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MASTER

AUTHOR(S): J. E. Dendy and J. M. Hyman

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MULTI-GRID AND ICCG FOR PROBLEMS WITH INTERFACES

by

J. E. Dendy, Jr. and J. M. Hyman*

In this paper we compare computation times for the multi-grid (MG) algorithm [1], the incomplete Cholesky conjugate gradient (ICCG) algorithm [4,5] and the modified ICCG (MICCG) algorithm [3] to solve elliptic partial differential equations of the form

$$\begin{aligned} -\nabla\cdot(D(x,y)\nabla U(x,y)) + \sigma(x,y)U(x,y) &= f(x,y)\ ,\ (x,y)\epsilon\Omega \\ \nu(x,y)\cdot D(x,y)\nabla U(x,y) + \gamma(x,y)U(x,y) &= 0\ ,\ (x,y)\epsilon\partial\Omega\ . \end{aligned} \tag{1}$$

Here Ω is a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega$, ν is the outward normal to $\partial\Omega$, D is possitive, and σ and γ are non-negative. Moreover, the functions D, σ, and f are allowed to be discontinuous across internal boundaries Γ of Ω. We make the assumption that U and $\mu\cdot(D\nabla U)$ are continuous at almost every $(x,y)\epsilon\Gamma$, and μ is a smooth vector normal to Γ.

The application of MG to (1) is discussed in [1]. The main conclusion in that paper was that the interpolation J^K_{K-1}, from grid G^{K-1} to G^K should preserve the flux $\mu\cdot(D\nabla U)$ across the interfaces Γ, and that the coarse grid operator L^{K-1} should then be taken to be $(J^K_{K-1})^* L^K J^K_{K-1}$. This implementation of MG is more robust than the proposals in [2].

To solve Ax=b by ICCG, where A is a symmetric positive definite sparse matrix, one approximates A in factored form as LL^T where L is lower triangular and sparse. The sparseness structure $L+L^T$ is at least that of A. Following the notation of [3], we use ICCG(n) and MICCG(n) to denote how many additional diagonals, n, L has than the lower triangular part of A. In ICCG, the entries of LL^T appearing outside the sparseness structure of $L+L^T$ are taken to be zero; in MICCG they are summed [3] and added to the diagonal. Both methods are accelerated with conjugate

*Los Alamos Scientific Laboratory, University of California, MS-233, Los Alamos, N.M. 87545.

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gradient. For smooth coefficient problems the work to solve to a given level of error is $O(N^{3/2})$ for ICCG and $O(N^{5/4})$ for MICCG, where N is the number of unknowns. For MG the work is O(N).

In general, we have found the MICCG and ICCG algorithms more robust than the MG for general positive definite systems. Also, ICCG and MICCG can be implemented as reliable black boxes that work for a wide range of problems. For MG, unfortunately, this is not yet the case. MG must often be tuned to the structure of the problem at hand. A major advantage of the MG algorithm is that the structure of the problem can be exploited to significantly reduce the solution time. Problems arising from difference equations are usually rich in structure and the Brandt MG program structure is natural not only for the solution of the difference equation but also for the local mesh refinement. An added bonus of the MG structure is that it can also be used to solve companion problems (such as the eigenvalue problem) that often arise in conjunction with Eq. 1. Indeed, as many as 90% of the multi-group diffusion problems (of which (1) is a special, one group, case) solved are eigenvalue problems.

The first example is the equation

$$\Delta u - u = f \text{ on } \Omega = (0,1)X(0,1)\ ,\ u = g \text{ on } \partial\Omega\ ,$$

where f and g are chosen so that $u = \log[(x+10^{-10})^2 + (y+10^{-10})^2]$. The fourth order nine point box is used, and ICCG(0) and MICCG(0) are compared to MG, using $u \equiv 1$ as initial guess and solving until the discrete

Entries indicate CDC 7600 C.P.U. time in seconds	Numbers of unknowns	MG $u \equiv 1$ guess to 10^{-6}	ICCG $u \equiv 1$ guess to 10^{-6}	MICCG $u \equiv 1$ guess to 10^{-6}
Problem 1	11x11	.04	.04	.03
	23x23	.14	.21	.19
	47x47	.50	1.50	1.08
	93x93	4.29	23.12	10.04

L^2 norm of the residual is less than 10^{-6}. A MG work unit (SOR sweep) took 0.0013, 0.0045 or 0.016 seconds on an 11x11, 23x23 or 47x47 point grid, respectively. The ICCG and MICCG algorithms took 0.0033, 0.011 or 0.041 seconds per iteration on the same grids.

Examples 2-5 are all simplified versions of the complicated mosaics that appear in neutron diffusion problems. In these examples we use either a second order five point difference operator, described in [1], or a piecewise bilinear nine point finite element method. The ICCG(1) and MICCG(1) algorithms are used for the five point star and the ICCG(0) and MICCG(0) algorithms are used to solve the nine point operator.

Example 2: $\Omega = (0,24)x(0,24)$,

$$\frac{\partial u}{\partial \nu} = \begin{cases} -u/2D \text{ on } y = 24 \text{ or } x = 24 \\ 0, \text{ otherwise} \end{cases}$$

$$D(x,y) = \begin{cases} 1, \text{ if } (x,y) \in [0,12)x[0,12) \cup (12,20]x(12,20] \\ 1000, \text{ otherwise} \end{cases}$$

Example 3: $\Omega = (0,16)x(0,16)$, $\frac{\partial u}{\partial \nu}$ as in Ex. 2

$$D(x,y) = \begin{cases} 1000 \text{ if } (x,y) \in (0,5)x[5,7) \cup (5,7)x[0,7) \\ 1, \text{ otherwise} \end{cases}$$

Example 4: $\Omega = (0,16)x(0,16)$, $\frac{\partial u}{\partial \nu}$ as in Ex. 2

$$D(x,y) = \begin{cases} 1000, \text{ if } (x,y) \in (1,3)x[0,5) \cup (3,7)x(3,5) \\ \qquad \cup (5,11)x(5,7) \cup (9,15)x(7,9) \\ \qquad \cup (13,15)x(9,13) \cup (15,16]x(11,13) \\ 1, \text{ otherwise.} \end{cases}$$

Example 5: $\Omega = (0,24)x(0,24)$ $\quad \frac{\partial u}{\partial \nu} = -\frac{1}{2D} u$ on all sides

$$D(x,y) = \begin{cases} 1000, \text{ if } (x,y) \in (11,13)x(11,13) \\ 1, \text{ otherwise.} \end{cases}$$

In all these examples f=0 when D=1, and f=1 when D=1000. Comparisons are presented for two cases, $\sigma = 0$ and $\sigma = 1/3D$, on the next page.

The MG algorithm employed differs from that described in [1] in that it uses one full MG cycle [2] to first solve to truncation error. The

Entries indicate CDC 7600 C.P.U. time in seconds	Number of unknowns	MG T.E.	ICCG T.E.	MICCG T.E.	MG T.E. to 10^{-6}	ICCG T.E. to 10^{-6}	MICCG T.E. to 10^{-6}	MG Total	ICCG 0 guess to 10^{-6}	MICCG 0 guess to 10^{-6}
Problem 2	13x13	.01	.02	.02	.03	.04	.04	.04	.04	.05
finite difference	25x25	.04	.12	.11	.11	.25	.25	.15	.27	.28
$\sigma \equiv 0$	49x49	.12	.77	.60	.48	1.75	1.23	.60	1.89	1.61
Problem 2	13x13	.01	.02	.02	.04	.04	.04	.05	.05	.05
finite element	25x25	.04	.17	.14	.15	.27	.23	.19	.29	.30
$\sigma \equiv 0$	49x49	.14	.88	.71	.69	1.95	1.29	.83	2.31	1.87
Problem 2	13x13	.01	.02	.02	.03	.04	.04	.04	.04	.04
finite difference	25x25	.03	.09	.09	.10	.22	.19	.13	.23	.25
$\sigma = \frac{1}{3D}$	49x49	.12	.54	.58	.38	1.52	1.06	.50	1.55	1.42
Problem 2	13x13	.01	.02	.02	.03	.04	.04	.04	.04	.04
finite element	25x25	.04	.10	.11	.14	.24	.21	.18	.26	.27
$\sigma = \frac{1}{3D}$	49x49	.13	.60	.58	.65	1.65	1.24	.78	1.74	1.58
Problem 3, $\sigma \equiv 0$	17x17	.02	.04	.03	.05	.08	.06	.07	.08	.08
finite difference	33x33	.05	.29	.16	.17	.53	.37	.22	.55	.47
Problem 3, $\sigma \equiv 0$	17x17	.02	.05	.03	.06	.08	.07	.08	.08	.09
finite element	33x33	.06	.36	.23	.19	.60	.38	.25	.60	.55
Problem 3, $\sigma = 1/3D$	17x17	.02	.03	.02	.05	.06	.05	.07	.05	.05
finite difference	33x33	.06	.18	.12	.16	.35	.26	.22	.36	.31
Problem 3, $\sigma = 1/3D$	17x17	.02	.03	.03	.06	.06	.06	.08	.06	.06
finite element	33x33	.06	.21	.17	.19	.40	.28	.25	.39	.36
Problem 4, $\sigma \equiv 0$	17x17	.02	.06	.03	.06	.08	.09	.08	.09	.09
finite difference	33x33	.05	.42	.16	.19	.62	.37	.24	.64	.49
Problem 4, $\sigma \equiv 0$	17x17	.02	.22	.21	.06	.10	.08	.08	.10	.09
finite element	33x33	.06	.18	.18	.21	.63	.44	.28	.64	.57
Problem 4, $\sigma = 1/3D$	17x17	.02	.05	.02	.05	.07	.05	.07	.07	.06
finite difference	33x33	.05	.31	.13	.14	.46	.28	.19	.47	.34
Problem 4, $\sigma = 1/3D$	17x17	.02	.06	.03	.05	.08	.06	.07	.08	.06
finite element	33x33	.06	.34	.17	.16	.50	.31	.22	.52	.39
Problem 5, $\sigma \equiv 0$	25x25	.03	.07	.03	.08	.17	.11	.11	.16	.14
finite difference	49x49	.12	.54	.26	.27	1.10	.61	.39	1.17	.82
Problem 5, $\sigma \equiv 0$	25x25	.04	.08	.07	.09	.16	.10	.13	.17	.15
finite element	49x49	.14	.57	.32	.28	1.28	.64	.42	1.24	.89
Problem 5, $\sigma = 1/3D$	25x25	.03	.06	.04	.06	.07	.07	.09	.09	.09
finite difference	49x49	.12	.29	.22	.19	.60	.36	.31	.60	.43
Problem 5, $\sigma = 1/3D$	25x25	.04	.06	.05	.05	.08	.07	.09	.09	.08
finite element	49x49	.13	.32	.22	.21	.67	.36	[illegible]	[illegible]	[illegible]

first comparison takes this MG solution at the truncation error level and uses it as in initial guess to continue solving to a discrete L^2 residual norm of 10^{-6}; this comparison mimics having a good initial guess, as one would have in each step of a time dependent problem. The second uses $u \equiv 0$ as an initial guess. Note that MICCG does much better with a good initial guess.

The clearest conclusion from these examples is that MICCG seems preferable to ICCG, especially since if one is already using ICCG, then one can change just one line of code to obtain MICCG. Another conclusion is that MG is most impressive when solving to truncation error. We emphasize that MG has a mechanism for doing this which ICCG and MICCG do not have. When solving to smaller tolerances, the gains for MG are less impressive, especially in the $\sigma = 1/3D$ case, although for the harder problems 2 and 4, they are almost even for this case.

For problems with little structure and for one shot calculations we recommend ICCG over MG and MICCG over ICCG. For problems that are done many times, it is worth investing the effort to study method like MG.

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