# ORGDP

**OAK RIDGE GASEOUS DIFFUSION PLANT**

**MARTIN MARIETTA**



K/DSRD-113

# HISTORICAL SUPPORT FOR A MIXED LAW LANCHESTRIAN ATTRITION MODEL: HELMBOLD'S RATIO

D. S. Hartley III
K. L. Kruse



OPERATED BY
MARTIN MARIETTA ENERGY SYSTEMS, INC.
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

K/DSRD--113

DE90 007109

# HISTORICAL SUPPORT FOR A MIXED LAW LANCHESTRIAN ATTRITION MODEL: HELMBOLD'S RATIO

D. S. Hartley III
K. L. Kruse

Date Published– November 1989

Research sponsored by the
The Joint Staff/J-8
under Interagency Agreement 1950-1612-A1

Prepared by
Data Systems Engineering Organization
Data Systems Research and Development Program

Located at
OAK RIDGE GASEOUS DIFFUSION PLANT
Oak Ridge, Tennessee 37831
operated by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
for the
U.S. DEPARTMENT OF ENERGY
under contract DE-AC05-84OR21400

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

## ABSTRACT

This is the first in a series of reports on the breakthrough research in historical validation of attrition in conflict. Significant defense policy decisions, including weapons acquisition and arms reduction, are based in part on models of conflict. Most of these models are driven by their attrition algorithms, usually forms of the Lanchester square and linear laws. None of these algorithms have been validated.

Helmbold defined the "activity ratio" to be the ratio of the Lanchester coefficients in the pair of differential equations of the Lanchester square law of attrition. He derived an equivalence between this ratio and a ratio containing the initial and ending force sizes, herein called the Helmbold ratio, and demonstrated a relationship between the Helmbold ratio and the initial force ratio in a large number of historical battles. This paper reexamines the implications of this relationship and concludes that its existence, rather than being supportive of the Lanchester square law, is supportive of a mixed law lying between the Lanchester linear law and a Lanchester logarithmic law. It is shown that the Helmbold relationship can discriminate between several attrition formulations; however, while this is a necessary condition, it is not sufficient to conclude that data fitting the relationship were caused by a given attrition formulation. The conclusion is that the data are not fine enough to determine the differential form of the attrition equations but do lead to a statistical statement about the outcomes of battles.

# 1. INTRODUCTION

Lanchester's equations of combat, or Lanchester's laws, are models of attrition in combat. There are two sets of equations: the first is based on an aimed fire model of combat under which the rate of attrition on one side is proportional to the number of troops on the other side; the second is based on an unaimed fire model of combat in which the rate of attrition on a side is proportional to the product of the numbers of troops on each side.

The first pair of equations [Eq.(1)] is called the "square law," because its integrated equation (2) contains $x^2$ and $y^2$, where $x_0$ and $y_0$ are the initial strengths of the two forces (attacker and defender, respectively) and $x$ and $y$ are their strengths at some later time. Table 1 presents a glossary of mathematical expressions used in these equations (and other equations in this paper), for convenient reference. The concept for this law is that each firer on side Y will pick targets on side X and try to kill that target. As long as there are targets on side X, the rate of attrition on side X will depend on the number of firers on side Y and that side's success rate, $D$ .

$$dx/dt = -Dy \qquad (1)$$
$$dy/dt = -Ax$$

$$D/A = ( x_0^2 - x^2 ) / ( y_0^2 - y^2 ) \qquad (2)$$

The second pair of equations [Eq.(3)] is called the "linear law," because its integrated equation [Eq.(4)] contains $x$ and $y$ as linear terms. The concept is that each firer on side Y will shoot into the general area of side X. The attrition on side X will depend on the number of firers on side Y, the number of targets on side X, and the success rate, $D'$ .

$$dx/dt = -D'xy \qquad (3)$$
$$dy/dt = -A'xy$$

$$D'/A' = ( x_0 - x ) / ( y_0 - y ) \qquad (4)$$

An additional set of equations [Eq.(5)] is shown below with the integrated equation [Eq.(6)]. [The function ln() represents the natural logarithm.] This pair will be useful for the discussions later and will be termed the logarithmic law.

$$dx/dt = -D''x^2y \qquad (5)$$
$$dy/dt = -A''xy^2$$

$$D''/A'' = ( \ln(x_0) - \ln(x) ) / ( \ln(y_0) - \ln(y) ) \qquad (6)$$

While the Lanchestrian laws are generally stated as pairs of differential equations, most simulations of warfare evaluate their results by using a corresponding difference equation approximation. The methodology consists of modeling the course of a battle with

**Table 1. Glossary of mathematical expressions**

| Term | | Explanation |
|---|---|---|
| $x_0$ | = | initial number of men on side X (attacker side) |
| $y_0$ | = | initial number of men on side Y (defender side) |
| $x_0/y_0$ | = | initial **force ratio** "forrat" |
| $D$ , $D'$, etc. | = | coefficient used to define the rate of change (attrition) of the X forces |
| $A$ , $A'$ , etc. | = | coefficient used to define the rate of change (attrition) of the Y forces |
| $D/A$ | = | **activity ratio** |
| $x$ | = | number of men on side X at time t |
| $y$ | = | number of men on side Y at time t |
| $\frac{(x_0^2 - x^2)}{(y_0^2 - y^2)}$ | = | **Helmbold's ratio** "helmrat" |
| casx | = | casualties on side X |
| casy | = | casualties on side Y |
| casx/casy | = | **exchange ratio** "exchrat" |
| $x_r$ | = | remaining number of men on side X |
| $y_r$ | = | remaining number of men on side Y |
| $\frac{(x_0 + x_r)/2}{(y_0 + y_r)/2}$ | = | **average forces ratio** |
| $SSE$ | = | sum of squared errors |
| $TSS$ | = | total sum of squares |
| $R^2$ | = | 1 - ($SSE/TSS$ ), indicates how much of the variation is explained by model |

discrete time steps, $\Delta t$ . The time step is made small enough that the corresponding change in the force sizes, $\Delta x$ and $\Delta y$ , approximate the differentials, $dx$ and $dy$ . Hence, the values given in Eqs.(1), (3), or (5) are used for $\Delta x / \Delta t$ and $\Delta y / \Delta t$ .

It has been shown[1] that there exists at least one historical battle for which neither pair of equations is capable of predicting the attrition results. However, the logic which led to the formulation of the laws is so compelling that many attempts have been made to find some confirmation of some part of the concept. In fact, many computer models of warfare use extensions of these laws for their attrition submodel.

Dr. Robert Helmbold collected and analyzed data on a large number of battles, looking for trends and relationships. He published a sequence of papers[2,3,4] containing his results. Section 2 of this paper discusses one of his results. Section 3 uses simulations to show that neither the square law nor the linear law in the forms described here would produce this result.

The conclusion from Helmbold's data is that, in spite of the massive variability of combat data (including differing types of combat), there are some effects that may be consistently predicted from the battle's initial conditions. If this conclusion can be confirmed with other data, useful results can be derived from this relationship between initial conditions and outcomes.

## 2. A RESULT FROM HELMBOLD

Helmbold began by assuming the attrition for the battles can be modeled with the square law. To obtain a fit between the assumed square law and the data, he introduced the notion of force-ratio dependent *A* and *D* coefficients. The analysis later in this paper shows a different interpretation of the results, in that it does not require a square law assumption; however, the relationship between the Helmbold ratio (defined in Sect. 2.2) and the force ratio still exists.

Section 2.1 describes some key concepts for understanding the differences between this work and Helmbold's work and for understanding what is assumed to be true that may be false. Section 2.2 describes Helmbold's data and the result. Section 2.3 sets forth the generalized Helmbold relationship and defines the questions that must be answered concerning the relationship. The thought experiments of Sect. 2.3 set the stage for the more detailed stochastic investigations of Sect. 3.

### 2.1 THE CONSTANT FALLACY AND VARIABLE CONSTANTS

The key to understanding the difference between Helmbold's results and the results in this paper requires some fine distinctions in definitions and clear statements of assumptions. One of the key distinctions resides in what Helmbold calls the Constant Fallacy (R. L. Helmbold, U.S. Army Concepts Analysis Agency, Bethesda, MD, letter to D.S. Hartley III, May 26, 1989).

Equations (1-6) contain several explicit constants ($D$ , $A$ , $D'$ , etc.) and some not so clearly explicit constants (i.e., the exponents of the variables $x$ and $y$ ). The statement that a given law is valid for all battles contains the statement that the exponents of the variables are universal constants. That is, if the square law were claimed, then the exponent of the $y$ variable on the right side of the first equation in Eq.(1) is 1.0 for all battles and the exponent of the $x$ variable as a factor of the right side of the first equation is 0.0, implicitly. These "constants" are claimed to be constant for all battles. However, the $D$ and $A$ constants of Eq.(1) need not be universal constants. Indeed, standard usage of the laws in models and a reading of the definitions usually given for the square law, etc., requires that $D$ and $A$ be free to vary from battle to battle and are constant only within the duration of the battle (and sometimes not even then). Such "constants" are "particular" constants. Assuming that $D$ and $A$ are universal constants is what Helmbold calls the Constant Fallacy.

Any efforts to derive information from historical data based on this assumption are flawed to the extent of the dependence of the results on the assumption. The work in this paper resembles some work that may be flawed in this manner; however, care has been taken to ensure that the Constant Fallacy assumption is avoided.

Two assumptions are made. These assumptions are **not** based on facts; they are assumptions.

The first assumption is that there is a law of attrition. This means that it is assumed that the exponents, whatever they may be are universal constants. (Actually, there is room for some variation, of unknown magnitude. That is, it is conceivable that an exponent's basic value is 1.0, with some battles having an exponent of, for example, 1.01.) This assumption is immediately suspect, as the philosophical explanation of the differences between the square law and the linear law invoke proposed different attrition methodologies for different circumstances, each equally believable. The assumption must rest on either the invalidity of these methodologies in actual combat, or in a disappearance of their distinguishable effects in combat at some sufficiently aggregated level.

The second assumption is the one that differentiates the conclusions of this paper from those of Helmbold. Helmbold assumes the square law and finds the requirement for the coefficients to include a factor based on the initial force ratio in order to find a fit to the historical data. We do not assume the square law; however, there is an **almost** equivalent assumption made. We assume that the coefficients are not dependent on the initial force ratio.

Within the limits of this data the two assumptions are equivalent. We replace the **initial** force ratio in the fitting process with the **previous** force ratio. This has two effects. The first is that we have both $x$ and $y$ variables in the right hand sides of both differential equations, yielding a mixed law. The second effect is that for this data, the results are equivalent, because the only **previous** force ratio available is the **initial** force ratio. However, for sets of data with intermediate force ratios during the course of each battle, the two assumptions are no longer equivalent. The test between the two assumptions must wait for either perfect data about the course of one battle or imperfect data on the courses of a sufficient number of battles to overcome the imperfection of the data.

## 2.2 HELMBOLD'S RELATIONSHIP

Helmbold derived Eq.(2) from the square law equations [Eq.(1)]. Helmbold called the ratio of coefficients, $D/A$ , the activity ratio. We shall call the right side of Eq.(2) the Helmbold ratio, whether or not it equals $D/A$ . For a battle governed by Eq.(1), Eq.(2) states the equality of the activity ratio and the Helmbold ratio.

One of the relationships that Helmbold found to be significant was the relationship between the logarithm of the activity ratio and the logarithm of the initial force ratio, $x_0/y_0$ . Not only was this relationship apparent in the 92 (land) battles in his first volume of analysis[2], but it was also there in the 83 (land) battles of the second volume[3] (Helmbold reduced the 83 battles to 81 because of doubts about the data for two battles), and in the Battle of Britain data (air battles)[4]. Even more significant, the slope coefficient, $\alpha$, of $\ln(x_0/y_0)$ and the intercept, $\beta$, of the regressions are close in value. The value of $\alpha$ will be shown to be particularly significant. Therefore, it is important to note that the $\alpha$ values of the 83 and 92 land battles are not significantly different from the $\alpha$ of the Battle of Britain data. The $\alpha$ value of the combined data is different from 0.0, 1.0 and 2.0 at the 99% level of significance.

The regression results that define the Helmbold relationship for each set of data are shown in Table 2. Table 2 also shows the Helmbold relationship parameters for the combined data, both including and excluding the two suspect battles of the 83. The combined data used in this paper will include the two battles unless otherwise specified.

Table 2. Historical data Helmbold Relationship results

| Historical case | α | β | $R^2$ |
|---|---|---|---|
| 92 Battles | 1.3 | 0.21 | 0.54 |
| 83 Battles | 1.7 | -.16 | 0.70 |
| 83 Battles (-2) | 1.5 | -.03 | 0.63 |
| Battle of Britain | 1.5 | 0.24 | 0.67 |
| Combined data (-) | 1.4 | 0.12 | 0.60 |
| Combined data | 1.5 | 0.05 | 0.65 |

These relationships are shown graphically in Figs. 1, 2, and 3. The two battles that Helmbold omitted from the 83 are shown in the extreme lower left in Fig. 2. The computer variables LFORRAT and LHELMRAT refer to the natural logarithms of the force ratio and Helmbold ratio, respectively. The scales are identical for ease of comparison, and each graph shows the regression line from Table 2 and the 95% confidence limits for the sample points. These limits imply that all of the data might be from the same population, with the implication that whatever "law" may be governing the results may be consistent across samples and between land and air engagements.

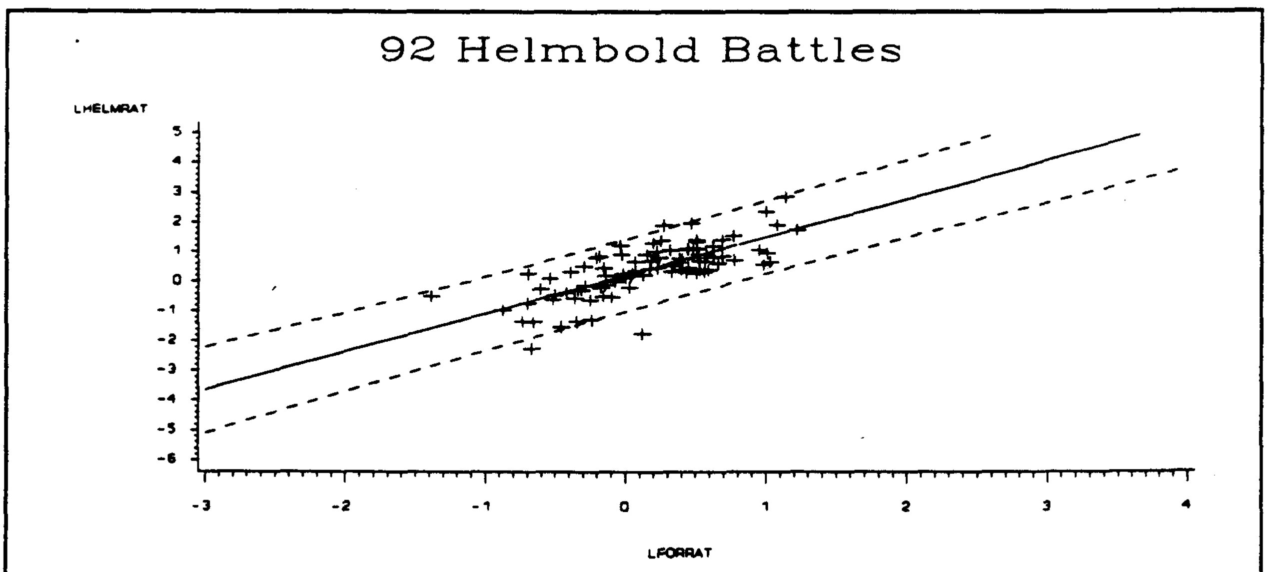


Fig. 1. 92 land battles ln(helmrat) vs ln(forrat).

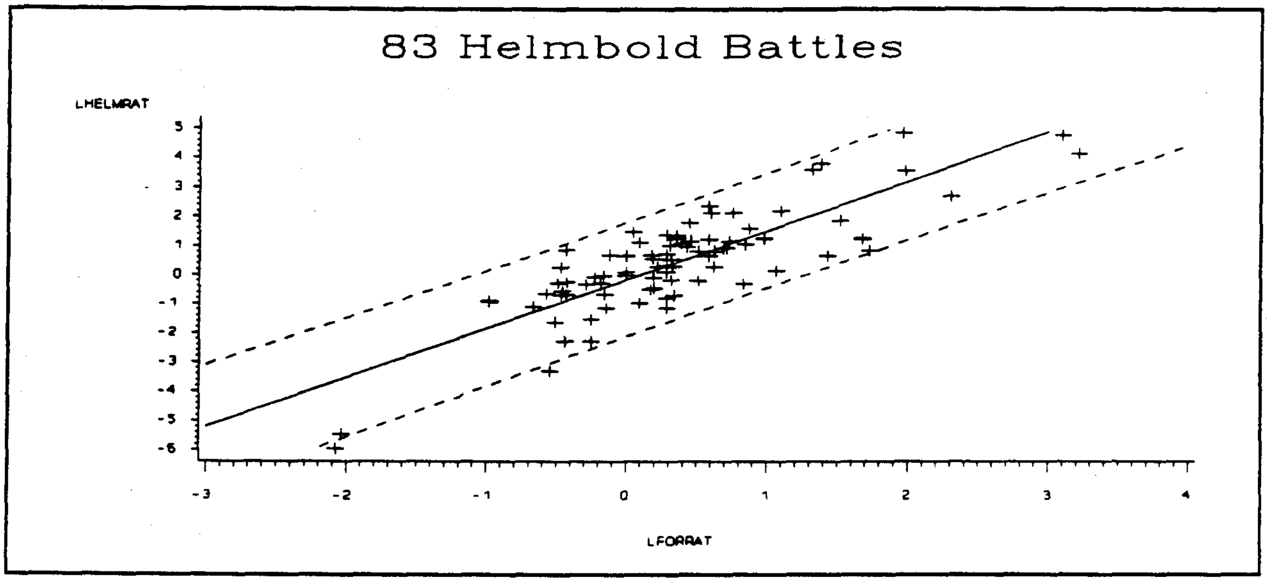


Fig. 2. 83 land battles ln(helmrat) vs ln(forrat).

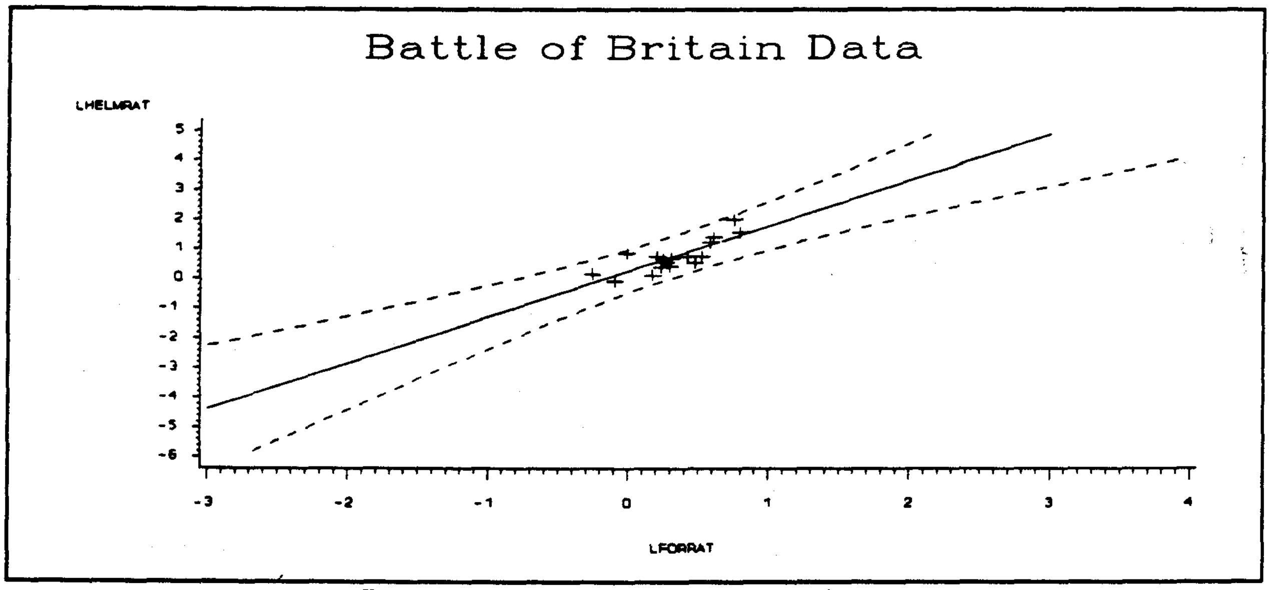


Fig. 3. Air battles ln(helmrat) vs ln(forrat).

Helmbold assumed a square law relationship for the battles and stated the relationship as a fit between the activity ratio and the force ratio. However, as neither *D* nor *A* had historically recorded values, he computed the activity ratio from the Helmbold ratio using Eq.(2). Helmbold expressed the results of this fit for the 92 battles as:

$$\ln(D/A) = 1.26\ln(x_0/y_0) + 0.23 + \text{error}. \tag{7}$$

(Our regression on Helmbold's data yielded a slope coefficient of 1.29 and an intercept of 0.21, with an $R^2$ of 0.54.)

## 2.3 THOUGHT EXPERIMENTS

Because the battles Helmbold studied have not been shown to follow the square law, the relationship discovered by Helmbold is better expressed using the Helmbold ratio. Replacing the experimental coefficients in Eq.(7) with $\alpha$ and $\beta$, the idealized form of Helmbold's relationship (with a zero error term) is:

$$\ln[(x_0^2 - x^2) / (y_0^2 - y^2)] = \alpha \ln(x_0 / y_0) + \beta. \tag{8}$$

From its formulation, in which $x_0$ appears in the numerator and $y_0$ in the denominator on both sides of the equation, Helmbold's relationship might be a nearly tautological relationship, producing large $R^2$ values for any data. For Helmbold's relationship to be useful, it should be restrictive; that is, a random collection of values for the $x$ and $y$ values should be expected to have an $R^2$ of zero when Eq.(8) is fit to the collection of data. If good fits can be found for arbitrary sets of values of the variables, then the relationship tells us nothing about warfare. However, if the sets of variable values for which good fits can be found are extremely restrictive, then the existence of such a relationship for historical data becomes important. Moreover, if the sets can be classified in military terms by the values of the coefficients, the relationship becomes even more important.

If all battles ended with the final strengths of both sides on the order of 1% of their respective initial strengths, then Helmbold's ratio would approximate $x_0^2/y_0^2$. Its natural log would then be about twice the natural log of the force ratio:

$$\ln[(x_0^2 - x^2) / (y_0^2 - y^2)] \approx \ln(x_0^2/y_0^2) = 2\ln(x_0/y_0). \tag{9}$$

This would be the kind of tautological situation mentioned above. Although this experiment does not rest on a valid assumption (we know that most battles do not annihilate both sides), it does illustrate the technique of approximation which will be used in more realistic experiments.

First let us consider the implications of a perfect data fit. Equation (8) may be recast as:

$$(x_0^2 - x^2) / (y_0^2 - y^2) = e^{\beta}(x_0/y_0)^{\alpha} \tag{10}$$

(where e is 2.718..., the base of the natural logarithm). If we let casx and casy be the casualties on sides X and Y, respectively:

$$\begin{aligned} \text{casx} &= x_0 - x \\ \text{casy} &= y_0 - y, \end{aligned} \tag{11}$$

then Eq.(10) may be rewritten as:

$$(\text{casx}/\text{casy})[(2x_0 - \text{casx}) / (2y_0 - \text{casy})] = e^{\beta}(x_0/y_0)^{\alpha}. \tag{12}$$

We are now prepared to compare three basic hypothetical cases, governed by the square, linear, and logarithmic laws, respectively, and to determine the restrictions placed by these laws on the coefficients $\alpha$ and $\beta$ in Eq.(12).

For our first hypothetical case, let us suppose that the data consists of the results from a group of battles, each with identical pairs of square law coefficients governing the attrition, and each lasting only a very short time. Because the time is so short, casx and casy are very small fractions of $x_0$ and $y_0$, respectively. Hence, we have:

$$\text{casx} = -D\,y_0 \qquad (13)$$
$$\text{casy} = -A\,x_0\,.$$

$$(D\,y_0 / A\,x_0)(2x_0 / 2y_0) = e^{\beta}(x_0/y_0)^{\alpha} \qquad (14)$$

$$D/A = e^{\beta}(x_0/y_0)^{\alpha}. \qquad (15)$$

If we assume that $D$ , $A$ , $x_0$ , and $y_0$ are independent, so that $D/A$ and $x_0/y_0$ are independent, we conclude:

$$\alpha = 0 \qquad (16)$$
$$\beta = \ln(D/A).$$

The assumption that the activity ratio and the force ratio are independent is not mathematically required. In fact, Helmbold's statement of his results is that they are not independent. It is certainly conceivable that there is an underlying military reality such that battles are joined only when conditions exist that cause this dependence to hold; however, the practical considerations of warfare make this appear unlikely. If Y beat X in a battle and X spent the next year training (increasing $A$ ), then the two fought another battle with the same size forces, it is not reasonable that the mere fact that X increased $A$ would cause Y's coefficient against X, $D$ , to increase proportionally. We will show later in the paper that Lanchester theoretical considerations also preclude this dependence. Therefore, we shall assume that $D$ , $A$ , $x_0$ , and $y_0$ are independent.

For our second hypothetical case, let us assume that we have a set of battle data, obeying the linear law, and having a perfect regression as shown in Eq.(12). We may proceed in a manner analogous to our derivation of Eq.(16). Assuming the time is short so that casx and casy are very small fractions of $x_0$ and $y_0$ , respectively, we have:

$$\text{casx} = -D'\,x_0\,y_0 \qquad (17)$$
$$\text{casy} = -A'\,x_0\,y_0$$

$$(D'\,x_0\,y_0 / A'\,x_0\,y_0)(2x_0/2y_0) = e^{\beta}(x_0/y_0)^{\alpha} \qquad (18)$$

$$D'/A' = e^{\beta}(x_0/y_0)^{\alpha-1}. \qquad (19)$$

Again, if the force ratio may vary independently of the coefficients, we conclude:

$$\alpha = 1 \qquad (20)$$
$$\beta = \ln(D'/A').$$

For our third hypothetical case, let us assume that we have a set of battle data, obeying the logarithmic law [Eq.(5)], and having a perfect regression as shown in Eq.(12). Assuming the time is short so that casx and casy are very small fractions of $x_0$ and $y_0$, respectively, we have:

$$\text{casx} = -D'' x_0^2 y_0 \qquad (21)$$
$$\text{casy} = -A'' x_0 y_0^2$$

$$( D'' x_0^2 y_0 / A'' x_0 y_0^2 )( 2x_0 /2y_0 ) = e^{\beta}( x_0 /y_0 )^{\alpha} \qquad (22)$$

$$D''/A'' = e^{\beta}( x_0 /y_0 )^{\alpha-2}. \qquad (23)$$

Again, if the force ratio may vary independently of the coefficients, we conclude:

$$\alpha = 2 \qquad (24)$$
$$\beta = \ln( D'' /A'' ).$$

At this point, given our assumptions about a perfect fit and the independence of the activity ratios and the force ratios, we have the possibility of classifying the type of attrition of a set of battles by the value of the coefficient $\alpha$. We now proceed with the results of some stochastic simulations to examine the effects of groups of battles with varying sets of coefficient values and less than perfect fits and to verify our independence assumption.

## 3. SIMULATION PLAN AND RESULTS

All of the simulations are variations on a basic simulation plan. Each simulation constructs a group of battles (usually 100) and is replicated 50 times. Each battle within a group has randomly chosen initial forces for each side. (The forces are chosen to be normally distributed with mean of 1000 and standard deviation of 200. The normal distribution was chosen simply as convenient distribution with a strong central tendency.) All of the battles within a simulation have the same attrition coefficient pairs for the two sides and use the same attrition method.

The simulations are divided into four major sets on the basis of the attrition method to be used: square law, linear law, logarithmic law, or mixed law (labeled, respectively, S1-S9, L1-L8, Lo1-Lo8, and M1). Within each set, the simulations differ in their application of the law.

The first simulation of each set is a straightforward application of the basic law (using the difference equations approximation of the law). Each battle is evaluated for one attrition iteration--a nominal day of battle. This first simulation sets a base line for comparison. Each of the following simulations within a set is a variation designed to test the effect of factors which would be expected to have affected historical battles or to test analysis techniques. Tables 3, 5, and 7 label each simulation and give the values for the variables used in each. The remainder of this section discusses the purposes for the simulations within each set.

The first comparison-to-reality factor to be tested is the effect of random statistical variations in the values of *D* and *A*, the attrition coefficients. According to Lanchester theory, these are what make one battle different from another one having the same force sizes. It is therefore to be expected that a group of historical battles will be made up of battles with differing pairs of attrition coefficients. Since *D* and *A* are hypothetical constructs, having an uncertain existence, we have no knowledge of the range or distribution of their values. The normal distribution provides a convenient first approximation in this situation.

The second and third simulations of each set test the results of allowing moderate and small variations respectively in the *D* and *A* values. These variations are implemented by randomly selecting the *D* and *A* values so that the mean values match those of the first simulation and the standard deviations are 10% and 5% of the means for *D* and A, respectively, for the moderate cases, and 1% and 0.5% of the means in the small variation cases. The size of the standard deviations were selected by trial and error so that the results of the analyses would be in the neighborhood of the results seen by Helmbold in his analysis of real battle data. Restrictions are applied to the casualty generating program to prevent casualties exceeding force size, being negative, etc.

The first factor tested for analytical reasons (that is, to clarify a question of analytical technique) is the effect of choosing a single attrition iteration for the simulations as opposed to some larger number of iterations. The fourth simulation of each set is identical to the first except that each battle proceeds for 5 days. The difference equations are evaluated for five iterations, with the first being the same as the single iteration of hypothetical case one. The fifth simulation confirms the effects of moderate *D* and *A* variations upon 5-d battles.

The second comparison-to-reality factor to be tested is the effect of having battles within a group lasting varying lengths of time. (This differs from the thrust of the fourth and fifth simulations in that they tested whether it makes a difference to look at the battles after day z as opposed to day 1; whereas the thrust here is to examine the effect of having variations in battle length within a group of battles.) The sixth simulation of each set consists of battles with fixed *D* and *A* values and varying lengths. These variations are implemented by randomly selecting the length of each battle so that the mean is 2-d with a standard deviation of 1-d. The number of days is rounded to an integer value and restrictions are applied to prevent zero or negative days. (While the base distribution used for the number of days to be used is a normal distribution, the truncation of the distribution due to the closeness of the mean to zero and the clumping due to rounding to integral values makes the final distribution merely one with a central tendency, which was all that was desired.)

The second factor tested for analytical reasons is the effect of dividing the attrition results of each battle by the length of the battle. This procedure is an attempt to allow the analyst to compare "apples with apples" (see Fain's "casualties per day")[5], which will be later shown to introduce problems of its own. The seventh simulation implements this test.

Each of the first seven simulations has been derived from the assumption that real battles between any pair of forces may have any set of coefficients, forcing the $\alpha$ parameter of the regression to the value in the thought experiment (e.g., zero for the square law). However, given a set of combat data yielding a good regression with a nonzero $\alpha$ value, we might conclude that there is a nonmathematical (real world) constraint on the pairings of force ratios and Helmbold ratios which allows a square law attrition, but does not force a zero $\alpha$. (Similar conclusions might be justified for the linear law and the logarithmic law when the value of $\alpha$ does not equal that produced for the law in the thought experiment).

Fortunately, there is a theoretical test of this question. By our assumption of a battle obeying one of the laws for attrition, we may produce a set of data points all arrived at using common coefficients, yet having varying force ratios. Each data point is defined by 1-d of a battle-to-annihilation: it uses that day's casualties for casx and casy and the previous day's ending force sizes as the $x_0$ and $y_0$ values. The *D* and *A* values stay constant over the days of the battle; but the force ratio changes for the square law, linear law, and logarithmic law as defined in Eq.(1), (3), and (5). An alternative version of the logarithmic law, the Eq.(25), does not have this property if $D = A$. These equations have the same integral Eq.(6) as the Eq.(5) and are often referred to as the logarithmic law.

$$dx/dt = -D'' x \qquad (25)$$
$$dy/dt = -A'' y$$

The Eq.(5) follow the same pattern defined by Willard[6] and used by Fain[5] in their search for a mixed law fitting combat history and are used here. Both versions yield regressions with an $\alpha$ value of 2.0.

The eighth simulation of each set makes the test discussed above. The group of battles in the simulation consists of each of the daily battles of one battle fought to annihilation of one or the other side. Each of these battles starts with the ending forces of the previous day.

The ninth simulation (existing only in the square law set) tests the conjecture that the problems seen with the results of the eighth simulation were caused by using the inexact, difference equation approach. The ninth simulation uses the same solutions as the square law equations.

Sections 3.1, 3.2, and 3.3 discuss the results of the three sets of simulations. Tables 4, 6, and 8 contain statistics from the analyses of the simulations. The statistics give data about the $R^2$, $\alpha$, and $\beta$ values for the regressions performed on each replication of each simulation. For each simulation, the minimum, maximum, and mean values for each statistic are shown, as well as the value of the standard deviation of the statistic within the replications.

## 3.1 SQUARE LAW SIMULATION RESULTS

The square law simulations were produced using $D$ =.02 and $A$ =.01 in Eq.(1), as shown in the title of Table 3. The analysis consists of a regression of the attrition and force data using Eq.(10), with results shown in Table 4, and analysis of the graphs of various pairs of variables.

**Table 3. Square law simulation descriptions**

Base value $D$ =.01, $A$ =.02, $x_0$ =1000, $y^0$ =1000, 1 $\sigma$ for $x^0$ and $y^0$ = 200

| Label | 1 $\sigma$ $D$ and 1 $\sigma$ $A$ Variation/Base | Days of battle | 1 $\sigma$ days | Comments |
|---|---|---|---|---|
| S1 | 0.0 | 1 | 0 | 1 day, no variation in $D/A$ or days |
| S2 | 0.001 | 1 | 0 | moderate $D$ and $A$ variation |
| S3 | 0.0001 | 1 | 0 | small $D$ and $A$ variation |
| S4 | 0.0 | 5 | 0 | 5 day, no variation in $D/A$ or days |
| S5 | 0.001 | 5 | 0 | moderate $D$ and $A$ variation |
| S6 | 0.0 | 2 | 1 | moderate day variation |
| S7 | 0.0 | 2 | 1 | moderate day variation, attrit/days |
| S8 | 0.0 | | | battle to end, difference equations |
| S9 | 0.0 | | | battle to end, exact solution |

Table 4. Square law Helmbold relationship results

| | $R^2$ | | | | $\alpha$ | | | | $\beta$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Label | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ |
| S1 | 0.92 | 1.00 | 0.99 | 0.01 | 0.02 | 0.02 | 0.02 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| S2 | 0.00 | 0.07 | 0.01 | 0.02 | -.11 | 0.10 | 0.01 | 0.04 | 0.67 | 0.72 | 0.69 | 0.01 |
| S3 | 0.05 | 0.34 | 0.17 | 0.07 | 0.01 | 0.03 | 0.02 | 0.00 | 0.68 | 0.69 | 0.69 | 0.00 |
| S4 | 0.95 | 1.00 | 0.99 | 0.01 | 0.02 | 0.02 | 0.02 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| S5 | 0.00 | 0.06 | 0.01 | 0.01 | -.04 | 0.10 | 0.02 | 0.03 | 0.67 | 0.71 | 0.69 | 0.01 |
| S6 | 0.86 | 1.00 | 0.99 | 0.02 | 0.02 | 0.02 | 0.02 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| S7 | 0.00 | 0.18 | 0.04 | 0.04 | -.01 | 0.00 | 0.00 | 0.00 | 0.69 | 0.70 | 0.69 | 0.00 |

For the first simulation (S1), the results are as predicted by the first thought experiment. The $R^2$ for the regression is nearly 1.00 in all replications, with $\alpha = 0.02$ (near 0) and $\beta = \ln(D/A) = \ln(2) = 0.69$. Figures 4 through 7 plot the relationships of some of the variables for one representative replication out of the 50.

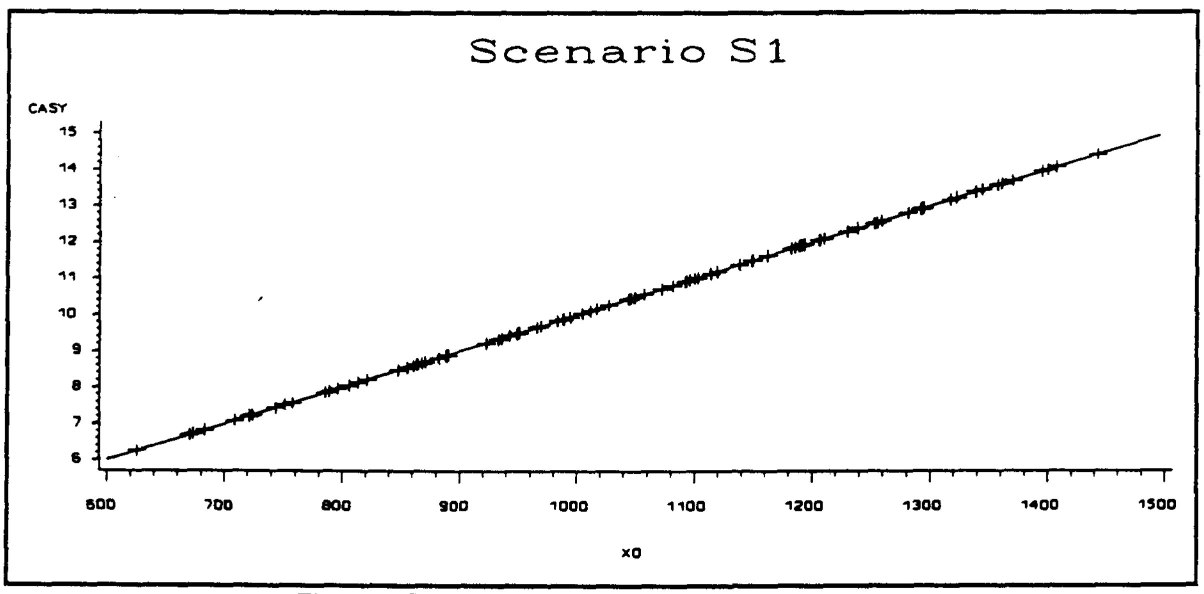


Fig. 4. S1 casualties vs opposing force size.

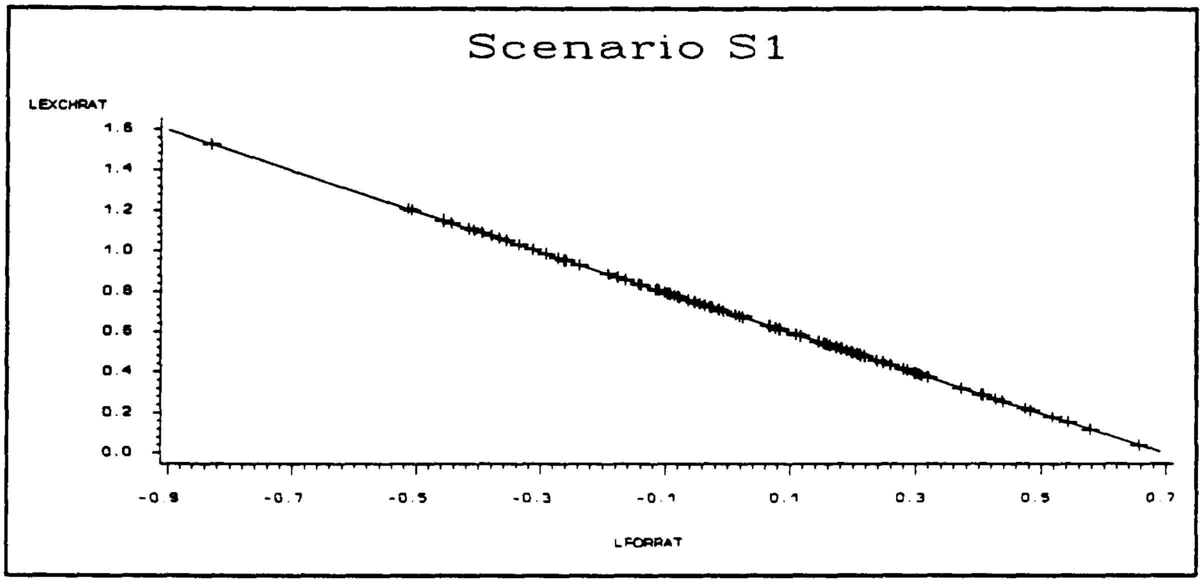


Fig. 5. S1 ln(exchrat) vs ln(forrat).

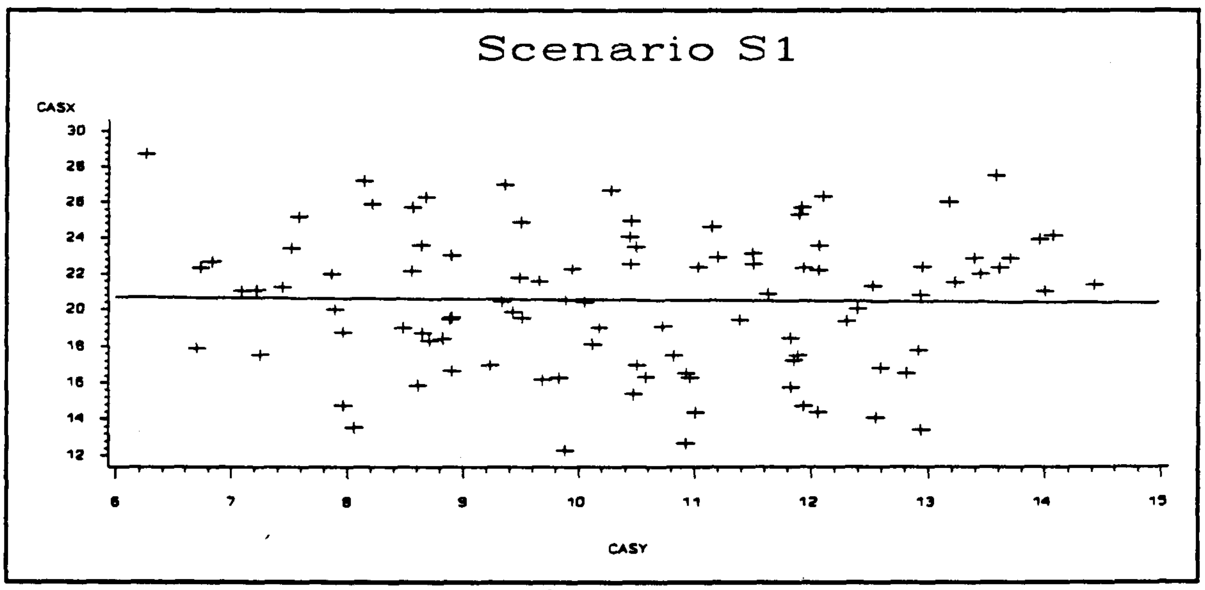


Fig. 6. S1 casx vs casy.

Figure 4 illustrates the basic structure of a square law simulation: the casualties on one side are a linear function of the other side's force size. Figure 5 illustrates a consequence of the basic structure of a square law simulation: the logarithm of the exchange ratio is a linear function of the logarithm of the force ratio. Figure 6 illustrates that taking fixed percentages of the randomly generated force sizes for each side yields a zero-correlation situation between the casualties for each side.

Figure 7 illustrates Helmbold's relationship for this simulation: the logarithm of the Helmbold ratio is nearly a linear function of the logarithm of the force ratio. (The small slope is exaggerated by the scale of the plot to emphasize the shape of the function.)

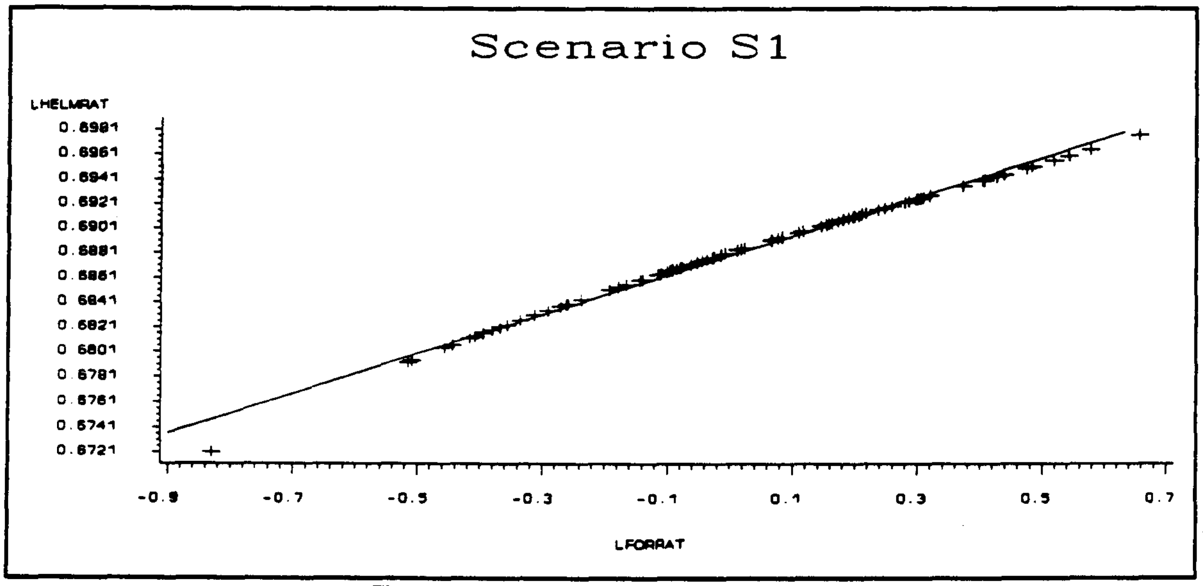


Fig. 7. S1 ln(helmrat) vs ln(forrat).

The fact that the relationship shown in Fig. 7 is not exactly linear will be addressed in simulation S9. The range of the $R^2$ values of 0.92 to 1.00 indicates that some runs are less nearly linear than others. The mean of 0.99 and standard deviation of 0.01 implies that most of the runs are nearly linear. In scenario S1, all of the runs had identical $\alpha$ and $\beta$ values, with corresponding standard deviations, $\sigma$, of zero.

Figure 8 shows the spread of values that the variation of *D* and *A* of the second simulation (S2) induces in the relationship between casualties and opposing force size. (Compare this to Fig. 4.) Figure 9 shows the similar effect on the Helmbold relationship. The $R^2$ for the replications were all essentially zero; however, the calculated $\alpha$ and $\beta$ values had means that agree with the results from the first simulation. That is all of the $\alpha$ values were close to zero, with a mean of zero, and all of the $\beta$ values were close to ln(2), with a mean of ln(2).

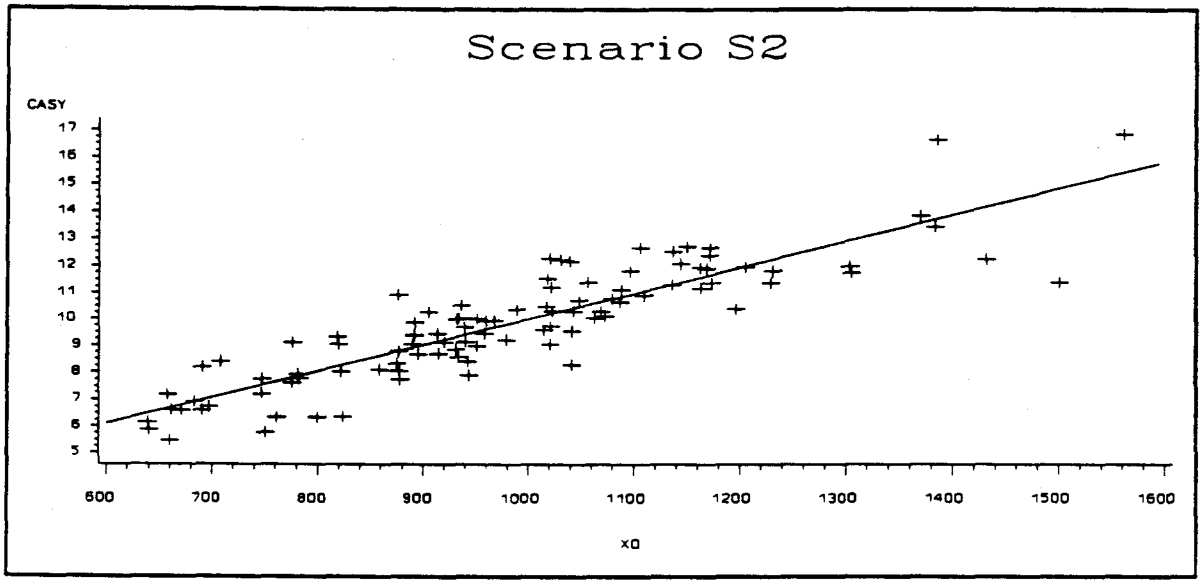


Fig. 8. S2 casualties vs opposing force size.

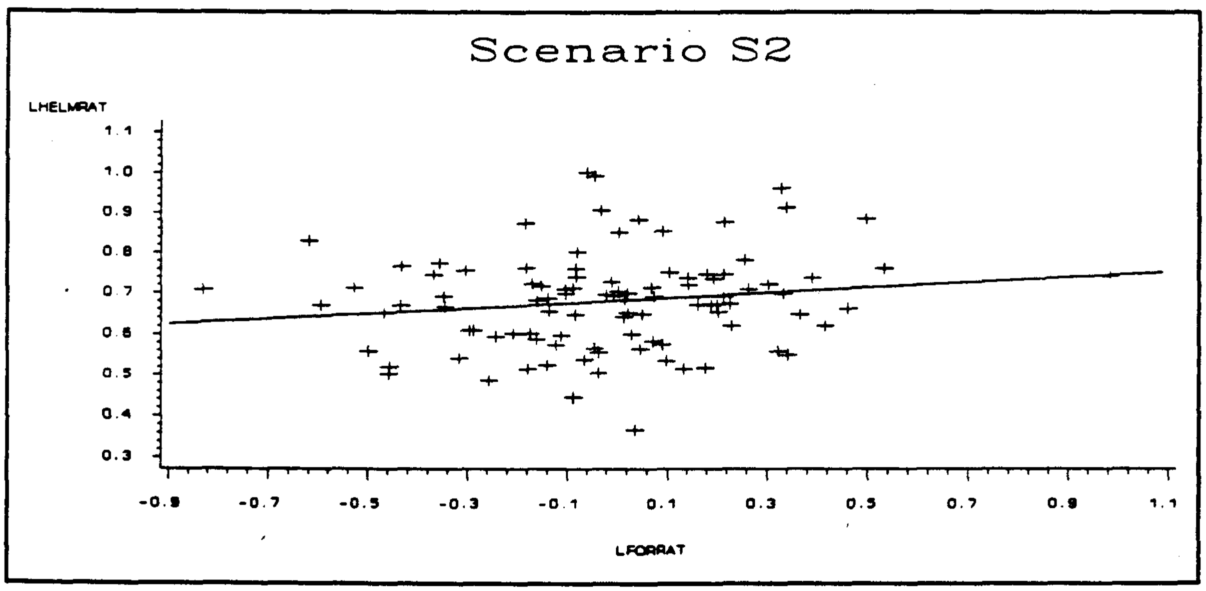


Fig. 9. S2 ln(helmrat) vs ln(forrat).

The third simulation, S3, was produced in the same manner as the second, the only difference being that the standard deviations for the coefficient samples were smaller. The chief output differences were that the $R^2$ values were greater than zero, ranging from 0.05 to 0.34 with a mean of 0.17, and the standard deviations of the $\alpha$ and $\beta$ values were zero, rather than 0.04 and 0.01, respectively. Figures 10 and 11, corresponding to Figs. 8 and 9 for S2, show the results for one replication, with the plots being closer approximations to lines.

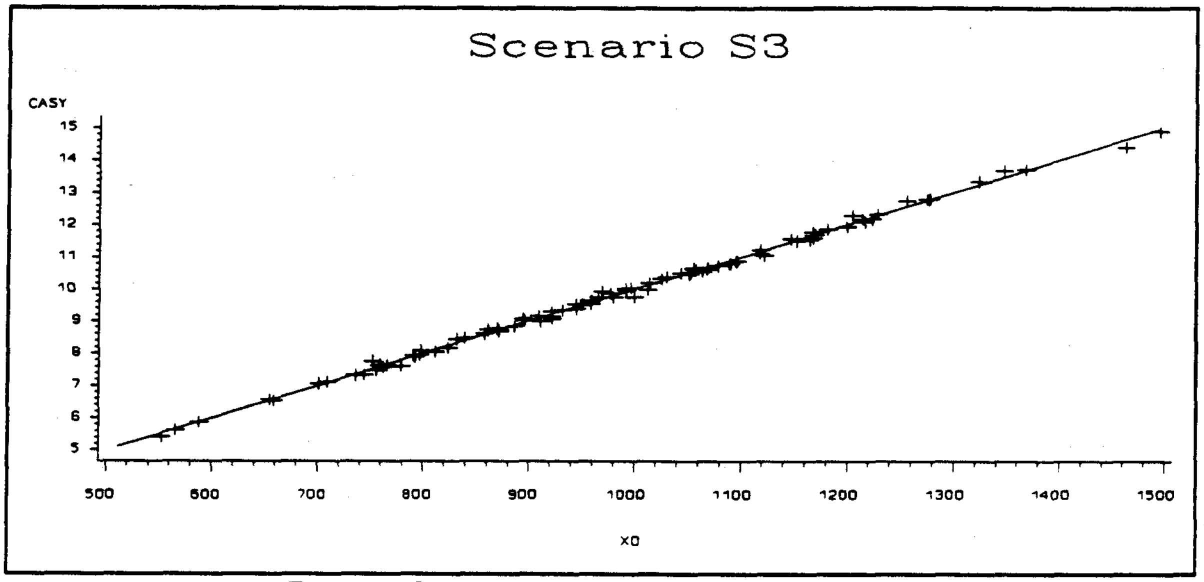


Fig. 10. S3 casualties vs opposing force size.

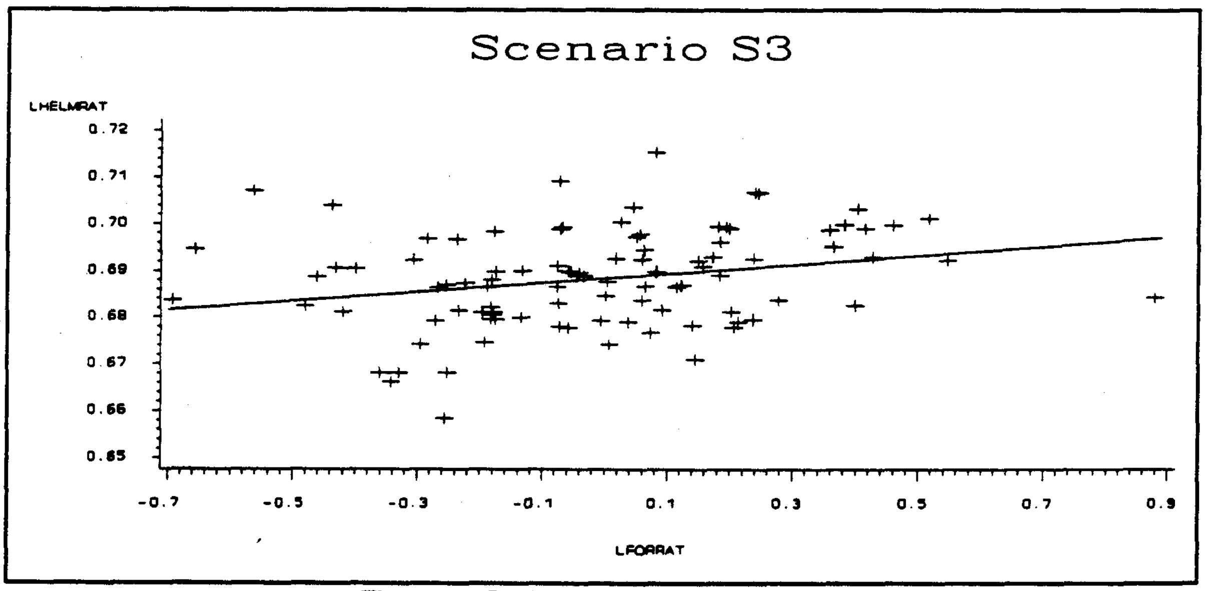


Fig. 11. S3 ln(helmrat) vs ln(forrat).

Figures 12 and 13 for simulation S4 correspond to Figs. 4 and 7 of S1, Figs. 8 and 9 of S2, and Figs. 10 and 11 of S3. The statistical results shown in Table 4, as well as the graphical results, were not significantly different from those of the first simulation. That is, if all the battles have exactly the same activity ratios, it does not effect the Helmbold relationship much whether the casualties were computed after 1-d (one iteration with given $D$ and $A$ ) or after 5-d (five iterations with the same coefficients). This graph displays the same near linearity as that shown in Fig. 7.

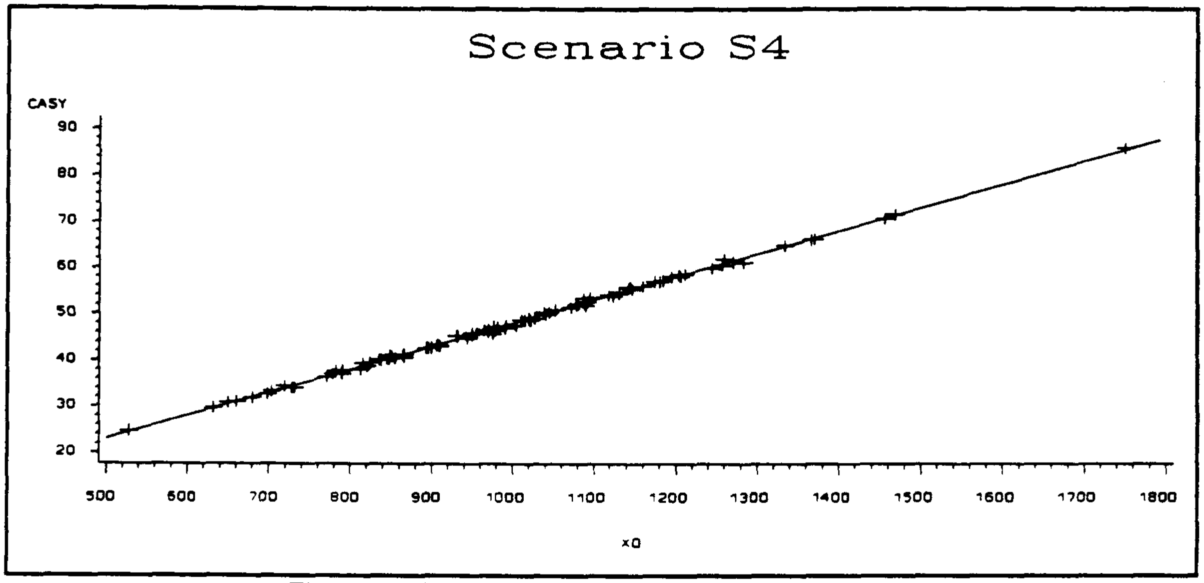


**Fig. 12. S4 casualties vs opposing force size.**

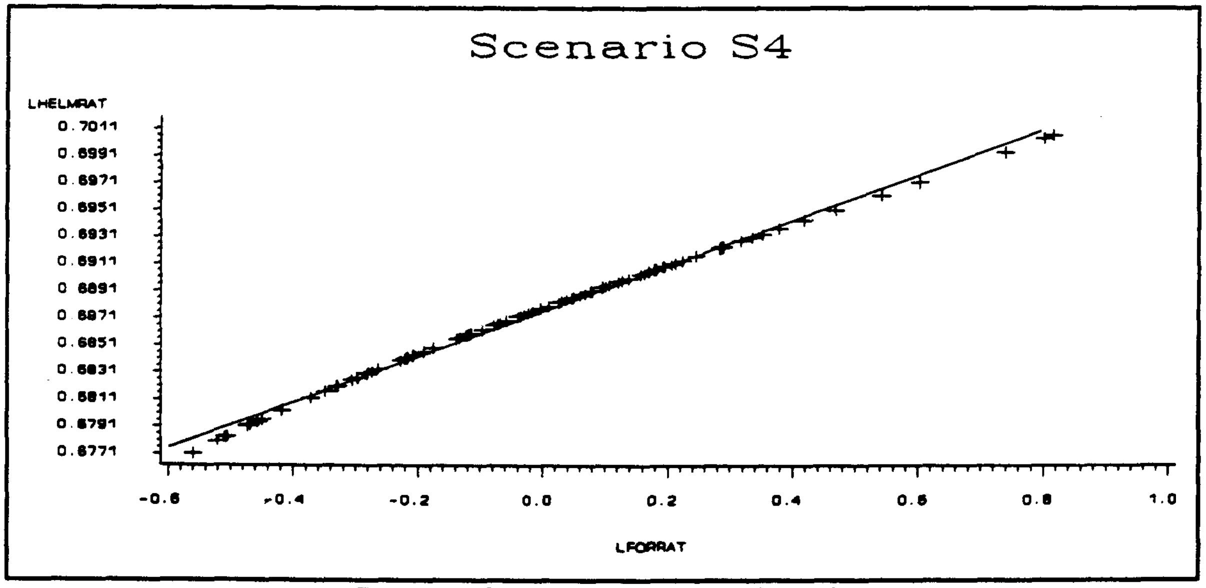


**Fig. 13. S4 ln(helmrat) vs ln(forrat).**

The fifth simulation, S5, is a combination of the second simulation and the fourth simulation: the coefficients have moderately varying values and the battles proceed for 5-d (five iterations). Figures 14 and 15 correspond to Figs. 8 and 9 of S2. The statistics are comparable to the results of simulation two.

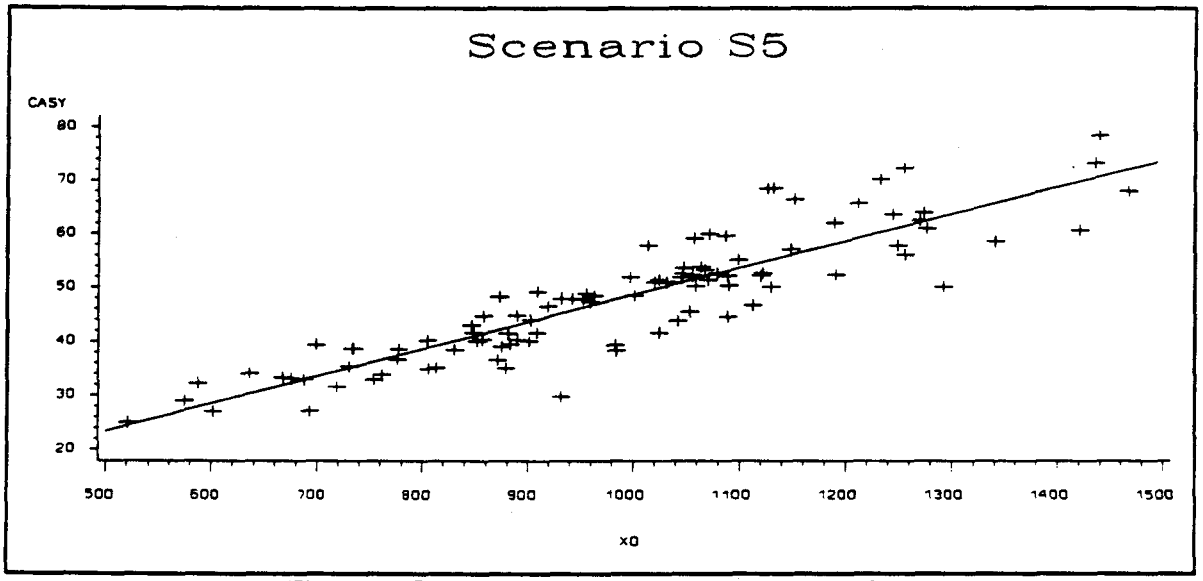


**Fig. 14. S5 casualties vs opposing force size.**

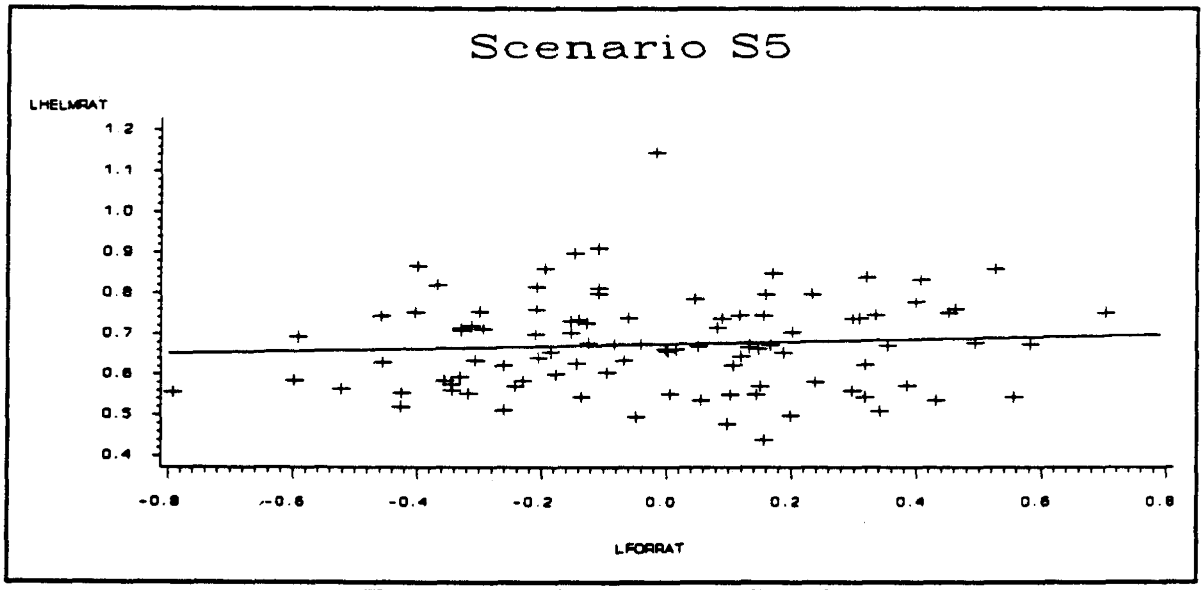


**Fig. 15. S5 ln(helmrat) vs ln(forrat).**

We see that a small amount of variation in the *D* and *A* values dramatically reduces the fit of the regression for a square law simulation and, correspondingly, that a relatively good fit must not have too large an amount of variation in these coefficients. Also, the analysis results are not particularly affected by the number of iterations, as long as the number is constant. (Simulation S6 examines the effects of battle data with varied lengths.)

In addition, the variation in *D* and *A* values did not affect the ability of the regression to detect the correct values of $\alpha$ and $\beta$ as much as it affected the goodness of fit.

Simulation S6 is a variation on the fourth simulation: the coefficients are constant, but the iterations are variable. Two results were interesting in the graphs. First the plots of casualties vs opposing force initial size (Fig. 16) are much more scattered, as expected, but the scattering has a pattern. The data are quantized, with the battles of each different length producing a line of data points, corresponding to the single line produced when all battles were of the same length. (The data points are plotted using the value of the iteration to illustrate this point. The line gives the regression for the entire set of data. More will be covered on this point in the discussion of simulation S7.)

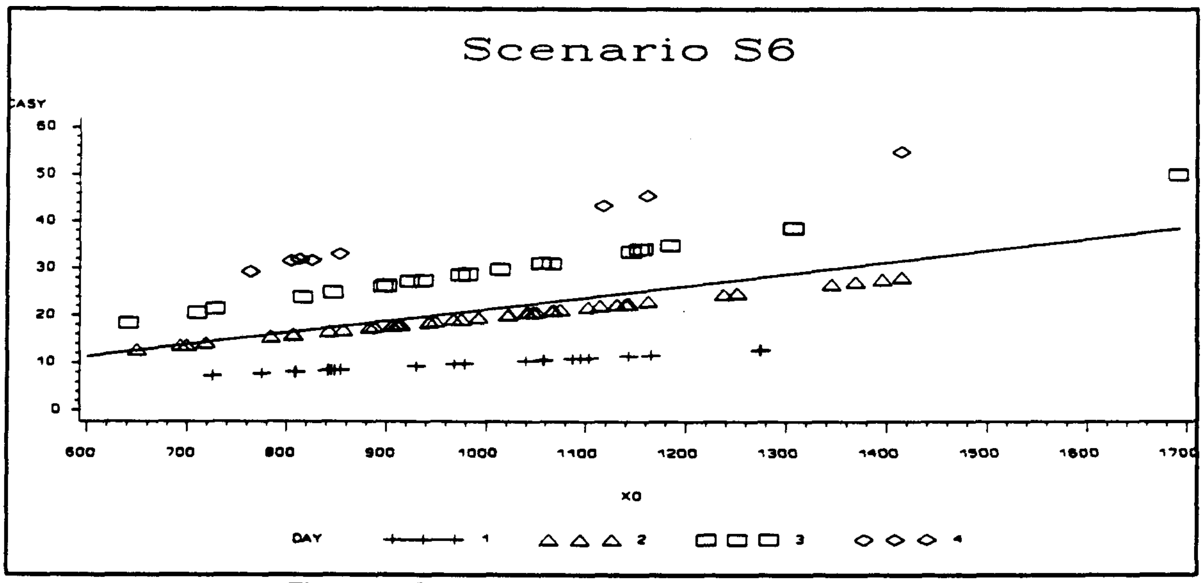


Fig. 16. S6 casualties vs opposing force size.

The second plot of interest (Fig. 17) is the graph comparing the casualties from each side. In the previous cases, the plots were typical zero correlation scatter plots (like Fig. 6). Here, a definite correlation can be seen. This is caused by size exclusion considerations. That is, those battles of 1-d produced no battles with casualties greater than 15 for Y or 30 for X. For battles of 2-d, the upper limit is greater, but there is also a lower limit. This pattern leads to the exclusion of data points from the upper left corner and the lower right corner of the graph, hence a correlation of casualties results naturally. (Comparing many battles with a wide diversity of sizes will also result in this effect, as most battles will occur between forces of similar sizes, with battles between large forces generally incurring more casualties than battles between smaller forces. This type effect will be examined more fully in a later paper concerning the constraint model of warfare.)

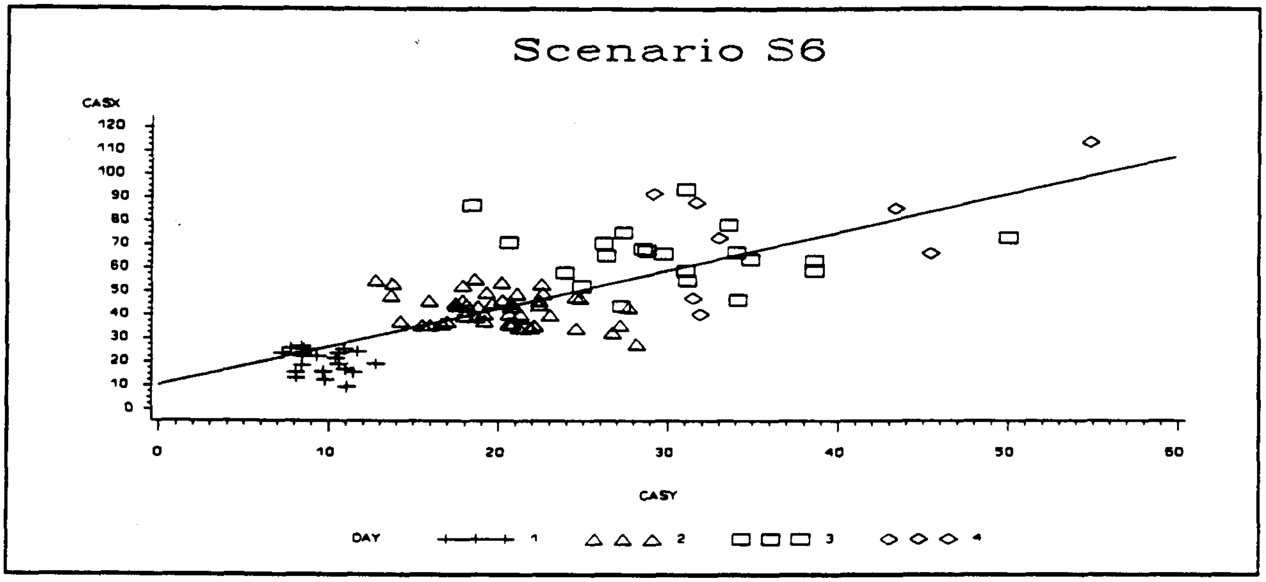


Fig. 17. S6 casx vs casy.

The most interesting point concerning these plots, is that the regression was not much affected by either the dispersion in the casualty vs initial force graphs or the correlation of the opposing casualty figures. Figure 18 plots the Helmbold relationship for one of the replications. The data points for the differing battle lengths are indicated, showing no perceptible segregations. The resulting $R^2$ and $\alpha$ and $\beta$ statistics are identical with those of the first simulation, the one with no variation. We conclude that small variations in the length of battles within a group, all with consistent activity ratios, do not obscure the Helmbold relationship. This result is not apparent from the equations.

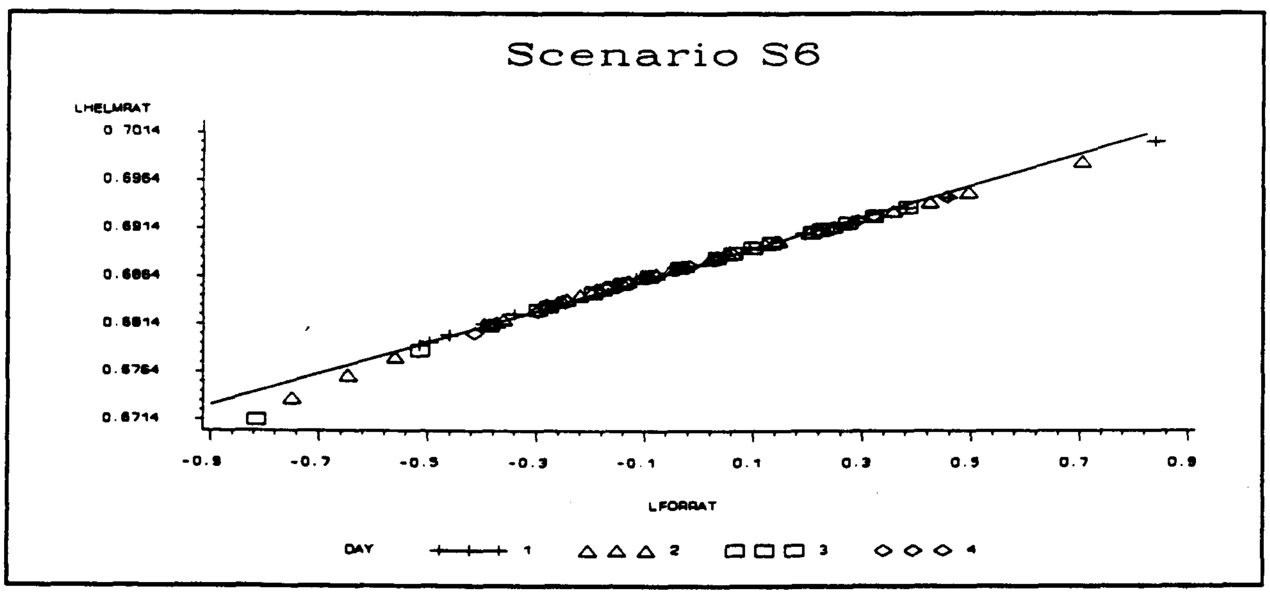


Fig. 18. S6 ln(helmrat) vs ln(forrat).

Simulation S7 is useful in two ways: (1) it illustrates a possible flaw in an obvious data normalization technique, and (2) the understanding of this problem leads to better understanding of the quantized results above. In this simulation, the casualties are derived as in the sixth simulation; however, they are normalized by dividing by the number of days in the battle. The idea is that now the comparisons will be of "apples with apples," rather than of "apples with oranges." Figures 19 and 20, corresponding to Figs. 16 and 17 seem to bear out this idea; they look like those of the first simulation, with no segregations by battle length.

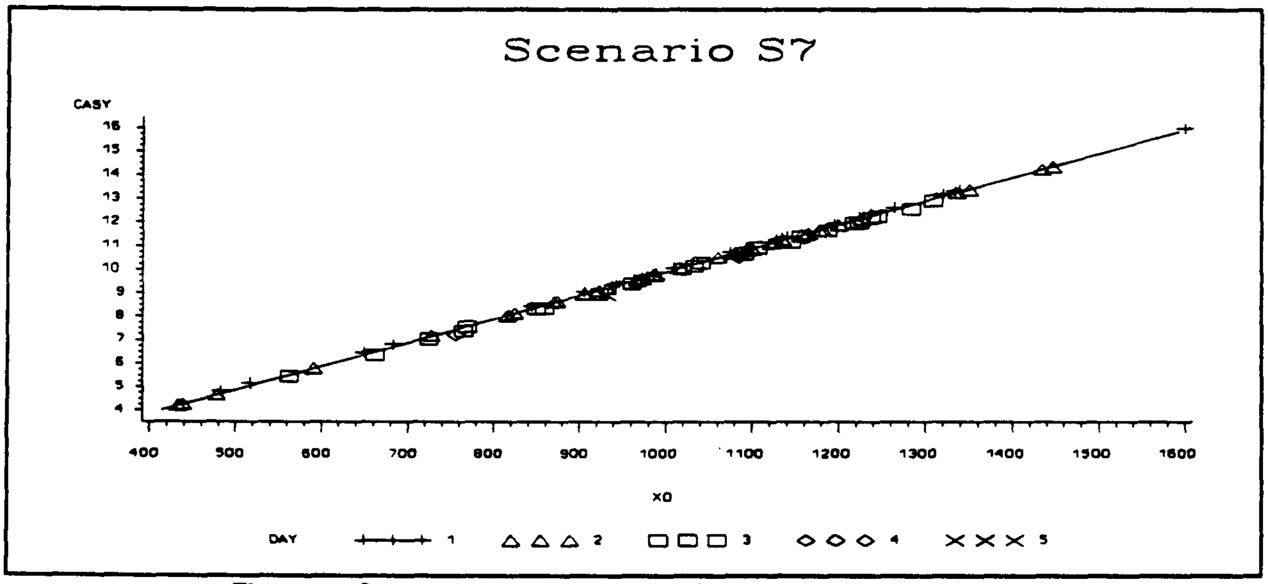

Fig. 19. S7 casualties (normalized) vs opposing force size.

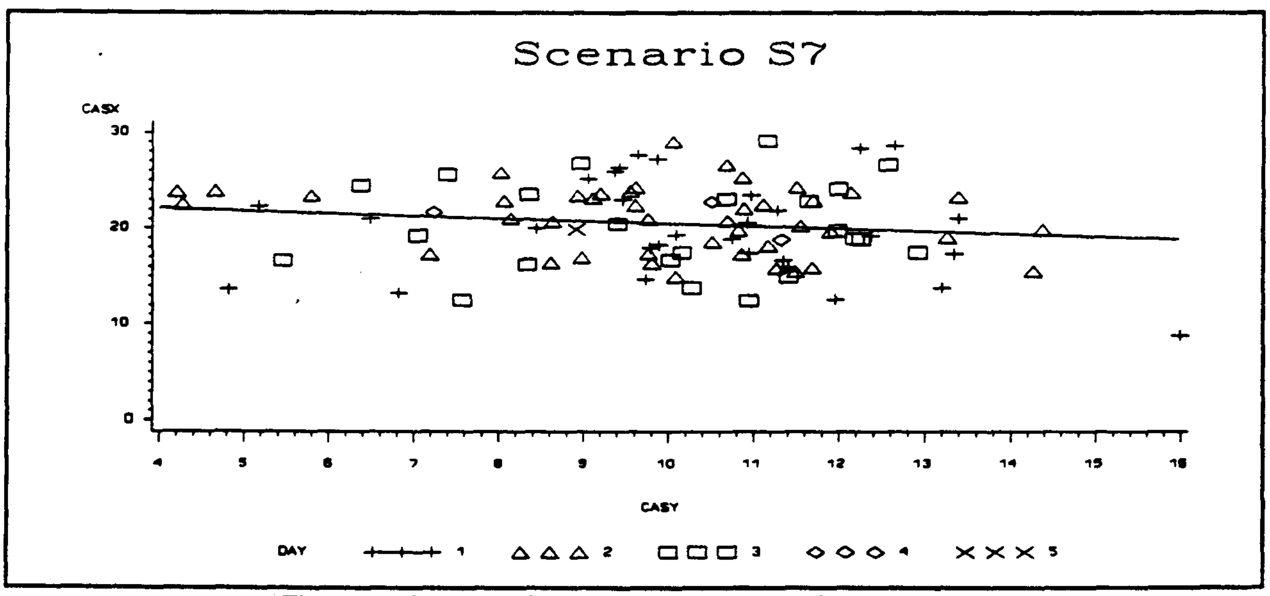

Fig. 20. S7 casx (normalized) vs casy (normalized).

Figure 21, the plot of the log of the Helmbold ratio vs the log of the initial force ratio presents cause for concern. Rather than being linear, it is star shaped. Further examination shows that each line making up the star is caused by the quantization of the data. (Again, the data points show the iteration number [battle length].) Because the Helmbold ratio is formed as the product of the exchange ratio and the average forces ratio [alternative expression of the second factor of the left side of Eq.(12)], its logarithm is the sum of their logarithms. The positive values of each are almost entirely negated by corresponding negative values of the other.

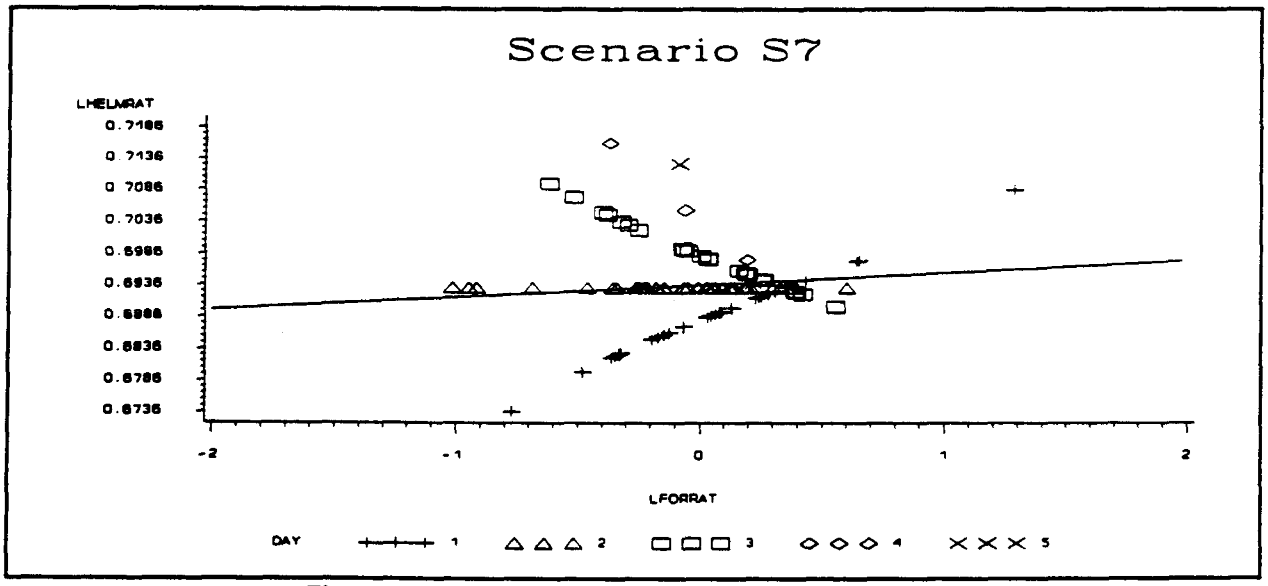


Fig. 21. S7 ln(helmrat) (normalized) vs ln(forrat).

The residual is approximately 0.69. (Notice that the spread on the y-axis in fig. 21 is very small compared to that in previous figures.) However, the residuals for each set of battles of a given length form a distinct line. The cause for this is easily seen by calculating the mathematical expressions for several days' ending forces given by the difference equation approach.

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= x_0' - Dy_0 \\ x_2 &= x_1 - Dy_1 \\ &= x_0 - Dy_0 - D\,(y_0 - Ax_0\,) \\ &= x_0 - 2Dy_0 + DAx_0 \\ x_3 &= x_0 - 3Dy_0 + 3DAx_0 - D\,^2Ay^0. \end{aligned} \tag{26}$$

Thus, when we divide the casualties for 2-d battle by two, we get $Dy_0 - DAx_0\,/2$, which is slightly different from the first day's casualties of $Dy_0$. Similarly, the 3-d casualties divided by three are slightly different from either of the others. Even though the casualties for Y undergo the same type of behavior, the differences do not cancel, so the exchange ratio and the average forces ratio vary with the number of the day. It can be shown that the result of dividing the casualties for 2-d is the same as a better approximation to the first day's casualties and, hence, has a slope ($\alpha$ value) closer to zero than the slope from the first day casualties approximation. (There is another factor acting on the results which will be discussed in simulation 9.) Thus the data quantization in the Helmbold relationship is due to the normalization method.

Simulation S8 verifies that a battle which is fought to the end (or 100 iterations) using the square law exhibits the same characteristics as a group of battles in S1. Simulation S8 uses the same difference equation approach used in the previous simulations. The result yields regressions with near zero $\alpha$ values, as expected.

Figure 22 shows the expected linear relationship between casualties and opposing forces and Fig. 23 shows the nonlinear relationship between the two sides' casualties, resulting from one side winning. This in turn results in a variation over time of the force ratios. The maintenance of a constant activity ratio with a varying force ratio validates our initial assumption of the independence of these factors.

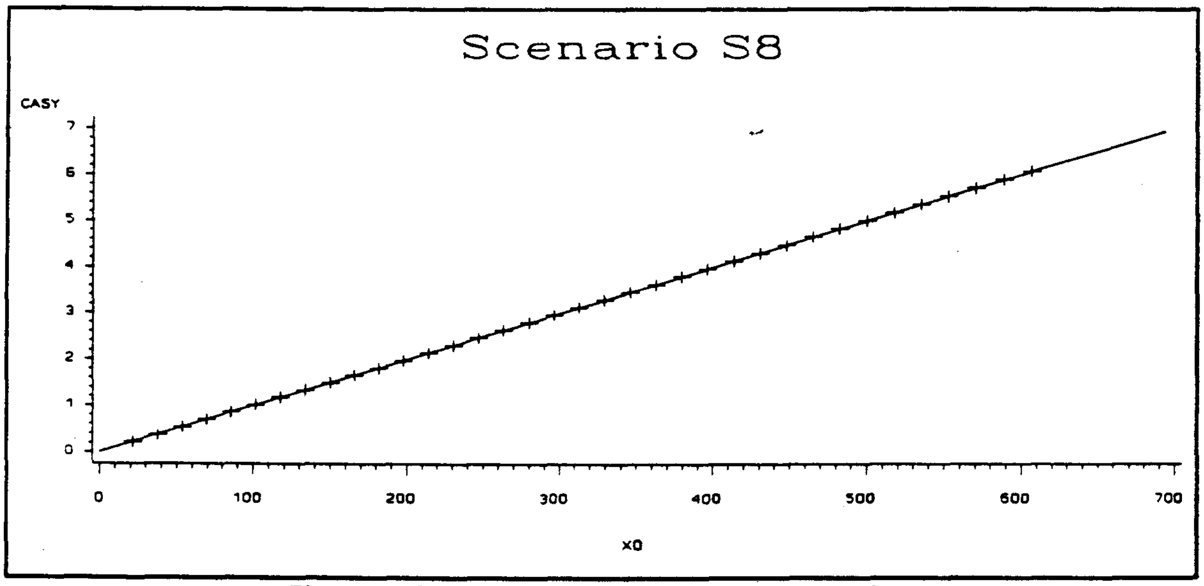


**Fig. 22. S8 casualties vs opposing force size.**

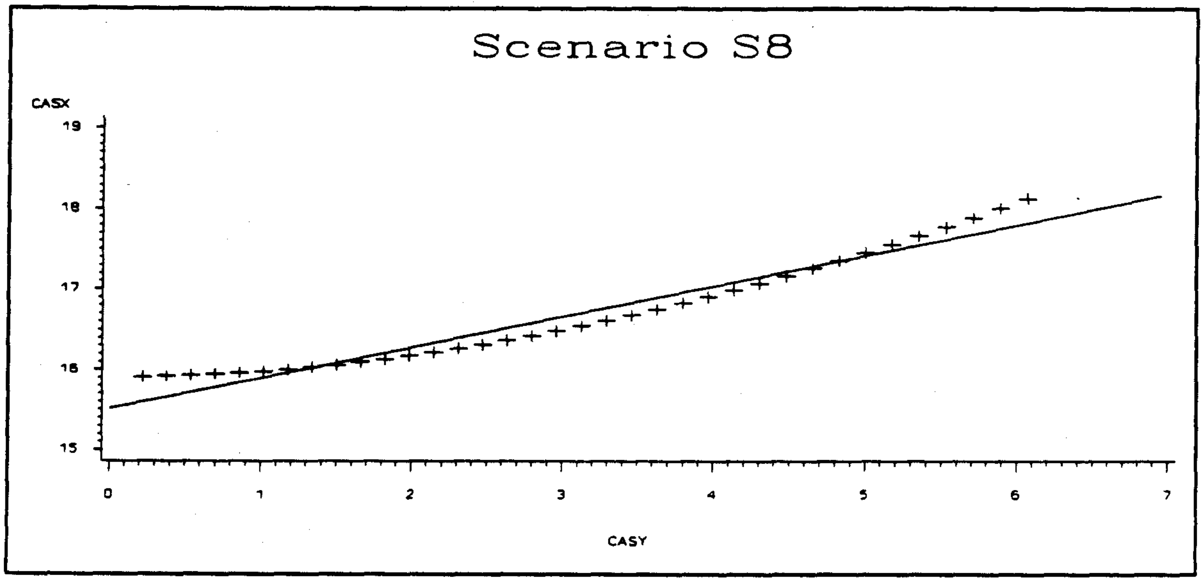


Fig. 23. S8 casx vs casy.

However, Fig. 24, the graph of the Helmbold relationship is clearly non-linear. The derivation of the formulation for the Helmbold ratio for the square law requires that the Helmbold ratio, for this situation and the fixed coefficient simulations above, be a constant, namely *D/A* -- the activity ratio. Not only should the relationship be linear, it should be constant valued.

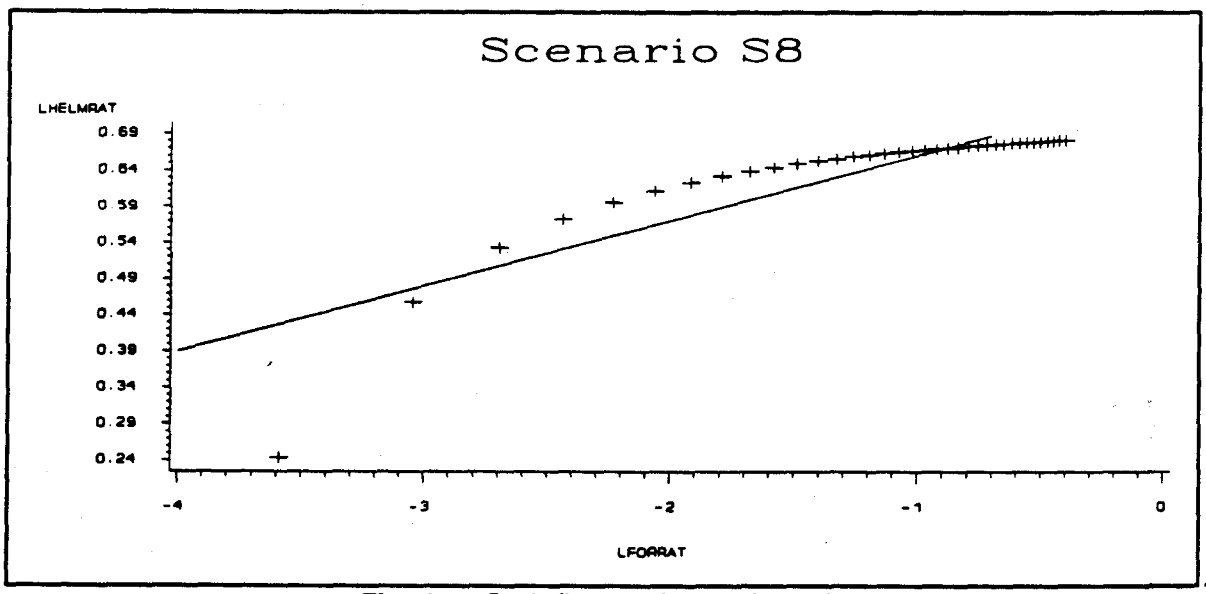


Fig. 24. S8 ln(helmrat) vs ln(forrat).

Simulation S9 was designed to see if this nonlinearity is caused by using the difference equation approximation, rather than the exact solution. Simulation S9 repeats S8 with the modification that the exact square law attrition equations for the remaining forces (involving hyperbolic functions) is used in place of the difference equations. For day *t* these are shown below.

$$x_r = x_0\,(\cosh(\phi t) - \theta \sinh(\phi t)) \qquad (27)$$
$$y_r = y_0\,(\cosh(\phi t) - (1/\theta)\sinh(\phi t))$$

$$\text{casx} = x_0 - x_r \qquad (28)$$
$$\text{casy} = y_0 - y_r\,,$$

where

$$\phi = \sqrt{(DA)} \qquad (29)$$
$$\theta = (x_0/y_0)\sqrt{(D/A)}.$$

The result is that the value of the Helmbold ratio becomes a constant. The regression results in a perfect fit; however, because of the definition of $R^2$ ($R^2$ = model sum of squares/total sum of squares and model sum of squares is the sum of squares of the deviations from the mean, zero in the case of a constant function and a perfect fit), the value of $R^2$ is zero. Figures 25-27 illustrate three of the relationships of interest.

Figure 25 shows the linear relationship between casualties and the opposing force size. Figure 26 shows the nonlinear relationship between the casualties on each side. In this replication, the defender, *y* , wins and his daily casualties drop to zero.

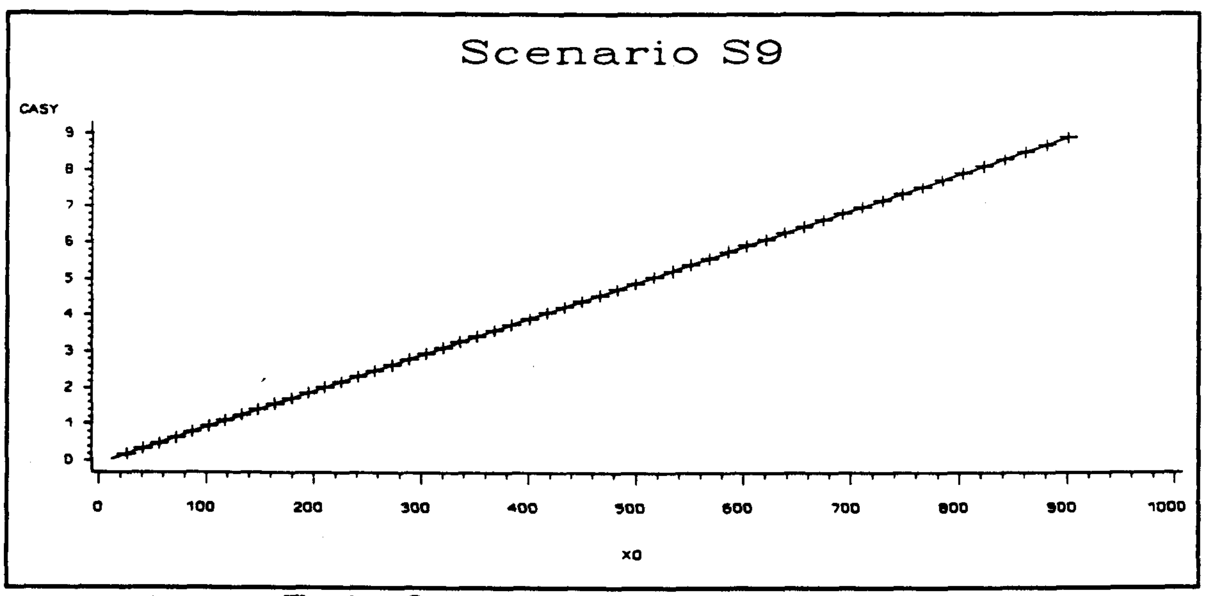


**Fig. 25. S9 casualties vs opposing force size.**

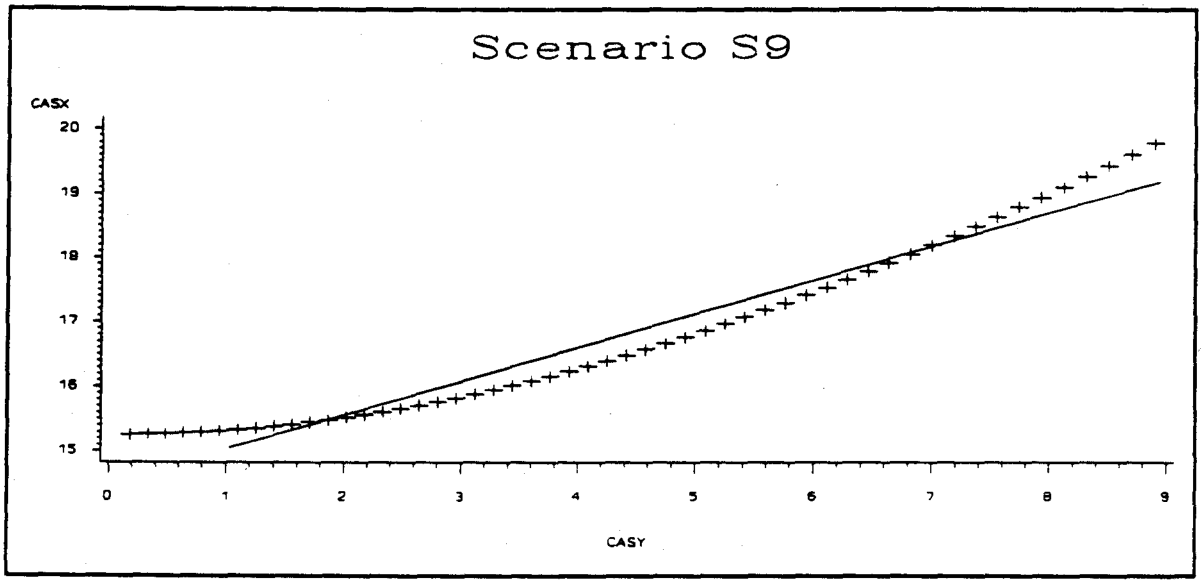


Fig. 26. S9 casx vs casy.

Figure 27 shows that using the exact solution also removes the very small deviation from linearity in the Helmbold relationship, seen in the graphs in the constant coefficient simulations above. It also removes the small deviation of $\alpha$ from zero.

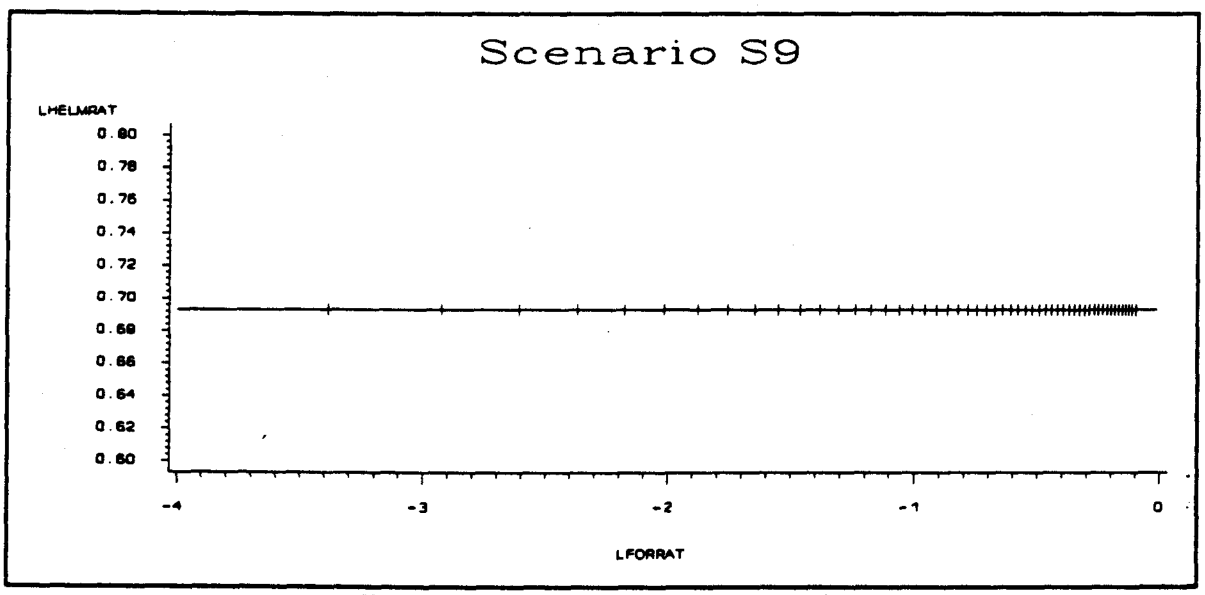


Fig. 27. S9 ln(helmrat) vs ln(forrat).

As long as the relative magnitude of the casualties of a side compared to its own current force size is small, the error of the difference equation solution as compared to the exact solution is negligible. Battles which are fought to annihilation, or near that point, pass the level at which the differences are negligible. (This subject will be discussed at greater length in Sect. 3.4.)

The conclusion is that the starting force sizes and the $D$ and $A$ values are independent. Hence any set of battles obeying the square law with moderate $D$ and $A$ variations and moderate variations in the number of days of battle will tend to have an $\alpha$ value near zero and a $\beta$ value near the ln[avg($D/A$ )]. (Naturally, a sample taken from the population of all possible battles might have any $\alpha$ or $\beta$ value.)

### 3.2 LINEAR LAW SIMULATION RESULTS

While Helmbold's derivation of the equivalence of the activity ratio and Helmbold's ratio [see Eq.(2)] depends on a square law formulation, this only means that the activity ratio need not equal the Helmbold ratio; it does not preclude the Helmbold ratio from being related to the force ratio. The linear law simulations examine the form of the relationship if a linear law attrition is assumed. These simulations were produced using $D$ =.00002 and $A$ =.00001 in Eq.(3), as shown in Table 5. These values produce casualties of the same order of magnitude as those in the square law cases. The analysis consists of a regression of the attrition and force data using Eq.(10), with results shown in Table 6, and analysis of the graphs of various pairs of variables.

**Table 5. Linear law simulation descriptions**

Base value $D$ =.00001, $A$ =.00002, $x_0$ =1000, $y^0$ =1000, 1 $\sigma$ for $x^0$ and $y^0$ = 200

| Label | 1 $\sigma$ $D$ and 1 $\sigma$ $A$ Variation/Base | Days of battle | 1 $\sigma$ days | Comments |
|---|---|---|---|---|
| L1 | 0.0 | 1 | 0 | 1 day, no variation in $D/A$ or days |
| L2 | 0.000001 | 1 | 0 | moderate $D$ and $A$ variation |
| L3 | 0.0000001 | 1 | 0 | small $D$ and $A$ variation |
| L4 | 0.0 | 5 | 0 | 5 day, no variation in $D/A$ or days |
| L5 | 0.000001 | 5 | 0 | moderate $D$ and $A$ variation |
| L6 | 0.0 | 2 | 1 | moderate day variation |
| L7 | 0.0 | 2 | 1 | moderate day variation, attrit/days |
| L8 | 0.0 | | | battle to end, difference equations |

Table 6. Linear law Helmbold relationship results

| | $R^2$ | | | | $\alpha$ | | | | $\beta$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Label | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ |
| L1 | 1.00 | 1.00 | 1.00 | 0.00 | 1.01 | 1.01 | 1.01 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| L2 | 0.79 | 0.92 | 0.88 | 0.03 | 0.93 | 1.07 | 1.00 | 0.04 | 0.66 | 0.71 | 0.69 | 0.01 |
| L3 | 1.00 | 1.00 | 1.00 | 0.00 | 1.00 | 1.02 | 1.01 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| L4 | 1.00 | 1.00 | 1.00 | 0.00 | 1.03 | 1.04 | 1.03 | 0.00 | 0.67 | 0.67 | 0.67 | 0.00 |
| L5 | 0.83 | 0.95 | 0.89 | 0.03 | 0.96 | 1.15 | 1.03 | 0.04 | 0.64 | 0.71 | 0.67 | 0.01 |
| L6 | 1.00 | 1.00 | 1.00 | 0.00 | 1.01 | 1.02 | 1.02 | 0.00 | 0.68 | 0.68 | 0.68 | 0.00 |

Simulation L1 is the simplest linear law simulation, having constant *A* and *D* values and one iteration. The $R^2$ values (in Table 6) are all 1.00; the $\alpha$ values are all 1.01 (cf the theoretical value of 1.00); and the $\beta$ values are all 0.69 [$= \ln(2) = \ln(D/A)$].

Figures 28-31 illustrate the results of simulation L1. The linear relationship (in Fig. 28) between the casualties on the two sides is characteristic of a linear law structure. Notice in Fig. 29 that the enforced relationship between the casualties on one side and the product of the force sizes of the two sides means that there cannot be a linear relationship between the casualties on one side and the force size of the other as is found in square law situations.

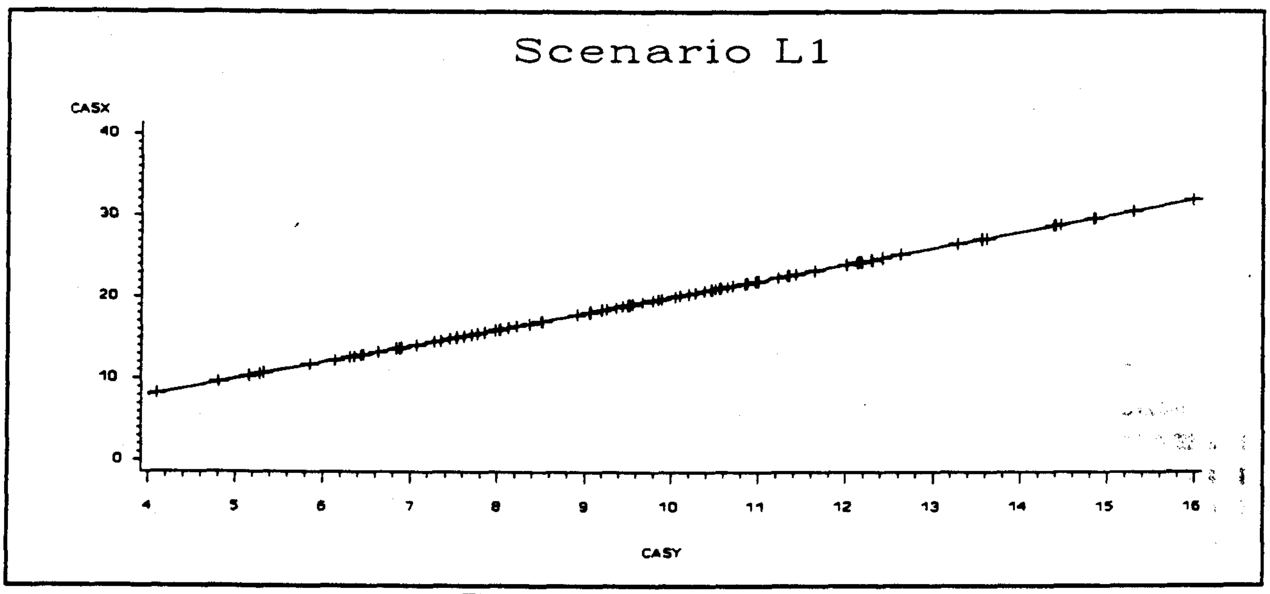


Fig. 28. L1 casx vs casy.

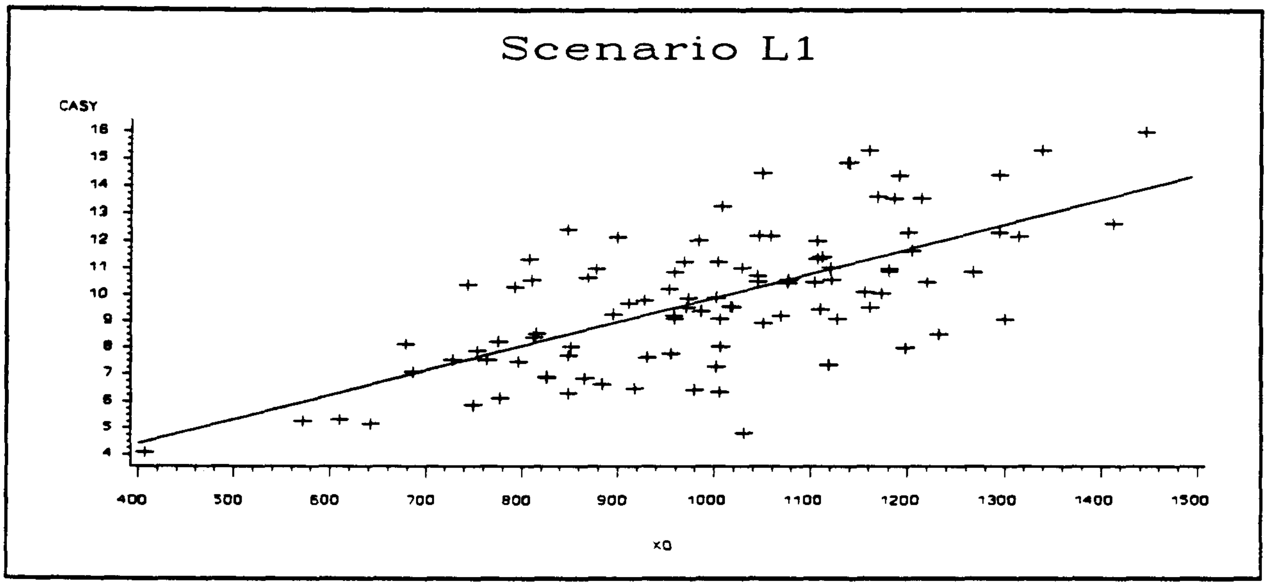


Fig. 29. L1 casualties vs opposing force size.

The linear relationship shown in Fig. 28 implies that the exchange ratio is constant over all the battles, hence the logarithm of the exchange ratio, plotted in Fig. 30, must also be constant. Because the Helmbold ratio is formed as the product of the exchange ratio, and the average forces ratio and its logarithm is the sum of their logarithms, it is the average forces ratio which drives the Helmbold relationship, shown in Fig. 31. The $R^2$ for the regression is 1.00 in all replications, with $\alpha = 1$ and $\beta = \ln(D/A)$.

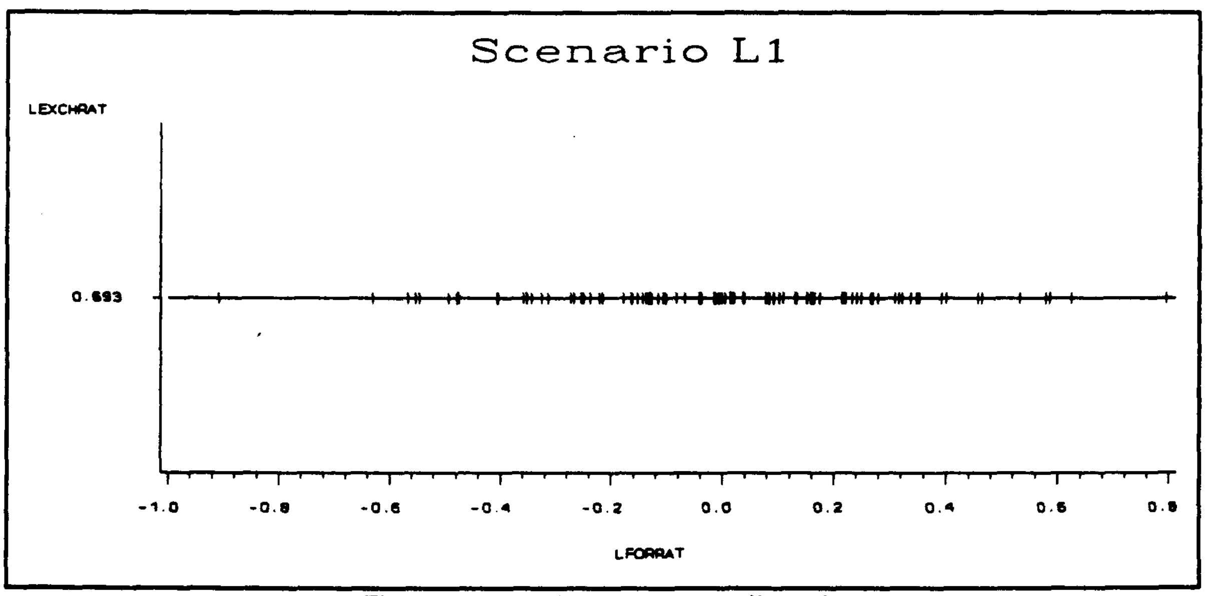


Fig. 30. L1 ln(exchrat) vs ln(forrat).

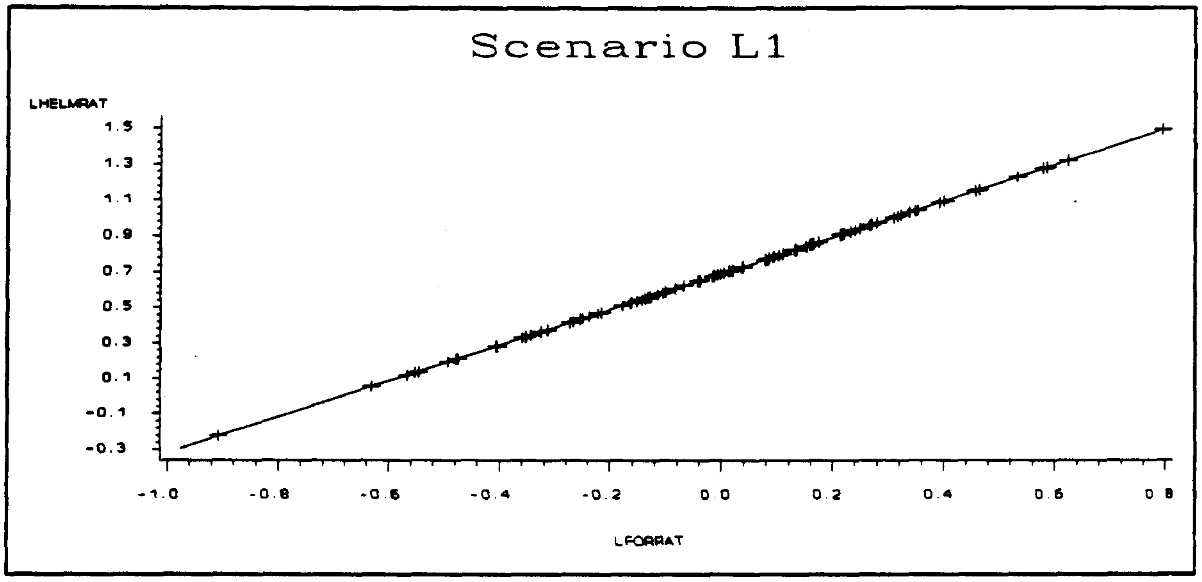


Fig. 31. L1 ln(helmrat) vs ln(forrat).

Figures 32 and 33 show that introducing a variation in the activity ratios within the set of battles (simulation L2) can destroy the constant value of the exchange ratio and confuse the linear relation between the casualty figures. However, the Helmbold relationship, shown in Fig. 34, still exists. The $R^2$ for all the replications were respectably high (unlike in simulation S2). Additionally, the $\alpha$ and $\beta$ values had means which agree with those of L1.

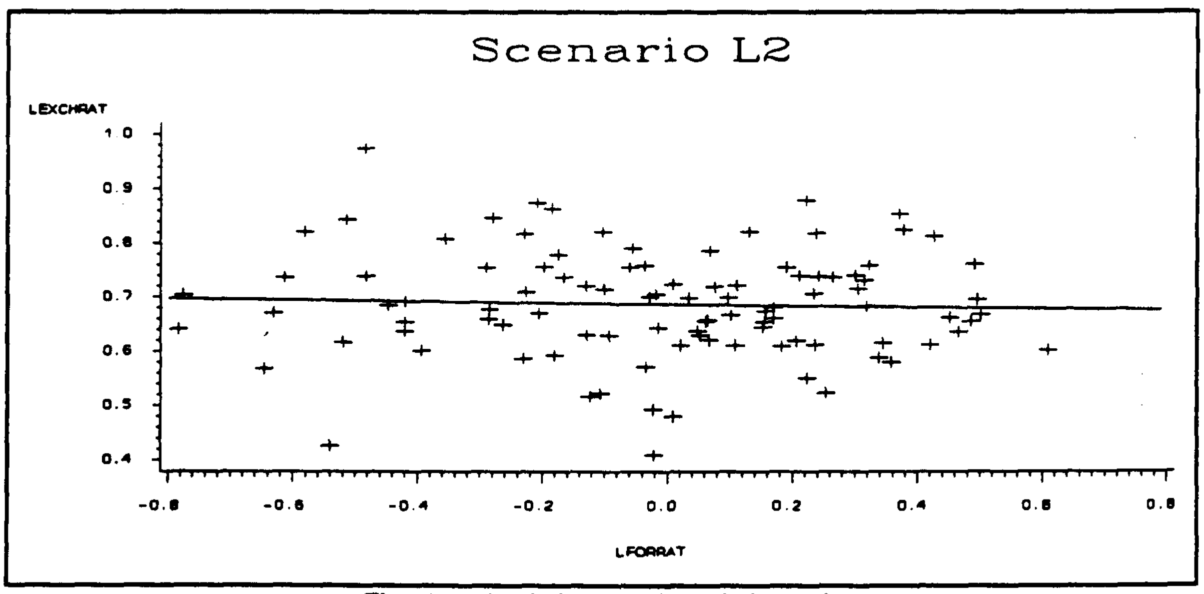


Fig. 32. L2 ln(exchrat) vs ln(forrat).

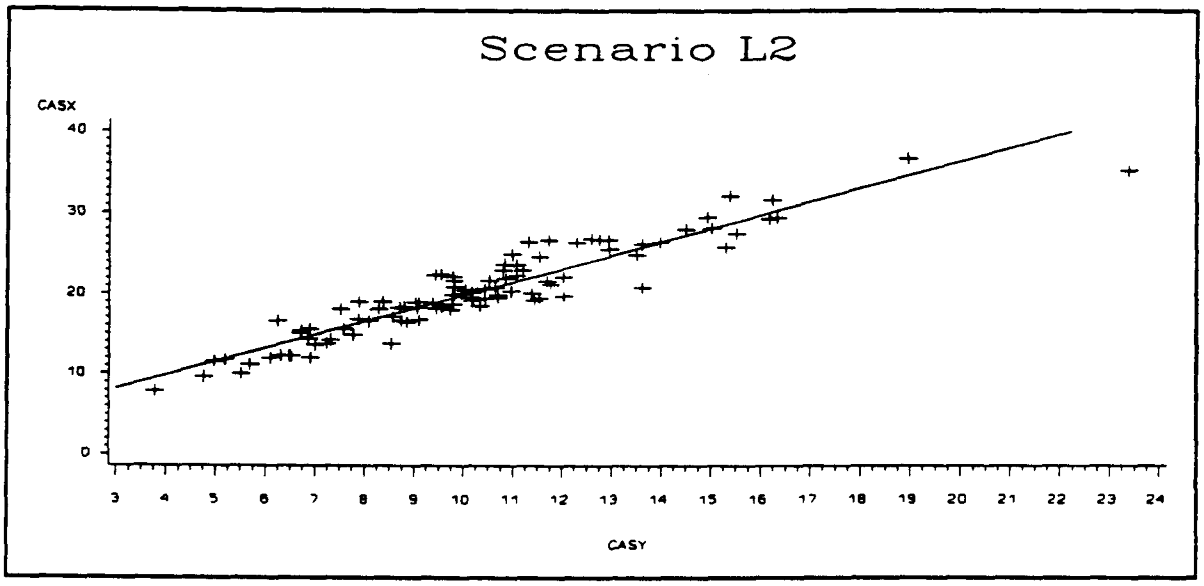


Fig. 33. L2 casx vs casy.

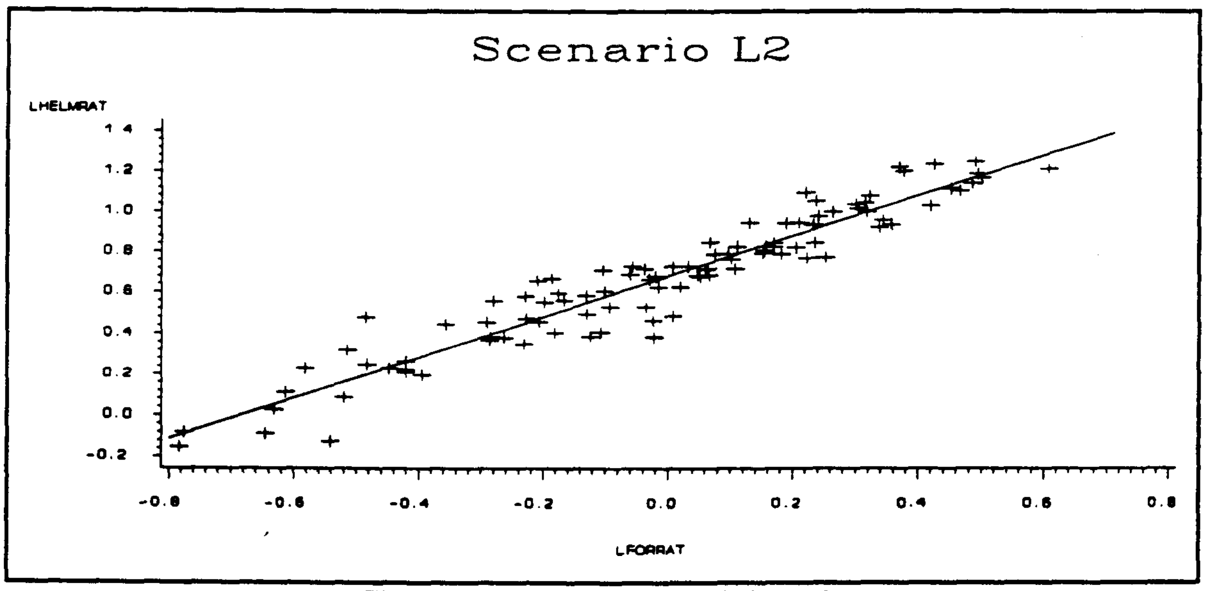


Fig. 34. L2 ln(helmrat) vs ln(forrat).

Simulation L3, with smaller variations of *D* and *A* than L2, shows results almost as good as those of L1. Figures 35 and 36 illustrate that reducing the variability of the activity ratios improves the linear relationship of the casualty figures and the Helmbold relationship. The $R^2$ and $\beta$ values (Table 6) duplicate those of simulation L1 and the $\alpha$ values are nearly identical.

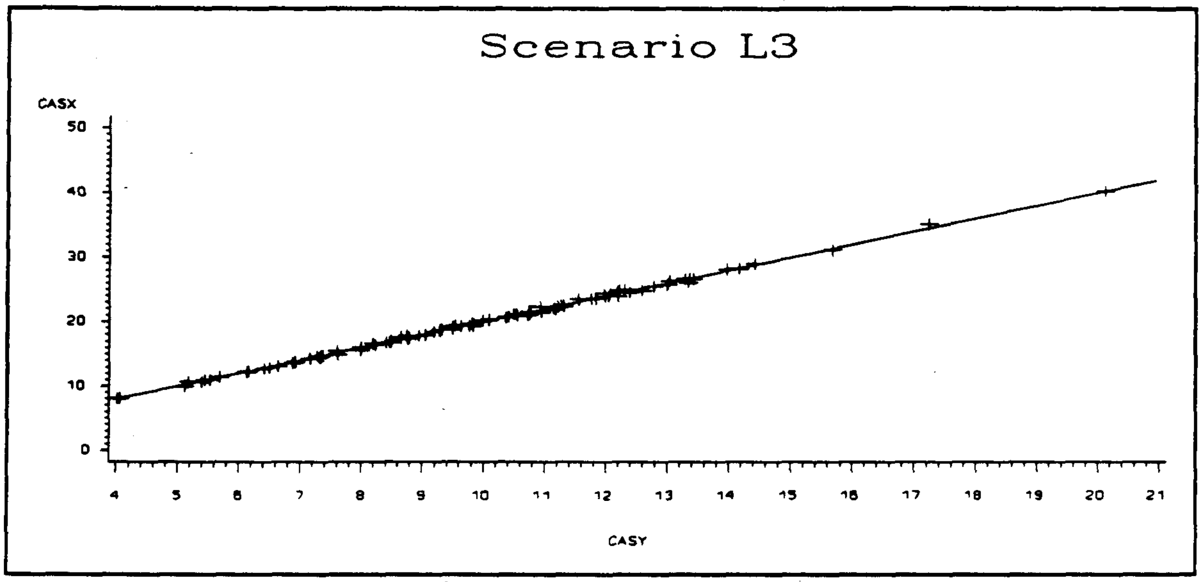


Fig. 35. L3 casx vs casy.

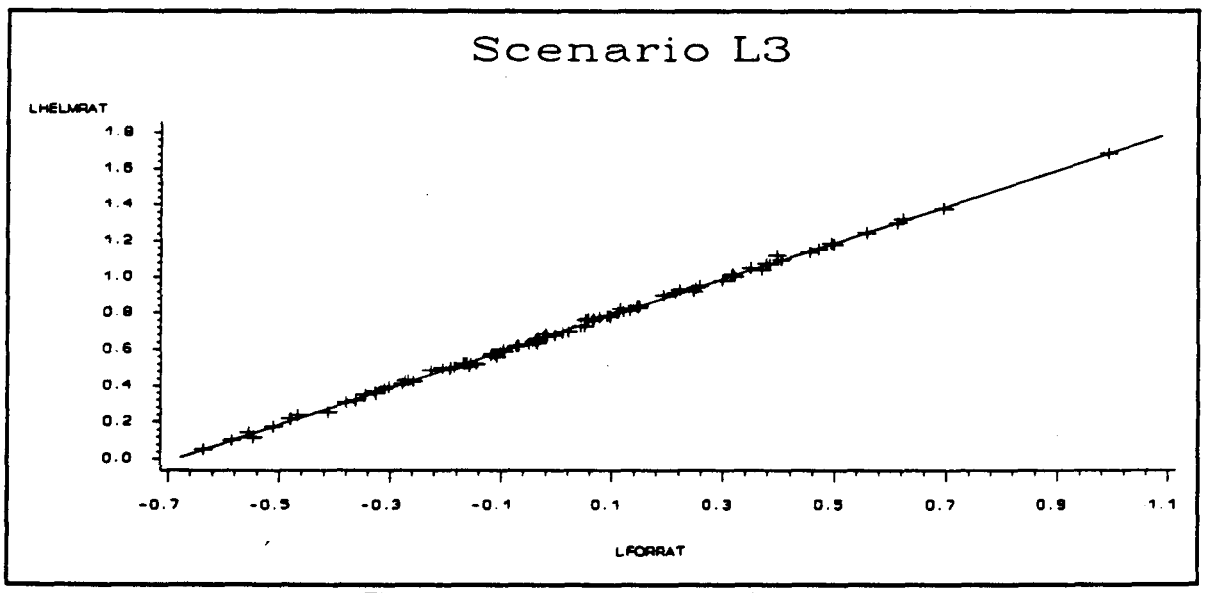


Fig. 36. L3 ln(helmrat) vs ln(forrat).

Simulations L4 and L5 exhibit the same lack of effect of five iterations (days of battle) versus one iteration that was seen in the square law cases. Figure 37 for L4 resembles the corresponding figure for L1 and Fig. 38 resembles that for L2. The $R^2$, $\alpha$, and $\beta$ values for L4 and L5 also resemble those for L1 and L2 in which the L1 and L4 values are uniform across the replications and the L2 and L5 values show small variations in the $\alpha$ and $\beta$ values and large variations in the $R^2$ values.

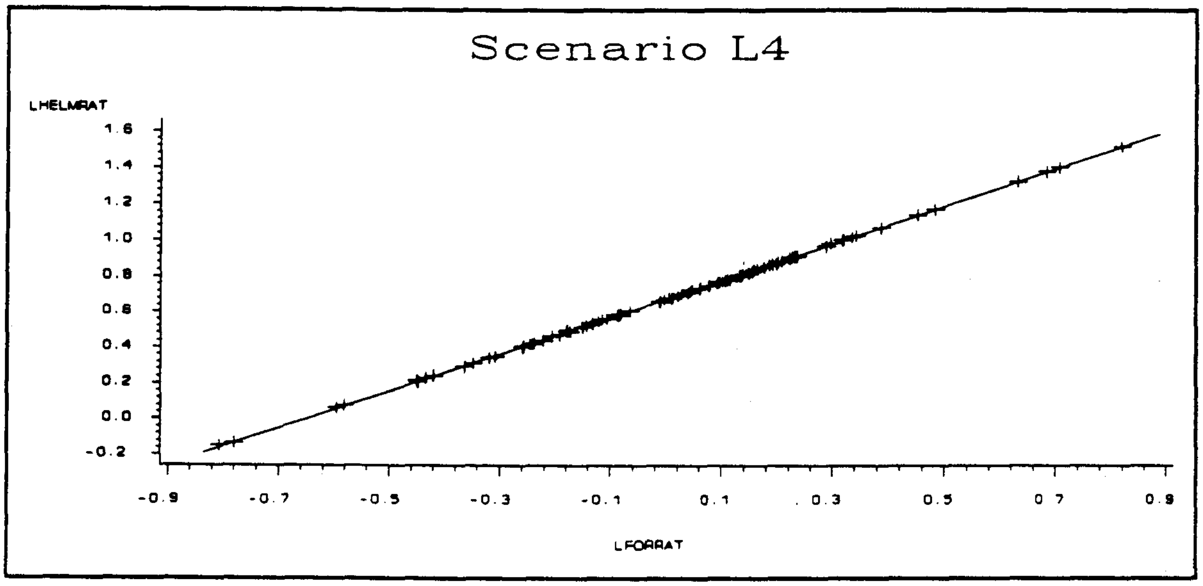


Fig. 37. L4 ln(helmrat) vs ln(forrat).

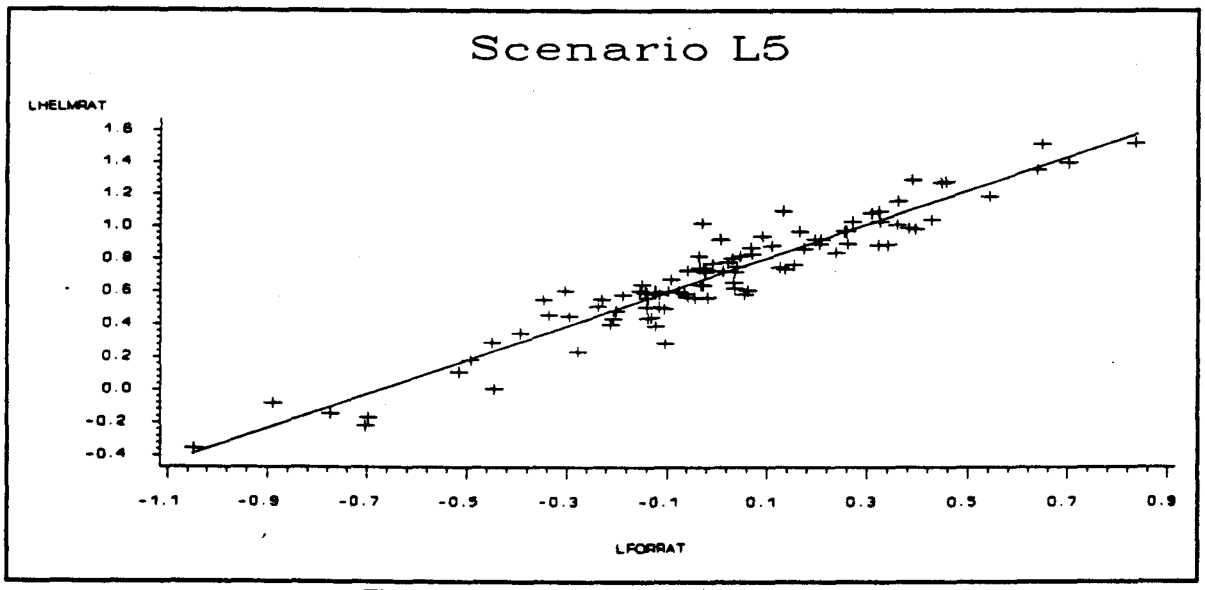


Fig. 38. L5 ln(helmrat) vs ln(forrat).

Simulation L6, which tests the effects of varying lengths of battle within the group of battles, does not exhibit the same variability of results seen in S6. The $R^2$, $\alpha$, and $\beta$ values in Table 6 and the plot shown in Fig. 39 illustrate this. The lack of variation from simulation L1 is due to the basic linearity of the relation between the casualties on each side: a segregation along the line of the battles will not affect the results.

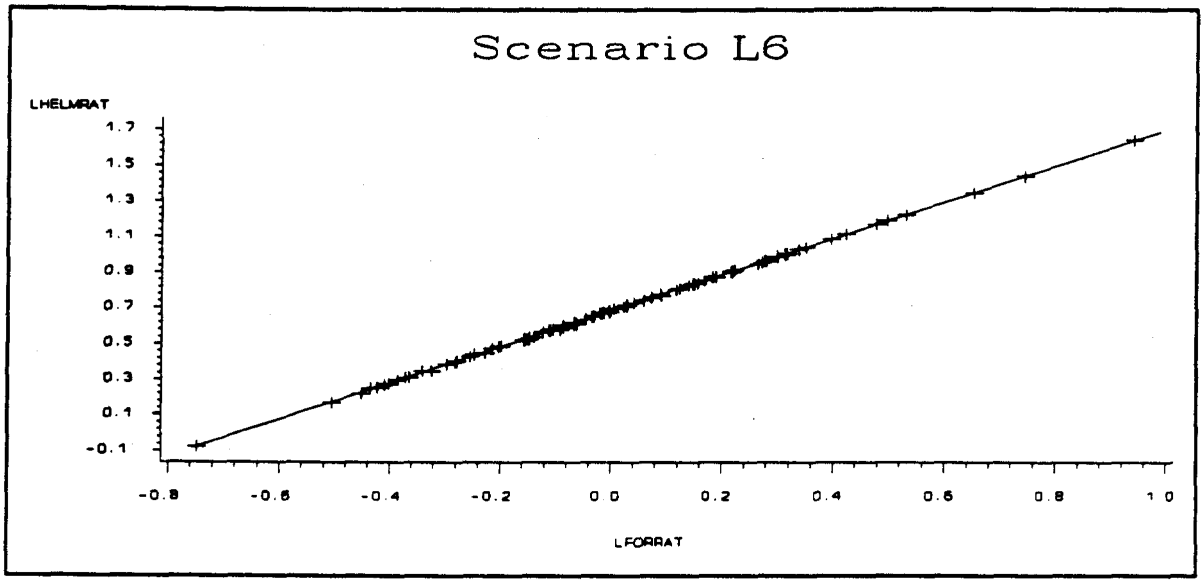


Fig. 39. L6 ln(helmrat) vs ln(forrat).

The same situation holds in simulation L7. When the L6 simulation is normalized, using the iteration value, no anomalies appear. Figure 40 and the values in Table 6 illustrate the results.

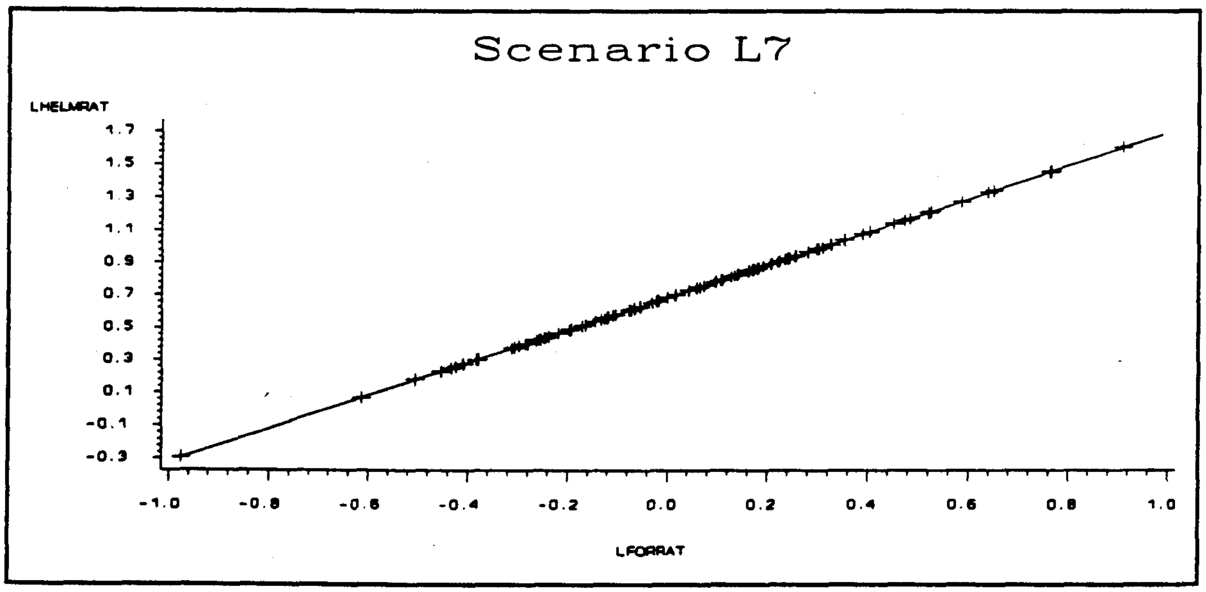


Fig. 40. L7 ln(helmrat) vs ln(forrat).

Simulation L8 examines the course of linear law battles in the same way S8 examines square law battles. Because the two sides' casualties have a linear relationship, the exchange ratio is constant over the spectrum of force ratios (see Figs. 41 and 42), and because the value of the exchange ratio is *D/A* , no power of the force ratio other than 1.0 will make Eq.(12) true (as shown in Fig. 43). The tables show values very similar to simulation L1.

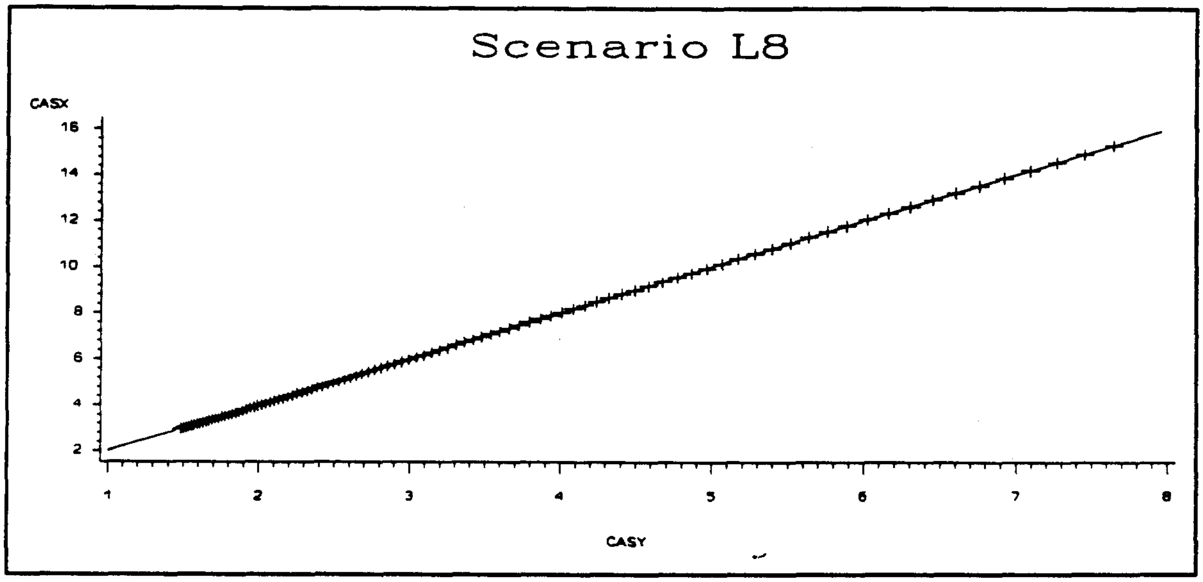


Fig. 41. L8 casx vs casy.

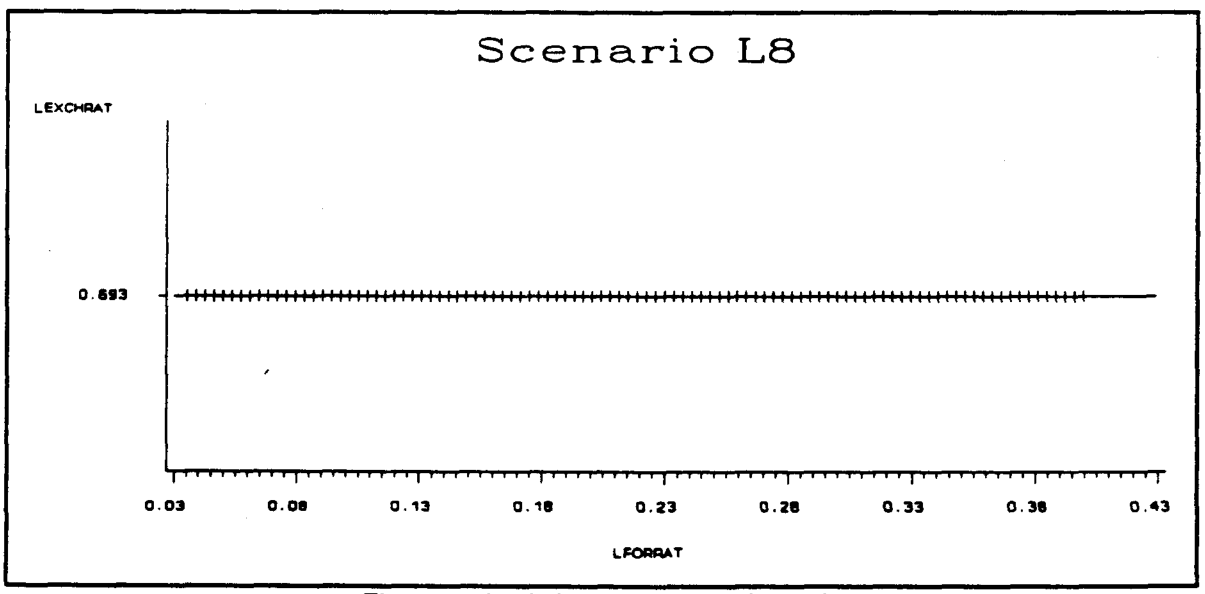


Fig. 42. L8 ln(exchrat) vs ln(forrat).

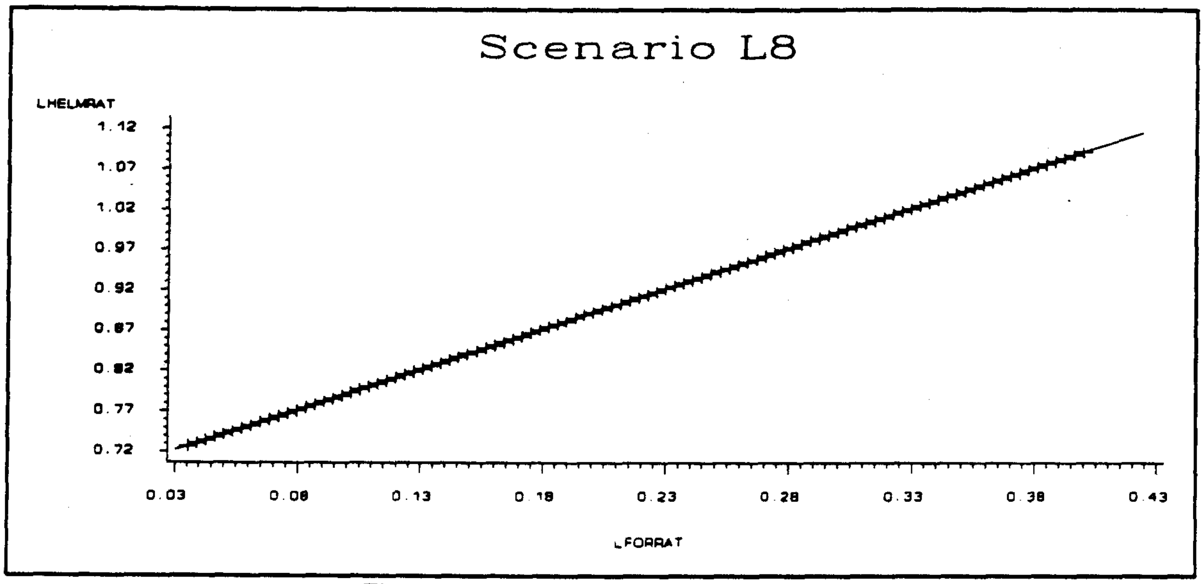


Fig. 43. L8 ln(helmrat) vs ln(forrat).

The conclusion is that the starting force sizes and the *D* and *A* values are independent under linear law assumptions. Hence any set of battles obeying the linear law with moderate *D* and *A* variations and moderate variations in the number of days of battle will tend to have an $\alpha$ value near one and a $\beta$ value near the ln[avg(*D*/*A* )]. (Naturally, a sample taken from the population of all possible battles might have any $\alpha$ or $\beta$ value.)

## 3.3 LOGARITHMIC LAW SIMULATION RESULTS

The $\alpha$ value for linear law simulations ($\alpha$=1) is closer to that found for the historical battles ($\alpha$ = 1.3 - 1.7) than is the $\alpha$ for the square law ($\alpha$ = 0), so we are encouraged to extend our search further in the same direction. This leads to the logarithmic law [see Eq.(5)]. These simulations were produced using *D* = .00000002 and *A* = .00000001, as shown in the header of Table 7. These values produce casualties of the same order of magnitude as those in the square law cases. The analysis consists of a regression of the attrition and force data using Eq.(10), with results shown in Table 8, and analysis of the graphs of various pairs of variables.

Table 7. Logarithmic law simulation descriptions

| Label | $1\ \sigma\ D$ and $1\ \sigma\ A$ Variation/Base | Days of battle | $1\ \sigma$ days | Comments |
|---|---|---|---|---|
| Base value $D$ =.00000001, $A$ =.00000002, $x_0$ =1000, $y^0$ =1000, $1\ \sigma$ for $x^0$ and $y^0$ = 200 | | | | |
| Lo1 | 0.0 | 1 | 0 | 1 day, no variation in *D/A* or days |
| Lo2 | 0.000000001 | 1 | 0 | moderate *D* and *A* variation |
| Lo3 | 0.0000000001 | 1 | 0 | small *D* and *A* variation |
| Lo4 | 0.0 | 5 | 0 | 5 day, no variation in *D/A* or days |
| Lo5 | 0.000000001 | 5 | 0 | moderate *D* and *A* variation |
| Lo6 | 0.0 | 2 | 1 | moderate day variation |
| Lo7 | 0.0 | 2 | 1 | moderate day variation, attrit/days |
| Lo8 | 0.0 | | | battle to end, difference equations |

Table 8. Logarithmic law Helmbold relationship results

| | $R^2$ | | | | $\alpha$ | | | | $\beta$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Label | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ | Min | Max | Mean | $1\sigma$ |
| Lo1 | 1.00 | 1.00 | 1.00 | 0.00 | 2.00 | 2.00 | 2.00 | 0.00 | 0.69 | 0.69 | 0.69 | 0.00 |
| Lo2 | 0.95 | 0.98 | 0.96 | 0.01 | 1.88 | 2.07 | 1.99 | 0.04 | 0.67 | 0.73 | 0.69 | 0.01 |
| Lo3 | 1.00 | 1.00 | 1.00 | 0.00 | 1.99 | 2.01 | 2.00 | 0.00 | 0.68 | 0.69 | 0.69 | 0.00 |
| Lo4 | 1.00 | 1.00 | 1.00 | 0.00 | 1.99 | 2.01 | 2.00 | 0.00 | 0.65 | 0.65 | 0.65 | 0.00 |
| Lo5 | 0.95 | 0.98 | 0.97 | 0.01 | 1.92 | 2.08 | 2.00 | 0.03 | 0.63 | 0.68 | 0.66 | 0.01 |
| Lo6 | 1.00 | 1.00 | 1.00 | 0.00 | 1.99 | 2.01 | 2.00 | 0.00 | 0.67 | 0.68 | 0.68 | 0.00 |
| Lo7 | 1.00 | 1.00 | 1.00 | 0.00 | 2.00 | 2.00 | 2.00 | 0.00 | 0.68 | 0.68 | 0.68 | 0.00 |
| Lo8 | 1.00 | 1.00 | 1.00 | 0.00 | 1.99 | 2.00 | 1.99 | 0.00 | 0.68 | 0.69 | 0.69 | 0.00 |

Comparison of Figs. 44-47 with the corresponding figures for S1 (Figs. 4-7) and L1 (Figs. 28-31) illustrates the similarities and differences among logarithmic law battles and linear and square law battles. Figure 44 shows some correlation between the casualties on the two sides, but not as much as in the linear law simulations. Figure 45 shows some correlation between the casualties on one side and the force size of the other, but not as much as in the square law simulations.

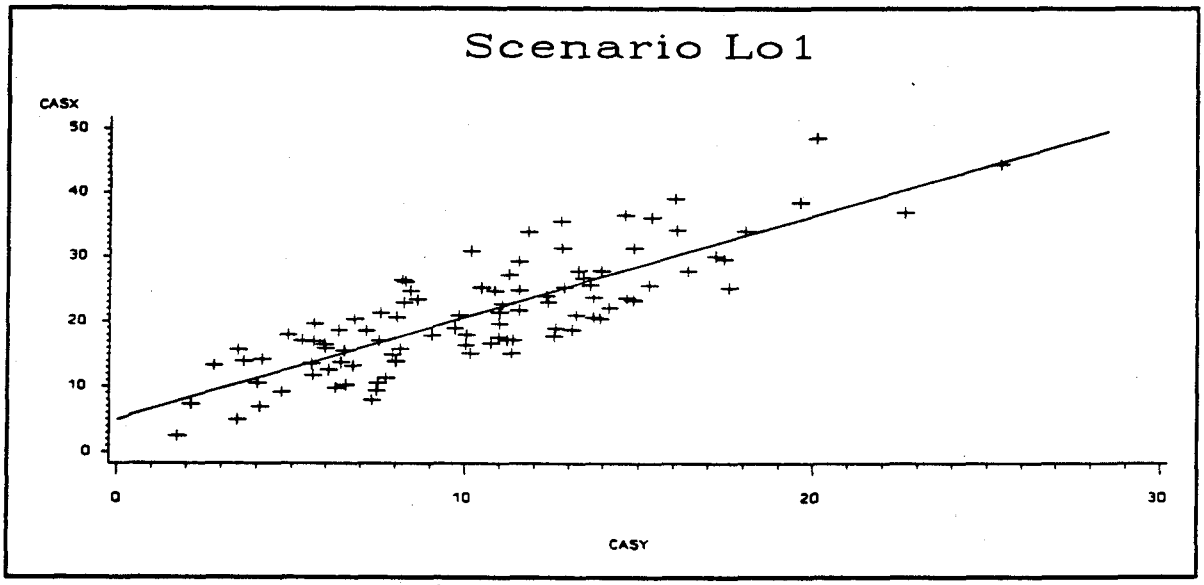


Fig. 44. Lo1 casx vs casy.

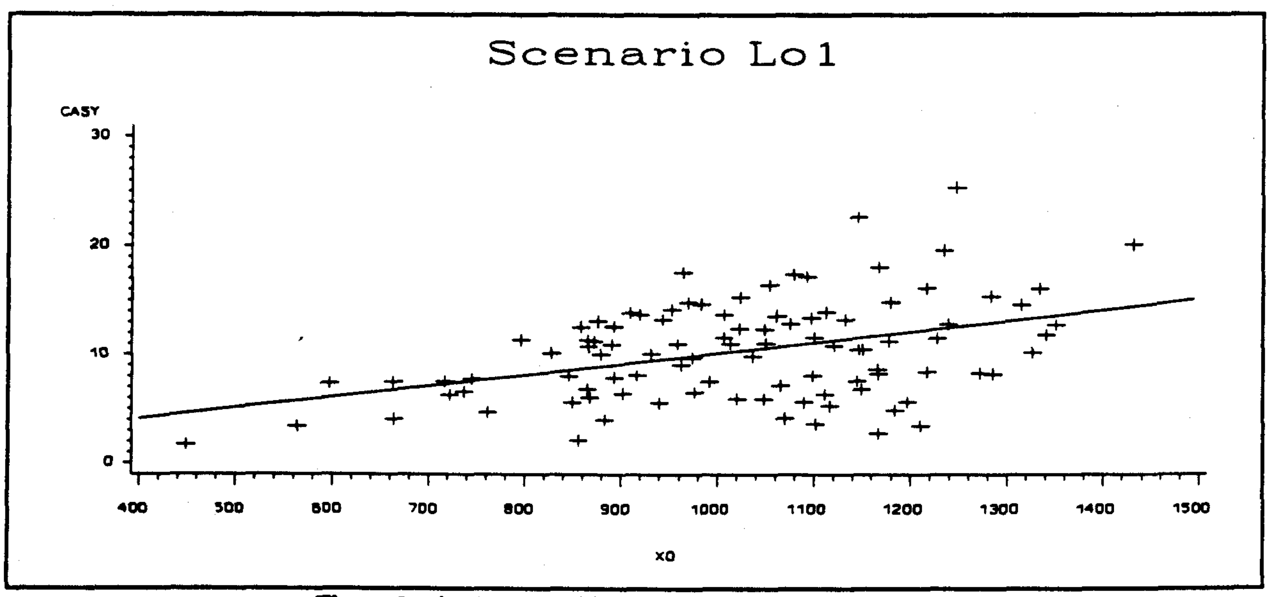


Fig. 45. Lo1 casualties vs opposing force size.

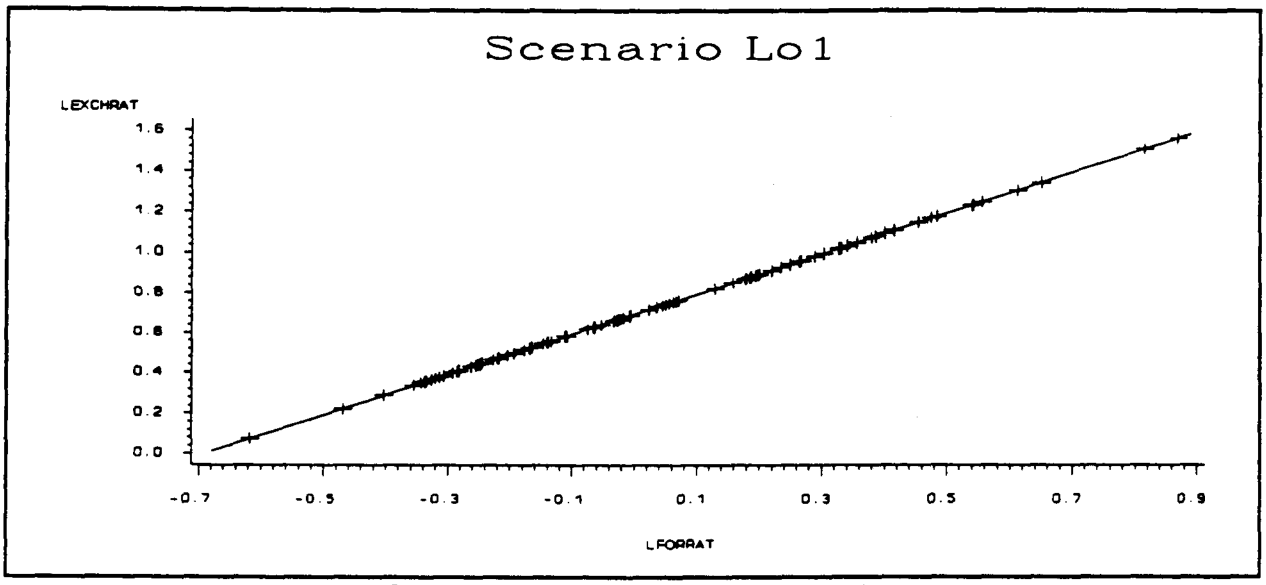


Fig. 46. Lo1 ln(exchrat) vs ln(forrat).

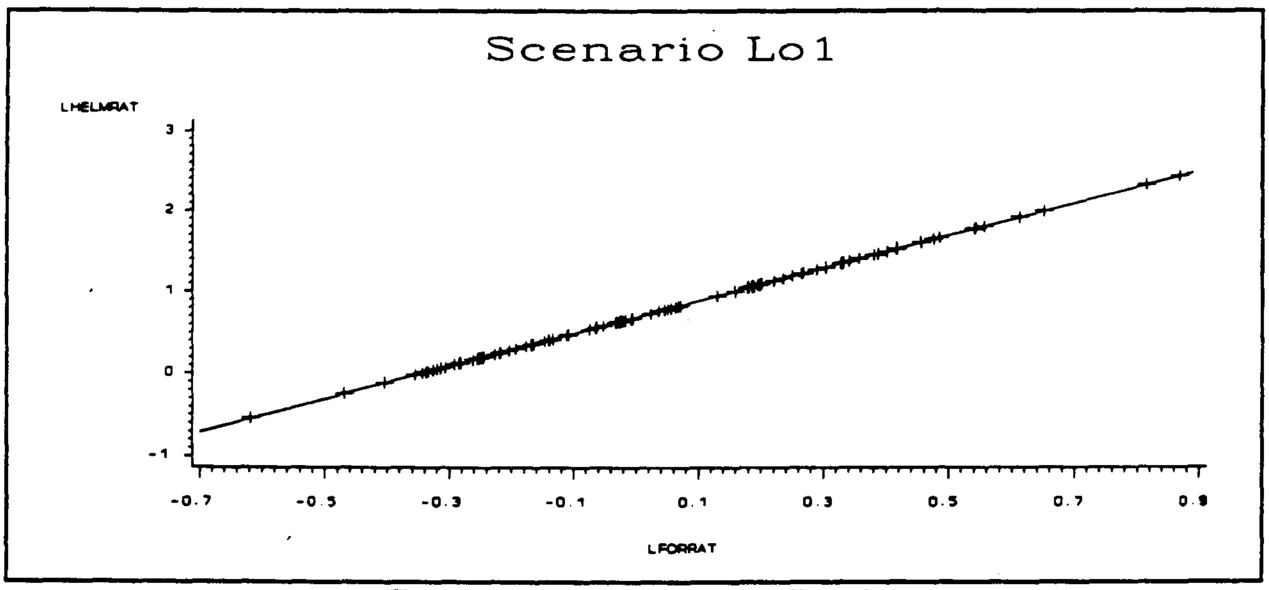


Fig. 47. Lo1 ln(helmrat) vs ln(forrat).

Figure 5 shows a decreasing slope in the relationship between ln(force ratio) and ln(exchange ratio) for the square law; Fig. 30 shows a zero slope for the linear law; and Fig. 46 shows a positive slope for the logarithmic law.

Figure 7 would show a 0.0 slope for the Helmbold relationship for the square law, if corrected by the exact solution; Fig. 31 shows a slope of 1.0 for the Helmbold relationship for the linear law; and Fig. 47 shows a slope of 2.0 for the logarithmic law. As indicated in Tables 7 and 8, the results of the simulations parallel those of the other two cases. For this reason, figures for simulations Lo2-Lo8 are not included.

The conclusion is that the starting force sizes and the $D$ and $A$ values are independent under logarithmic law assumptions. Hence any set of battles obeying the logarithmic law with moderate $D$ and $A$ variations and moderate variations in the number of days of battle will tend to have an $\alpha$ value near two and a $\beta$ value near the ln[avg($D/A$ )]. (Naturally, a sample taken from the population of all possible battles might have any $\alpha$ or $\beta$ value.)

### 3.4 MIXED LAW SIMULATIONS

In order to obtain $\alpha$ values between 1.0 and 2.0, some sort of mixed law simulation must be used. The equations in Eq.(30) use a linear mix of Eqs.(3) and (25), and the equations in Eq.(31) use an exponential formulation. The formulation in Eq.(30) is of the same form as the mixed laws described by Taylor[7], while that of Eq.(31) has similarities to the generalizations of Helmbold and Weiss described by Taylor[7].

$$\begin{aligned} dx/dt &= -D'xy - D''x \\ dy/dt &= -A'xy - A''y \end{aligned} \tag{30}$$

$$\begin{aligned} dx/dt &= -D''''x^{\alpha}y \\ dy/dt &= -A''''xy^{\alpha} \end{aligned} \tag{31}$$

The first formulation yields an $\alpha$ value between 1.0 and 2.0, depending on the relation of the two coefficients in each equation. The second formulation yields a regression with an $\alpha$ value given by the exponent $\alpha$. For $\alpha$ = 0.0, the square law results; for $\alpha$ = 1.0, the linear law results; for $\alpha$ = 2.0, the logarithmic law results; for intermediate values of $\alpha$, the corresponding mixed law regressions are achieved. The formulation of the mixed law given in the equations in Eq.(31) is similar to those of Willard[8] and Fain[1] and more convenient for generalization, and, therefore, is the one used here.

Figures 48-51 allow a comparison of the similarities and differences among the linear, logarithmic, and mixed laws. Figure 48 shows a closer correspondence between the opposing casualties than does the logarithmic law, but not the absolute linear relationship shown by the linear law. Figure 49 shows more correlation between casualties and the opposing force size than does the logarithmic law, but not as much.

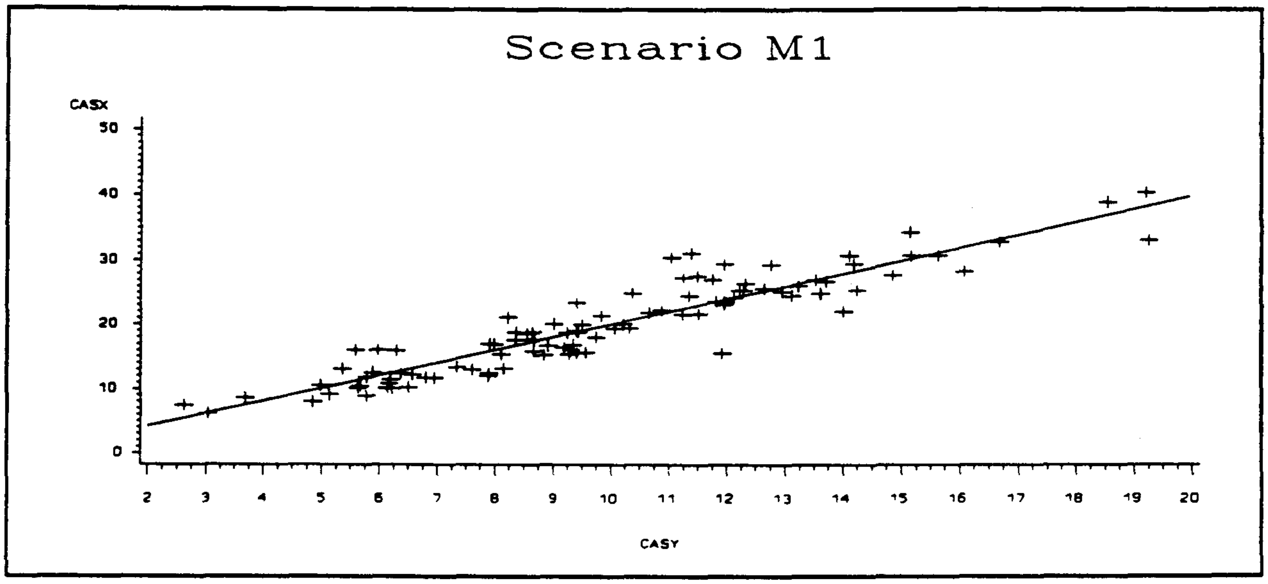


Fig. 48. M1 casx vs casy.

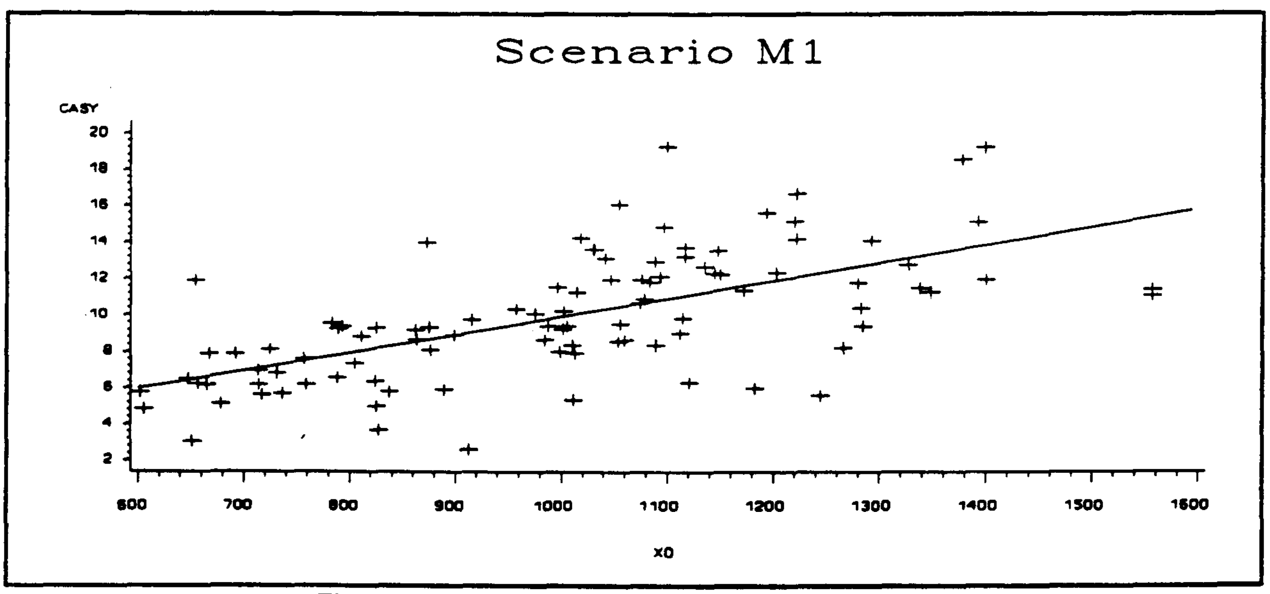


Fig. 49. M1 casualties vs opposing force size.

Figures 50 and 51 could be graphs for either the linear or the logarithmic laws, except for the slopes of the lines. The fit for the data shown in the figures gives an $R^2$ of 1.00, an $\alpha$ of 1.50 and a $\beta$ of 0.67, as expected from the *D* and *A* values of 0.00000063 and 0.00000032 and the difference equations involving the first power of one force and the 1.5 power of the other. By extension, a mixed law having any desired $\alpha$ value may be constructed.

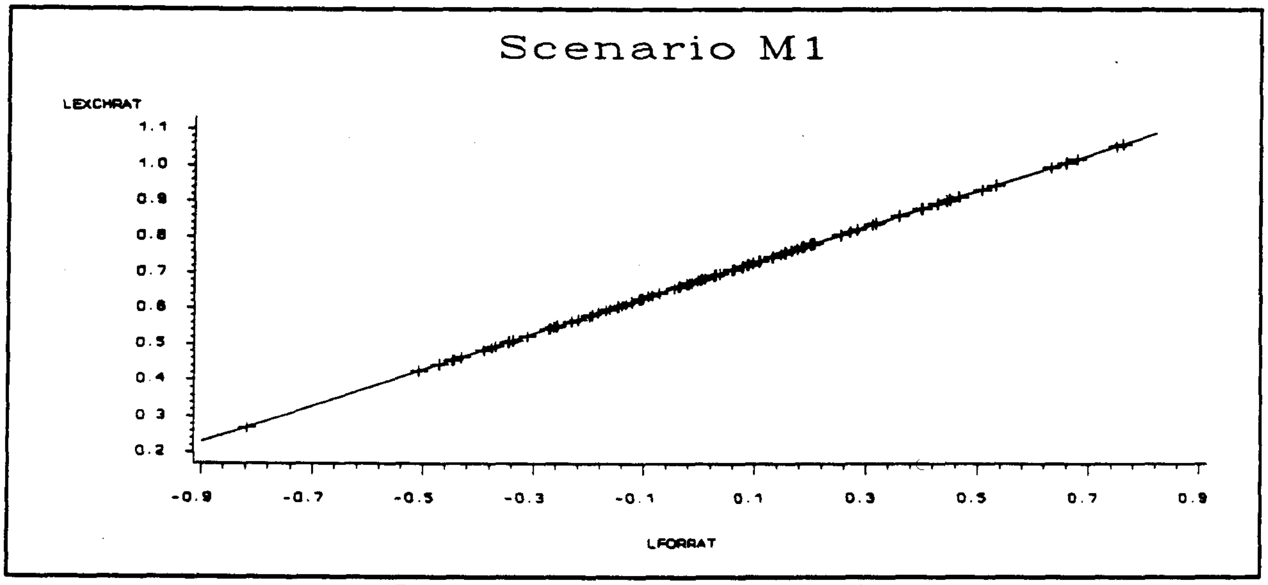


Fig. 50. M1 ln(exchrat) vs ln(forrat).

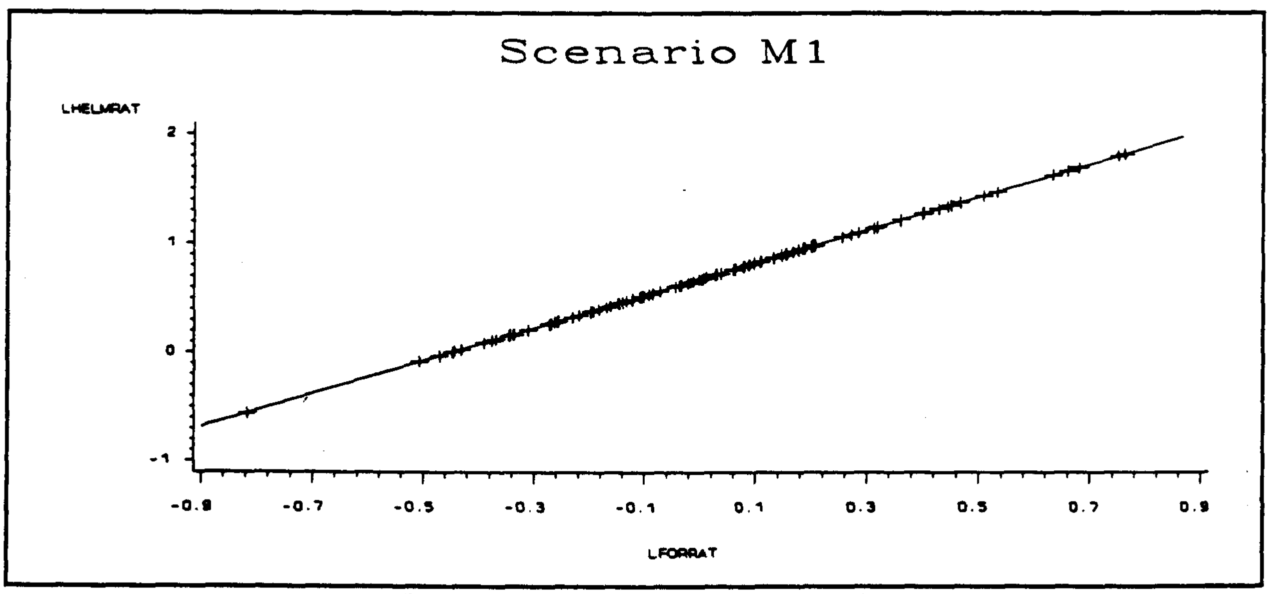


Fig. 51. M1 ln(helmrat) vs ln(forrat).

### 3.5 IMPLICATIONS OF THE SIMULATIONS

The regression that fits Helmbold's relationship to simulated battle data is very good at detecting a Lanchester type attrition law when it exists, even with noise produced by variations in activity ratio and battle length. It appears that its accuracy becomes less sensitive to this noise as the underlying law approaches the logarithmic law. For a square law situation, an $R^2 > 0.5$ could imply battles with very close activity ratios and lengths. For a logarithmic law situation, the same $R^2$ may imply a wide dispersion of values. As with any statistical analysis, the number of data points influences the confidence interval for any conclusion. A result formed from a large number of battles, or replicated by other sets of battles, gives more confidence than one derived from a single small set of battles.

Two technical points on analysis were found: 1. The use of improper normalization may destroy the fit if the underlying law is nearly a square law. 2. There is a problem with running a battle simulation to annihilation of one side using the difference equations approximation, because near the end of the battle the cumulative error of approximation compared to exact solution grows too large. (This latter problem also exists if the $D$ and $A$ values are too large, compared to the $x$ and $y$ values, as the casualties after one, or a few, iterations will approach annihilation for one side.) As a result, simulations using the difference equation approach should cut off before annihilation. This should not be an undesirable restriction on models, because it is most likely that in actual battles the losing commander will try to avoid fighting to annihilation. Historical examples to the contrary can be found; however, for conservative reasons, most simulations should not assume best case or worst case results. In the search for attrition patterns in historical data, perhaps we should omit battles with more than 30% casualties to remove casualties from routs, etc., which may have differing attrition patterns.

These simulations cause us to conclude that the 92 land battles, the 83 land battles, and the air battles from the Battle of Britain were not subject to the pure square law, linear law, or logarithmic law attrition equations, since $\alpha$ was not found to be 0.0, 1.0, or 2.0. [This does not preclude the battles from being composed of engagements obeying one of these laws or from having combinations of the effects of these laws taking place. All that is claimed is that the battles, taken as a whole, do not obey any of the integral laws (square, linear, or logarithmic).]

Helmbold's statement of his results is subject to misinterpretation because of his assumption of a square law attrition for the historical battles. The relationship he found is that the logarithms of the Helmbold ratios for the historical battles were fit nicely by a linear function of the logarithms of the initial force ratios for the battles. This coincides with an assumed mixed law attrition function (mixed between the linear and the logarithmic laws), with some randomness producing an error term. Such an attrition function has the property that its activity ratio is independent of the initial force ratio.

Helmbold[8] extends his square law approach by developing Lanchester coefficients with embedded initial force ratio factors, with a variable exponent. The military rationale is that there is a law of diminishing returns for the effect of large force ratios on the outcome of a battle. The mathematical effect is to produce an analog of the equations in Eq.(31). The trajectories of the attrition (that is, the plot of cumulative casualties over time) are different; but the integrated equations are identical. Helmbold's equations have the advantage of having known exact solutions; however, the linear law produced by

Helmbold's variable exponent is different from the classical linear law. The equations in Eq.(31) do not have known exact solutions but do agree with the classical square and linear laws. Helmbold's casting of his equations as square law variants obscures the fact that they are really examples of mixed law formulations and that the data fit a mixed law between the logarithmic and the linear, not the square law.

## 4. CONCLUSIONS

The evidence of the stochastic simulations is that a set of battles governed by a particular Lanchestrian attrition law will exhibit a characteristic slope when the natural logarithm of the Helmbold ratio is plotted against the natural logarithm of the force ratio. This slope is zero for a square law, one for a linear law, and two for a logarithmic law. Intermediate slopes are obtainable from mixed laws. The consistency of the coefficient ratios within the set of battles will generally govern the spread of the plot.

This result is true, given the caveat that it is possible for a sample to be taken from a set of battles governed by one law that exhibits the slope of another law. The $\alpha$ value spreads and standard deviations in Tables 4, 6, and 8 provide data for estimating the likelihood of such a result. A 10% misestimate of the slope is unlikely for moderate variations of the coefficient ratios. The $R^2$ values in the historical data cases may indicate larger variations in the $D$ and $A$ values than were allowed in the Lanchestrian simulations. Thus, the historical $\alpha$ values of 1.3, 1.5, and 1.7 are not likely to be the result of samples from either the square, linear, or logarithmic laws; however, the possibility exists.

The question remains as to the interpretation which should be placed on discovering that the data for a set of real world battles has a relationship similar to that produced by a mixed law (between linear and log) mathematical attrition formula. The exclusion of the square law, linear law, and logarithmic law does not prove a mixed law cause. Also, the coincidence of results does not guarantee a coincidence of causes; there may be other mechanisms that could generate these results.

Even if we assume that actual battles are governed by a mixed linear-logarithmic attrition law, this does not give enough information to predict the trajectories of the forces over time (many sets of derivative functions yield the same integral function). Still, the knowledge that the data do not fit a square law or a linear law pattern is useful. Knowing the pattern that the data do fit may lead to other avenues of exploration.

Helmbold's results may be useful in rough predictions of battle outcomes. For models with a low resolution of details, a mixed law formulation may be entirely appropriate for determining battle outcomes. Such a model might use Helmbold's data fit ($\alpha$ = 1.3 to 1.7, $\beta$ = 0.2) and the difference equations corresponding to the equations in Eq.(31). A stopping rule would also be required, either one based on the variable values or provided by the players in interactive games through their instructions to units about breaking off contact, etc.

A second avenue of exploration is a search for alternative mechanisms to explain the pattern seen in the historical data. The unplanned correlation between the casualties on the two sides that resulted in the investigation of multi-iteration square law battles suggests that the investigation of military constraints on battles may be fruitful. Such constraints as an attacker's decision to attack only if his force ratio is better than 3 to 1 may constrain the results independently of the attrition mechanism. This approach will be the subject of a future paper.

We conclude by remarking that patterns in data have been pursued fruitfully in the physical sciences from their inceptions. The underlying assumption is that the patterns are not randomly generated, but are caused. The patterns shown in Figs. 1-3 are definite and a mixed linear-logarithmic attrition law would generate such patterns.

## REFERENCES


1. Hartley III, D. S., *Non-Validation of Lanchester's Original Equations in the Inchon-Seoul Campaign of the Korean War*, ORNL/DSRD/TM-12, Oak Ridge National Laboratory, Oak Ridge, Tennessee, January 1988.

2. Helmbold, R. L., *Historical Data and Lanchester's Theory of Combat*, CORG-SP-128, Combat Operations Research Group, DTIC AD 480 975, 1961.

3. Helmbold, R. L., *Historical Data and Lanchester's Theory of Combat Part II*, CORG-SP-190, Combat Operations Research Group, DTIC AD 480 109, 1964.

4. Helmbold, R. L., *Air Battles and Ground Battles--A Common Pattern?*, DTIC AD 718 975, 1971.

5. Fain, J. B., "The Lanchester Equations and Historical Warfare: An Analysis of Sixty World War II Land Engagements," *History, Numbers, and War* 1(1), 34-52 (1977).

6. Helmbold, R. L., "Solution of a Generalized Version of Lanchester's Equations (Taylor-Helmbold Equations)," presented at ORSA/TIMS Meeting, St. Louis, October 1987.

7. Taylor, J. G., *Lanchester Models of Warfare, Volume I*, Military Applications Section, ORSA, 1983.

8. Willard, D., *Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905*, RAC-TP-74, Research Analysis Corporation, DTIC AD 297 375 L, 1962.