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A NOTE ON ZENO'S PARADOX IN QUANTUM THEORY

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Abstract: A decaying quantum system, if observed very frequently in order to ascertain whether or not it is still undecayed, will not decay at all. The derivation of this effect--known, e.g., as Zeno's paradox--has been criticized recently. It has been argued that measurements performed in a very short time interval Δt produce states with a very large energy uncertainty ΔE, and that Zeno's paradox disappears if this is taken into account. By construction of an explicit counterexample we prove, however, that there is no energy-time uncertainty relation of the required kind; therefore, the criticism mentioned is unjustified.

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1. Introduction: Decaying States and Zeno's Paradox

Consider a decaying state, described by a certain unit vector f in the Hilbert space $\mathcal{H}$ of the decaying system, whose time evolution is described by a Hamiltonian H. If left undisturbed, the system at time t is in the (Schrödinger) state $e^{-iHt}f$, provided it starts in the undecayed state f at time t = 0. Therefore the probability of finding the system still undecayed at time t is

$$p(t) = |(f,e^{-iHt}f)|^2. \qquad (1)$$

If, instead, the system is not left undisturbed, but a series of measurements are made at equidistant times $t_1 = \frac{t}{n}$, $t_2 = \frac{2t}{n}$ ... $t_n = t$, each one designed to check whether or not the system is still undecayed, one easily obtains the expression

$$P_n(t) = |(f,e^{-iHt/n}f)|^{2n} = (p(\tfrac{t}{n}))^n \qquad (2)$$

for the probability to find the system undecayed at each one of those measurements. In order to derive (2), one has to assume that the measurements are "ideal" in the usual sense; i.e., at each time t_i of a measurement the state is assumed to "collapse" from $f(t_i) = e^{-iHt/n}f$ to f with the probability $|(f,f(t_i))|^2$, which is also the probability of finding the system undecayed.[1]

If the free decay law p(t) is a pure exponential, $p(t) = e^{-\lambda t}$, Eq. (2) leads to the same exponential decay law for the system observed at intermediate times, independent of the number n of intermittent measurements. It is known, however, that the free decay law (1) cannot be a pure exponential for all times t > 0, although usually it may be approximated quite well by an exponential during a time interval of the order of several or even many life-times $\tau = \frac{1}{\lambda}$.[2] Deviations from exponentiality necessarily occur for very large as well as for very small times. For large times they are a consequence of the semiboundedness (positivity) of physically acceptable Hamiltonians H, which implies that for large t the free decay law (1) is given by some inverse power rather than an exponential of t, and the decay is thus slower than exponential. At small times, all physically realizable decays will also be slower than exponential, if one tacitly assumes that in any realizable state f the average $\langle E \rangle$ as well as the spread ΔE of the energy should be finite.[3] This means[4] that f, and consequently $e^{-iHt}f$, is in the domain of definition D_H of the Hamiltonian H, implying that p(t) is continuously differentiable for all t, with $\dot{p}(0) = 0$:

$$\dot{p}(t) = \frac{d}{dt} [(f,e^{-iHt}f)(f,e^{iHt}f)]$$
$$= i[(f,e^{-iHt}f)(Hf,e^{iHt}f) - (Hf,e^{-iHt}f)(f,e^{iHt}f)]$$

is continuous in t since $e^{\pm iHt}$ is weakly continuous, and $\dot{p}(0) = 0$. Therefore p(t), for $t \to 0$, approaches $p(0) = 1$ with zero slope, and is thus bigger than any exponential for small t. Since $P_n(t) \equiv p(t)$ for $p(t) = e^{-\lambda t}$ only, the free decay law p(t) and the "survival probability" $P_n(t)$ are necessarily different for all physically realizable decaying states.

Keeping the time interval δt between successive measurements fixed, one may rewrite the survival probability (2) in the form

$$P_{\delta t}(n \cdot \delta t) = (p(\delta t))^n . \quad (3)$$

(As P now depends on δt, we use this as subscript.) If δt is neither too large nor too small, so that $p(\delta t)$ is well approximated by an exponential $e^{-\lambda \cdot \delta t}$ with the "natural" decay constant λ of the state f as determined by H, i.e., the internal dynamics of the decaying system, then (3) yields an exponential decay

$$P_{\delta t}(t) = P_{\delta t}(n \cdot \delta t) = e^{-\lambda n \cdot \delta t} = e^{-\lambda t} \quad (4)$$

with the natural decay constant λ for arbitrarily large times $t = n \cdot \delta t$. In suitably refined form, with random rather than uniformly distributed intermediary measurements, and a continuous time variable t instead of the discrete time $t = n \cdot \delta t$ considered here, this argument has been used to show that the deviations from exponentiality at large times t will not be observed in practice.[2] An unstable charged particle while moving in a Wilson chamber, for instance, is indeed not completely isolated but produces a recognizable track of droplets, each one indicating that it is still present, and thus constituting an intermittent measurement of the kind considered above. These measurements are just frequent enough to replace (1), with its nonexponential "tail," by the exponential decay law (4) for all times, including arbitrarily large ones. On the other hand, the intervals δt between successive measurements are still so large that the nonexponentiality of $p(\delta t)$ for small δt can be neglected.

However, if δt is further decreased, one finally reaches the region where $p(\delta t)$ is no longer given by $e^{-\lambda \cdot \delta t}$, and for sufficiently small δt it becomes strictly larger than this, as shown above. Representing $p(t)$ in the form

$$p(t) = e^{-\nu(t) \cdot t} \tag{5}$$

for all t, we thus find that

$$\nu(t) = - \frac{1}{t} \log p(t) \tag{6}$$

coincides with λ for not too small t, while

$$\nu(t) = - \frac{1}{t} \log p(t) < - \frac{1}{t} \log(e^{-\lambda t}) = \lambda \tag{6}$$

for sufficiently small t and, by l'Hospital's rule,

$$\lim_{t \to 0} \nu(t) = -\lim_{t \to 0} \frac{d}{dt} (\log p(t)) = - \frac{\dot{p}(0)}{p(0)} = 0 . \tag{7}$$

With (5), Eq. (3) still leads to an exponential decay law

$$P_{\delta t}(t) = e^{-\nu(\delta t) \cdot t} , \tag{8}$$

but with a decay "constant" $\nu(\delta t)$ which is not fixed by the internal dynamics of the decaying system only, but also depends on the frequency of observation. By (6), the decay (8) proceeds slower than the "natural" decay (4) for sufficiently small δt, and in the limit $\delta t \to 0$ of "continuous observation" Eqs. (7) and (8) imply $P_0(t) \equiv 1$, i.e., there is no decay any more. This phenomenon was discovered several times, more or less independently, by various authors during the last decades. It has been investigated rigorously and quite extensively, under much more general assumptions than those used here, in a recent paper by Misra and Sudarshan,[5] who called it the "Zeno's paradox" in quantum theory.

In actual experiments with decaying systems the mean frequency of successive measurements cannot be made arbitrarily large. In the example of a particle in a Wilson chamber mentioned above, this limitation is caused by the atomic structure of the vapor, leading to a minimum distance d in space, and thus to a minimum time lag $\delta t \gtrsim d/c$, between the successive formation of any two droplets. Numerical estimates[2] seem to indicate that practically realizable values of δt are still within or at most slightly outside the domain for which $p(\delta t)$ coincides with $e^{-\lambda \cdot \delta t}$. Therefore the dependence on δt of the decay "constant" $\nu(\delta t)$ in the observed decay law (8), if present at all, will be very weak, and thus hardly detectable.[6] In spite of this difficulty of a direct experimental test, Zeno's paradox is certainly of considerable theoretical interest, as it belongs to the many "nonclassical" features which, starting with Heisenberg's discovery of the uncertainty relations, have been proved to follow from the general assumptions of quantum theory.

In a recent paper, however, Ghirardi et al.[7] claim to prove that Zeno's paradox is not only difficult to observe, but rather that the arguments leading to it are in conflict with the very principles of quantum mechanics, and that by properly taking into account these principles--roughly speaking, some kind of time-energy uncertainty relation--the paradox disappears. After reviewing the crucial ingredient of this criticism in Section 2, we want to convince the reader in Section 3 that this criticism is unjustified, and that the above sketchy derivation of Zeno's paradox is basically correct.

2. An Objection Against the Derivation of Zeno's Paradox

A crucial assumption used to derive Eq. (2) is that each one of the measurements at times t_i projects out of the state $f(t_i)$ found at time t_i, again the original state f, or at least some state f' which is so close to f that it can be replaced by f in the calculation of the further behavior of the decaying system. As pointed out, e.g., in Ref. 2, the type of measurement which is appropriate for this purpose, and which is actually used in many experiments, is a suitable localization procedure. Indeed, the undecayed state f differs from the decayed states most significantly by the fact that no decay products are found within a relative distance greater than the range R of their mutual interactions in the first case, whereas in almost all decayed states this distance is greater than R. If one thus chooses for the property to be measured at times t_i a characteristic function of a suitable distance coordinate $\underset{\sim}{x}$ between decay products, e.g.,

$$Q(\underset{\sim}{x}) = \begin{cases} 1 & \text{for } |\underset{\sim}{x}| < R \\ 0 & \text{for } |\underset{\sim}{x}| > R \end{cases},$$

the measurements should in fact have the desired effect. In the configuration space representation, vectors $g \in \mathcal{H}$ are described by wavefunctions $g(\underset{\sim}{x},\ldots)$, the dots denoting all other coordinates besides $\underset{\sim}{x}$, whereas $Q(\underset{\sim}{x})$, acting multiplicatively on the wavefunctions, represents a projection operator Q on $\mathcal{H}$. (If, for instance, the decaying state is a virtual bound state of two particles interacting via a suitable potential, $\underset{\sim}{x}$ is simply the distance between the particles, and the remaining coordinate $\underset{\sim}{R}$ is the center of mass.) An ideal measurement of Q in a (normalized) state g yields a positive result, and transforms g into the new (again normalized) state $g' = NQg$, with the "transition probability"

$$w = (g,Qg) = (g,Q^2g) = \|Qg\|^2 = |N|^{-2} . \qquad (9)$$

The new wavefunction is given by

$$g'(\underset{\sim}{x},\ldots) = NQ(\underset{\sim}{x})g(\underset{\sim}{x},\ldots) . \qquad (10)$$

A slight modification is appropriate, however. There will be a contribution to the Hamiltonian H from the relative motion of decay products as described by the distance coordinate $\underset{\sim}{x}$. For relative distances $|\underset{\sim}{x}| \gtrsim R$ this motion is already free, so that the corresponding part of the Hamiltonian acts on wavefunctions in the region $|\underset{\sim}{x}| \gtrsim R$ like a free Hamiltonian, which is a differential operator in $\underset{\sim}{x}$. Due to the discontinuity of $Q(\underset{\sim}{x})$ at $|\underset{\sim}{x}| = R$, however, the new wavefunction g' given by (10) will not be differentiable even if g is. Thus, a measurement of Q will produce states which do not belong to the domain of definition of H, and which therefore are at least very inconvenient as an approximation for the original decaying state f. This difficulty is easily circumvented, if $Q(\underset{\sim}{x})$ is replaced by a smooth (real) function

$$A(\underset{\sim}{x}) = \begin{cases} 1 & \text{for } |\underset{\sim}{x}| < R \\ 0 & \text{for } |\underset{\sim}{x}| > R + \varepsilon \end{cases} , \quad 0 \le A(\underset{\sim}{x}) \le 1 \text{ for all } \underset{\sim}{x} ,$$

which has to be sufficiently often differentiable, but need not be specified further in the transition region $R < |\underset{\sim}{x}| < R + \varepsilon$, which moreover can be chosen arbitrarily narrow. Acting multiplicatively on wavefunctions, $A(\underset{\sim}{x})$ again defines a Hermitian operator A on $\mathcal{H}$ which, however, is now no longer a projection operator ($A^2 \neq A$). A "measurement" of A[8] gives a positive result, and transforms the original (normalized) wavefunction $g(\underset{\sim}{x},\ldots)$ into a new (normalized) one,

$$g'(\underset{\sim}{x},\ldots) = NA(\underset{\sim}{x})g(\underset{\sim}{x},\ldots) , \tag{11}$$

with the transition probability

$$w = (g,A^2g) = \|Ag\|^2 = |N|^{-2} . \tag{12}$$

By comparing (11) and (12) with the corresponding formulae (9) and (10), we see that for small enough ε "approximate" localization measurements (with A) give almost the same results for g' and w as "exact" ones (with Q). In particular, if the undecayed state f can be separated at all from the decay products by a localization measurement of the type considered, a "smooth" localization like (11) can be used as well.

In order to explain the main argument of Ref. 7 in a most simple way, we first calculate the second time derivative of p(t). Using the expression for $\dot{p}(t)$ already obtained, we find

$$\ddot{p}(t) = (f,He^{-iHt}f)(Hf,e^{iHt}f) - (f,e^{-iHt}f)(Hf,He^{iHt}f)$$
$$+ (f,He^{iHt}f)(Hf,e^{-iHt}f) - (f,e^{iHt}f)(Hf,He^{-iHt}f) .$$

Putting t = 0 we get from this

$$\ddot{p}(0) = -2[(Hf,Hf) - (f,Hf)^2] = -2(\Delta E)^2 .$$

For $f \in D_H$, therefore, ΔE is indeed finite, and p(t) is even twice continuously differentiable and behaves like

$$p(t) \approx 1 - (\Delta E)^2 t^2 \tag{13}$$

for small t. Assuming now that indeed each measurement, with sufficient accuracy, reproduces the original state f, the quantity entering (3) is

$$p(\delta t) \approx 1 - (\Delta E)^2 (\delta t)^2 \tag{14}$$

so that, for $t = n\cdot\delta t$ and $\Delta E\cdot\delta t << 1$, (3) leads to

$$P_{\delta t}(t) \approx 1 - n(\Delta E)^2(\delta t)^2 . \tag{15}$$

Keeping t fixed and increasing the frequency of successive measurements by a factor m, we have to replace δt by $\delta t/m$ and n by $m\cdot n$ in (15), and thus

$$P_{\delta t/m}(t) \approx 1 - m\cdot n(\Delta E)^2(\delta t/n)^2 = 1 - \frac{n}{m}(\Delta E)^2(\delta t)^2 .$$

For $m \to \infty$ we therefore obtain $P_{\delta t/m}(t) \to 1$, which is Zeno's paradox.

The duration Δt of each single measurement is obviously restricted by

$$\Delta t \leq \delta t ,$$

and thus goes to zero if δt does. Suppose now one could prove that any measurement of finite duration Δt produces a state whose energy spread ΔE is larger than a given function of Δt which goes to $+\infty$ for $\Delta t \to 0$. Then the above reasoning, which assumed ΔE to be independent of δt, would obviously be wrong, as one could no longer conclude from (14) that $p(\delta t) \to 1$ for $\delta t \to 0$. For instance, if there were an uncertainty relation of the usual type for

ΔE and Δt,

$$\Delta E \cdot \Delta t \gtrsim 1 , \tag{16}$$

the approximation (14) for $p(\delta t)$, and therefore any conclusion drawn from it, would become completely meaningless. But even a weaker estimate like

$$(\Delta E)^2 \cdot \Delta t \gtrsim \text{const.} \tag{17}$$

would spoil Zeno's paradox since then, for $\delta t \to 0$, $(\Delta E)^2$ could grow like $\frac{1}{\delta t}$, and $p(\delta t)$, according to (14), would then go to 1 linearly in δt, so that

$$P_{\delta t}(t) = P_{\delta t}(n \cdot \delta t) \approx (1 - \alpha \cdot \delta t)^n = (1 - \frac{\alpha t}{n})^n \approx e^{-\alpha t}$$

would be an exponential, entirely independent of δt, for sufficiently large $n = \frac{t}{\delta t}$.

Although the authors of Ref. 7 are aware of the fact that an uncertainty relation of the form and the physical meaning of (16) does not follow from quantum theory, they maintain that at least for localization measurements of the type (11) described above, an estimate for ΔE with the required properties can indeed be derived. After a somewhat lengthy--and at least partly quite heuristic--calculation, they actually arrive at an inequality of the form (17) (cf. Eq. (5.18) of their paper), and conclude from this that there is no Zeno's paradox for decaying states. We want to show in the following section, however, that this criticism is unjustified, since there are no principal restrictions whatsoever for the time duration Δt of localization measurements of the type described by Eq. (11).

3. Localization Measurements in Arbitrarily Short Times

In order to prove our last statement, it is sufficient to construct explicitly a quantum mechanical model for the localization experiment described by Eqs. (11) and (12), and to show that the duration Δt of this model experiment can be made arbitrarily small.

Since only the coordinate $\underset{\sim}{x}$ is involved in the measurement, we shall suppress all other coordinates and take the wavefunctions $\psi, \psi' \ldots$ to be functions of $\underset{\sim}{x}$ only. (To make this mathematically rigorous, we have to factorize the state space of the decaying system in the form

$$\mathcal{H} = L^2(\underset{\sim}{x}) \otimes L^2(\ldots)$$

into spaces of square-integrable functions of $\underset{\sim}{x}$ and of the other coordinates $(\ldots)$, respectively. As the second factor $L^2(\ldots)$ will not be affected by the interaction with the measuring apparatus, it can be omitted during the calculations and reinserted afterwards.) We assume, moreover, that all values of Δt considered here are already very small as compared to the internal time scale of the decaying system, so that any change of the wavefunction due to the Hamiltonian H of the isolated system can be neglected during time intervals of length Δt. We are thus dealing, as usual, with (practically) instantaneous measurements.

We take as a quantum mechanical model for the measuring apparatus a very simple quantum system with two-dimensional state space $\mathcal{H}_2$ and describe its states by column vectors as usual. Without interactions, moreover, all apparatus states shall be stationary, so that the free Hamiltonian of the apparatus may be chosen as zero. The state space for system plus apparatus is

$$\underset{\sim}{H} = L^2(\underset{\sim}{x}) \otimes H_2 \ .$$

As is well-known, vectors in $\underset{\sim}{H}$ may be represented by column vectors of the form

$$\underset{\sim}{g} = \begin{pmatrix} g_1(\underset{\sim}{x}) \\ g_2(\underset{\sim}{x}) \end{pmatrix} , \qquad g_1, g_2 \in L^2(\underset{\sim}{x}) \ .$$

In this representation, operators on $\underset{\sim}{H}$ become 2×2 operator matrices

$$\underset{\sim}{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \qquad A_{ik}\text{: operators on } L^2(\underset{\sim}{x}) \ ,$$

and[9]

$$(\underset{\sim}{g}, \underset{\sim}{g}') = (g_1, g_1')_{L^2} + (g_2, g_2')_{L^2} \ ,$$

$$\underset{\sim}{A}\underset{\sim}{g} = \begin{pmatrix} (A_{11}g_1)(\underset{\sim}{x}) + (A_{12}g_2)(\underset{\sim}{x}) \\ (A_{21}g_1)(\underset{\sim}{x}) + (A_{22}g_2)(\underset{\sim}{x}) \end{pmatrix} ,$$

etc. The interpretation of this formalism is also well-known and quite obvious; e.g., the state

$$\underset{\sim}{g} = \begin{pmatrix} g(\underset{\sim}{x}) \\ 0 \end{pmatrix} \equiv g(\underset{\sim}{x}) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (18)$$

means that the system has the wavefunction $g(\underset{\sim}{x})$, and the apparatus is in the "up" state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The interaction Hamiltonian $\underset{\sim}{H}_I$ on $\underset{\sim}{H}$ in chosen to be

$$\underset{\sim}{H}_I = \begin{pmatrix} 0 & -i\lambda s(\underset{\sim}{x}) \\ i\lambda s(\underset{\sim}{x}) & 0 \end{pmatrix} , \qquad (19)$$

where λ is a positive coupling constant and $s(\underset{\sim}{x})$ a smooth real "sensitivity function," acting multiplicatively on

$L^2(\underset{\sim}{x})$, with

$$s(\underset{\sim}{x}) = \begin{cases} 0 & \text{for } |\underset{\sim}{x}| < R \\ 1 & \text{for } |\underset{\sim}{x}| > R + \varepsilon \end{cases}, \quad 0 \leq s(\underset{\sim}{x}) \leq 1 \text{ for all } \underset{\sim}{x} . \tag{20}$$

Clearly Eq. (19) defines a Hermitean operator on $\underset{\sim}{H}$. Moreover, one can easily calculate (e.g., by a power series expansion) the unitary operators

$$e^{-i\underset{\sim}{H}_I t} = \exp\begin{pmatrix} 0 & -\lambda t s(\underset{\sim}{x}) \\ \lambda t s(\underset{\sim}{x}) & 0 \end{pmatrix} = \begin{pmatrix} \cos(\lambda t s(\underset{\sim}{x})) & -\sin(\lambda t s(\underset{\sim}{x})) \\ \sin(\lambda t s(\underset{\sim}{x})) & \cos(\lambda t s(\underset{\sim}{x})) \end{pmatrix}. \tag{21}$$

Now choose as initial apparatus state at the time t_i, at which the measurement begins, the "up" state, and denote the wavefunction of the system at that time by $g(\underset{\sim}{x})$, so that the initial combined state of system plus apparatus is given by (18). Assume, moreover, that the interaction (19) is "switched on" in the time interval between t_i and $t_i + \Delta t$, with Δt and λ related by

$$\lambda \cdot \Delta t = \frac{\pi}{2} . \tag{22}$$

Since the free time evolution of the wavefunction $g(\underset{\sim}{x})$ in this short time interval was assumed to be negligible, and there is no free time evolution of the apparatus state, the state of system plus apparatus at time $t_i + \Delta t$ follows from (18), (21) and (22) to be

$$e^{-i\underset{\sim}{H}_I \cdot \Delta t}\,\underset{\sim}{g} = \begin{pmatrix} A(\underset{\sim}{x})g(\underset{\sim}{x}) \\ B(\underset{\sim}{x})g(\underset{\sim}{x}) \end{pmatrix}$$

$$= A(\underset{\sim}{x})g(\underset{\sim}{x}) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B(\underset{\sim}{x})g(\underset{\sim}{x}) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{23}$$

with

$$A(\underset{\sim}{x}) = \cos(\frac{\pi}{2}\, s(\underset{\sim}{x})) \;, \qquad B(\underset{\sim}{x}) = \sin(\frac{\pi}{2}\, s(\underset{\sim}{x})) \;. \qquad (24)$$

The apparatus, by construction (cf. Eqs. (19) and (20)), was "sensitive" (i.e., $\underset{\sim}{H}_I \neq 0$) in the region $|\underset{\sim}{x}| > R$ only. A transition of the apparatus from the original "up" state to the "down" state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ thus means that it has responded to some part of $g(\underset{\sim}{x})$ in that region, i.e., it has registered the presence of "decay products." The part of $g(\underset{\sim}{x})$ in the region $|\underset{\sim}{x}| < R$, on the other hand, does not influence the apparatus, and thus leaves it in the initial "up" state. Therefore the apparatus state "up" is correlated, in the final state (23), with that part of the original wavefunction $g(\underset{\sim}{x})$ which represents (up to some small error, mainly due to the possible presence of some decay products even inside the region $|\underset{\sim}{x}| < R$) the still undecayed system. The apparatus remains in the "up" state, i.e., the system is found undecayed, with probability

$$w = \int |A(\underset{\sim}{x})\, g(\underset{\sim}{x})|^2 d\underset{\sim}{x} = (g, A^2 g)_{L^2} \;, \qquad (25)$$

as easily seen from (23). As the system is already decayed, and will thus not be investigated further, if the final apparatus state is "down," the selection of those cases in which the apparatus state is "up" finally leads from (23) to the state

$$\underset{\sim}{g}' = NA(\underset{\sim}{x})\, g(\underset{\sim}{x}) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

of system plus apparatus, and therefore to the new wavefunction

$$g'(\underset{\sim}{x}) = NA(\underset{\sim}{x})\, g(\underset{\sim}{x}) \qquad (26)$$

of the system, which has to be normalized again. The normalization constant N, obviously, has to satisfy

$$1 = \|g'\|^2 = |N|^2 \int |A(\underset{\sim}{x}) g(\underset{\sim}{x})|^2 d\underset{\sim}{x} = |N|^2 w ,$$

so that $|N|^{-2} = w$, in accordance with (12). The above physical interpretation also implies that w is the transition probability for the change of wavefunction $g(\underset{\sim}{x}) \to g'(\underset{\sim}{x})$ during the measurement.

From Eqs. (20) and (24) we find that $A(\underset{\sim}{x})$ as defined here has the properties required in Section 2; vice versa, given any $A(\underset{\sim}{x})$ with those properties, our model with

$$s(\underset{\sim}{x}) = \frac{2}{\pi} \operatorname{arc\,cos} A(\underset{\sim}{x})$$

will just lead to this particular function $A(\underset{\sim}{x})$. Thus Eqs. (25) and (26) correspond exactly to Eqs. (12) and (11) of Section 2, as they should.

The most important point is, however, that Eq. (22) does not imply any restrictions for Δt. A smaller Δt just requires a correspondingly larger coupling constant λ--which is indeed what one would intuitively expect. It is probably true, of course, that there are no arbitrarily strong interactions in nature. But this has nothing to do with the basic principles of quantum theory, and in particular there is no internal inconsistency of our model for arbitrary values of λ. And the proper working of our model for arbitrarily small Δt was the only thing we wanted to prove.[10]

One could still object that the "apparatus" studied here is much too simple as a model for real measuring instruments. In particular, a characteristic feature of real instruments is the irreversible "amplification" of the initial microscopic state changes produced by the observed system, which finally results in macroscopically different "pointer readings." However, this may be taken into account by considering our model apparatus as describing some microscopic "trigger" part of the apparatus only, which is embedded in the real macroscopic apparatus in such a way that the microscopically different states "up" and "down" initiate irreversible processes leading to macroscopically different final states. (A discussion of models for measuring instruments imitating such kind of irreversible behavior will be presented elsewhere.[11] In these models, moreover, the "triggering" is not--as in the present model--achieved by an explicitly time-dependent (artificially "switched") interaction, but rather by suitable scattering processes involving the decay products and the "trigger" part of the apparatus.) Although the time of interaction between the decaying system and this enlarged apparatus is still Δt as given by Eq. (22), the total duration of the measurement could then be defined so as to include also the duration $\Delta t'$ of the subsequent irreversible processes. Since, however, there is no limit in principle for the rapidity of irreversible processes either, $\Delta t + \Delta t'$ can still be made arbitrarily small.

Summarizing the present discussion, we are inclined to answer the question: "Is Zeno's paradox real?" in a somewhat cautious way[12]: "In principle, yes--but...!" Every reader is invited to replace the dots by arguments of his own, concerning the practical difficulty or impossibility of an experimental verification. What we insist on, however, is the first part of the answer.

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Footnotes and References:

1. "Ideal" measurements are a very special type of state changes due to measurements, and are not very likely to occur in practice. For more general state changes, see K. Kraus, Ann. Phys. (N.Y.) 64, 311-335 (1971).
2. For details see the recent review article by L. Fonda, G. C. Ghirardi, A. Rimini, Rep. Progr. Phys. 41, 587-631 (1978), and the literature cited there.
3. This assumption is always made in the rigged Hilbert space framework; cf. A. Bohm, The Rigged Hilbert Space in Quantum Mechanics, Lecture Notes in Physics 78, Springer 1978.
4. For ΔE to be finite it is not necessary that f belong to the domain of definition of H^2; cf. K. Kraus, J. Schröter, Int. J. Theor. Phys. 7, 431-442 (1973).
5. B. Misra, E.C.G. Sudarshan, J. Math. Phys. 18, 756-763 (1977). Most of the earlier work on this problem is quoted there. For a very elementary discussion, see A. M. Wolsky, Found. Phys. 6, 367-369 (1976).

6. The situation might be different in very dense (e.g., nuclear) matter, where effects related to Zeno's paradox could become important; cf. P. Valanju, E.C.G. Sudarshan, C. B. Chiu, University of Texas preprint, June 1979.
7. G. C. Ghirardi, C. Omero, A. Rimini, T. Weber, International Centre for Theoretical Physics preprint IC/79/43, Trieste, 1979.
8. In the general framework described in Ref. 1, this measurement is a "selective operation" corresponding to the "effect" $F = A^2$.
9. Scalar products of vectors in $L^2(\underset{\sim}{x})$ are indicated by a subscript L^2.
10. Contrary to the arguments presented in Ref. 7 (cf. Eq. (5.14)), there is also no relation whatsoever between Δt and the width ε of the "smoothing" region of $A(\underset{\sim}{x})$.
11. K. Kraus, University of Texas Preprint, in preparation.
12. Compare Radio Erewan.