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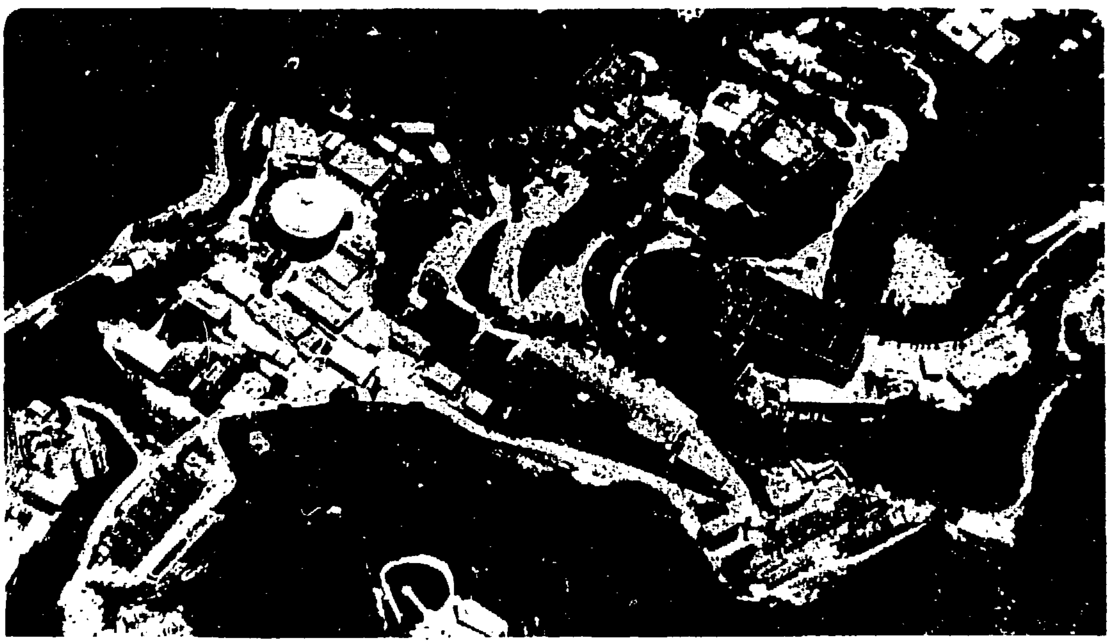

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The Geometry of the Virasoro Group for Physicists *

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Abstract

$Diff(S^1)$, the group of reparametrizations of the circle, is known as the Virasoro group in string theory. Reparametrizations keeping fixed a point of the circle form the quotient space $Diff(S^1)/S^1$. The geometry of this space is relevant for string theory and string field theory. We describe this space as an infinite dimensional complex manifold with a Kähler metric and compute its Riemann tensor and its Ricci tensor.

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1. Introduction

String theory[1] has spurred some interesting recent developments in infinite dimensional geometry. On the one hand, the study of Riemann surfaces of arbitrary genus has led to the idea that the respective moduli spaces should be considered as embedded in an infinite dimensional Grassmassian. On the other hand the study of string field theory is naturally connected with that of loop spaces. As Bowick and Rajeev[2] have emphasized, the geometry of the Virasoro group $Diff(S^1)$ and more precisely of the infinite dimensional complex manifold $Diff(S^1)/S^1$, is especially relevant for string field theory. The necessary mathematical tools were developed earlier by Dan Freed in his thesis.[3] The basic notions of differential geometry can be found in Ref. 4.

The purpose of these lectures is to describe the geometry of $Diff(S^1)/S^1$ in a way understandable to a physicist familiar with some string theory and with only the most basic tools of Riemannian geometry, as they are used in general relativity. This seems useful because Refs. 2 and its super extension, Ref. 5, are not easy reading. Furthermore, in both references certain misprints complicate the understanding of the material. We have simplified some of the derivations (e.g. the computation of the Ricci tensor) and hope to have achieved sufficient clarity to stimulate the reader to study the original literature. Our arguments are formally correct but not rigorous, in the sense that we have mostly ignored questions of convergence (this applies particularly to Sec. 4).

An important clarification was made to our subject by Pilch and Warner.[6] A related earlier mathematical paper by Segal[7] is also recommended reading.

2. The difference operators

Let $g_{\mu\nu}$ be the metric tensor of a Riemann manifold, whose points are labelled by coordinates x^μ. A vector field ξ of components $\xi^\mu(x)$ is a Killing vector (an isometry) if it leaves the metric invariant

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} = 0. \tag{2.1}$$

Here $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector ξ and $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$. Let $\Gamma_\lambda{}^\nu{}_\mu$ be the Christoffel connection

$$\Gamma_\lambda{}^\nu{}_\mu = \frac{1}{2} g^{\nu\rho} \left(\partial_\lambda g_{\mu\rho} + \partial_\mu g_{\lambda\rho} - \partial_\rho g_{\lambda\mu} \right) = \Gamma_\mu{}^\nu{}_\lambda. \tag{2.2}$$

It is easy to see that, if ξ is a Killing vector, then

$$\mathcal{L}_\xi \Gamma_\lambda{}^\nu{}_\mu \equiv \xi^\rho \partial_\rho \Gamma_\lambda{}^\nu{}_\mu + \partial_\lambda \xi^\rho \Gamma_\rho{}^\nu{}_\mu + \partial_\mu \xi^\rho \Gamma_\lambda{}^\nu{}_\rho - \partial_\rho \xi^\nu \Gamma_\lambda{}^\rho{}_\mu + \partial_\lambda \partial_\mu \xi^\nu = 0. \tag{2.3}$$

Consider the covariant derivative ∇_μ formed with Γ. For instance

$$\nabla_\lambda v^\mu = \partial_\lambda v^\mu + \Gamma_\lambda{}^\mu{}_\nu v^\nu. \tag{2.4}$$

As a consequence of (2.3) $\mathcal{L}_\xi$ commutes with ∇_μ

$$\mathcal{L}_\xi \nabla_\mu = \nabla_\mu \mathcal{L}_\xi \tag{2.5}$$

and with the operation of covariant differentiation $\nabla = dx^\mu \nabla_\mu$

$$[\mathcal{L}_\xi, \nabla] = 0. \tag{2.6}$$

Let $\eta^\mu(x)$ be another vector field and

$$\nabla_\eta = i_\eta \nabla = \eta^\mu \nabla_\mu \tag{2.7}$$

be the covariant differentiation along the field η. It is easy to see that (2.5) or (2.6) are equivalent to

$$[\mathcal{L}_\xi, \nabla_\eta] = \nabla_{\mathcal{L}_\xi \eta} = \nabla_{[\xi,\eta]}, \tag{2.8}$$

where we have used the usual bracket of two vectors

$$(\mathcal{L}_\xi \eta)^\mu = [\xi, \eta]^\mu = \xi^\lambda \partial_\lambda \eta^\mu - \eta^\lambda \partial_\lambda \xi^\mu. \tag{2.9}$$

Note that in (2.8) ξ must be a Killing vector but η is an arbitrary vector field.

The curvature of the connection Γ can be defined from the formula

$$R_{\xi,\eta} = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}. \tag{2.10}$$

In general, to compute the curvature, one needs the second derivatives of the metric tensor, or the first derivatives of Γ. However, if ξ and η are Killing vectors only Γ itself is needed, i.e. the first derivatives of the metric tensor. To show this we introduce the tensorial operators

$$\varphi_\xi = \mathcal{L}_\xi - \nabla_\xi. \tag{2.11}$$

In the difference the differentiation operators $\xi^\mu \partial_\mu$ cancel. Therefore φ_ξ operates as a matrix on tensors, with no differentiation of the tensor. For instance on a vector v^μ

$$\varphi_\xi\, v^\mu = (\varphi_\xi)^\mu{}_\lambda v^\lambda \tag{2.12}$$

where

$$(\varphi_\xi)^\mu{}_\lambda = -\partial_\lambda \xi^\mu - \xi^\nu \Gamma_\nu{}^\mu{}_\lambda = -\nabla_\lambda \xi^\mu. \tag{2.13}$$

We shall call derivations like φ_ξ "difference operators" as a reminder for formula (2.11). Now let ξ and η be Killing vectors. We have

$$[\varphi_\xi, \varphi_\eta] = [\mathcal{L}_\xi, \mathcal{L}_\eta] + [\nabla_\xi, \nabla_\eta] - [\mathcal{L}_\xi, \nabla_\eta] + [\mathcal{L}_\eta, \nabla_\xi]. \tag{2.14}$$

Using (2.8) and the identity

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} \tag{2.15}$$

one finds

$$\begin{aligned}[\varphi_\xi, \varphi_\eta] &= \mathcal{L}_{[\xi,\eta]} + [\nabla_\xi, \nabla_\eta] - 2\nabla_{[\xi,\eta]} \\ &= \varphi_{[\xi,\eta]} + [\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}.\end{aligned} \tag{2.16}$$

Comparing with (2.10) we see that the curvature is given by

$$R_{\xi,\eta} = [\varphi_\xi, \varphi_\eta] - \varphi_{[\xi,\eta]}. \tag{2.17}$$

We repeat: provided ξ and η are Killing vectors, the ξ, η component of the curvature can be computed at a point of the manifold in terms of the connection at that point, i.e. the first derivatives of the metric at that point (and the first derivative of the Killing vectors). If there are enough Killing vectors, (2.17) will give the entire Riemann tensor at a point. The Riemann tensor is then determined everywhere by means of the isometries of the manifold. Notice that, for any vector ξ,

$$\nabla_\xi g_{\mu\nu} = 0 \tag{2.18}$$

(metric compatability of the connection). For a Killing vector (2.1) is also valid and therefore

$$\varphi_\xi g_{\mu\nu} \equiv -(\varphi_\xi)^\rho{}_\mu\, g_{\rho\nu} - (\varphi_\xi)^\rho{}_\nu\, g_{\mu\rho} = 0. \tag{2.19}$$

This also follows from the explicit form (2.13) since a Killing vector satisfies

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.20}$$

Exercise. Verify (2.17) using directly the expression (2.13) and the identity

$$\nabla_\nu \nabla_\lambda \xi_\mu = -R_{\mu\lambda,}{}^\rho{}_\nu\, \xi_\rho \tag{2.21}$$

satisfied by a Killing vector. Observe that the Lie bracket (2.9) of two vectors can also be written as

$$[\xi, \eta]^\mu = \xi^\lambda \nabla_\lambda \eta^\mu - \eta^\lambda \nabla_\lambda \xi^\mu. \tag{2.22}$$

Consider now a complex manifold parametrized in some neighborhood by complex coordinates $z^n, \overline{z^n}$ and endowed with a hermitian metric $g_{m\bar{n}}$. Let $\xi = (\xi^n, \overline{\xi^n})$ be a real Killing vector. Then

$$\varphi_\xi\, g_{m\bar{n}} \equiv -(\varphi_\xi)^\ell{}_m\, g_{\ell\bar{n}} - \overline{(\varphi_\xi)^\ell{}_n}\, g_{m\bar{\ell}} = 0. \tag{2.23}$$

Therefore, using the inverse matrix $g^{\bar{m}n}$,

$$g_{m\bar{\ell}}\, g^{\bar{\ell}n} = g^{\bar{n}\ell} g_{\ell\bar{m}} = \delta_m{}^n, \tag{2.24}$$

we find

$$(\varphi_\xi)^m{}_\ell = -g_{\ell\bar{r}}\, \overline{(\varphi_\xi)^r{}_n}\, g^{\bar{n}m}. \tag{2.25}$$

If ξ is not a real vector, and its components are not related by complex conjugation, the linear combination

$$\eta = \alpha\xi + \bar{\alpha}\bar{\xi} \tag{2.26}$$

is a real vector, where α is an arbitrary complex number. The components of $\bar{\xi}$ are by definition the complex conjugates of those of ξ. Applying (2.25) to η and using the linearity of φ

$$\varphi_{a\xi_1+b\xi_2} = a\,\varphi_{\xi_1} + b\,\varphi_{\xi_2} \tag{2.27}$$

we can identify separately the coefficients of α and of $\bar{\alpha}$ to obtain

$$(\varphi_\xi)^\ell{}_m = -g_{m\bar{r}}\,(\varphi_\xi)^{\bar{r}}{}_{\bar{n}}\, g^{\bar{n}\ell} \tag{2.28}$$

where

$$(\varphi_\xi)^{\bar{r}}{}_{\bar{n}} = \overline{(\varphi_{\bar{\xi}})^r{}_n}. \tag{2.29}$$

3. The quotient space G/H.

Consider the Virasoro algebra

$$[L_a, L_b] = (a-b)L_{a+b} \tag{3.1}$$

where the indices a, b take all integer values $\lesseqgtr 0$. We take a representation in which the generators are operators satisfying the reality conditions

$$L_a^\dagger = L_{-a}. \tag{3.2}$$

By exponentiation this algebra generates the real Virasoro group G, a Lie group with infinitely many parameters whose elements can be represented, for instance, as

$$g = \exp[i\alpha^a L_a] \tag{3.3}$$

where the sum is over all integers a and the complex numbers α^a satisfy

$$\overline{\alpha^a} = \alpha^{-a}. \tag{3.4}$$

We are interested in studying the geometry of the homogeneous space G/H which is the quotient of the Virasoro group by its one parameter subgroup H generated by $L_0 = L_0^+$.

A standard way to parametrize the quotient space is known to physicists from the theory of nonlinear realizations. One writes a group element of G as the product of two exponentials

$$g = \exp\left[i \sum_{a\neq 0} \beta^a L_a\right] \exp[i\beta L_0]. \tag{3.5}$$

Every group element can be split uniquely in this way by factoring out an element of H on the right. The parameters $\beta^a, a \neq 0$ can be used as coordinates for the quotient space G/H, at least in the neighborhood of the origin. Here β is real and

$$\bar{\beta}^a = \beta^{-a}. \tag{3.6}$$

The action of an element g_1 of G on G/H is obtained as follows. One multiplies g from the left by g_1 and separates the result again as above:

$$g_1 g = \exp\left[i \sum_{a\neq 0} \beta'^a L_a\right] \exp[i\beta' L_0]. \tag{3.7}$$

The new coordinates $(a \neq 0)$

$$\beta'^a = \beta'^a(\beta^b, g_1) \tag{3.8}$$

give the point of G/H which is the transformed of β^b by g_1.

We shall use the notation

$$V = \exp\left[i \sum_{a\neq 0} \beta^a L_a\right] \tag{3.9}$$

where β^a satisfy (3.6). Then (3.7) can be abbreviated as

$$g_1 V = V' \exp[i\beta' L_0]. \tag{3.10}$$

V is unitary and the exponent in (3.9) does not contain L_0. Now (sum over a $\gtrless 0$)

$$V^{-1} dV = \omega^a L_a = \omega_+ + \omega_- + \omega_0 \tag{3.11}$$

is an element of the Lie algebra of G. The separation of the various parts of (3.11) is defined by (sum over $a > 0$)

$$\begin{aligned} \omega_+ &= \omega^a L_a \\ \omega_- &= \omega^{-a} L_{-a} \\ \omega_0 &= \omega^0 L_0. \end{aligned} \tag{3.12}$$

Since V is unitary,

$$\omega_+^\dagger = -\omega_-, \omega_0^\dagger = -\omega_0, \tag{3.13}$$

which means

$$\overline{\omega^a} = -\omega^{-a}, \overline{\omega^0} = -\omega^0. \tag{3.14}$$

These one–forms are defined on G/H. They depend only on the coordinates of G/H and their differentials.

Now, by exterior differentiation

$$d(V^{-1}dV) = -(V^{-1}dV)^2. \tag{3.15}$$

Therefore (sum over a $\gtrless 0$)

$$d(\omega^a L_a) = -(\omega^a L_a)^2. \tag{3.16}$$

Since one–forms anticommute, this implies

$$d\omega^a L_a = -\omega^a\omega^b L_a L_b = -\frac{1}{2}\omega^a\omega^b[L_a, L_b], \tag{3.17}$$

which gives, from (3.1) (sum over $b, c \gtrless 0$)

$$d\omega^a = -\frac{1}{2}(b-c)\delta_{b+c}{}^a\, \omega^b\omega^c. \tag{3.18}$$

These are the Cartan–Maurer equations. The forms ω^a satisfy these equations on G/H. They are not quite invariant. From (3.10) we find

$$\begin{aligned} V'^{-1}dV' &= e^{i\beta' L_0}V^{-1}g_1^{-1}d(g_1 V e^{-i\beta' L_0}) \\ &= e^{i\beta' L_0}V^{-1}dV e^{-i\beta' L_0} + e^{i\beta' L_0}de^{-i\beta' L_0} \end{aligned} \tag{3.19}$$

which means

$$\begin{aligned} \omega'_\pm &= e^{i\beta' L_0}\omega_\pm e^{-i\beta' L_0} \\ \omega'_0 &= \omega_0 - id\beta' L_0 \end{aligned} \tag{3.20}$$

or, in terms of components,

$$\begin{aligned} \omega^{a'} &= e^{-i\beta' a}\omega^a \\ \omega^{o'} &= \omega^o - id\beta'. \end{aligned} \tag{3.21}$$

Clearly the two–forms (no sum over $a > 0$)

$$\omega'^a\omega'^{-a} = \omega^a\omega^{-a} \tag{3.22}$$

are invariant for each value of a. Therefore

$$\frac{i}{2}\omega_2 = \sum_{a>0}\omega^a\omega^{-a}f(a) = \frac{1}{2}\sum_{a\gtrless 0}\omega^a\omega^{-a}f(a) \tag{3.23}$$

is an invariant two-form for any function $f(a)$ of the integer a, which satisfies

$$f(-a) = -f(a). \tag{3.24}$$

When is it closed? Using (3.18) we see that

$$\frac{i}{2} d\omega_2 = \frac{1}{2} \sum_{a,b,c} (b-c)\, \delta_{a+b+c,0}\, f(a)\, \omega^a \omega^b \omega^c. \tag{3.25}$$

This vanishes if

$$(b-c)f(a) + (c-a)f(b) + (a-b)f(c) = 0 \tag{3.26}$$

for

$$a + b + c = 0. \tag{3.27}$$

It is not difficult to see that the general solution of (3.26) and (3.27) is

$$f(a) = Aa^3 + Ba, \tag{3.28}$$

where A and B are constants.

Exercise. Show this. Also show that, for $f(a) = Ba$, there exists ω_1 such that $\omega_2 = d\omega_1$. On the other hand, for $f(a) = Aa^3$ there is no such ω_1, i.e. ω_2 is closed but not exact.

The closed two-form ω_2 given by (3.23) with $f(a)$ given by (3.28) can be taken as Kähler form on G/H, since, as we shall explain in the next section, G/H is a complex manifold.

4. Holomorphic coordinates for G/H

The method described in the previous section is perfectly satisfactory in general, but in our particular application it fails to make explicitly the very important fact that G/H is a complex manifold. The coordinates β^a and $\overline{\beta^a}$ $(a > 0)$ are not good complex coordinates. It is easy to see that they mix under the action of a general element g_1 of G. In order to render manifest the complex structure of G/H we introduce a further decomposition of the group element and write

$$\begin{aligned} V &= \exp\left[i \sum_{a\neq 0} \beta^a L_a\right] \\ &= \exp\left[i \sum_{a>0} z^a L_a\right] \exp\left[i \sum_{a>0} \mu^a L_{-a}\right] \exp[\rho L_0]. \end{aligned} \tag{4.1}$$

This should be possible, at least in a formal sense, in a neighborhood of the identity. Observe that the L_a for $a \leq 0$ form a subalgebra and generate a "subgroup" F such that

$$G \supset F \supset H. \tag{4.2}$$

The product of the last two exponentials in (4.1) represents an element of F. Since we are considering the real Virasoro group we cannot take z^a, μ^a and ρ as independent complex parameter. The relations they satisfy can be obtained in our representation of the generators by requiring that (4.1) be satisfied, i.e. that the right hand side be unitary and expressible as the left hand side, (without L_0 in the exponent). This gives μ^a and ρ as functions of z and $\bar{z}$. These functions can be computed as power series in $z, \bar{z}$. It is easy to see that the first terms in the expansion are

$$\mu^a(z,\bar{z}) = \overline{z^a} + \cdots \tag{4.3}$$

$$\rho(z,\bar{z}) = \sum_{a>0} a\,|z^a|^2 + \cdots \,. \tag{4.4}$$

We can take $z^a, \overline{z^a}$ as coordinates on G/H. The action of an element g_1 of G is obtained by multiplying (4.1) from the left by g_1 and splitting the result again as in (4.1). This means that z'^a is given by

$$\begin{aligned} g_1 \ \exp\left[i\sum_{a>0} z^a L_a\right] &= \\ = \exp\left[i\sum_{a>0} z'^a L_a\right] \exp\left[i\sum_{a<0} \tilde{\mu}^a L_{-a}\right] &\exp\left[i\tilde{\varphi} L_0\right]. \end{aligned} \tag{4.5}$$

The last two exponential factors in this formula can then be combined with the last two factors in (4.1), since all these factors are group elements of F. Formula (4.5) makes it clear that

$$z'^a = z'^a(z, g_1) \tag{4.6}$$

depend only on z^a and not on $\overline{z^a}$. The action of G on G/H is holomorphic and $z^a, \overline{z^a}$ are good complex coordinates for G/H. The transformation law of $\overline{z^a}$ is obtained from (4.6) by complex conjugation.

The Lie derivatives corresponding to the infinitesimal generators $iL_a (a \gtrless 0)$ are (on functions)

$$\mathcal{L}_a = \xi_a = \xi_a{}^n(z)\partial_n + \xi_a{}^{\bar{n}}\partial_{\bar{n}} \tag{4.7}$$

where the sum is over $n > 0$. Here $\partial_n = \frac{\partial}{\partial z^n}$, $\partial_{\bar{n}} = \frac{\partial}{\partial \bar{z}^n}$, and

$$\overline{\xi_a{}^n} = \xi_{-a}{}^{\bar{n}} \quad , \quad \overline{\xi_a} = \xi_{-a}. \tag{4.8}$$

They can be split into a $(1,0)$ part

$$\xi_z^{(1,0)} = \xi_a{}^n(z)\partial_n \tag{4.9}$$

and a $(0,1)$ part

$$\xi_a^{(0,1)} = \xi_a{}^{\bar{n}}(\bar{z})\partial_{\bar{n}} \tag{4.10}$$

which commute with each other:

$$\left[\xi_a^{(1,0)}, \xi_b^{(0,1)}\right] = 0. \tag{4.11}$$

The (1,0) parts and the (0,1) parts satisfy the same algebra

$$\left[\xi_a^{(1,0)}, \xi_b^{(1,0)}\right] = -i(a-b)\, \xi_{a+b}^{(1,0)} \tag{4.12}$$

$$\left[\xi_a^{(0,1)}, \xi_b^{(0,1)}\right] = -i(a-b)\, \xi_{a+b}^{(0,1)}. \tag{4.13}$$

Observe that the sign in (4.12) and (4.13) is opposite to that occurring in the algebra of iL_a. This is as it should be because we defined the action of the group by left multiplication.

It is easy to compute the first few terms of the infinitesimal transformations in power series of z. One finds for the Killing vectors (no sum over indices; $a > 0, z^n = 0$ for $n \leq 0$)

$$\begin{cases} \xi_0{}^n(z) & = -inz^n + \cdots \\ \xi_a{}^n(z) & = \delta_a{}^n + \frac{i}{2}(2a-n)z^{n-a} + \cdots \\ \xi_{-a}{}^n(z) & = -i(2a+n)z^{n+a} + \cdots) \,. \end{cases} \tag{4.14}$$

Note that, for $z = 0$, $\xi_0{}^n$ and $\xi_{-a}{}^n$ vanish, while $\xi_a{}^n = \delta_a{}^n$. Similarly, for $\bar{z} = 0$, $\xi_0{}^{\bar{n}}$ and $\xi_a{}^{\bar{n}}$ vanish, while $\xi_{-a}{}^{\bar{n}} = \delta_a{}^n$. We shall need these facts later.

Using (4.1) we can compute the differential forms given by (3.11). We introduce the abbreviations.

$$z \cdot L_+ = \sum_{a>0} z^a L_a \tag{4.15}$$

$$\mu \cdot L_- = \sum_{a>0} \mu^a L_{-a} \tag{4.16}$$

and write

$$\begin{aligned} V^{-1}dV &= e^{-\rho L_0} e^{-i\mu\cdot L_-} (e^{-iz\cdot L_+} d e^{iz\cdot L_+}) e^{i\mu\cdot L_-} e^{\rho L_0} \\ &\quad + e^{-\rho L_0} e^{-i\mu\cdot L_-} d(e^{i\mu\cdot L_-} e^{\rho L_0}). \end{aligned} \tag{4.17}$$

Clearly the last term is an element of the Lie algebra of F (generators $L_a, a \leq 0$) and contributes only to ω_0 and ω_-. Only the first term contributes to ω_+, which shows that ω_+ contains only the differentials dz^m, and not $d\overline{z^m}$. So, from (3.12) and (3.13)

$$\omega^a = dz^m \omega_m{}^a(z, \bar{z}) \tag{4.18}$$

$$\omega^{-a} = -d\overline{z^m}\, \overline{\omega_m{}^a(z, \bar{z})} \tag{4.19}$$

($a > 0$, sum over $m > 0$), while ω^0 contains both dz^m and $d\overline{z^m}$.

5. The Toeplitz operators.

We now wish to compute the difference operators corresponding to the Killing vectors ξ_a of Sec. 4. It is easy to work out their effect on the $(1,0)$ part of the vectors ξ_b themselves for $b > 0$ at the origin $z = \bar{z} = 0$. Let us denote by $\hat{\xi}$ the $(1,0)$ part of a vector ξ. Thus, if the components of ξ_a are

$$\xi_a = ({\xi_a}^n, {\xi_a}^{\bar{n}}), \tag{5.1}$$

those of $\hat{\xi}_a$ are

$$\hat{\xi}_a = ({\xi_a}^n, 0). \tag{5.2}$$

In the following all indices $a, b \cdots$ and $l, m, n, r, \cdots$ take only positive values unless explicitly indicated otherwise. We first observe that, at the origin,

$$\nabla_{\xi_0} \hat{\xi}_b = \nabla_{\xi_{-a}} \hat{\xi}_b = 0, \;\; z = \bar{z} = 0. \tag{5.3}$$

This is immediate because, at the origin $\xi_0 = 0$ and

$$\nabla_{\xi_{-a}} \hat{\xi}_b = ({\xi_{-a}}^m \nabla_m + {\xi_{-a}}^{\bar{m}} \partial_{\bar{m}}) \hat{\xi}_b. \tag{5.4}$$

At the origin ${\xi_{-a}}^m$ vanishes and in general $\hat{\xi}_b$ is independent of $\bar{z}$. As a consequence of (5.3) the difference operators (2.11) at the origin can be computed from the Lie derivatives alone. Thus

$$\begin{aligned} \varphi_{\xi_0} \hat{\xi}_b &= \mathcal{L}_{\xi_0} \hat{\xi}_b = \left[\xi_0, \hat{\xi}_b\right] \\ &= \left[\hat{\xi}_0, \hat{\xi}_b\right] = ib\, \hat{\xi}_b \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \varphi_{\xi_{-a}} \hat{\xi}_b &= \mathcal{L}_{\xi_{-a}} \hat{\xi}_b = \left[\xi_{-a}, \hat{\xi}_b\right] \\ &= \left[\hat{\xi}_{-a}, \hat{\xi}_b\right] = i(a+b)\, \hat{\xi}_{b-a}. \end{aligned} \tag{5.6}$$

At the origin $\hat{\xi}_{b-a}$ vanishes for $b \leq a$. The difference operators φ_{ξ_0} and $\varphi_{\xi_{-a}}$ operate within the space of vectors $\hat{\xi}_b$ with positive b. For this reason, in the present application, the difference operators are called Toeplitz operators, in analogy with operators occurring in the theory of Fourier series.

To obtain φ_{ξ_a} for $a > 0$ we shall make use of (2.28) and (2.29), which by (4.8) relates it to $\varphi_{\xi_{-a}}$. First we observe that the matrix elements of $\varphi_{\xi_{-a}}$ at the origin are

$$(\varphi_{\xi_{-a}})^m{}_\ell = i\delta_\ell{}^{a+m}(2a+m). \tag{5.7}$$

This follows immediately from (5.6) and the form of the Killing vectors at the origin (4.14). Our metric tensor is

$$g_{\ell \bar{r}} = f(\ell)\, \delta_{\ell \bar{r}} \tag{5.8}$$

with $f(\ell)$ given by (3.28). Therefore

$$\begin{aligned}(\varphi_{\xi_a})^m{}_\ell &= ig_{\ell\bar r}\,\delta_n{}^{a+r}g^{\bar n m}(2a+r)\\ &= if(m-a)\,\delta_\ell{}^{m-a}\,\frac{1}{f(m)}\,(a+m).\end{aligned}\tag{5.9}$$

This formula can be rewritten as

$$\varphi_{\xi_a}\,\hat\xi_b = i\frac{f(b)}{f(a+b)}\,(2a+b)\,\hat\xi_{a+b}.\tag{5.10}$$

(5.7) and (5.9) are correct also for $a=0$, in which case they agree.

Exercise. Verify (5.7) and (5.9) by computing the Toeplitz operators at the origin from their definition (2.11) as difference operators on arbitrary tangent vectors of type (1,0). This requires computing the connection at the origin from (A.17), which in turn requires the metric to the appropriate order. Notice, in contrast, the simplicity of Freed's method, which we have employed above.

In the following we shall simplify the notation and write φ_a for φ_{ξ_a} and R_{ab} for R_{ξ_a,ξ_b} for all a,b. We always work at the origin. It is ($a,b \gtrless 0$)

$$\varphi_{[\xi_a,\xi_b]} = -i(a-b)\varphi_{\xi_{a+b}} = -i(a-b)\varphi_{a+b}.\tag{5.11}$$

Therefore (2.16) gives

$$R_{ab,}{}^m{}_\ell = ([\varphi_a,\varphi_b] + i(a-b)\varphi_{a+b})^m{}_\ell.\tag{5.12}$$

It is easy to verify by matrix multiplication, using (5.7) and (5.9), that R_{ab} vanishes except when a and b are both different from zero and have opposite sign. This is expected because in general

$$R_{ab,}{}^m{}_s = (\xi_a{}^{\bar r}\xi_b{}^\ell - \xi_b{}^{\bar r}\xi_a{}^\ell)R_{\bar r\ell,}{}^m{}_s\tag{5.13}$$

and for all other cases the components of the Killing vectors vanish at the origin by (4.14). The non vanishing components ($a,b>0$)

$$R_{-a,b} = [\varphi_{-a},\varphi_b] - i(a+b)\varphi_{b-a}\tag{5.14}$$

can be computed from (5.7) and (5.9). The result is

$$\begin{aligned}R_{\bar r\ell,}{}^m{}_s = \Bigg[&-(2r+m)(m+r+\ell)\,\frac{f(m+r-\ell)}{f(m+r)}\\ &+(m+\ell)(2r+m-\ell)\,\frac{f(m-\ell)}{f(m)}\theta(m-\ell)\\ &+(r+\ell)\left\{\begin{matrix}2r-2\ell+m\\ (m+\ell-r)\,\dfrac{f(m-\ell+r)}{f(m)}\end{matrix}\right.\Bigg]\delta_{m+r,\ell+s}\qquad \begin{matrix}r\geq\ell\\ r\leq\ell\end{matrix}\end{aligned}\tag{5.15}$$

where

$$\theta(m) = \begin{cases} 1 & \text{for } m > 0 \\ 0 & \text{for } m \leq 0 \end{cases} .$$

Exercise. Verify that the Riemann tensor satisfies the symmetry condition

$$R_{\bar{r}\ell,}{}^{m}{}_{s} = R_{\bar{r}s,}{}^{m}{}_{\ell} \tag{5.16}$$

valid for a Kähler manifold. Hint: use the identity (3.26), (3.27) satisfied by $f(m)$. The expressions given in Refs. 2 and 5 are incomplete and do not satisfy (5.16).

To compute the Ricci tensor we set $\ell = m$ in (5.15) and sum over all positive values. For $\ell = m$ (no sum)

$$R_{\bar{r}m,}{}^{m}{}_{s} = \delta_{rs} \left[-(2r+m)(2m+r)\,\frac{f(r)}{f(m+r)} + (r+m) \begin{Bmatrix} 2r-m \\ (2m-r)\,\dfrac{f(r)}{f(m)} \end{Bmatrix} \right] \quad \begin{matrix} r \geq m \\ r \leq m \end{matrix} \tag{5.17}$$

The sum over m is

$$\sum_{m=1}^{\infty} \left[-(2r+m)(2m+r)\,\frac{f(r)}{f(m+r)} + (r+m)(2m-r)\,\frac{f(r)}{f(m)} \right] - \sum_{m=1}^{r} (r+m)(2m-r)\,\frac{f(r)}{f(m)} + \sum_{m=1}^{r} (r+m)(2r-m) . \tag{5.18}$$

For (3.28), with $A \neq 0$, the infinite sum converges, due to cancellations between the first and second term. Separately the two sums diverge only logarithmically. Therefore one can shift the variable in the first term. Setting $m' = m + r \to m$ the infinite parts cancel and one is left with

$$\sum_{m=1}^{r} (r+m)(2m-r)\frac{f(r)}{f(m)}. \tag{5.19}$$

This exactly cancels the term before the last in (5.18). Therefore (5.18) reduces to

$$\sum_{m=1}^{r} (r+m)(2r-m) = \frac{13}{6}r^3 - \frac{1}{6}r. \tag{5.20}$$

In conclusion, the Ricci tensor is given by

$$R_{s\bar{r}} = -R_{\bar{r}s} = -\delta_{rs}\left(\frac{13}{6}r^3 - \frac{1}{6}r\right). \tag{5.21}$$

The disappearance of the function $f(m)$ from the result is expected from the general structure of the Ricci tensor for a Kähler manifold.

Appendix. Basic formulas for Kähler manifolds.

We consider a complex manifold whose points are parametrized in some neighborhood by complex coordinates $z^m, \overline{z^m}$. Let there be a hermitean metric tensor

$$g_{\bar{n}m} = g_{m\bar{n}} = \overline{g_{n\bar{m}}}, \quad g_{mn} = g_{\bar{m}\bar{n}} = 0. \tag{A.1}$$

It is called a Kähler metric if it satisfies the vanishing curl conditions

$$\begin{aligned} \partial_\ell\, g_{m\bar{n}} &= \partial_m\, g_{\ell\bar{n}} \\ \partial_{\bar{\ell}}\, g_{m\bar{n}} &= \partial_{\bar{n}}\, g_{m\bar{\ell}}, \end{aligned} \tag{A.2}$$

where

$$\partial_\ell = \frac{\partial}{\partial z^\ell}, \quad \partial_{\bar{\ell}} = \frac{\partial}{\partial \overline{z^\ell}} . \tag{A.3}$$

The conditions (A.2) imply at least locally the existence of a real function $\mathcal{V}(z, \bar{z})$ such that

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}}\, \mathcal{V}. \tag{A.4}$$

$\mathcal{V}$ is called the Kähler potential and plays an important role in supersymmetry and supergravity theories. A change

$$\mathcal{V} \rightarrow \mathcal{V}' = \mathcal{V} + k(x) + \overline{k(z)} \tag{A.5}$$

is called a Kähler transformation. It leaves the metric invariant, because $k(z)$ depends only on z^m and $\overline{k(z)}$ only on $\overline{z^m}$. To the metric tensor (A.1) one associates a two-form

$$\omega = -2ig_{m\bar{n}}dz^m\overline{dz^n} \tag{A.6}$$

which is called the Kähler form of the manifold (we have omitted the wedge which indicates exterior product, but the differentials are understood to anticommute). The conditions (A.2) are equivalent to the statement that the Kähler form is closed

$$d\omega = 0 \tag{A.7}$$

while (A.4) gives

$$\omega = -2i\bar{\partial}\partial\,\mathcal{V} \tag{A.8}$$

where

$$\begin{aligned} d &= \partial + \bar{\partial}, \quad \partial = dz^m \partial_m, \quad \bar{\partial} = \overline{dz^m}\partial_{\bar{m}} \\ d^2 &= \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 . \end{aligned} \tag{A.9}$$

A vector of type $(1,0)$ has components $(v^\ell, 0)$, a vector of type $(0,1)$ has components $(0, u^{\bar{\ell}})$, a general vector is the sum $(v^\ell, u^{\bar{\ell}})$. Covariant derivatives are constructed by means of the connection coefficients $\Gamma_\ell{}^m{}_n$ and their complex conjugates

$$\Gamma_{\bar{\ell}}{}^{\bar{m}}{}_{\bar{n}} = \overline{\Gamma_\ell{}^m{}_n} . \tag{A.10}$$

These are the only non vanishing components of the connection coefficients, for instance

$$\Gamma_{\bar{\ell}}{}^{m}{}_{n} = 0. \tag{A.11}$$

Thus the covariant derivatives of a vector of type (1,0) are

$$\nabla_{\ell}\, v^m = \partial_{\ell}\, v^m + \Gamma_{\ell}{}^{m}{}_{n} v^n \tag{A.12}$$

$$\nabla_{\bar{\ell}}\, v^m = \partial_{\ell}\, v^m. \tag{A.13}$$

The metric compatibility condition states that

$$\nabla_{\ell}\, g_{m\bar{n}} \equiv \partial_{\ell}\, g_{m\bar{n}} - \Gamma_{\ell}{}^{r}{}_{m}\, g_{r\bar{n}} = 0 \tag{A.14}$$

and

$$\nabla_{\bar{\ell}}\, g_{m\bar{n}} \equiv \partial_{\bar{\ell}}\, g_{m\bar{n}} - \Gamma_{\bar{\ell}}{}^{\bar{r}}{}_{\bar{n}}\, g_{m\bar{n}} = 0. \tag{A.15}$$

These equations can be solved by using the inverse matrix $g^{\bar{n}\ell}$

$$g_{m\bar{n}}\, g^{\bar{n}\ell} = g^{\bar{\ell}n} g_{n\bar{m}} = \delta^{\ell}_{m} \tag{A.16}$$

and give

$$\Gamma_{\ell}{}^{r}{}_{m} = (\partial_{\ell}\, g_{m\bar{n}}) g^{\bar{n}r} \tag{A.17}$$

and

$$\Gamma_{\bar{\ell}}{}^{\bar{r}}{}_{\bar{m}} = g^{\bar{r}n} \partial_{\bar{\ell}}\, g_{n\bar{m}}. \tag{A.18}$$

From (A.2) we see that

$$\Gamma_{\ell}{}^{r}{}_{m} = \Gamma_{m}{}^{r}{}_{\ell} \tag{A.19}$$

and

$$\Gamma_{\bar{\ell}}{}^{\bar{r}}{}_{\bar{m}} = \Gamma_{\bar{m}}{}^{\bar{r}}{}_{\bar{\ell}} \tag{A.20}$$

(absence of torsion). Notice the relative simplicity of (A.17), (A.18) as compared with the general formula (2.2) for the Christoffel connection. A Kähler manifold is a Riemann manifold having a very particular structure.

One can write the above formulas in the notation of differential forms. Introduce the matrix one form

$$dz^{\ell} \Gamma_{\ell}{}^{r}{}_{m} = (\Gamma)^{r}{}_{m} \tag{A.21}$$

and use matrix notation for the metric tensor as well. Then the metric compatibility condition can be written as

$$dg - \Gamma^T g - g\bar{\Gamma} = 0. \tag{A.22}$$

This equation separates into two equations

$$\partial g - \Gamma^T g = 0 \tag{A.23}$$

(equivalent to (A.14)) and

$$\bar{\partial} g - g\bar{\Gamma} = 0 \tag{A.24}$$

(equivalent to (A.15)). Here Γ^T is the transposed of the matrix Γ and $\bar{\Gamma}$ the complex conjugate matrix. Finally (A.23) and (A.24) are solved by

$$\Gamma^T = (\partial g)g^{-1}, \quad \bar{\Gamma} = g^{-1}\bar{\partial} g \tag{A.25}$$

which are equivalent to (A.17) and (A.18) respectively.

To obtain the Riemann tensor, we first define the matrix valued Riemann two-form

$$R^r{}_m = (d\Gamma + \Gamma^2)^r{}_m. \tag{A.26}$$

From (A.25) we see that

$$\begin{aligned} R^T &= d\Gamma^T - (\Gamma^T)^2 = d(\partial g\, g^{-1}) - \partial g\, g^{-1}\partial g\, g^{-1} \\ &= \overline{\partial}\partial g\, g^{-1} + \partial g\, g^{-1}(\partial + \overline{\partial}) g\, g^{-1} - \partial g\, g^{-1}\partial g\, g^{-1} \\ &= \overline{\partial}\partial g\, g^{-1} + \partial g\, g^{-1}\overline{\partial} g\, g^{-1} \\ &= \overline{\partial}(\partial g\, g^{-1}) = \overline{\partial}\Gamma^T . \end{aligned} \tag{A.27}$$

This gives the components

$$\begin{aligned} R_{\bar{\ell}n,}{}^r{}_m &= \partial_{\bar{\ell}}(\partial_n\, g_{m\bar{s}}\, g^{\bar{s}r}) = \partial_{\bar{\ell}}\, \Gamma_n{}^r{}_m \\ &= (\partial_{\bar{\ell}}\partial_n\, g_{m\bar{s}} - \partial_n\, g_{m\bar{t}}\, g^{\bar{t}u}\, \partial_{\bar{\ell}}\, g_{u\bar{s}})g^{\bar{s}r}. \end{aligned} \tag{A.28}$$

Lowering the index r we obtain

$$R_{\bar{\ell}n,\bar{r}m} \equiv g_{s\bar{r}} R_{\bar{\ell}n,}{}^s{}_m = \partial_{\bar{\ell}}\partial_n\, g_{m\bar{r}} - \partial_n\, g_{m\bar{t}}\, g^{\bar{t}u}\, \partial_{\bar{\ell}}\, g_{u\bar{r}}. \tag{A.29}$$

From the first line of (A.28) we see that the Kähler condition (A.2) implies the symmetry relation

$$R_{\bar{\ell}n,}{}^r{}_m = R_{\bar{\ell}m,}{}^r{}_n. \tag{A.30}$$

Also, (A.29) shows that the hermiticity relation

$$\overline{R_{\bar{\ell}n,\bar{r}m}} = R_{\bar{n}\ell,\bar{m}r} \tag{A.31}$$

is satisfied. The only other nonvanishing components of the Riemann tensor are obtained by complex conjugation, *e.g.*

$$\overline{R_{\bar{\ell}n,\bar{r}m}} = R_{\ell\bar{n},r\bar{m}}, \tag{A.32}$$

or using the antisymmetry condition

$$R_{\ell\bar{n},r\bar{m}} = -R_{\bar{n}\ell,r\bar{m}} = -R_{\ell\bar{n},\bar{m}r}. \tag{A.33}$$

We finally come to the Ricci tensor. It is defined as usual

$$R_{\bar{\ell}m} = R_{\bar{\ell}n,}{}^{n}{}_{m}\,; \tag{A.34}$$

however, for a Kähler manifold, we see from the symmetry condition (A.30) that it is also obtained by summing over the last two indices

$$R_{\bar{\ell}m} = R_{\bar{\ell}m,}{}^{n}{}_{n}. \tag{A.35}$$

Using this formula and (A.28) one finds

$$\begin{aligned} R_{\bar{\ell}m} &= \partial_{\bar{\ell}}(\partial_m\, g_{n\bar{s}}\, g^{\bar{s}n}) \\ &= \partial_{\bar{\ell}}\partial_m\, \log\det g_{n\bar{s}} \equiv -R_{m\bar{\ell}}. \end{aligned} \tag{A.36}$$

The other components of the Ricci tensor vanish

$$R_{\ell m} = R_{\bar{\ell}\bar{m}} = 0. \tag{A.37}$$

One also defines the Ricci two-form

$$\begin{aligned} \rho &= -2iR_{m\bar{\ell}}dz^m d\bar{z}^{\bar{\ell}} \\ &= -2i\bar{\partial}\partial\, \log\det g_{n\bar{s}}\,. \end{aligned} \tag{A.38}$$

It is obvious that it has the very important property of being closed,

$$d\rho = 0, \tag{A.39}$$

i.e. the Ricci tensor satisfies vanishing curl conditions analogous to those satisfied by the Kähler metric.

Formulas (A.36) and (A.38) show that the Ricci tensor of the manifold can be interpreted as the curvature of a line bundle, the metric for the line bundle being given by the determinant of the metric tensor of the manifold. This means that the transformation functions for the line bundle are the Jacobians of the coordinate transformations. It is the bundle of scalar densities. Ref. 6 exploits this connection and uses directly vacuum line bundles thereby avoiding the route through the Riemann tensor. We note that in Refs. 2 and 5 the Ricci tensor is computed from the Riemann tensor by using (A.35). Our computation in Sec. 5 using (A.34) appears somewhat simpler.

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