A UNIVERSAL FORMULA FOR THE QUASISTATIC SECOND-ORDER DENSITY PERTURBATION BY A COLD MAGNETOPLASMA WAVE*

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Abstract

Using the general expression for the ponderomotive Hamiltonian, we obtain the quasi-static quasi-neutral density change caused by the ponderomotive force of a cold magnetoplasma wave of arbitrary frequency and polarization:

$$\delta n(\underline{x}) = -\frac{|\tilde{E}(\underline{x})|^2 - |\tilde{B}(\underline{x})|^2}{4\pi(T_e + T_i)} .$$

This formula agrees with and extends previous results for unmagnetized and magnetized plasma.

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In studying the modulation of a finite-amplitude plasma wave, a number of authors have calculated the quasi-static quasi-neutral second-order density perturbation produced by the ponderomotive force of the modulation. With the representation

$$\phi(\underline{x},t) = \phi(\underline{x}) \exp(-i\omega t) + \text{c.c.} \tag{1}$$

for a longitudinal magnetoplasma wave, the result[1]

$$\delta n(\underline{x}) = -\frac{|\nabla\phi(\underline{x})|^2}{4\pi(T_e + T_i)} \tag{2}$$

has been obtained by Morales and Lee[2] for lower-hybrid waves, and by Shukla[3] for electron magnetoplasma waves. The former authors remarked on the identity of formula (2) with the familiar expression for Langmuir wave modulation in unmagnetized plasma.

It is natural to inquire into the universality of formula (2). In this paper, we show that it does indeed apply to <u>any</u> longitudinal <u>cold</u>-plasma wave (for a single ion species[4]); i. e., the three solutions[5] $\omega(\theta)$ of $\varepsilon_L(\omega,\theta) = 0$, where $\varepsilon_L \equiv \hat{k} \cdot \underset{\sim}{\varepsilon}(\omega) \cdot \hat{k}$.

More importantly, we show that formula (2) can be simply generalized to apply to a cold plasma wave of <u>any</u> polarization, i. e., to a wave with non-zero $\nabla \times \underline{E}$. Here we use a local plane-wave[6] representation

$$\underline{E}(\underline{x},t) = \underline{\tilde{E}}(\underline{x}) \exp(i\underline{k}\cdot\underline{x} - i\omega t) + \text{c. c.}, \tag{3a}$$

with
$$\underline{\tilde{B}}(\underline{x}) = (c/\omega)\underline{k} \times \underline{\tilde{E}}. \tag{3b}$$

The generalization, derived below, is

$$\delta n(\underline{x}) = -\frac{|\underline{\tilde{E}}(\underline{x})|^2 - |\underline{\tilde{B}}(\underline{x})|^2}{4\pi(T_e + T_i)} \tag{4}$$

We note first that it reduces to (2) when $\underline{\tilde{B}} = 0$. Secondly, for the transverse unmagnetized case, where

$$|\underline{\tilde{B}}|^2 = (kc/\omega)^2 |\underline{\tilde{E}}|^2 = (1 - \omega_p^2/\omega^2) |\underline{\tilde{E}}|^2,$$

formula (4) becomes $\delta n/n = -(e^2/m\omega^2) |\underline{\tilde{E}}|^2/(T_e + T_i)$, the familiar result.[7]

Formula (4) can be used for <u>any</u> cold-magnetoplasma wave, e. g., lower hybrid in the electromagnetic region[8], fast-magnetosonic-whistler[9], Alfvén[10], ordinary and extraordinary, etc., so long as (3b) is a valid approximation. (When it is not, use formula (10) below.)

Our derivation begins with the standard expression[11] for the quasi-static density perturbation, of species s, caused by the ponderomotive potential energy $\Psi_s(\underline{x})$ of an oscillation center[12] and by the self-consistent electric potential $\Phi(x)$:

$$\frac{\delta n_s(\underline{x})}{n_s^o} = -\frac{\Psi_s(\underline{x}) + e_s\Phi(\underline{x})}{T_s} . \tag{5}$$

For two species (electrons and singly-charged ions), we impose quasi-neutrality ($\delta n_e = \delta n_i$, $n_e^o = n_i^o$) to eliminate Φ, and obtain the relation

$$\frac{\delta n(\underline{x})}{n^o} = -\frac{\Psi_e(\underline{x}) + \Psi_i(\underline{x})}{T_e + T_i} . \tag{6}$$

Our expression for $\Psi_s(\underline{x})$ is based on a useful relation[13] for the ponderomotive Hamiltonian[14] of an oscillation center. In the cold-species limit, Eq. (3) of Ref. (13) reduces to

$$n_s(\underline{x})\,\Psi_s(\underline{x}) = -(4\pi)^{-1}\,\underline{E}^*(\underline{x}) \cdot \underset{\sim}{\chi}_\omega^s(\underline{x}) \cdot \underline{E}(\underline{x}), \tag{7}$$

with the representation $\underline{E}(\underline{x},t) \equiv \underline{E}(\underline{x}) \exp(-i\omega t) + \text{c. c.}$, where $\underset{\sim}{\chi}_\omega$ is the well-known[15] cold-species susceptibility. (We note that $\underset{\sim}{\chi}$ is proportional to density, so that Ψ is density-independent; but the dependence of $\underset{\sim}{\chi}$ on possibly nonuniform magnetic field $\underline{B}_o(\underline{x})$ appears in Ψ.)

Inserting (7) into (6), we have

$$\delta n\,(\underline{x}) = \frac{\underline{E}^*(\underline{x}) \cdot (\underset{\sim}{\chi}_\omega^e + \underset{\sim}{\chi}_\omega^i) \cdot \underline{E}(\underline{x})}{4\pi(T_e + T_i)} . \tag{8}$$

Now we use the field equation

$$(\underset{\sim}{\chi}_\omega^e + \underset{\sim}{\chi}_\omega^i) \cdot \underline{E}(\underline{x}) = - \underline{E}(\underline{x}) + (ic/\omega)\nabla \times \underline{B}(\underline{x}), \tag{9}$$

where $\underline{B}(\underline{x}) = (c/i\omega)\,\nabla \times \underline{E}(\underline{x})$, to obtain

$$\delta n(\underline{x}) = - \frac{|\underline{E}(\underline{x})|^2 - |\underline{B}(\underline{x})|^2 - (c/\omega)\mathrm{Im}\ \nabla \cdot \underline{E}^*(\underline{x}) \times \underline{B}(\underline{x})}{4\pi(T_e + T_i)} . \tag{10}$$

Finally, for a local plane wave, with $\underline{E}(\underline{x}) \equiv \tilde{\underline{E}}(\underline{x}) \exp i\underline{k} \cdot \underline{x}$ and (3b), one may drop the complex Poynting term in (10), as higher order in $k\nabla \ln \tilde{E}$; the result is then Eq. (4).

Two points should be kept in mind in applying (4): second-order magnetic perturbations may be of significance[16]; and the quasi-static assumption may be invalid.[17]

Footnotes and References

1. The numerical factor in the denominator is sometimes given incorrectly as 8π. If, instead of (1), one uses $\phi(\underline{x},t) = \text{Re}\, \phi(\underline{x}) \exp(-i\omega t)$, the factor should be 16π.

2. G. J. Morales and Y. C. Lee, Phys. Rev. Lett. 35, 930 (1975).

3. P. K. Shukla, J. Plasma Phys. 18, 249 (1977).

4. For more than one ion species, formulas (2) and (4) generalize to less beautiful forms.

5. We note that for the lowest-frequency solution (ion-cyclotron wave), the cold plasma model may be invalid.

6. More correctly, $\exp i\, \underline{k} \cdot \underline{x} \rightarrow \exp i\, \Theta(\underline{x})$, with $\underline{k}(\underline{x}) \equiv \nabla\Theta$.

7. P. Kaw, G. Schmidt, and T. Wilcox, Phys. Fluids 16, 1522 (1973).

8. G. J. Morales, Phys. Fluids 20, 1164 (1977).

9. R. L. Berger and F. W. Perkins, Bull. APS Oct. 1977; K. H. Spatschek, M. Y. Yu, and P. K. Shukla, J. Geophys. Res. 81, 1413 (1976).

10. J. A. Tataronis and W. Grossmann, Nuc. Fusion 16, 667 (1976); D. J. Kaup and A. C. Newell, Bull. APS Oct. 1977.

11. In a Vlasov treatment, T_s represents the effective temperature of velocity distribution parallel to the magnetic field, if the wave is localized. For a cavity mode, the temperature is not an adiabatic invariant.

12. G. Schmidt, Physics of High Temperature Plasmas, Sec. 2-9.

13. J. R. Cary and A. N. Kaufman, Phys. Rev. Lett. 39, 402 (1977).

14. R. Dewar, Phys. Fluids 16, 1102 (1973); S. Johnston, Phys. Fluids 19, 93 (1976).

15. T. Stix, The Theory of Plasma Waves, Sec. 1-2.

16. A. N. Kaufman and L. Stenflo, Phys. Scripta 11, 269 (1975); M. Porkolab and M. V. Goldman, Phys. Fluids 19, 879 (1976).

17. N. R. Pereira, R. N. Sudan and J. Denavit, Phys. Fluids 20, 271 (1977).