SIMILARITY METHODS FOR REACTIVE FLOW

by

J.D. Logan and J.J. Perez

Technical Report No. 55

FINAL REPORT

SIMILARITY METHODS FOR REACTIVE FLOW

by

J.D. Logan and J.J. Perez

Technical Report No. 55

Prepared for:

The University of California
Lawrence Livermore Laboratory
P.O. Box 808
Livermore, Ca. 94550

Attn: H. Pitkin, L-127
Technical Contact: R.C. Weingart, L-35
ERDA Contract 7173303
August 1977

1. Introduction

The purpose of this report is to show, using similarity or group theoretic methods, that the one-dimensional, time-dependent, Lagrangian, equations of reactive flow can be reduced, via a similarity transformation, to a system of ordinary differential equations, from which a class of invariant solutions can be determined.

The present work differs from that of Cowperthwaite [1] in that we actually assume a form for the reaction rate and treat the entire system rather than calculate the reaction rate from pressure or particle velocity gage data. Sternberg [2] has also investigated similarity solutions in reactive flow, but his calculations are in Eulerian coordinates and involve different reaction rates.

In Lagrangian form, the equations of hydrodynamics can be written

$$\frac{\partial v}{\partial t} = v_0 \frac{\partial u}{\partial h} \qquad \text{(conservation of mass)} \qquad (1)$$

$$\frac{\partial u}{\partial t} = -V_0 \frac{\partial p}{\partial h} \qquad \text{(conservation of momentum)} \qquad (2)$$

$$\frac{\partial p}{\partial t} = -\frac{kp}{v}\frac{\partial v}{\partial t} + \frac{\Gamma q}{v}\frac{\partial \lambda}{\partial t} \qquad \text{(conservation of energy)} \qquad (3)$$

where t is time, h is Lagrangian position, u is particle velocity, p is pressure, v is specific volume, and λ is the reaction coordinate. The reactants and the products are assumed to have the same polytropic equation of state with the same polytropic index k. The parameters Γ and q are the Grüneisen constant and the specific heat of reaction, respectively (see Appendix I for a derivation of eqn (3)).

We assume that the chemical reaction is governed by the equation

$$\frac{\partial\lambda}{\partial t} = L(1 - \lambda)^{\alpha}p^{\beta} \tag{4}$$

where L, α, and β are constants. Hence, we are assuming a pressure dependent rate law.

Finally, the shock discontinuity is governed by the Rankine-Hugoniot jump conditions

$$v_0(D - u_1) = v_1D \tag{5}$$

$$p_1 - p_0 = \frac{1}{v_0} Du_1 \tag{6}$$

$$e_1 - e_0 = \frac{1}{2} (p_1 + p_0)(v_0 - v) \tag{7}$$

where D is the time-dependent shock velocity and e is the specific internal energy. The subscript zero denotes the thermodynamic and hydrodynamic values ahead of the shock, and the subscript one denote those values immediately behind the shock.

2. Invariance Criteria

For convenience, we introduce the notation

$$x^1 = t,\ x^2 = h,\ u^1 = u,\ u^2 = p,\ u^3 = v,\ u^4 = \lambda$$

$$p_i^k = \partial u^k/\partial x^i$$

The partial differential equations (1) - (4) can then be rewritten as

$$A_1 \equiv p_1^3 - v_0 p_2^1 = 0 \tag{1'}$$

$$A_2 \equiv p_1^1 + v_0 p_2^2 = 0 \tag{2'}$$

$$A_3 \equiv p_1^2 + \frac{ku^2}{u^3} p_1^3 - \frac{rq}{u^3} p_1^4 = 0 \tag{3'}$$

$$A_4 \equiv p_1^4 - L(1 - u^4)^\alpha (u^2)^\beta = 0 \tag{4'}$$

We seek a *one-parameter family of transformations* (a one-parameter local Lie group) of the form

$$\bar{x}^i = x^i + \varepsilon \xi_x^i(x^i, u^\ell), \quad i = 1,2 \tag{8}$$

$$\bar{u}^k = u^k + \varepsilon \xi_u^k(x^i, u^\ell), \quad k = 1,\ldots,4 \tag{9}$$

under which (1)' - (4)' are invariant in the following sense. We say that (1)' - (4)' is constantly conformally invariant under (8) - (9) if

$$\frac{\partial}{\partial \varepsilon} A_i(\bar{x}^j, \bar{u}^\ell, \bar{p}_j^\ell)\Big|_{\varepsilon=0} = \alpha_i A_i(x^j, u^\ell, p_j^\ell) \tag{10}$$

for $i = 1,2,3,4$. We note that (9) requires that solutions get mapped to solutions, but the equations are not "form" invariant. The α_i are constants which depend on the transformation.

It is clear that in order to calculate the left-hand-side of (10) we must determine how the derivatives p_j^ℓ transform. It is well-known (see Ovsjannikov [3] or Bluman and Cole [4]) that

$$\overline{p}_i^k = p_i^k + \varepsilon \xi_{pi}^k (x^j, u^\ell, p_j^\ell) \tag{11}$$

where

$$\xi_{pi}^k = \frac{\partial \xi_u^k}{\partial x^i} + \frac{\partial \xi_u^k}{\partial u^\ell} p_i^\ell - \frac{\partial \xi_x^j}{\partial x^i} p_j^k - \frac{\partial \xi_x^j}{\partial u^\ell} p_j^k p_i^\ell \tag{12}$$

Consequently, the invariance condition (10) may be written

$$\frac{\partial A_i}{\partial x^j} \xi_x^j + \frac{\partial A_i}{\partial u^k} \xi_u^k + \frac{\partial A_i}{\partial p_j^k} \xi_{pj}^k = \alpha_i A_i \tag{13}$$

for $i = 1,\dots,4$ (no sum on i). From these four equations, which are first-order quasilinear partial differential equations, we can determine the group generators ξ_u^k and ξ_x^i. Once the group is known, a similarity variable will be determined from the invariant surface conditions (Refs. [3], [4]).

3. Invariance of Eqns. (1) - (3).

For the mass equation, (13) becomes

$$\frac{\partial A_1}{\partial p_1^3} \xi_{p1}^3 + \frac{\partial A_1}{\partial p_2^1} \xi_{p2}^1 = \alpha_1 (p_1^3 - v_0 p_2^1)$$

Substituting ξ^3_{p1} and ξ^1_{p2} from (12) gives

$$\frac{\partial\xi^3_u}{\partial x^1} + \frac{\partial\xi^3_u}{\partial u^\ell} p^\ell_1 - p^3_j \frac{\partial\xi^j_x}{\partial x^1} - p^3_j p^\ell_1 \frac{\partial\xi^j_x}{\partial u^\ell}$$
$$- v_0\left(\frac{\partial\xi^1_u}{\partial x^2} + \frac{\partial\xi^1_u}{\partial u^\ell} p^\ell_2 - p^j_1 \frac{\partial\xi^j_x}{\partial x^2} - p^1_j p^\ell_2 \frac{\partial\xi^j_x}{\partial u^\ell}\right) = \alpha_1 p^3_1 - \alpha_1 v_0 p^1_2$$

Equating the coefficients of the p^k_j *to zero gives, if we introduce the* notation

$$\xi^1_x = T,\ \xi^2_x = H,\ \xi^1_u = U,\ \xi^2_u = P,\ \xi^3_u = V,\ \xi^4_u = \Lambda\ ,$$

the eight equations

$$\left.\begin{aligned} &V_u + v_0 T_h = 0,\quad V_p = 0,\quad V_v - T_t = \alpha_1,\quad V_\lambda = 0 \\ &U_u - H_h = \alpha_1,\quad U_p = 0,\quad H_t + v_0 U_v = 0,\quad U_\lambda = 0 \end{aligned}\right\} \qquad (14)$$

The coefficient of the constant term is

$$V_t - v_0 U_h = 0 \qquad (15)$$

Setting the coefficients of the sixteen terms involving $p^i_k p^j_\ell$ to zero gives

$$\left.\begin{aligned} &T_u = 0,\quad T_p = 0,\quad T_v = 0,\quad T_\lambda = 0 \\ &H_u = 0,\quad H_p = 0,\quad H_v = 0,\quad H_\lambda = 0. \end{aligned}\right\} \qquad (16)$$

Now we write down (13) for $i = 2$, i.e. the momentum equation. Using the fact, from (16), that

$$\frac{\partial \xi_x^i}{\partial u^\ell} = 0, \quad i = 1,2;\ l = 1,\ldots,4, \tag{17}$$

we see that the quadratic terms $p_j^k p_i^\ell$ are not present and (13) becomes

$$\frac{\partial \xi_u^1}{\partial x^1} + \frac{\partial \xi_u^1}{\partial u^\ell} p_1^\ell - p_j^1 \frac{\partial \xi_x^j}{\partial x^1}$$
$$+ v_0\left(\frac{\partial \xi_u^2}{\partial x^2} + \frac{\partial \xi_u^2}{\partial u^\ell} p_2^\ell - p_j^2 \frac{\partial \xi_x^j}{\partial x^2}\right) = \alpha_2 p_1^1 + \alpha_2 v_0 p_2^2 \ .$$

Equating to zero the constant term and the coefficients of the p_i^k we obtain

$$U_t + v_0 P_h = 0 \tag{18}$$

and

$$\left.\begin{array}{llll} U_u - T_t = \alpha_2, & U_p - v_0 T_h = 0, & U_v = 0, & U_\lambda = 0 \\ H_t - v_0 P_u = 0, & P_p - H_h = \alpha_2, & P_v = 0, & P_\lambda = 0 \end{array}\right\} \tag{19}$$

Combining (14), (15), and (16) with (18) and (19) we get, in summary,

$$\begin{array}{c} T = T(t), \quad H = H(h), \quad U = U(t,h,u), \quad P = P(t,h,p) \\ V = V(t,h,v), \quad \Lambda = \Lambda(t,h,u,p,v,\lambda) \end{array} \tag{20}$$

satisfying

$$V_t - v_0 U_h = 0 \tag{21}$$

$$U_t + v_0 P_h = 0 \tag{22}$$

$$V_v - T_t = U_u - H_h = \alpha_1 \tag{23}$$

$$P_p - H_h = U_u - T_t = \alpha_2 \tag{24}$$

At the present time it is possible to obtain some information from (20) - (24), but first we shall study the invariance of the energy equation. For $i = 3$, equation (13) becomes

$$\frac{\partial A_3}{\partial u^2}\xi_u^2 + \frac{\partial A_3}{\partial u^3}\xi_u^3 + \frac{\partial A_3}{\partial p_1^2}\xi_{p1}^2 + \frac{\partial A_3}{\partial p_1^3}\xi_{p1}^3 + \frac{\partial A_3}{\partial p_1^4}\xi_{p1}^4$$

$$= \alpha_3(p_1^2 - k\frac{u^2}{u^3}p_1^3 - \frac{rq}{u^3}p_1^4)$$

Upon substituting for the ξ_{p1}^2, ξ_{p1}^3, and ξ_{p1}^4 from (12) and using (17), we obtain

$$+\frac{kp_1^3\xi_u^2}{u^3} - \frac{ku^2}{(u^3)^2}p_1^3\xi_u^3 + \frac{rq}{(u^3)^2}\xi_u^3 p_1^4$$

$$+\frac{\partial\xi_u^2}{\partial x^1} + \frac{\partial\xi_u^2}{\partial u^\ell}p_1^\ell - \frac{\partial\xi_x^j}{\partial x^1}p_j^2$$

$$+\frac{ku^2}{u^3}(\frac{\partial\xi_u^3}{\partial x^1} + \frac{\partial\xi_u^3}{\partial u^\ell}p_1^\ell - \frac{\partial\xi_x^j}{\partial x^1}p_j^3)$$

$$-\frac{rq}{u^3}(\frac{\partial\xi_u^4}{\partial x^1} + \frac{\partial\xi_u^4}{\partial x^\ell}p_1^\ell - \frac{\partial\xi_x^j}{\partial x^1}p_j^4)$$

$$= \alpha_3(p_1^2 + k\frac{u^2}{u^3} p_1^3 - \frac{\Gamma q}{u^3} p_1^4)$$

Equating the coefficients of p_2^1, p_2^2, p_2^3, and p_2^4 equal to zero gives us no new information. However, setting the constant term to zero as well as the coefficients of p_1^1, p_1^2, p_1^3 and p_1^4 gives

$$P_t + \frac{kp}{v} V_t - \frac{\Gamma q}{v} \Lambda_t = 0 \tag{25}$$

$$\Lambda_u = 0 \tag{26}$$

$$P_p - T_t - \frac{\Gamma q}{V} \Lambda_p = \alpha_3 \tag{27}$$

$$-\frac{V}{v} + \frac{P}{p} + V_v - T_t - \frac{\Gamma q}{kp} \Lambda_v = \alpha_3 \tag{28}$$

$$\frac{V}{v} + T_t - \Lambda_\lambda = -\alpha_3 \tag{29}$$

Equations (21) through (29) give a system of PDEs for the generators T, H, U, P, V, and Λ of the transformations (8) - (9). We shall now solve these ten PDEs to obtain the generators.

Subtracting (24) from (23) we get

$$T_t - H_h = \alpha_1 - \alpha_2 = \text{constant},$$

from which we conclude that

$$T_t = a_1, \quad H_h = a_1 - \alpha_1 + \alpha_2$$

where a_1 is a constant. Hence, letting

$$b_2 = a_1 - \alpha_1 + \alpha_2 \tag{30}$$

we get

$$\begin{aligned} T &= a_1 t + a_6 \\ H &= b_2 h + b_6 \end{aligned} \tag{31}$$

Therefore, from (24),

$$U = (a_1 + \alpha_2)u + f(t,h) \tag{32}$$

where f is an arbitrary function. Similarly, from (23) and (24) we get

$$P = (b_2 + \alpha_2)p + g(t,h) \tag{33}$$

$$V = (a_1 + \alpha_1)v + r(t,h) \tag{34}$$

where g and r are arbitrary functions. Now we substitute the information (30) - (34) into (25) - (29). In particular, (27) thru (29) become

$$\Lambda_p = (b_2 + \alpha_2 - \alpha_3 - a_1) \frac{v}{\Gamma_q} \tag{27'}$$

$$\frac{\Gamma_q}{kp}\Lambda_v = (b_2 + \alpha_2 - \alpha_3 - a_1) - \frac{r(t,h)}{v} + \frac{g(t,h)}{p} \qquad (28)'$$

$$\Lambda_\lambda = 2a_1 + \alpha_1 + \alpha_3 + \frac{r(t,h)}{v} \qquad (29)'$$

Taking $\partial/\partial\lambda$ of (28)' and $\partial/\partial v$ of (29)' and setting the mixed partials equal, we get $r(t,h) \equiv 0$. Thus

$$V = (a_1 + \alpha_1)v \qquad (35)$$

Thus, (21) gives the fact that U is independent of h, i.e.,

$$U = (a_1 + \alpha_2)u + f(t). \qquad (36)$$

Equation (22) then gives

$$P = (b_2 + \alpha_2)p + g(t) - \frac{f'(t)}{v_0}\ h \qquad (37)$$

From (29)',

$$\Lambda = (2a_1 + \alpha_1 + \alpha_3)\lambda + s(t,h,v,p) \qquad (38)$$

Now, we can write α_3 in terms of the other constants as follows. From (27)' we get

$$\Lambda_{pv} = \frac{b_2 + \alpha_2 - \alpha_3 - a_1}{\Gamma_q}$$

From (28)' we get

$$\Lambda_{vp} = \frac{(b_2 + \alpha_2 - \alpha_3 - a_1)k}{\Gamma q}$$

Setting the mixed partials equal we get

$$\alpha_3 + a_1 - \alpha_2 - b_2 = 0$$

or

$$\alpha_3 = b_2 + \alpha_2 - a_1$$

Thus, from (27)' we get

$$\Lambda_p = 0$$

Thus, from (28)', using (39) and (30) we get

$$\Lambda = 2(b_2 + \alpha_1) + \frac{k}{\Gamma_q}(g(t) - \frac{f'(t)h}{v_0})v + s(t,h)$$

Substituting this last relation along with (37) into (25) gives $\partial s/\partial t = 0$ or $s = s(h)$. No additional simplifications are possible, and so we summarize our result in the following: *The most general transformation of the form (8), (9) under which the equations (1) - (3) are constantly conformally invariant is defined by the generators*

$$T = a_1 t + a_6$$

$$H = b_2 h + b_6$$

$$U = (a_1 + \alpha_2)u + f(t)$$

$$P = (b_2 + \alpha_2)p + g(t) - \frac{f'(t)}{v_0} h \qquad (40)$$

$$V = (a_1 + \alpha_1)v$$

$$\Lambda = 2(b_2 + \alpha_1)\lambda + \frac{kv}{\Gamma_q}\left(g(t) - \frac{f'(t)}{v_0} h\right) + s(h)$$

where f, g, and s are arbitrary functions, $b_2 = a_1 - \alpha_1 + \alpha_2$.

4. Invariance of the Jump Conditions

Before treating the equation for reaction rate, we first will investigate the invariance of the Rankine-Hugoniot jump conditions. Our purpose in following this approach lies in the notion that we later may wish to investigate other reaction rates. Thus we attempt to complete the solution to that point.

By requiring the boundary conditions to be invariant, we will obtain a further refinement of the transformations. First, we assume the strong shock condition which, for a polytropic gas, takes the form

$$D = \frac{k+1}{2} u, \qquad (41)$$

Then, the jump conditions can be written

$$(k - 1)v_0 = (k + 1)v_1 \tag{5'}$$

$$p_1 = \frac{k + 1}{2v_0} u_1^2 \tag{6'}$$

$$e_1 - e_0 = \frac{v_0}{k + 1} \; p_1 \tag{7'}$$

From (5)' we have

$$(k+1)\overline{v}_1 - (k-1)v_0 = (k+1)(v_1 + \varepsilon(a_1 + \alpha_1)v_1) - (k-1)v_0$$

$$= (k+1)v_1 - (k-1)v_0 + \varepsilon(k+1)(a_1+\alpha_1)$$

Hence, (5)' is invariant if

$$a_1 = -\alpha_1 \tag{42}$$

Similarly, from (6)',

$$\overline{p}_1 - \frac{k + 1}{2v_0} \overline{u}_1^2 = \{p_1 + [(b_2 + \alpha_2)p_1 + g(t) - \frac{f'(t)}{v_0} h] \varepsilon \}$$

$$- \frac{k + 1}{2v_0} \{u_1 + \varepsilon[(a_1 + \alpha_2)u_1 + f(t)]\}^2$$

$$= [2\varepsilon(a_1 + \alpha_2) + 1] \; (p_1 - \frac{k + 1}{2v_0} u_1^2)$$

$$+ \varepsilon(g(t) - \frac{f'(t)}{v_0} h - \frac{k + 1}{v_0} u_1 f(t)) + O(\varepsilon^2),$$

where we have used (42) and (30). Hence, (6)' will be invariant if

$$g(t) \equiv 0,\ f(t) \equiv 0\ .$$

Consequently, we may rewrite (40) as

$$\begin{aligned} T &= a_1 t + a_6 \\ H &= b_2 h + b_6 \\ U &= (b_2 - a_1)u \\ P &= 2(b_2 - a_1)p \\ V &= 0 \\ \Lambda &= 2(b_2 - a_1)\lambda + s(h). \end{aligned} \tag{43}$$

The energy jump condition (7)' gives no new information. For,

$$e = e_1^0 + \lambda\Delta e^0 + \frac{pv}{k-1}$$

is the equation of state of the mixture, and, substituting this into (7)' using $p_0 = 0$ gives ($\lambda = 0$ at the shock)

$$\frac{p_1 v_1}{k-1} - \frac{v_0 p_1}{k+1} = 0 .$$

Thus,

$$\frac{\bar{p}_1 \bar{v}_1}{k-1} - \frac{v_0 \bar{p}_1}{k+1} = \frac{(p_1 + 2(b_2 - a_1) p_1 \varepsilon) v_1}{k-1} - \frac{v_0 (p_1 + 2(b_2 - a_1) p_1 \varepsilon)}{k+1}$$

$$= [1 + 2(b_2 - a_1)] \ \varepsilon \ \{ \frac{p_1 v_1}{k-1} - \frac{v_0 p_1}{k+1} \} .$$

Therefore, the transformation whose generators are given by (43) leaves the hydrodynamics equations and the jump conditions constantly conformally *invariant.*

5. Invariance of the Rate Equation

The condition that the rate equation (4)' be invariant is that

$$-\beta L(1-u^4)^\alpha (u^2)^{\beta-1} \xi_u^2 + \alpha L (u^2)^\beta (1-u^4)^{\alpha-1} \xi_u^4 + \xi_{p1}^4$$

$$= \alpha_4 (p_1^4 - L(1-u^4)^\alpha (u^2)^\beta)$$

Substituting the expression ξ_{p1}^4 from (11) and setting to zero the coefficients of the $p_j^i \, p_\ell^k$ gives the following system of partial differential equations in terms of the unindexed variables:

$$\frac{\partial \Lambda}{\partial t} - \beta L(1-\lambda)^\alpha p^{\beta-1} P + \alpha L p^\beta (1-\lambda)^{\alpha-1} \Lambda + \alpha_4 L (1-\lambda)^\alpha p^\beta = 0 \quad (44)$$

$$\frac{\partial \Lambda}{\partial u} = 0 \qquad \frac{\partial \Lambda}{\partial p} = 0 \qquad \frac{\partial \Lambda}{\partial v} = 0 \qquad (45)$$

$$\frac{\partial\Lambda}{\partial\lambda} - \frac{\partial T}{\partial t} - \alpha_4 = 0 \qquad (46)$$

$$\frac{\partial H}{\partial t} = 0 \qquad (47)$$

Combining these equations with eqns. (40) gives a general family of transformations under which (1) through (4) are invariant. First eqn. (46) implies that

$$2(b_2 + \alpha_1) - a_1 - \alpha_4 = 0$$

or

$$\alpha_4 = 2b_2 - 3a_1 \qquad (48)$$

Also. from eqn. (44) we obtain

$$\frac{\partial\Lambda}{\partial t} - L(1-\lambda)^{\alpha}p^{\beta}[\frac{\beta P}{p} - \frac{\alpha\Lambda}{1-\lambda} - \alpha_4] = 0$$

Since $\partial\Lambda/\partial t = 0$ we must have

$$\frac{\beta P}{p} - \frac{\alpha\Lambda}{1-\lambda} - \alpha_4 = 0$$

or, from (43), with $s(h) = -2(b_2-a_1)$,

$$\alpha_4 = 2(b_2-a_1)(\alpha+\beta) \qquad (49)$$

Consequently, we have the following infinitesimal transformations under which equations (1) through (4) and jump conditions (5) through (7) are invariant:

$$
\begin{aligned}
T &= a_1 t + a_6 \\
H &= b_2 h + b_6 \\
U &= (b_2 - a_1) u \\
P &= 2(b_2 - a_1) p \\
V &= 0 \\
\Lambda &= 2(b_2 - a_1)(\lambda - 1)
\end{aligned}
\tag{50}
$$

6. Similarity Solutions

To obtain the form of the similarity variable and the similarity solution we write down the invariant surface condition (see Bluman - Cole [4]):

$$T \frac{\partial u}{\partial t} + H \frac{\partial u}{\partial h} = U \tag{51}$$

$$T \frac{\partial p}{\partial t} + H \frac{\partial p}{\partial h} = P \tag{52}$$

$$T \frac{\partial v}{\partial t} + H \frac{\partial v}{\partial h} = V \tag{53}$$

$$T\frac{\partial\lambda}{\partial t} + H\frac{\partial\lambda}{\partial h} = \Lambda \qquad (54)$$

Equation (51) becomes

$$(a_1 t + a_6)u_t + (b_2 h + b_6)u_h = (b_2 - a_1)u$$

The characteristic equations are

$$\frac{dt}{a_1 t + a_6} = \frac{dh}{b_2 h + b_6} = \frac{du}{(b_2 - a_1)u}$$

The first pair of equations can be integrated to give

$$\ln(a_1 t + a_6)^{b_2} = \ln[\hat{\eta}(b_2 h + b_6)^{a_1}] ,$$

where $\hat{\eta}$ is a constant; hence

$$\frac{(b_2 h + b_6)^{a_1}}{(a_1 t + a_6)^{b_2}} = \text{constant}$$

or

$$\frac{\frac{b_2}{b_6} h + 1}{(\frac{a_1}{a_6} t + 1)^{b_2/a_1}} = \eta = \text{constant} \qquad (55)$$

η is the similarity parameter. The equation

$$\frac{dt}{a_1 t + a_6} = \frac{du}{(b_2 - a_1)u}$$

can now be integrated to give

$$\ln(a_1 t + a_6)^{b_2 - a_1} = \ln((b_2 - a_1)u)^{a_1} = \ln \tilde{u}(\eta),$$

where $\tilde{u}$ is an arbitrary function. This leads to

$$u(t,h) = \left(\frac{a_1}{a_6} t + 1\right)^{\frac{b_2}{a_1} - 1} \tilde{u}(\eta). \qquad (56)$$

Similarly, solving (52) and (54) gives

$$p(t,h) = \left(\frac{a_1}{a_6} t + 1\right)^{2\left(\frac{b_2}{a_1} - 1\right)} \tilde{p}(\eta) \qquad (57)$$

and

$$\lambda(t,h) = 1 + \left(\frac{a_1}{a_6} t + 1\right)^{2\left(\frac{b_2}{a_1} - 1\right)} \tilde{\lambda}(\eta) \qquad (58)$$

The characteristic system for equation (53) is

$$\frac{dt}{a_1 t + a_6} = \frac{dh}{b_2 h + b_6} = \frac{dv}{0}$$

Again, the first equation has first integral η given by (55). Then,

$v = \tilde{v}(\eta)$ is a first integral also since $dv = 0$. Thus,

$$v(t,h) = \tilde{v}(\eta). \tag{59}$$

Now, we note that the shock path is described by $\eta = 1$ from (46) since $(t,h) = (0,0)$ implies $\eta = 1$, and since the shock path must be a similarity curve. That is, the shock path is given by

$$\frac{a_1}{a_6} t + 1 = (\frac{b_2}{b_6} h + 1)^{a_1/b_2} \tag{60}$$

If $a_1 = b_2$, then (48) contradicts (49) and therefore constant velocity shocks are not possible. If $a_1 > b_2$ we obtain decelerating shocks and for $a_1 < b_2$ we obtain accelerating shocks (see Fig. 1).

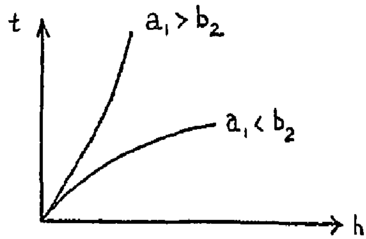

Figure 1

The shock velocity D can be computed from (60) to get

$$D = \frac{dh}{dt} = \frac{b_6}{a_6} (\frac{a_1}{a_6} t + 1)^{\frac{b_2}{a_1} - 1} \tag{61}$$

If, at $t = 0$, $h = 0$ we denote the initial values by u_i, v_i, p_i, and $\lambda_i = 1$, then we note that (since $\eta = 1$ at $(0,0)$)

$$p(0,0) = p_i = \tilde{p}(1).$$

$$u(0,0) = u_i = \tilde{u}(1)$$

$$v(0,0) = v_i = \tilde{v}(1)$$

$$\lambda(0,0) = \lambda_i = 1 = \tilde{\lambda}(1) \ .$$

Letting

$$\tilde{p}(\eta) = p_i\hat{p}(\eta), \quad \tilde{u}(\eta) = u_i\hat{u}(\eta)$$

$$\tilde{v}(\eta) = v_i\hat{v}(\eta), \quad \tilde{\lambda}(\eta) = \hat{\lambda}(\eta),$$

Then the solutions can be written

$$p = p_i\left(\frac{a_1}{a_6}t + 1\right)^{2\left(\frac{b_2}{a_1} - 1\right)} \hat{p}(\eta) \tag{62}$$

$$u = u_i\left(\frac{a_1}{a_6}t + 1\right)^{\frac{b_2}{a_1} - 1} \hat{u}(\eta) \tag{63}$$

$$v = v_i\hat{v}(\eta) \tag{64}$$

$$\lambda = \left(\frac{a_1}{a_6}t + 1\right)^{2\left(\frac{b_2}{a_1} - 1\right)} \hat{\lambda}(\eta) + 1 \tag{65}$$

Here, $\hat{p}(1) = 1$, $\hat{u}(1) = 1$, $\hat{v}(1) = 1$, and $\hat{\lambda}(1) = -1$. It will be shown in the next section that the $\hat{p}$, $\hat{u}$, $\hat{v}$, and $\hat{\lambda}$ can be determined from a system of ODEs.

7. Reduction of the PDEs to ODEs

If we substitute (62) - (65) into the PDEs (1) - (4) we will obtain a system of ODEs for $\hat{p}$, $\hat{u}$, $\hat{v}$, and $\hat{\lambda}$. First, substituting (63) and (64) into (1) we get

$$\eta\frac{d\hat{v}}{d\eta} = -\frac{a_6 v_0 u_i}{b_6 v_i}\frac{d\hat{u}}{d\eta} \tag{66}$$

Substituting (63) and (62) into (2) we get

$$\eta\frac{d\hat{u}}{d\eta} - (1 - \frac{a_1}{b_2})\hat{u} = \frac{a_6 v_0 p_i}{b_6 u_i}\frac{d\hat{p}}{d\eta} \tag{67}$$

Then substituting (62), (65), and (64) into (3) we get

$$\eta\frac{d\hat{p}}{d\eta} + 2\hat{p}\frac{a_1}{b_2}(1 - \frac{b_2}{a_1}) = -\frac{K\hat{p}}{\hat{v}}\eta\frac{d\hat{v}}{d\eta} + \frac{\Gamma q}{v_i \hat{v} p_i}(\eta\frac{d\hat{\lambda}}{d\eta} + 2\frac{a_1}{b_2}\hat{\lambda}(1 - \frac{b_2}{a_1})) \tag{68}$$

Finally, substituting (62) and (65) into (4) gives

$$\eta\frac{d\hat{\lambda}}{dt} = 2\frac{a_1}{a_6}(1 - \frac{a_1}{b_2})\hat{\lambda} - \frac{La_1}{b_2}\hat{\lambda}^{\alpha}p^{\beta}p_i^{\beta} \tag{69}$$

Therefore, equations (66) - (69) can be solved to determine the exact similarity solutions.

REFERENCES

1. M. Cowperthwaite, "Characterization of Initiation and Detonation by Lagrange Gage Technology", SRI Final Report (July 1977).

2. H. M. Sternberg, "Similarity Solutions for Reactive Shock Waves", Quarterly J. Mech. and Applied Math. 23, 77-99 (1970).

3. L. V. Ovsjannikov, "Group Properties of Differential Equations"; translation of "Gruppovye Svoystva Differentsialny Uravneni" Novosibirsk, U.S.S.R. (1962).

4. G. W. Bluman and J. D. Cole, "Similarity Methods for Differential Equations", Springer-Verlag, New York (1974).

Acknowledgment

The authors greatly benefited from many conversations with M. Cowperthwaite at Stanford Research Institute. They would also like to thank D. Weingart of Lawrence Livermore Laboratory for support.

Appendix 1. Derivation of the Energy Equation (3)

The EOS is given by

$$e = e(p,v,\lambda),$$

From which it follows that

$$\frac{\partial e}{\partial t} = \left(\frac{\partial e}{\partial p}\right)_{v,\lambda} \frac{\partial p}{\partial t} + \left(\frac{\partial e}{\partial v}\right)_{p,\lambda} \frac{\partial v}{\partial t} + \left(\frac{\partial e}{\partial \lambda}\right)_{p,v} \frac{\partial \lambda}{\partial t}$$

But also from conservation of energy we have

$$\frac{\partial e}{\partial t} = - p \frac{\partial v}{\partial t}$$

Thus, using

$$\frac{\Gamma}{v} = \left(\frac{\partial p}{\partial e}\right)_{v,\lambda} \quad , \quad q = - \left(\frac{\partial e}{\partial \lambda}\right)_{p,v} \quad ,$$

we get

$$\frac{\partial p}{\partial t} = - \frac{\Gamma}{v} \left(p + \left(\frac{\partial e}{\partial v}\right)_{p,\lambda}\right) \frac{\partial v}{\partial t} + q \frac{\partial \lambda}{\partial t}$$

But the EOS of the mixture is given by

$$e = e_1^0 + \lambda q + \frac{pv}{k-1}$$

and so

$$\Gamma = k - 1 \quad \text{and} \quad \left(\frac{\partial e}{\partial v}\right)_{p,\lambda} = \frac{p}{k-1}$$

Hence

$$\frac{\partial p}{\partial t} = -\frac{kp}{v}\frac{\partial v}{\partial t} + \frac{\Gamma q}{v}\frac{\partial \lambda}{\partial t}$$