the transformed partial differential equation contains a term $\frac{\partial u'}{\partial x'}$. But from Eq. (2c),

$$\frac{\partial u'}{\partial x'} = \frac{\partial}{\partial x'} [u(x,t) + \varepsilon U(x,t,u)]$$

$$= \frac{\partial u}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x'} + \varepsilon \left[\frac{\partial U}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial U}{\partial t}\frac{\partial t}{\partial x'} \right.$$

$$\left. + \frac{\partial U}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x'}\right) \right] . \qquad (3)$$

Now for $\frac{\partial x}{\partial x'}$ and $\frac{\partial t}{\partial x'}$, use Eqs. (2a-b) to obtain

$$\frac{\partial x}{\partial x'} = 1 - \varepsilon \left[\frac{\partial X}{\partial x} + \frac{\partial X}{\partial u}\frac{\partial u}{\partial x}\right] + 0(\varepsilon^2) \quad . \tag{4}$$

and

$$\frac{\partial t}{\partial x'} = - \varepsilon \left[\frac{\partial T}{\partial x} + \frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\right] + 0(\varepsilon^2) \quad . \tag{5}$$

By substituting Eqs. (4) and (5) into Eq. (3) and simplifying, we obtain

$$\frac{\partial u'}{\partial x'} = \frac{\partial u}{\partial x} + \varepsilon \left[\frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial u} - \frac{\partial X}{\partial x}\right)\frac{\partial u}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial X}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial T}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right] + 0(\varepsilon^2) \quad . \tag{6}$$

Correspondingly, any second derivatives such as $\frac{\partial^2 u'}{\partial x'^2}$ become very involved. The system of partial differential equations (1) is written in a completely vectorized form. Since MACSYMA knows the chain rule and can simplify and factor, it is quite feasible for MACSYMA to reliably transform complicated systems of partial differential equations in several independent variables.

3. Programming Considerations

In this section we describe the implemention of the STS on the MACSYMA system. The basic mode of operation is that a reference version of the STS is modified as required with the TECO editor to solve the problem at hand, and then the modified version is executed while in MACSYM using the BATCH command. Once program execution is initiated with the BATCH command, no further user interaction is required.

Input procedures for the STS are described as follows. Once a user obtains a reference copy of the STS program in his user space, a listing of the program should be made while in TECO. In the program listing the portions of the code which must be changed from one problem to another are preceded by a single asterisk and a blank line and are followed by a blank line and a double asterisk. There are three places in the STS program where the user must modify the text to solve a new problem. First, a title should be specified which identifies the system of partial differential equations being considered. Second, the user must specify the number of equations (i.e., the number of dependent variables u_j), NEQ, the spatial dimension of the problem, NX (NX $\geq$ 1), and whether or not there are time derivatives (NT = 0, no time derivatives; NT = 1, there are time derivatives). Finally, the user must specify the nonzero coefficients a_{ij}, b_{ij}, c_{ijk}, d_{ijkl}, and e_i of Eq. (1). Before assigning the subscript i to the coefficients, the equations should be ranked in ascending order according to the number of terms in the equations. For example, the equation with the fewest terms goes first, etc. After these changes are made in the reference STS program, the updated version is stored with TECO as a new file, to be excuted later with BATCH. In this way a user might build a library of STS programs appropriate to different systems of equations.

The output of the STS is in two stages. In the first stage the BATCH command causes listing and execution of the code. During this stage some of the intermediate calculations are printed out, so that the progress of the calculation can be inspected as it proceeds. This portion of the computing time takes about 10 minutes, nearly independent of the system of equations considered. The first intermediate printout is a listing of the functional dependencies of the functions involved in the infinitesimal transformations (2). The STS program is vectorized, so the variables $(x_k, t, u_j, X_k, T, U_j)$ are replaced with the program variables (X[K],T,U[J],FX[K],FT, FU[J]), respectively. The second intermediate printout is a listing of the system of equations to be solved. This serves as a check that the equations were ordered properly and that the coefficients were entered correctly. Next the linearized-in-epsilon versions of the new coefficients of the transformed partial differential equations are displayed. Recall that the original system of equations must be used to guarantee that the S_m's that appear in $\vec{H}$ are independent. The fourth intermediate printout lists NEQ of the S_m's which the STS has chosen to eliminate from $\vec{H}$ by using the original system of equations. The solutions of the original equations for the NEQ S_m's is then listed in the last intermediate printout.

The second stage of output in the STS is the displaying of the components of $\vec{H}$, from which the determining equations are derived. Each component of $\vec{H}$ is factored conveniently with respect to all the S_m's; the determining equations can be read off immediately as the coefficients of the independent terms involving the S_m's. The amount of time spent computing $\vec{H}$ depends significantly on the system of equations solved, as well as on the current load on the MACSYMA system. A very rough estimate of the computing time to calculate $\vec{H}$ is

$$T\ (\text{min}) = \text{NEQ}\ (\text{NX}+1)\ (\text{NX}/2 + 2). \tag{7}$$

4. An Example

As an example of the use of the STS, consider the classical Blasius problem from fluid mechanics; solve the problem of two-dimensional, steady state, incompressible, viscous, laminar flow in a boundary layer. The equations to be solved are

$$\frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y = 0 \tag{8a}$$

$$v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x = \nu \frac{\partial^2 v_x}{\partial y^2} , \tag{8b}$$

where $v = (v_x, v_y, 0)$ is the fluid velocity, and ν is the coefficient of viscosity. A semi-infinite plate exists in the $x - z$ plane for $x \geq 0$, and far from the plate, $y \to \pm\infty$, v_x

assumes a constant value, u_o . The viscous boundary conditions are:

$$v_x(y = \pm\infty) = u_o \tag{9a}$$

$$v_x(y = 0) = v_y\ (y = 0) = 0\ . \tag{9b}$$

Using arguments of dimensional analysis, one concludes that (Landau and Lifshitz [2]) the new independent similarity variable is

$$\zeta = y\ \sqrt{u_o/x} \tag{10a}$$

and that v_x and v_y are of the forms

$$v_x = u_o\ f_1(\zeta) \tag{10b}$$

$$v_y = \left(\zeta/y\right)\ f_2\ (\zeta)\ . \tag{10c}$$

Let us now obtain these same results as well as the general infinitesimal similarity group by means of the STS. In a program listing of the STS we first insert in the appropriate place a title for problem being solved. Next we insert on separate lines in the STS:

```
NEQ:2$
NX:2$          (11)
NT:0$.
```

Finally we insert the nonzero coefficients:

```
C[1,1,1]:1$

C[1,2,2]:1$

C[2,1,1]:U[1]$                                          (12)

C[2,1,2]:U[2]$

D[2,1,2,2]: - NU$ ,
```

where NU is the coefficient of viscosity, ν. After this version of the STS is stored in TECO and executed in MACSYM with BATCH, the printouts begin.

The components of $\vec{H}$ are printed out, and the determining equations can be read-off by inspection (see the Appendix). They are (letting U[1] → u, U[2] → v, FU[1] → U, and FU[2] → V)

$$\begin{aligned}
&X_v = X_u = X_y = X_{vv} = Y_{vv} = Y_v \\
&\quad = U_{vv} = u_v = Y_u = 0 \\
&V_u - Y_x = V_y + U_x = 0 \\
&Y_y - V_v - X_x + U_u = U - uX_x + 2uY_y = 0 \\
&\nu\ (-U_{uu} + 2Y_{uy}) + 2vY_u = -\ \nu U_{yy} + vU_y + uU_x = 0 \\
&\nu\ (Y_{yy} - 2U_{uy}) + vY_y - uY_x + V = 0\ .
\end{aligned} \tag{13}$$

This set of differential equations for X,Y,U, and V as functions of x,y,u and v is overdetermined.

The integrations involved is solving Eqs. (13) are straightforward. The result is that the general infinitesimal similarity group corresponding to Eqs. (8) is

$$\begin{aligned} X(x,y,u,v) &= (a + 2b)x + c \\ Y(x,y,u,v) &= by + g(x) \\ U(x,y,u,v) &= au \\ V(x,y,u,v) &= ug_x - bv \end{aligned} \tag{14}$$

where a,b, and c are arbitrary constants, and g(x) is an arbitrary function of x alone.

Now we find the subgroup of (12) that leaves invariant the boundary curves (9):

$$\text{Invariance of } y = 0 \Rightarrow y' = 0 \Rightarrow Y\ (x, y = 0, u, v) = 0$$

$$\Rightarrow g(x) \equiv 0 \ . \tag{15a}$$

From the transformations (2a), the boundary curves $y = \pm \infty$ are trivially invariant.

$$\text{Invariance of } x = 0 \Rightarrow x' = 0 \Rightarrow X\ (x = 0, y, u, v) = 0$$

$$\Rightarrow c = 0 \ . \tag{15b}$$

$$\text{Invariance of } x > 0 \Rightarrow x' > 0 \Rightarrow X\ (x > 0, y, u, v) > 0$$

$$\Rightarrow 2b + a > 0 \ . \tag{15c}$$

Equations (15) will restrict the group (14) so that the boundary curves are left invariant. Now we further restrict the group (14) so that the boundary conditions are invariant:

$$u(y = 0) = 0 \Rightarrow u'(y = 0) = 0 \Rightarrow U(x, y = 0, u=0, v) = 0$$

$$\Rightarrow \text{ no restriction on } a . \tag{16a}$$

Similarly, $v(y = 0) = 0$ is trivially satisfied

$$\Rightarrow \text{ no restriction on } b . \tag{16b}$$

$$u(y = \pm \infty) = u_o \Rightarrow U(x, y = \pm \infty, u = u_o, v) = 0$$

$$\Rightarrow a = 0 \tag{16c}$$

By combining Eqs. (15-16) the final subgroup of Eqs. (7-8) that leaves invariant the boundary data is

$$\begin{aligned} X(x,y,u,v) &= 2bx \\ Y(x,y,u,v) &= by \\ U(x,y,u,v) &= 0 \\ V(x,y,u,v) &= -bv. \end{aligned} \tag{17}$$

The first invariant surface condition is

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{du}{U} , \quad \text{or,} \quad \frac{dx}{2bx} = \frac{dy}{by} = \frac{du}{o} . \tag{18}$$

Integration of the first equality in (18) is

$$\zeta = {}^{y}/\sqrt{x} = \text{constant}. \tag{19a}$$

Integration of the second equality in (18)

$$u = F_1(\zeta) . \tag{19b}$$

The second invariant surface condition is

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dv}{V} , \quad \text{or,} \quad \frac{dx}{2bx} = \frac{dy}{by} = \frac{dv}{bv} . \tag{20}$$

Integration of the second equality in (20) gives

$$v = F_2(\zeta) /y . \tag{19c}$$

Equations (19) are the same results as Eqs. (10) obtained by Landau and Lifshitz. Furthermore, the infinitesimal group (14) is the general infinitesimal group corresponding to Eqs. (8). Thus the similarity solution corresponding to any other boundary conditions can be found by analyzing the group (14).

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References

1. G. W. Bluman and J. D. Cole, Similarity Methods for Differential Equations, (Springer-Verlag, New York, 1974), Part II.

2. L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Addison-Wesley Publishing Co., Reading, Massachusetts, 1959) pp. 145-150.

Appendix

In this appendix we illustrate how the determining equations are obtained from a printout of the functions $\vec{H}$. As an example we consider the system of partial differential equations (8), so there are two components $H_i, i = 1$ and 2.

The STS factors each H_i to produce a polynomial in the S_m's, where the factoring is performed first with respect to S_1, then with respect to S_2, and so on. That is, H_i first is factored to produce a polynomial in S_1 . Then the coefficient of each power of S_1 is factored to produce a polynomial in S_2, etc. In this way, once a particular product of powers of the S_m's appears in H_i, that same combination of the S_m's never appears again. Consequently the coefficients of the independent terms of H_i (i.e., the determining equations) can be identified by inspection.

The printout of $\vec{H}$ corresponding to the partial differential equations (8) is

COMPONENT NUMBER 1 OF THE EXPRESSION H FROM WHICH THE DETERMINING EQUATIONS ARE DETERMINED IS:

$$ZEROP = ((- \frac{d}{dU_2} FX_2 - \frac{d}{dU_1} FX_1)*S_4^2 + (\frac{d}{dX_2} FX_2 - \frac{d}{dU_2} FU_2 - \frac{d}{dX_1} FX_1 + \frac{d}{dU_1} FU_1)*S_4$$

$$+ ((- \frac{d}{dU_2} FX_2 - \frac{d}{dU_1} FX_1)*S_{13} - \frac{d}{dX_1} FX_2 + \frac{d}{dU_1} FU_2)*S_7$$

$$+ (- \frac{d}{dX_2} FX_1 + \frac{d}{dU_2} FU_1)*S_{13} + \frac{d}{dX_2} FU_2 + \frac{d}{dX_1} FU_1)*EPSLON$$

COMPONENT NUMBER 2 OF THE EXPRESSION H FROM WHICH THE DETERMINING EQUATIONS ARE DETERMINED IS:

$$ZEROP = ((- 2*(\frac{d}{dU_2} FX_1)*NU*S_4 + 2*(\frac{d}{dU_1} FX_1)*NU*S_7 + 2*(\frac{d}{dX_2} FX_1)*NU)*S_2$$

$$+ (\frac{d^2}{dU_2^2} FX_1)*NU*S_4^3 + ((\frac{d^2}{dU_2^2} FX_2 - 2*(\frac{d^2}{dU_1\, dU_2} FX_1))*NU*S_7$$

$$+ (- 2*(\frac{d^2}{dU_2\, dX_2} FX_1) - \frac{d^2}{dU_2^2} FU_1)*NU - 2*U_1*(\frac{d}{dU_2} FX_2) + U_2*(\frac{d}{dU_2} FX_1))*S_4^2$$

$$+ ((- 2*(\frac{d^2}{dU_1\, dU_2} FX_2) + \frac{d^2}{dU_1^2} FX_1)*NU*S_7^2$$

$$+ ((- 2*(\frac{d^2}{dU_2\, dX_2} FX_2) + 2*(\frac{d^2}{dU_1\, dX_2} FX_1) + 2*(\frac{d^2}{dU_1\, dU_2} FU_1))*NU - U_2*(\frac{d}{dU_2} FX_2)$$

$$+ 2*U_1*(\frac{d}{dU_1} FX_2))*S_7 - U_1*(\frac{d}{dU_2} FX_1)*S_{13} + (\frac{d}{dU_2} FX_1)*NU*S_{14}$$

$$+ (\frac{d^2}{dX_2^2} FX_1 + 2*(\frac{d^2}{dU_2\, dX_2} FU_1))*NU + 2*U_1*(\frac{d}{dX_2} FX_2) - U_2*(\frac{d}{dX_2} FX_1)$$

$$- U_1*(\frac{d}{dX_1} FX_1) - U_2*(\frac{d}{dU_2} FU_1) + FU_1)*S_4 + (\frac{d^2}{dU_1^2} FX_2)*NU*S_7^3$$

$$+ ((2*(\frac{d^2}{dU_1\, dX_2} FX_2) - \frac{d^2}{dU_1^2} FU_1)*NU + 2*U_2*(\frac{d}{dU_1} FX_2))*S_7^2$$

$$+ (- U_1*(\frac{d}{dU_2} FX_2)*S_{13} + (\frac{d}{dU_2} FX_2)*NU*S_{14} + (\frac{d^2}{dX_2^2} FX_2 - 2*(\frac{d^2}{dU_1\, dX_2} FU_1))*NU$$

$$+ U_2*(\frac{d}{dX_2} FX_2) - U_1*(\frac{d}{dX_1} FX_2) + FU_2)*S_7 + U_1*(\frac{d}{dU_2} FU_1)*S_{13} - (\frac{d}{dU_2} FU_1)*NU*S_{14}$$

$$- (\frac{d^2}{dX_2^2} FU_1)*NU + U_2*(\frac{d}{dX_2} FU_1) + U_1*(\frac{d}{dX_1} FU_1))*EPSLON$$

(D88) BATCH DONE

(C89)

The functions H_i can be simplified considerably by making use of the following procedure: if all the independent coefficients of the S_m's that contain only a single term are set to zero, these conditions on the groups can be used immediately to discard many of the remaining terms of $\vec{H}$. For instance, in H_2 we note that

$$\frac{d}{dU_2} FX_1 = \frac{d}{dU_1} FX_1 = \frac{d}{dX_2} FX_1 = \frac{d^2}{dU_2^2} FX_1 \tag{A.1}$$

$$= \frac{d^2}{dU_1{}^2} FX_2 = \frac{d}{dU_1} FX_2 = \frac{d}{dU_2} FX_2 = \frac{d}{dU_2} FU_1 = 0 \ .$$

By using these relationships many of the remaining terms of H_i can be crossed out immediately from the printout. For example, by using Eqs. (A.1) in H_1 the terms involving $S_4{}^2$, $S_{13}S_7$, and S_{13} vanish. By equating to zero the coefficients of the remaining terms of H_1 , those involving S_4, S_7, and the constant term, we obtain the following determining equations

$$\frac{d}{dX_2} FX_2 - \frac{d}{dU_2} FU_2 - \frac{d}{dX_1} FX_1 + \frac{d}{dU_1} FU_1 = 0 \quad ,$$

$$- \frac{d}{dX_1} FU_2 + \frac{d}{dU_1} FU_2 = 0 \quad ,$$

and

$$\frac{d}{dX_2} FU_2 + \frac{d}{dX_1} FU_1 = 0 \quad .$$

Similar reductions occur in the size of H_2 by applying Eqs. (A.1).