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On the Calculation of Similarity Solutions of Partial Differential Equations

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ON THE CALCULATION OF SIMILARITY SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

When a partial differential equation in two independent variables is invariant to a group G of stretching transformations, it has similarity solutions that can be found by solving an ordinary differential equation. Under broad conditions, this ordinary differential equation is also invariant to another stretching group G′, related to G. The invariance of the ordinary differential equation to G′ can be used to simplify its solution, particularly if it is of second order. Then a method of Lie's can be used to reduce it to a first-order equation, the study of which is greatly facilitated by analysis of its direction field. The method developed here is applied to three examples: Blasius's equation for boundary layer flow over a flat plate and two nonlinear diffusion equations, $cc_t = c_{zz}$ and $c_t = (cc_z)_z$.

1. INTRODUCTION

Self-similar functions preserve their forms under certain scale transformations of their magnitudes and arguments. For example, the function $c = \exp(-r^2/4t)/(4\pi t)^{3/2}$ preserves its form under the group of transformations $r' = \lambda r$, $t' = \lambda^2 t$, $c' = \lambda^{-3} c$, where λ is any positive number. This function, as the reader may recognize, is the instantaneous point source solution of the spherically symmetrical diffusion equation $c_t = c_{rr} + 2c_r/r$. Its self-similarity arises from the invariance of the diffusion equation to the wider group of transformations $r' = \lambda r$, $t' = \lambda^2 t$, $c' = \mu c$, where λ and μ are positive numbers independent of each other. Such self-similar solutions to partial differential equations are important because they are much easier to find than other solutions: when the partial differential equation has two independent variables, its self-similar solutions can be found by solving an *ordinary* differential equation.

According to Ames,[1] the study of self-similar solutions began in 1894 with Boltzman,[2] who studied the diffusion equation $c_t = [D(c)c_x]_x$. As Ames also points out, Blasius's[3] 1908 solution of the boundary layer flow over a flat plate is self-similar. So are the 1941 solutions of Taylor[4] and von Neumann[5] for the pressure and flow fields created in air by a point explosion.

Birkhoff[6] (1950) was the first explicitly to mention that invariance of a partial differential equation to a group of transformations (the word "group" now being used in its rigorous sense) could be used to find self-similar solutions. Since the time of Birkhoff's hint, self-similar solutions have been published to problems in diffusion, heat and mass transfer, hydrodynamics, shock propagation, solid mechanics, plasma physics, and applied superconductivity.[7]

Most applications involve one-parameter groups of stretching transformations, i.e., transformations in which images are formed by multiplication by powers of the group parameter. Quite often, the ordinary differential equation to which the problem reduces is also invariant to a stretching group. This is no coincidence, and the conditions for it are explained below (Sect. 2). The group invariance of the ordinary differential equation can be exploited in two ways, the discussion of which forms the bulk of this paper. First, if the solution of the ordinary differential equation satisfies two-point boundary conditions, we can use the group invariance to find it without trial and error. This is illustrated below with Blasius's problem of boundary layer flow over a flat plate (Sect. 4). Second, if the ordinary differential equation is of second order, we can find, with the help of the group, new independent and dependent variables whose use reduces the equation to one of first order.[8] This first-order equation can be analyzed by studying its direction field.[9] The advantages of this reduction are illustrated below for the nonlinear diffusion equations $cc_t = c_{zz}$ and $c_t = (cc_z)_z$ (Sects. 5-9). These examples make clear how important the group invariance of the ordinary differential equation is to the calculation of self-similar solutions.

2. GROUP INVARIANCE OF THE ORDINARY DIFFERENTIAL EQUATION

Suppose we have a partial differential equation with one dependent variable c and two independent variables z and t. Suppose the partial differential equation is invariant to the family of one-parameter groups of transformations:

$$\left.\begin{aligned} z' &= \lambda z \\ t' &= \lambda^{\beta} t \\ c' &= \lambda^{\alpha} c \end{aligned}\right\} \quad 0 < \lambda < \infty \, . \tag{1}$$

(Note that we lose no generality by taking the exponent of λ in the first line equal to 1.)

If the partial differential equation is written in the primed variables and the substitution (1) made, it quite often happens that each term in the partial differential equation is multiplied by a power of λ. The exponents of λ in these multipliers are linear combinations of α, β, and 1. Invariance means that the exponents of all terms are equal, and this equality leads to one or two independent linear equations in α and β. (There cannot be more than two independent linear equations because that would mean there is no solution for α and β and contradict the assumed invariance of the partial differential equation.) If there are two linear relations, α and β are uniquely determined, and the family of groups (1) reduces to a single group. If there is one linear relation, say,

$$M\alpha + N\beta = L \, , \tag{2}$$

then the partial differential equation is invariant to a one-parameter family of one-parameter groups. The groups of the family are labeled by α or β; the transformations of each group are labeled by λ. It is this second case that interests us.

To find self-similar solutions we look for solutions invariant to the group (1). The most general invariant relation connecting the variables c, z, and t can be written:[10]

$$\frac{c}{t^{\alpha/\beta}} = y\left(\frac{z}{t^{1/\beta}}\right), \tag{3}$$

where y is an arbitrary function. Substitution of (3) into the partial differential equation gives an ordinary differential equation that the function y(x) must satisfy (here x is an abbreviation for the argument $z/t^{1/\beta}$). The values of α and β are selected so that the boundary conditions can be satisfied. For example, if $c(0,t)$ = a prescribed constant, $\alpha = 0$ and $\beta = L/N$. If $\int_{-\infty}^{+\infty} c\,dz$ = a prescribed constant, $\alpha = -1$ and $\beta = (L + M)/N$. Denote by α_o and β_o the values, satisfying (2), so selected. Then

$$c = t^{\alpha_o/\beta_o}\, y\left(\frac{z}{t^{1/\beta_o}}\right) \tag{4}$$

is the most general relation connecting c, a, and t that is invariant to group (1) of the family for which $\alpha = \alpha_o$ and $\beta = \beta_o$.

If we transform $c(z,t)$ given by (4) according to other groups of the family (1) for which $\alpha \neq \alpha_o$ and $\beta \neq \beta_o$, its image $c'(z',t')$ must also satisfy the partial differential equation because the latter is invariant to *all* groups of the family (1). We find that

$$c' = \lambda^{(\alpha\beta_o - \alpha_o\beta)/\beta_o}\,(t')^{\alpha_o/\beta_o}\, y\left[\lambda^{(\beta\ \beta_o)/\beta_o}\,\frac{z'}{(t')^{1/\beta_o}}\right]. \tag{5}$$

It is easy to verify that

$$\frac{\alpha\beta_o - \alpha_o\beta}{\beta_o - \beta} = \frac{L}{M}, \tag{6}$$

regardless of the values of α and β as long as α and β satisfy (2). Therefore, if we set $\lambda^{(\beta-\beta_o)/\beta_o} = \mu$ and drop the primes, (5) becomes

$$c = t^{\alpha_o/\beta_o} \mu^{-L/M} y\left(\mu \frac{z}{t^{1/\beta_o}}\right) = t^{\alpha_o/\beta_o} \mu^{-L/M} y(\mu x) \ . \tag{7}$$

The function $c(z,t)$ given by (7) is also a solution of the partial differential equation and is furthermore invariant to (1) when $\alpha = \alpha_o$ and $\beta = \beta_o$.

The one-parameter family of functions of x, $\mu^{-L/M} y(\mu x)$, appearing in (7), is the family of images of $y(x)$ under the one-parameter group

$$\left.\begin{aligned} y' &= \mu^{L/M} y \\ x' &= \mu x \end{aligned}\right\} 0 < \mu < \infty \ . \tag{8}$$

[For $y'(x') \equiv \mu^{L/M} y(x) = \mu^{L/M} y(x'/\mu)$. If we replace μ by $1/\mu$, this becomes $y'(x') = \mu^{-L/M} y(\mu x')$.] A one-parameter family whose curves transform into each other under a group is said to be invariant to the group. Each function $y(x)$ satisfying (4) generates an invariant one-parameter family of functions $y(x)$ also satisfying (4), the invariance of the family being with respect to the group (8).

Suppose that the ordinary differential equation for $y(x)$ is of nth order. The solutions of such an equation form an n-parameter family of curves. From what we have just seen, this n-parameter family must decompose into an (n - 1)-parameter set of one-parameter families, each of which is invariant to (8). But then the entire n-parameter family is invariant to (8). This means that the differential equation for y is also invariant to (8), which is what we wanted to prove.

3. EXCEPTIONAL SOLUTIONS OF SECOND-ORDER EQUATIONS

Lie[8] has given a prescription for finding the most general second-order differential equation invariant to the group (8). If we introduce a group invariant u(x,y) as a new independent variable and a first-differential invariant $v(x,y,\dot{y})$ as a new dependent variable, the most general second-order equation for y in terms of x has the form dv/du = G(u,v), where G(u,v) is some function of u and v. An invariant and a first-differential invariant of (8) are

$$\left.\begin{array}{l} u = \dfrac{y}{x^{a}} \\[2ex] v = \dfrac{\dot{y}}{x^{a-1}} \end{array}\right\} a = \frac{L}{M} . \tag{9}$$

Differentiation gives

$$du = (v - au)\left(\frac{dx}{x}\right), \tag{10a}$$

$$dv = \left[\frac{\ddot{y}}{x^{a-2}} - (a - 1)v\right]\left(\frac{dx}{x}\right) \tag{10b}$$

$$= [F(u,v) - (a - 1)v]\left(\frac{dx}{x}\right) \tag{10c}$$

because Lie's theorem tells us that $\ddot{y}/x^{a-2}$ must be a function F(u,v) of u and v. Thus

$$\frac{dv}{du} = \frac{F(u,v) - (a - 1)v}{v - au} . \tag{11}$$

Any integral curve v(u) of (11) represents a first-order differential equation for y in terms of x, i.e., a one-parameter family of integral curves y(x) that transform into one another under the group (8). In addition to these one-parameter families, there are exceptional solutions arising from the singular points of (11). These exceptional solutions $y(x)$ correspond to constant values u_o and v_o of u and v. Constant values of u_o and v_o mean that as x and y vary, u and v remain fixed at u_o and v_o. Thus $du = dv = 0$, and we see then from (10) that u_o and v_o must satisfy the equations

$$F(u_o,v_o) = (a - 1)v_o \tag{12a}$$

and

$$v_o = au_o \ . \tag{12b}$$

The only solutions of (12), of course, are the singular points of (11), assuming $F(u,v)$ is not itself singular. Thus the solution $y = u_o x^a$ corresponds to the singular point (u_o,v_o).

4. EXAMPLE: BLASIUS'S EQUATION

Blasius's equation for the stream function c of the boundary layer developing along a flat plate can be written

$$c_z c_{zt} - c_t c_{zz} = c_{zzz} \ , \tag{13}$$

where z measures distance transverse to the plate and t measures distance along the plate. (Special units have been chosen in which the kinematic viscosity and the mean stream velocity are both equal to one.) Equation (13) is invariant to the group (1) if $\alpha - \beta = -1$, i.e., if $M = 1$, $N = -1$, and $L = -1$. The boundary conditions of Blasius's problem are:

$$c(0,t) = 0 \ , \tag{14a}$$

$$c_z(0,t) = 0 \ , \tag{14b}$$

$$c_z(\infty,t) = 1 \ , \tag{14c}$$

$$c_t(z,0) = 0 \ . \tag{14d}$$

Equations (14a) and (14b) become $y(0) = 0$ and $\dot{y}(0) = 0$, respectively. In order to satisfy (14c) we must have $\alpha = \alpha_o = 1$, $\beta = \beta_o = 2$. Then (14c) becomes $\dot{y}(\infty) = 1$. Equation (14d) becomes $\lim_{x\to\infty}(y - x\dot{y}) = 0$ or what is the same thing, $\lim_{x\to\infty}(y/x) = A =$ a constant; this condition is the same as the condition $\dot{y}(\infty) = 1$ if A is chosen to be one. The differential equation for y turns out to be

$$2\dddot{y} + y\ddot{y} = 0 \ , \tag{15a}$$

and the boundary conditions again are

$$y(0) = \dot{y}(0) = 0 \tag{15b}$$

and

$$\dot{y}(\infty) = 1 \; . \tag{15c}$$

The group (8) has the form

$$\left.\begin{aligned} y' &= \frac{y}{\mu} \\ x' &= \mu x \end{aligned}\right\} \tag{16}$$

in this example. It is easy to verify that (15a) is invariant to (16).

The boundary conditions (15b) and (15c) refer respectively to $x = 0$ and $x = \infty$. Neither (15b) nor (15c) alone is sufficient to allow numerical solution of (15a). Ordinarily, we would assume a value of $\ddot{y}(0)$, integrate (15a) to large x, find $\dot{y}(\infty)$, correct $\ddot{y}(0)$, and repeat. We can avoid repetition by using the relation $\dot{y}' = \dot{y}/\mu^2$ to find the value of μ that will make $\dot{y}'(\infty) = 1$. Then using (12) we can find $y'(x')$ by scaling the function $y(x)$ calculated in the first numerical integration. If high accuracy is sought, much repetition can be avoided in this way.

5. EXAMPLE: THE EQUATION $cc_t = c_{zz}$

The equation $cc_t = c_{zz}$ arises in the problem of transient heat transfer from a heated surface to a single-phase, near-critical fluid and in the problem of the expulsion of fluid from a long, slender, heated tube. In both of these problems the boundary and initial conditions are $c(z,0) = 0$, $c(\infty,t) = 0$, and $c_z(0,t) = -b$, a prescribed constant. These are the boundary conditions we shall consider here.

The partial differential equation is invariant to the group (1) if $\alpha - \beta = -2$, i.e., if $M = 1$, $N = -1$, and $L = -2$. In order to satisfy the boundary condition that $c(0,t)$ is constant, α must be chosen equal to $\alpha_o = 1$; then $\beta = \beta_o = 3$. For convenience, we introduce a factor of $\sqrt{3}$ into the definition of x, i.e., take $c = t^{1/3}\, y(z/\sqrt{3}\, t^{1/3}) = t^{1/3}\, y(x)$. Then

$$\ddot{y} = y(y - x\dot{y}) \tag{17a}$$

and

$$y(\infty) = 0, \quad \dot{y}(0) = -b\sqrt{3}\ . \tag{17b}$$

The group (8) now takes the form

$$\left.\begin{aligned} y' &= \mu^{-2}\, y \\ x' &= \mu x \end{aligned}\right\}, \tag{18}$$

and, as expected, the ordinary differential equation (17a) is invariant to it. According to the theorem of Lie,[8] if we introduce as new independent and dependent variables a group invariant $u(x,y)$ and a first-differential group invariant $v(x,y,\dot{y})$, Eq. (17a) will become a first-order differential equation for v in terms of u. Analysis of the direction field of this first-order equation can tell us much about the solutions of (18a); in this problem it will give us the value of $y(0)$ at the cost of a single integration.

The choice of u and v is not unique; a convenient choice is

$$u = x^2 y \tag{19a}$$

and

$$v = x^2(y - x\dot{y}) \ . \tag{19b}$$

Using (17) we find that

$$\frac{dv}{du} = \frac{v(2 - u)}{3u - v} \ . \tag{20}$$

6. THE DIRECTION FIELD OF EQ. (20)

Shown in Fig. 1 is the direction field of Eq. (20). The slope dv/du vanishes on the lines L_1: $v = 0$ and L_2: $u = 2$ and is infinite on the line L_3: $v = 3u$. There are two singular points, 0: (0,0) and P: (2,6). The singular point P is a saddle point; the origin 0 is a node. Traversing P are two separatrices S_1 and S_2. The separatrix S_1 also traverses the singular point 0. Some typical integral curves are also shown in Fig. 1.

The family of integral curves of Eq. (17) we are seeking transforms into itself under the transformations of the group (18). In other words, the image of each curve of the family is another curve of the family. The family thus corresponds to a *single* integral curve in the (u,v) plane. Furthermore, this single curve must pass through the origin 0 because u and v both approach zero as x approaches zero with y and $\dot{y}$ remaining finite. Of the integral curves passing through the origin, the separatrix S_1 is the one we want. For in the neighborhood of the singular point P, $y \cong 2/x^2$, and this is a satisfactory asymptotic behavior for the integral curve we are seeking.

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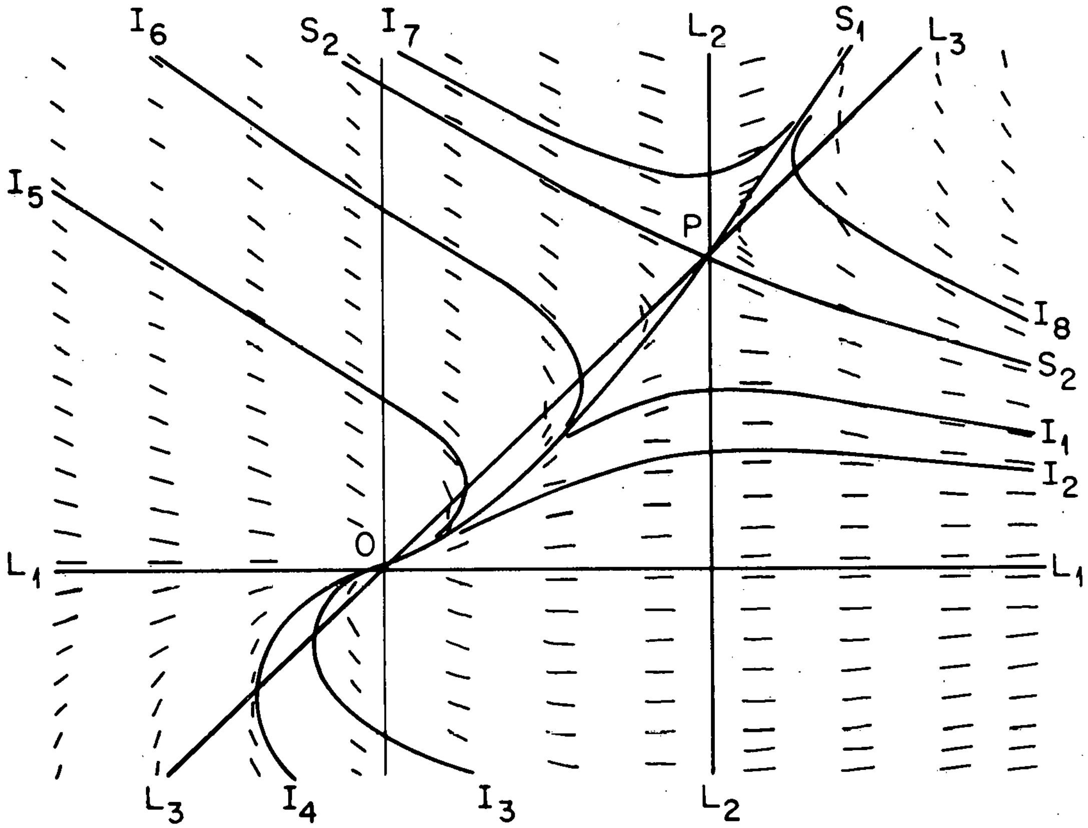

Fig. 1. The direction field of Eq. (20). O and P are the singular points, S_1 and S_2 are the separatrices, and the curves marked with I are integral curves.

7. CALCULATION OF THE SEPARATRIX

Near the origin where $u \ll 2$, Eq. (20) can be approximated by the homogeneous equation $dv/du = 2v/(3u - v)$. By "homogeneous" we mean that the differential equation is invariant to the group $v' = \lambda v$, $u' = \lambda u$. The separatrix S_1, being an invariant curve of the group, must have the form $v = mu$. Substitution into the homogeneous differential equation shows that $m = 1$.

If we substitute $v = u + w$ into (20), then to lowest order $dw/du = 3w/2u$, so that $w = Cu^{3/2}$. This suggests that v may be expanded in powers of $u^{1/2}$ near the origin. If we set

$$v = u + Cu^{3/2} + Du^2 + Eu^{5/2} + Fu^3 + Gu^{7/2} + Hu^4 + Ju^{9/2} + Ku^5 + \dots, \tag{21}$$

substitute into (20), clear fractions, and equate coefficients of equal powers of u, we get

$$D = \frac{3}{2} C^2 - 1 , \quad E = \frac{1}{2}\left(\frac{7}{2} CD - C\right) , \quad F = \frac{1}{3} (4CE + 2D^2 - D) ,$$

$$G = \frac{1}{4} \left[\frac{9}{2}(DE + FC) - E \right] , \quad H = \frac{1}{5}\left[\frac{5}{2} E^2 + 5(FD + GC) - F\right] ,$$

$$J = \frac{1}{6} \left[\frac{11}{2}(FE + GD + CH) - G \right] ,$$

$$K = \frac{1}{7} [3F^2 + 6(GE + HD + CJ) - H] . \tag{22}$$

From Eq. (22) it is clear that once we fix C, all the higher coefficients in (21) are determined. Since all the integral curves passing through the origin are tangent to one another (and to the line $v = u$), it is clear that they are distinguished from one another by the value of C. Finally,

$$C = \lim_{u \to o} \left(\frac{v - u}{u^{3/2}} \right) = - \frac{\dot{y}(0)}{[y(0)]^{3/2}} . \tag{23}$$

A similar procedure near the singular point P: (2,6) gives for the separatrix S_1

$$t = As + Bs^2 + Rs^3 + Ss^4 + \ldots, \tag{24a}$$

$$t = v - 6, \quad s = u - 2 , \tag{24b}$$

$$A = \frac{1}{2}(3 + \sqrt{33}) , \quad B = \frac{A}{3A - 6} ,$$

$$R = \frac{B(1 - 2B)}{4A - 9} , \quad S = \frac{L(1 - 5B)}{5A - 12} . \tag{24c}$$

We find the value of C on S_1 by using (24) to advance a short distance along S_1 away from P. Then we integrate (20) numerically, advancing along S_1 towards 0. When we get close to 0, we match the numerical solution to the series (21) by choosing C correctly. In this way, with a single numerical integration, we find

$$C = 0.932 . \tag{25}$$

This numerical integration, as well as all others mentioned later, was performed by the fourth-order Runge-Kutta method on a programmable desk calculator (Hewlett-Packard 97). Once the value of C is in hand, calculation of y is an easy matter because consistent initial values can be obtained from (23). Figure 2 shows the curve of y versus x for which $y(0) = 1$. As expected, $y \sim 2/x^2$ for large x. Also shown is the following simple analytic approximation y:

$$y \cong \left(1 + Cx + \frac{x^2}{2}\right)^{-1} , \tag{26}$$

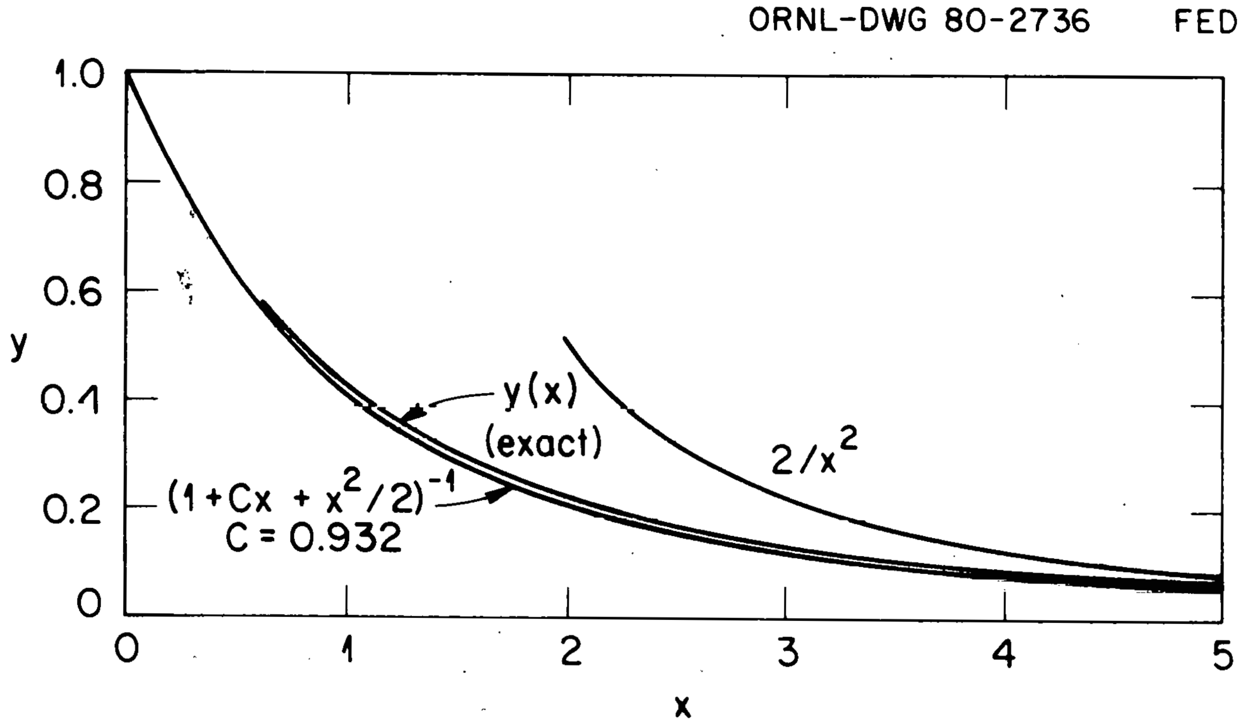

Fig. 2. The function y(x) in Sect. 7 for the case y(0) = 1.

for which $y(0) = 1$ and $\dot{y}(0) = -C$. It follows from (18) that in general

$$y \cong \lambda^2 \left(1 + \lambda Cx + \frac{\lambda^2 x^2}{2}\right)^{-1} . \qquad (27)$$

Finally,

$$\frac{c(0,t)}{t^{1/3}} = 1.51\ b^{2/3} . \qquad (28)$$

8. GENERALIZATION TO $c^n c_t = c_{zz}$

The partial differential equation $c^n c_t = c_{zz}$ can be treated in the same way when $n \neq 1$ as when $n = 1$. In general, $n\alpha - \beta = -2$ so that $M = n$, $N = -1$, and $L = -2$. To satisfy the boundary condition $c_z(0,t) = -b$, a prescribed constant, we must take $\alpha = \alpha_o = 1$ and $\beta = \beta_o = n + 2$. If we set $c = t^{1/\beta_o}\, y\left(z/\sqrt{\beta_o}\, t^{1/\beta_o}\right)$, we find $\ddot{y} = y^n(y - x\dot{y})$, $\dot{y}(0) = -b\sqrt{\beta_o}$, and $y(\infty) = 0$ as before. As expected, the differential equation for y is invariant to the group $y' = \mu^{-2/n}\, y$, $x' = \mu x$. If we choose $u = x^{2/n}\, y$ and $v = x^{2/n}(y - x\dot{y})$ we find

$$\frac{dv}{du} = \frac{v(2 - nu^n)}{(n + 2)u - nv}\ . \tag{29}$$

The direction field of (26) is similar to that of Fig. 1. There is a right triangle formed by lines L_1, L_2, and L_3 in the (u,v) plane like the one formed by L_1, L_2, and L_3 in Fig. 1. On L_3: $(n + 2)\, u = nv$ the slope is infinite; on L_2: $u = (2/n)^{1/n}$ and L_1: $v = 0$ the slope is zero. The two vertices on the line L_2 are singular points. These points are O: (0,0) and P: $[(2/n)^{1/n}, (1 + 2/n)(2/n)^{1/n}]$. They are joined by a separatrix that near the origin has the form

$$v = u + C_n u^{(n+2)/2} + \ldots \tag{30}$$

so that

$$C_n = \lim_{u \to o} \left[\frac{v - u}{u^{(n+2)/2}}\right] = \frac{-\dot{y}(0)}{[y(0)]^{(n+2)/2}}\ . \tag{31}$$

Finally, the separatrix near P corresponds to asymptotic behavior of y(x) of the form

$$y \sim \left(\frac{2}{n}\right)^{1/n} x^{-2/n} , \tag{32a}$$

$$\dot{y} \sim -\left(\frac{2}{n}\right)^{1+1/n} x^{-1-2/n} . \tag{32b}$$

When n = 2, I used the power series method to calculate C_2 and found $C_2 = 0.777$. When n = 0, the ordinary differential equation $\ddot{y} = y - x\dot{y}$ is solved by

$$y = e^{-x^2/2} - x \int_x^{\infty} e^{-x^2/2}\, dx . \tag{33}$$

It follows from (33) that $y(0) = 1$ and $\dot{y}(0) = -\sqrt{\pi/2}$. Thus $C_o = \sqrt{\pi/2} = 1.253$. Finally,

$$\frac{c(0,t)}{t^{1/(n+2)}} = \left(\frac{\sqrt{n+2}}{C_n}\right)^{2/(n+2)} b^{2/(n+2)} \tag{34}$$

Knowing C_0, C_1, and C_2, we can interpolate to find C_n for intermediate values. It is worth remarking that when $n \geq 2$, $\int_0^{\infty} y dx$ diverges, a fact which may be of importance in applications.

9. THE EQUATION $c_t = (cc_z)_z$

This partial differential equation $c_t = (cc_z)_z$ is invariant to the group (1) if $\alpha + \beta = 2$, i.e., if M = 1, N = 1, and L = 2. If we set

$$c = t^{\alpha/\beta}\, y\left(\frac{z}{t^{1/\beta}}\right) , \tag{35}$$

we find the following ordinary differential equation for y:

$$\beta(y\dot{y})^{\bullet} = \alpha y - x\dot{y} . \tag{36}$$

Equation (36) is invariant to the group $y' = \mu^2 y$, $x' = \mu x$ as expected. If we set $u = y/x^2$ and $v = \dot{y}/x$ we obtain the first-order equation:

$$\frac{dv}{du} = \frac{\alpha u - v - \beta v^2 - \beta vu}{\beta u(v - 2u)} . \tag{37}$$

The choice of α and β depends on the boundary conditions. Some typical boundary conditions together with the corresponding values of α and β are as follows:

1. Clamped temperature: $c(0,t)$ constant $\alpha = 0$, $\beta = 2$
2. Clamped flux: $(cc_z)_{z=0}$ constant $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$
3. Ramped temperature: $c(0,t) \sim t$ $\alpha = 1$, $\beta = 1$
4. Point source: $\int_{-\infty}^{+\infty} c\,dz$ constant $\alpha = -1$, $\beta = 3$

Let us first consider case 4, for which (37) becomes

$$\frac{dv}{du} = -\frac{(u + v)(1 + 3v)}{3u(v - 2u)} . \tag{38}$$

The direction field of (38) is shown in Fig. 3. The slope dv/du vanishes on the lines L_1: $u + v = 0$ and L_2: $v = -1/3$ and is infinite on the lines L_3: $u = 0$ and L_4: $v = 2u$. There are three singular points, O: (0,0), P: (0,-1/3), and Q: (-1/6,-1/3). It is immediately evident from the figure that $v = -1/3$ solves (38). Thus

$$\dot{y} = -\frac{x}{3} \tag{39a}$$

and

$$y = \frac{(x_o^2 - x^2)}{6}, \tag{39b}$$

a solution found earlier by Pattle.[11] Solution (39) enables us to satisfy the requirement that y be zero at infinity by making y vanish for $x \geqslant x_o$. For $x < x_o$, we use (39b). Interestingly, there is no solution y(x) for which y and $\dot{y}$ approach zero continuously as x approaches infinity. Such a solution would correspond to an integral curve in the (u,v) plane passing through the origin. However, none of the integral curves that do so ever attains the limit $u \to \infty$.

Let us now turn to case 2. Then (37) becomes

$$\frac{dv}{du} = \frac{u - 2v - 3v^2 - 3uv}{3u(v - 2u)}. \tag{40}$$

Figure 4 shows the direction field of (40). The slope dv/du = 0 on the curve C: $u = v(2 + 3v)/(1 - 3v)$. (The curve C has two branches, one of which is shown in Fig. 4. The other branch is in the second quadrant and is of no concern to us here.) The slope $dv/du = \infty$ on the lines L_1: $u = 0$ and L_2: $v = 2u$. There are three singular points, O: (0,0), P: (0,-2/3), and Q: (-1/6,-1/3), and two separatrices, S_1 and S_2: $v = u/2$. The singular point Q leads to the exceptional solution $y = -x^2/6$ and the separatrix S_2 leads to the family of solutions $y = \text{constant}\ \sqrt{x}$. Neither of the solutions is of any use to us here since neither is ever zero for any $x > 0$.

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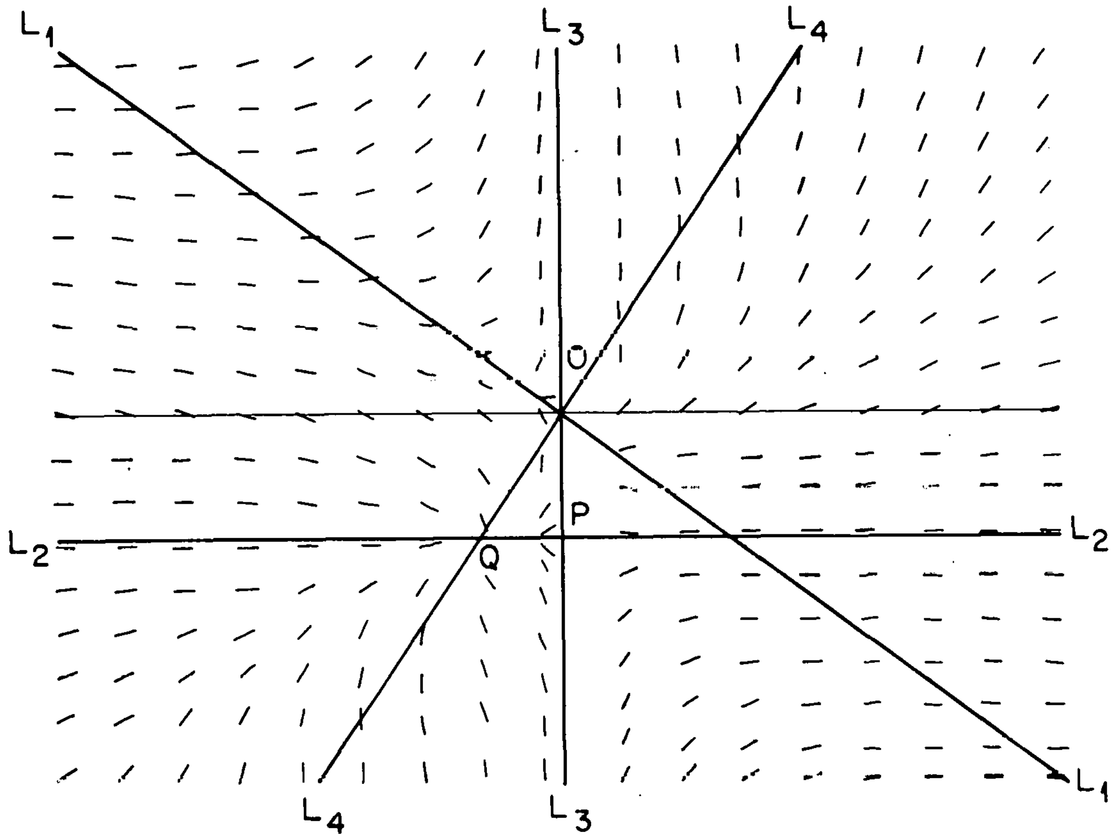

Fig. 3. The direction field of Eq. (38).

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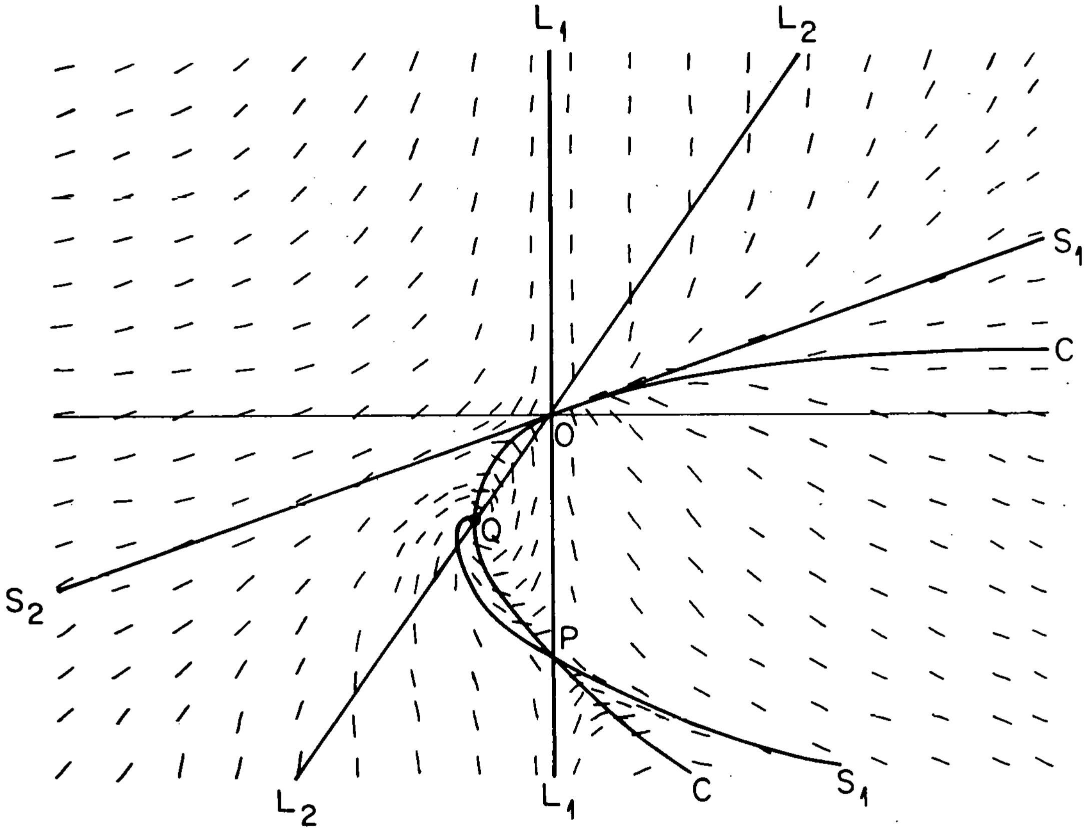

Fig. 4. The direction field of Eq. (40).

There is no solution for which y and $\dot{y} \to 0$ as $x \to \infty$. Such a solution must correspond to an integral curve in the (u,v) plane passing through the origin. All such curves are tangent to S_2 near the origin and thus behave asymptotically like $\sqrt{x}$ for large x. If we look for solutions y(x) that have a root at some $x = x_o$, then at x_o, $y = 0$ and $\dot{y} < 0$ so that $u = 0$, $v < 0$. The only point that fills the bill is the singular point P, and the family of curves y(x) we are seeking corresponds to the separatrix S_1. The value of v at the point P gives the slope $\dot{y}(x_o) = -2x_o/3$, and knowing this slope, we can undertake a numerical integration to find y(x). The results of such an integration for the case $x_o = 1$ are shown in Fig. 5. Other cases can be obtained by transformation with the group $y' = \mu^2 y$, $x' = \mu x$.

When u is large and positive, the separatrix S_1 has the asymptotic form $v = -C\sqrt{u}$, where C is a constant. If we substitute the definitions of u and v in this equation, we find $\dot{y} = -C\sqrt{y}$. Since $u \to \infty$ as $x \to 0$, this means

$$\frac{-\dot{y}(0)}{\sqrt{y(0)}} = C \ . \tag{41}$$

The numerical integration of (36) used to draw Fig. 5 gave C = 0.679; a numerical integration of (40) starting on S_1 near P gave the same result.

At this point we break off further discussion of examples.

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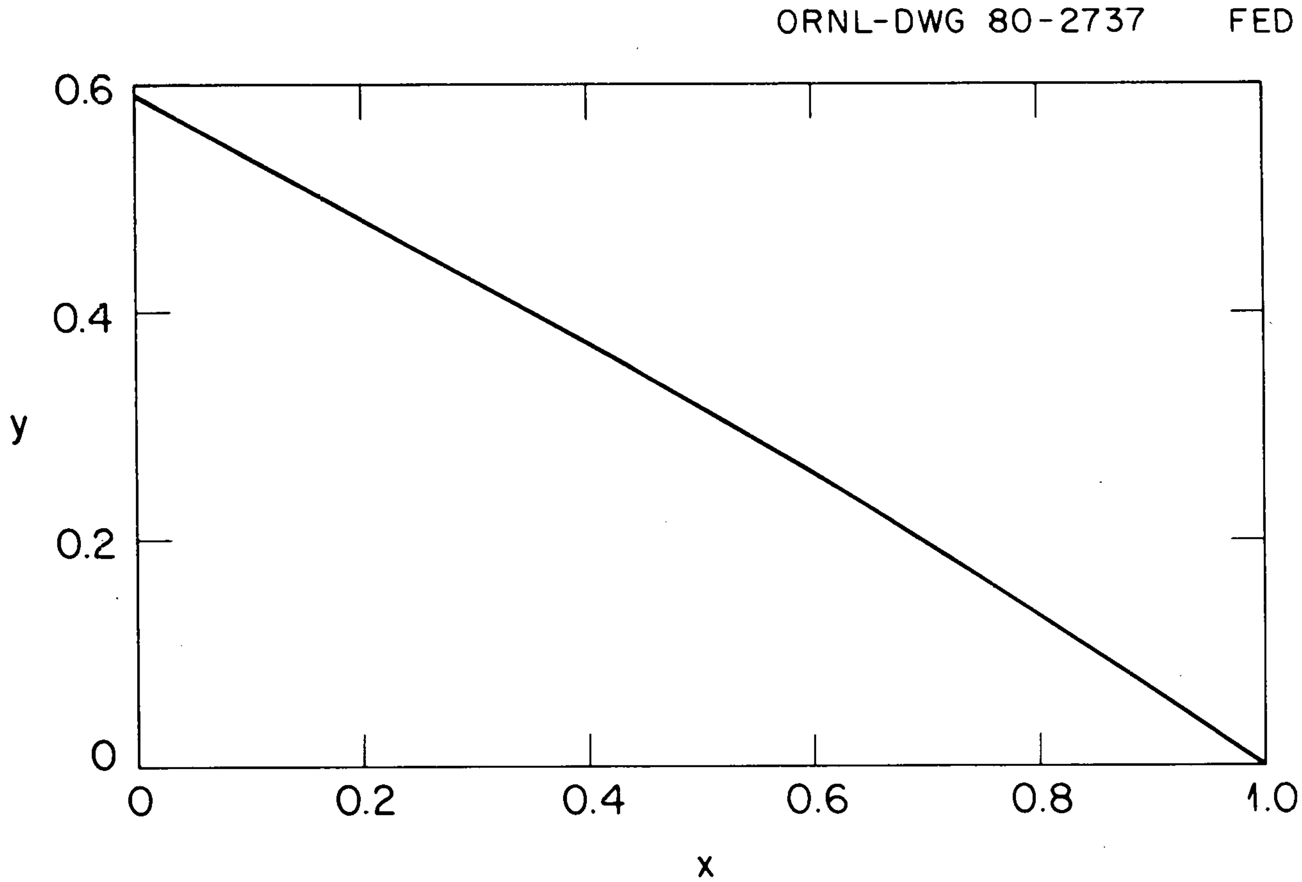

Fig. 5. The solution y(x) of Eq. (36) for case 2: $\alpha = 1/2$, $\beta = 3/2$ when $x_o = 1$.

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