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Introduction to the Linear Theory of Tearing Instabilities

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Prepared for
U.S. Department of Energy
Assistant Secretary for Energy Technology
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Introduction to the Linear Theory of Tearing Instabilities

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I. Introduction

Revived interest calls for an introductory survey of tearing instability theory. Here, we try to make the relevant literature more widely and painlessly accessible, by summarizing the basic notions of the theory from an elementary point of view. We take advantage of the fact that most tearing theories, despite their often qualitatively different predicted growth rates and tearing layer widths, are very similar. Thus one can proceed surprisingly far in understanding tearing mode theory, without being committed to a specific model or ordering.

A complete review is not attempted. In particular, we restrict our attention to tokamak-fusion applications, and recent nonlinear tearing studies are virtually ignored.

The reasons why tearing instabilities might bear importantly on tokamak performance are considered in Sec. II. Sec. III describes the mechanism of tearing, and Sec. IV outlines the method by which this mechanism is analyzed. We close in Sec. V with a survey of typical growth rate predictions.

Although the notion of magnetic field-line tearing apparently originated in the astrophysical literature,[1] most fusion-oriented theories of tearing can be traced to the famous 1963 paper by Furth, Killeen and Rosenbluth (FKR).[2] FKR treated a number of resistive instabilities, in the limit of vanishing gyration radius. But it was recognized[3] that finite gyroradius effects, and other corrections to Ohm's law, would become significant as the plasma temperature and electrical conductivity increased, and such corrections were studied in several papers, by Coppi and collaborators,[4] and others,[5,6] during the following two years.

The subsequent five or six year lapse in tearing-related publications was perhaps a consequence of the apparently esoteric, "theoretical" nature of the modes, and their relatively slow growth rates (proportional to a fractional power of the resistivity). The revival of interest, beginning in 1971 and continuing to the present, was stimulated mainly by the tokamak experimental program: it was noticed[7] that certain tokamak phenomena, universally observed,[8] and seemingly associated with grave effects on confinement, had significant features in common with the predicted properties of tearing instabilities. Thus, in particular, the (now "classical") FKR stability criterion was studied in detail for various tokamak current profiles.[9] The relation between tearing modes and the ideal MHD kink was also clarified[10] (the experimental results having also stimulated new interest in ideal effects[11,12]). But most theoretical work during the 1970's has been concentrated within three areas: (i) Attempts to extend tearing mode theory to the long mean free path regimes of modern tokamak experiments.[13,14,15] This effort has produced, inter alia, substantial unification of previous work.[14,15] (ii) Investigations of the effects of toroidal curvature, both in the context of resistive fluid theory,[16] and also, most recently, in the long mean free path limit.[17] Such effects turn out to be quite important, as is also the case for ideal kink modes.[18] (iii) Studies, both analytical[19,20] and numerical,[21,22,23] of the nonlinear properties of resistive instabilities. The objectives here include theoretical understanding of the so-called disruptive instability,[24,25] the consequences of magnetic island formation,[26] and the effects of tearing instabilities on transport.[27,28]

II. Significance of tearing instabilities

A. Confinement

Plasma confinement in an axisymmetric, toroidal system[29] depends upon the existence of nested surfaces of constant pressure p, almost each of which is covered ergodically by a single field line:

$$\underset{\sim}{B} \cdot \nabla p = 0. \tag{1}$$

Loss of confinement then requires some scattering mechanism to allow particle diffusion across the surfaces. But this radial motion is very slow: of order $(\nu/\Omega)^2$, where ν is the 90^o-scattering frequency and $\Omega = eB/mc$ the gyrofrequency, compared to particle motion along the magnetic field.

Much more rapid loss of confinement can result from destruction of the "flux surfaces" themselves, due to radial perturbations of the field. Consider a circular-cross-section tokamak, with toroidal coordinates (r, θ, φ): r is the minor radius, and $\theta(\varphi)$ is the poloidal (toroidal) angle. The unperturbed state is characterized by

$$\partial p_o / \partial \varphi = 0 = B_{or} . \tag{2}$$

Then Eq. (1) with $B_\theta \neq 0$ implies that p_o depends only on r. For simplicity, we consider perturbations $(\tilde{p}, \tilde{B})$ which maintain Eq. (1), while breaking the symmetry of Eq. (2) (as would correspond to low frequency excitations, or "neighboring equilibria"). Periodicity allows us to expand

$$-i\tilde{B}_r = \sum_{m,n} \psi_{mn} \, e^{i(m\theta - n\varphi)} , \tag{3}$$

$$\tilde{p} = \sum_{m,n} \tilde{p}_{mn} \, e^{i(m\theta - n\varphi)} ,$$

and to express the linear version of Eq. (1) as

$$\tilde{p}_{mn} = -(r\, p_o'/B_{o\theta})\, \psi_{mn}/(m-nq), \tag{4}$$

where

$$q(r) \equiv r\, B_{o\varphi}/R\, B_{o\theta} \tag{5}$$

is the safety factor of the unperturbed system, and finite aspect ratio corrections are omitted. Evidently, the linear theory breaks down on flux surfaces satisfying

$$|(r\, p_o'/p_o)\,(\psi_{mn}/B_{o\theta})| \geq |m-nq(r)|\,. \tag{6}$$

For chosen integers m and n, Eq. (6) will describe a three-dimensional region,

$$|r - r_s| \leq \Delta_{mn}, \tag{7}$$

where $r_s(m,n)$ labels the appropriate rational flux surface,

$$q(r_s) = m/n \tag{8}$$

and the radial width, Δ_{mn}, can be seen to depend upon the shear, q'/q, and upon the magnitude of the field perturbation at r_s. Inside this singular region, the perturbed constant pressure surfaces need not resemble their unperturbed counterparts. Furthermore, if Eq. (6) is satisfied for one pair (m,n), it will in general be satisfied for numerous pairs (m',n'), with $m'/n' \simeq m/n$, so that the pressure surfaces inside Δ_{mn} can become chaotically scrambled: one has in this case a "flux-volume", in which the concept of flux surfaces is irrelevant, and confinement is locally absent.

On the other hand, for a fixed, reasonably smooth field perturbation $|B_r/B_\theta| \ll 1$, the width Δ_{mn} decreases sharply with increasing m or n. [This follows from the presumed convergence of the series in Eq. (3).] Thus the presence of high-order rational values of q, and the fact that every number is nearly rational, are not physically important. One expects significant destruction of the irrational flux surface having $q = (1.1)^{\frac{1}{2}}$, but not, usually, of the rational surface having $q = 23/16$.

Hence we may restrict our attention to reasonably small values of m and n, so that on "most" flux surfaces, Eq. (6) is not satisfied. Such flux surfaces suffer only mild deformation, without topological change. The situation for both types of surface is depicted in Fig. 1.

These remarks are made precise in a theorem ("KAM") first conjectured by Kolmogorov, and later proved by Arnol'd and Moser, in the context of ergodic theory. An illuminating discussion has been presented by Walker and Ford.[30] The significance of the KAM theorem to plasma confinement was pointed out by Grad,[31] while detailed studies of flux surface destruction have been presented by, among others, Rosenbluth et al.,[32] and, most recently, Rechester and Stix.[26]

Ideal MHD theory[11] predicts that the most dangerous modes in a tokamak have toroidal mode number $n = 1$, so that flux surface destruction, and local loss of confinement, seems most likely to occur near flux surfaces having small integral safety factors: $q = 1, 2, 3 \ldots$. But ideal MHD also predicts that

$$B_r(r_s) = 0 \tag{9}$$

in which case Eq. (6) cannot be satisfied and the flux surface topology is preserved!

This comment brings us, finally, to the subject of tearing, since the crucial property of a tearing mode is the presence of a radial magnetic field perturbation on rational flux surfaces. Thus any observed flux surface destruction (if it results from an instability rather than lack of equilibrium[31]) can be associated with something akin to a tearing instability.

It follows that tearing instabilities could provide fundamental limitations on tokamak confinement: they could limit, for example, the maximum achievable $\beta \equiv 8\pi p/B^2$. When the singular regions of Eq. (7) are well separated, the resulting local confinement loss would appear in the observed pressure or temperature profiles [e.g., a central flattening of $p(r)$, if $q \simeq 1$ on axis]. Much more serious, presumably even "disruptive", consequences would be observed in the case of overlap between singular regions corresponding to different values of q. Such possibilities have encouraged theoretical and experimental interest in the tearing mode.

To avoid misunderstanding, we remark here that the linear analysis of tearing, to be considered in Sec. IV, differs from Eqs. (2)-(6), in that singular perturbations do not occur. In fact the crucial feature of every tearing analysis consists of finding and including small corrections to the operator $\underset{\sim}{B} \cdot \nabla$, and thereby resolving the singularity in equations analogous to Eq. (4). Of course our comments concerning Eq. (6) remain applicable, so long as the corrections are indeed small.

B. Observations on tokamaks

Magnetic probe measurements on a number of tokamak plasmas have revealed sinusoidally oscillating magnetic field perturbations, with the spatial structure,[7,8,33]

$$\tilde{B} \propto \exp(im\theta - \varphi), \quad m = 1,2,3,\ldots$$

By comparing the times of onset of such "Mirnov oscillations" with the time-evolution of the safety factor $q(r,t)$, it has been confirmed that a particular mode number m occurs only when q achieves the value of m at some radius inside the plasma. This behavior coincides with that of both tearing modes and ideal internal kink modes, although the former seem more likely to be unstable.[9]

The frequency of the Mirnov oscillations is usually identified with the electron diamagnetic frequency, the growth rate being typically smaller. This fact is also consistent with modern theories of the tearing mode.

Some experiments indicate an association between Mirnov oscillations and local flattening of the electron temperature profile, near the appropriate rational surface r_s.[34] (This effect is most clearly visible for modes with $m \geq 2$. A similar flattening of, for example, the plasma density, is not ruled out, but more difficult to detect.) Since profile-flattening is an obvious consequence of local flux surface destruction, these observations are especially suggestive.

Mirnov oscillations are relatively harmless by themselves. However, they frequently appear as precursors, either to internal "mini-disruptions", manifested in saw-tooth oscillations of the x-ray-detected electron temperature, or to major disruptions, resulting in abrupt collapse of the discharge ("disruptive instability"). Although patently nonlinear, both kinds of disruption have explained in terms of magnetic field line tearing. For example, nonlinear growth and decay of magnetic islands could be associated with mini-disruptions,[21,22,25] while the major disruption might involve sudden overlap of islands growing from initially distinct rational flux surfaces.[25,26]

It should be emphasized that all explanations of Mirnov oscillations and disruptions remain tentative, and several theories of the phenomena have been proposed which make no reference to the tearing mechanism. For example, the oscillations have been interpreted in terms of neighboring helical equilibria,[13] and in terms of temperature-gradient driven drift waves;[35]

profile flattening near the magnetic axis (for $q = m = 1$) has been explained in terms of a rotational interchange mode;[36] and a modified neoclassical theory has been shown to predict disruptive effects, similar to those which are observed.[37] If tearing theories seem prominent among the various candidates for explaining Mirnov phenomena, it is probably because such theories hope to relate all the observations to a single, relatively simple idea: changes in the topology of the magnetic field.

Finally, we point out that the great wealth of detailed experimental information on Mirnov oscillations and disruptions has been barely touched upon here. It is known, for example, that the sawtooth signal changes phase across the mode-rational flux surface, that the major disruption is accompanied by a negative voltage spike, and so on. Thus any proposed theoretical picture of the oscillations and disruptions is subject to an unusually rich assortment of experimental tests. Since a successful theory of the Mirnov phenomena would bear on several questions crucial to the performance of a tokamak reactor (such as transport scaling laws, or the previously mentioned question of β limitations), the opportunity offered to theorists is clear.

III. Tearing Instability Mechanism

A. Magnetic Shear

The importance of magnetic shear with regard to flux surface destruction has already been noted. In addition, conventional tearing instabilities are driven by the magnetic field energy associated with shear. Hence we begin by considering an unperturbed field $\underset{\sim}{B}_o(r)$ whose direction depends upon radius. Then an interior radius, r_*, and a fixed unit vector $\hat{n}$ can always be chosen such that the field component $\underset{\sim}{B}_n = \hat{n}(\underset{\sim}{B}_o \cdot \hat{n})$ changes sign at r_s,

$$\underset{\sim}{B}_o(r_*) \cdot \hat{n} = 0. \tag{10}$$

The lines of the $\underset{\sim}{B}_n$-field, in a radial neighborhood of r_*, are depicted in Fig. 2.

The minimum field energy configuration (vacuum field) is roughly characterized by the absence of shear:

$$\underset{\sim}{B} \cdot \hat{n} = 0, \quad \text{for all } r.$$

The configuration of Fig. 2 can relax to this minimum energy state by means of field annihilation: field lines above (below) r_* migrate downwards (upwards) so as to annihilate their oppositely directed counterparts. It is instructive to distinguish two mechanisms available for such migration: convection and diffusion.

These mechanisms are clearly visible in a resistive MHD model, in which we assume

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B}/c = \eta \underset{\sim}{J}, \quad \nabla \cdot \underset{\sim}{V} = 0, \tag{11}$$

where η is the (spatially constant) resistivity, $\underset{\sim}{J}$ is the plasma current, $\underset{\sim}{V}$ the flow velocity and $\underset{\sim}{E}$ the electric field. In slab geometry, Maxwell's equations then yield the familiar linearized relation

$$\frac{d\tilde{B}}{dt} = -\frac{a^2}{\tau_s}\nabla^2\tilde{B} + i k_{\parallel} B_o \tilde{V} \qquad (12)$$

where a is the plasma radius, $k_{\parallel}$ is the parallel wave vector and

$$\tau_s \equiv 4\pi a^2 / c^2 \eta \qquad (13)$$

is the resistive skin-time. For a nominal ordering we have

$$\nabla \sim a^{-1} \sim k_{\parallel} \qquad (14)$$

and $V \sim v_A \equiv (B_o{}^2/4\pi\rho)^{\frac{1}{2}}$, the Alfvén speed. Then the first, diffusive term on the right-hand side of Eq. (12) corresponds to the time scale $\tau_s \sim 10^{-1}$ sec., while the second, convective term corresponds to the scale $\tau_A \simeq a/v_A \sim 10^{-8}$ sec. (These numerical estimates correspond very roughly to typical tokamak parameters.) Thus equilibrium magnetic field diffusion, characterized by $\nabla B \sim B/a$, is extremely slow, and significantly rapid field annihilation must involve the convective term.

However, a crucial property of ideal ($\tau_s \to \infty$) MHD convection must be recalled: if $\tilde{\xi}$ is the plasma displacement, then the radial field perturbation B_r satisfies

$$B_r / B_o = i k_{\parallel} \xi_r . \qquad (15)$$

This can be seen from Eq. (12) or, more perspicuously, from Fig. 3; the essential point is that the field is "frozen" to the plasma fluid, in the ideal case.

B. Annihilation and reconnection

With these remarks in mind, we return to Fig. 2, and consider how we would perturb this field to most quickly relax the shear. To produce rapid, Alfvénic excitation, we would "pluck" the rubber-band-like $\tilde{B}_n$-lines periodically along $\hat{n}$; to produce field annihilation, we must pluck them anti-symmetrically with respect to the radius r_*, so that a field line below r_* is drawn up at the same positions, along the $\hat{n}$-direction, that the corresponding field line above r_* is drawn down. The resulting kink-like perturbation is depicted in Fig. 4. Note that annihilation and reconnection, or "tearing", at r_* has altered the magnetic field topology: magnetic islands have appeared. It is clear that this topological change occurs even for an infinitesimal perturbation, i.e., even in linear theory, so long as the perturbation has the geometry we have described.

A simple characterization of this geometry is also clear from Fig. 4: the wave vector $\underset{\sim}{k}$ lies along the direction $\hat{n}$. It follows that the field component $\underset{\sim}{B}_n$ is proportional to $k_\parallel$, whence

$$k_\parallel (r_*) = 0, \tag{16}$$

by the definition of r_*. In an axisymmetric torus we must have $\underset{\sim}{k} = \hat{\theta}\, m/r - \hat{\varphi}\, n/R$, where $\hat{\theta}$ and $\hat{\varphi}$ are the oovious unit vectors, R is the major toroidal radius, and m and n are the poloidal and toroidal mode numbers respectively, so that Eq. (16) imples

$$q(r_*) = m/n \; .$$

Thus the radius, r_*, must be identified with one of the radii r_s of Eq. (6), if the tearing mechanism is to function.

A crucial conclusion follows from this identification and from inspection of Fig. 4: the tearing mode requires

$$B_r(r_s) \neq 0. \tag{17}$$

in contradiction to Eq. (15). In other words, significant decoupling of the plasma fluid and the perturbed magnetic field lines must occur at the rational flux surface. Of course, the decoupling need not in general result from resistivity.

It is clear that the departure from ideal behavior described by Eq. (17) is only locally significant: at most radii $r \neq r_s$, the second term of Eq. (12) easily dominates the first. Thus the tearing mode appears as a kink-instability, slightly modified near rational surfaces. For a complementary description, emphasizing diffusion, we may consider the narrow region near r_s,

$$|r - r_s| \leq \lambda << a \qquad (18)$$

in which Eq. (14) does not pertain, and the two terms on the right-hand side of Eq. (12) are comparable. In this "tearing layer", one expects (from, for example, Fig. 4) that λ would replace a as a measure of the radial scale length:

$$\frac{a^2}{\tau_s} \nabla^2 \underset{\sim}{B} \sim \left(\frac{a}{\lambda} \right)^2 \frac{\underset{\sim}{B}}{\tau_s} \gg B/\tau_s \qquad (19)$$

Detailed analysis, to be outlined in the following Section, reveals a somewhat more complicated situation; in many cases only one factor of (a/λ) actually appears, and the skin time τ_s need not always be resistive. But the essential point is generally valid: enhanced radial gradients in the tearing layer yield relatively rapid, <u>localized</u> diffusion of the perturbed field. By identifying such diffusion with the tearing instability, we can correctly estimate its growth rate, and we omit only its non-local, kink-like, "wake".

In summary, the tearing instability takes advantage of field-line tension, through the Alfvén mechanism, to allow shear-relaxation to proceed on time scales much shorter than the skin time. The instability can occur only in the presence of rational flux surfaces, in the close vicinity of which the (small) equilibrium field component $B_n \simeq B_{\theta o}(1-nq/m)$ is annihilated and reconnected. The tearing of B_n-lines requires non-ideal behavior, $B_r(r_s) \neq 0$, and can be considered analogous to localized magnetic field diffusion. But the tearing mode itself is not localized: it is a convection-diffusion hybrid which appears, far from the rational surface, similar to the kink mode.

IV. Linear analysis of tearing instabilities

A. Boundary Layer Problem

The previous two Sections indicate the importance of modes having the following properties:

(i) a kink-like structure, far from the rational flux-surface (i.e., for $k_{\parallel} \sim a^{-1}$);

(ii) $B_r\left(r_s\right) \neq 0$;

(iii) low frequency: ω or $\gamma \equiv \mathrm{Im}(\omega)$ should be small compared to $k_{\perp} v_A$, as in the observed tokamak oscillations;

(iv) $\gamma > 0$ in parameter ranges of interest to present or planned tokamak experiments.

It seems convenient to identify any mode having properties (i) and (ii) as a tearing mode; with this definition, most known tearing modes have been found to satisfy property (iii). Property (iv) is more delicate, in that the stability or consistency criteria for a given, theoretically proposed, tearing mode often depend strongly on fine details of the equilibrium. We shall not use property (iv) restrictively here.

We begin our outline of the linear analysis by considering the kinked region, exterior to the tearing layer. Here, the slow plasma motion indicated by property (iii) allows us to neglect, as a first approximation, inertial and viscous effects in the equation of motion. Thus we consider a perturbed equilibrium, with approximately scalar pressure:

$$c \nabla \tilde{p} = \tilde{\underset{\sim}{J}} \times \underset{\sim}{B}_0 + \underset{\sim}{J}_0 \times \tilde{\underset{\sim}{B}} . \tag{20}$$

Equation (20), together with the Maxwell equations

$$\nabla \cdot \underset{\sim}{B} = 0, \quad \nabla \times \underset{\sim}{B} = 4\pi \underset{\sim}{J} / c,$$

provides a closed, homogeneous system for the seven unknown quantities $\tilde{p}$, $\tilde{\underset{\sim}{J}}$ and $\tilde{\underset{\sim}{B}}$. In cylindrical geometry, the system may algebraically be reduced to a second order differential equation which involves only the radial field perturbation. This is the so-called Newcomb equation:[38]

$$k_{\parallel} (r\,\psi'/k^2)' - (\psi/B_o)\,[\,r\,(k_{\parallel} B_o)'/k^2]' - r\,g\,\psi = 0, \tag{21}$$

with

$$g \equiv k_{\parallel} \frac{k^2 r^2 - 1}{k^2 r^2} - \frac{2n^2}{k^2 r^2} \left(\frac{r}{R}\right)^2 \left[\frac{k_{\perp}}{k^2 r^2} - \frac{4\pi p_o'}{B_o^2}\right]. \tag{22}$$

Here,

$$\psi = -\,i\,\tilde{B}_r \tag{23}$$

is the conventional measure of the field perturbation (or axial vector potential), the primes denote radial derivatives, and the wave vector components are given by

$$\begin{aligned} k_{\parallel}(r) &= \frac{m}{r}\frac{B_\theta}{B_o} - \frac{n}{R}\frac{B_z}{B_o}, \\ k_{\perp}(r) &= \frac{m}{r}\frac{B_z}{B_o} + \frac{n}{R}\frac{B_\theta}{B_o}, \end{aligned} \tag{24}$$

whence

$$k^2 r^2 = m^2 + (r/R)^2\, n^2, \tag{25}$$

with $(m,n) = 1, 2, 3, \ldots$. Axial periodicity is assumed, so that the cylinder models a torus with major radius R.

The first noteworthy feature of Eq. (21) concerns its derivation: such standard assumptions of ideal MHD as

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B}/c = 0, \tag{26}$$

$$p\,\rho^{-\gamma} = \text{const.},$$

and so on, have not been used, and need not pertain. The question of incompressibility is similarly irrelevant. The fact that Eq. (21), which is conventionally derived from the MHD energy principle, does not in fact depend upon MHD assumptions, is significant, since the observed oscillations seem too slow to be correctly described by an MHD model.

Of course, the most important feature of the Newcomb equation is that $k_{\parallel}$, which appears in the coefficient of the highest order, ψ'', term, may in general vanish at some radius r_s

$$k_{\parallel}(r_s) = 0. \tag{27}$$

If Eq. (21) is presumed valid in the close vicinity of r_s, then any acceptable solution ψ must satisfy

$$\psi(r_s) = 0 \tag{28}$$

in contradiction to property (ii). Examples of modes satisfying Eqs. (21) and (28) are the local, Suydam modes, and the non-local, ideal-MHD kink modes (which are referred to as "internal" or "external", depending upon whether r_s lies in the plasma, or in a vacuum region surrounding the plasma, respectively.[9]) The relation between the existence of such solutions and MHD stability was studied in detail by Newcomb.[38]

More generally, and more realistically, we must acknowledge that Eq. (21) may be a poor approximation in the region close to r_s. The description of this region by a more accurate differential equation, which is not singular at r_s, leads to the concept of a "tearing layer", as described in Section III.

To anticipate salient features of the tearing layer equation, note that it is $\psi'' \sim \psi/k_{\parallel}$, rather than ψ itself, which becomes large as r approaches r_s. To "turn-over" ψ'' - to round off the tip of its spike at r_s - a curvature term of the form $(\psi'')'' \sim \psi''/\lambda^2$, where λ is the layer width of Eq. (18), is required. Such a term will presumably come from including, inter alia, $\omega/k_\perp v_A$ corrections to the force balance equation. On the other hand, terms in this equation which are small in λ/a may be omitted. Thus we expect the tearing layer to be described in general by an equation of the form

$$\psi'''' + \alpha(\omega)\, h(r)\, (\psi'' + \ldots) = 0\,(\lambda/a) \tag{29}$$

The final step in the linear analysis of tearing instabilities consists of treating Eqs. (21) and (29) as a standard boundary layer problem: one seeks the regular solutions of Eq. (29) which smoothly join, as $|r - r_s|$ becomes large, onto the corresponding solutions of Eq. (21). Of course the latter must also satisfy boundary conditions at $r = 0$ and $r = a$. Applied to Eq. (29), the regularity and "matching conditions" in general restrict $\alpha(\omega)$ to a discrete set of values,

$$\alpha(\omega) = \alpha_j\,, \quad j = 1, 2, \ldots \tag{30}$$

where the α_j typically depend upon asymptotic properties, for $r \to r_s$, of the exterior solutions, i.e., the solutions to Eq. (21).

Equation (30) represents the tearing mode dispersion relation. We next turn our attention to a concrete example.

B. Classical tearing mode

Noting that Eq. (20) imples Eq. (1), and recalling that Eq. (1) leads to singular perturbations, $\tilde{p} \propto (k_{\parallel})^{-1}$, we conclude that small corrections to Eq. (20) will be important when $k_{\parallel}$ is small. The simplest such correction comes from inertia:

$$\rho\, \partial \underset{\sim}{V}/\partial t = -\rho\omega^2\, \underset{\sim}{\xi}\,,$$

where $\underset{\sim}{\xi}$ is the plasma displacement, and ρ the equilibrium mass-density. We add the inertial term to Eq. (20), which is then used to compute the perturbed current perpendicular to $\underset{\sim}{B}_o$. As a result we find

$$\tilde{J}_r = (c/B_o)\,(\rho\omega^2\,\xi_\perp - ik_\perp\,\tilde{p})$$

where $\xi_\perp \equiv (\underset{\sim}{B}_o \times \hat{r}/B_o)\cdot\underset{\sim}{\xi}$ and $k_\perp \equiv (\underset{\sim}{B}_o \times \hat{r}/B_o)\cdot\underset{\sim}{k}$, as in Eq. (24). Next, we approximate for $\lambda/a \ll 1$, by retaining only the highest order radial derivatives of perturbed quantities. Then, since $\underset{\sim}{\nabla}\cdot\tilde{\underset{\sim}{B}} = 0$, Ampere's law has the parallel component

$$4\pi\,\tilde{J}_\parallel/c \cong \psi''/k_\perp$$

and the quasineutrality condition,

$$0 = \underset{\sim}{\nabla}\cdot\underset{\sim}{J} \cong J_r' + ik_\parallel\,J_\parallel$$

takes the form

$$k_\parallel\,B_o\,\psi'' + 4\pi\,\rho\omega^2\,(ik_\perp\,\xi_\perp)' \cong 0. \qquad (31)$$

Here we have omitted a contribution from the $\tilde{p}$-term in $\tilde{J}_r$, using $\tilde{p} \cong -\xi_r\,p_o'$ and $\xi_r/\xi_\perp = 0(\lambda/a)$, as will be seen presently. We note, however, that the pressure perturbation is important for certain low-frequency, drift-tearing modes, considered in Sec. V. It can also affect the classical tearing mode, when toroidicity is included.[16] Physically, Eq. (31) states that the crucial accelerating force is field-line tension, $(\underset{\sim}{B}\cdot\nabla\underset{\sim}{B})_r \sim ik_\parallel\,B_oB_r$, in qualitative agreement with the discussion of Sec. III.

Note that the first term of Eq. (31) is simply the highest-order derivative term of the Newcomb equation. The second term in Eq. (31) will be seen to resolve the singularity at r_s, after we have obtained an equation for $\underset{\sim}{\xi}$.

The classical prescription for $\underset{\sim}{\xi}$ is provided by Ohm's law and incompressibility: Eqs. (11). Thus the radial component of Eq. (12) relates ξ_r to ψ, while

$$\nabla \cdot \underset{\sim}{\xi} = 0 \simeq i\, k_\perp\, \xi_\perp + \xi_r{}' + 0(\lambda/a)$$

allows Eq. (31) also to be written in terms of ψ and ξ_r. We then obtain the equations[2]

$$k_\parallel\, B_o\, \psi'' - 4\pi\rho\, \omega^2\, \xi_r{}'' = 0 \tag{32}$$

$$(a^2/\tau_s)\, \psi'' + i\, \omega\, (\psi - k_\parallel\, B_o\, \xi_r) = 0 \tag{33}$$

which evidently may be combined to yield a fourth order differential equation, the tearing layer equation of Eq. (29), for ψ.

Note that the second term of Eq. (33) is proportional to the parallel electric field,

$$i\,\omega\,(\psi - k_{\parallel} B_0\, \xi_r) \simeq c\, k_{\perp}\, E_{\parallel}\ , \qquad (34)$$

if a small term, $c\,\eta k_{\parallel} J_{\perp}$, is omitted. For ordinary resistive diffusion, $\psi'' \sim \psi/a^2$, and Eq. (33) states that $E_{\parallel} \sim \psi/c\, k_{\perp} \tau_s$ is very small, as in ideal MHD. Thus the tearing layer may be characterized as that region in which $E_{\parallel}$ significantly exceeds its resistive diffusion value; recall Eq. (19).

Because we have omitted all finite Larmor radius effects, and because the simple form of Ohm's law we have used applies only for large collision frequency, ν, Eqs. (32) and (33) have a limited range of validity. In particular they require

$$\nu \gg \omega \gg \omega_{*}\ , \qquad (35)$$

where ω_* is a diamagnetic frequency. It is significant that the three frequencies appearing in Eq. (35) are in fact roughly comparable in many present tokamak experiments. On the other hand, several more recent studies, which allow $\omega \sim \omega_*$ and $\omega \sim \nu$, yield tearing layer equations nearly identical in form to Eqs. (32) and (33); only the coefficients are changed. We therefore regard the equations as archetypal, and consider next some properties of their solution.

C. Matching Condition

To estimate the growth rate, $\gamma = -i\omega$, and layer width, λ, of the classical tearing mode, we assume both terms in each of Eqs. (32) and (33) are comparable, and expand

$$k_{\parallel} \simeq k_{\parallel}'\ (r - r_s)$$
$$\sim k_{\parallel}'\,\lambda\ .$$

In the most obvious ordering, we would further estimate $\xi_r'' \sim \xi_r/\lambda^2$ and

$$\psi'' \sim \psi/\lambda^2. \tag{36}$$

Then Eq. (33) yields

$$\gamma \sim (a/\lambda)^2 \tau_s^{-1},$$

corresponding to localized diffusion, while Eq. (32) yields

$$\gamma \sim k_\parallel v_A \sim k_\parallel' \lambda v_A,$$

corresponding to Alfvén convection (v_A is the Alfvén speed). After eliminating λ we find,[39] with $\tau_A \equiv a/v_A$,

$$\gamma^3 \sim (k_\parallel' a^2)^2 / (\tau_s \tau_A^2), \tag{37}$$

a result which confirms several tearing mode features anticipated in Sec. III: the instability is driven by shear, and has the form of a diffusion-convection hybrid.

However, Eqs. (36) and (37) do not describe the tearing modes most frequently considered in the literature. The point is that, unless $\psi(r_s)$ is small, Eq. (36) corresponds to an apparent (i.e., when viewed with respect to the scale length $a \gg \lambda$) discontinuity in the radial field perturbation. It can be seen from Fig. 4 that such a discontinuity is not required, nor even helpful, for the tearing instability mechanism to operate. In a less extreme case, ψ itself is continuous, and only its slope, ψ', changes appreciably across the tearing layer. The apparent discontinuity in slope is measured by

$$\Delta' \equiv [\psi'(r_s+\epsilon) - \psi'(r_s-\epsilon)] / \psi(r_s) \tag{38}$$

where $\lambda \ll \epsilon \ll a$, and Eq. (36) becomes in this case

$$\psi'' \sim \Delta' \psi(r_s)/\lambda. \tag{39}$$

The nominal ordering, $\Delta' a \sim 1$, implies that $\psi(r) \simeq \psi(r_s)$ throughout the tearing layer;[2] this "constant-ψ approximation" is frequently used, though occasionally disputed, to simplify the solutions of Eq. (32) and (33).

Without making any assumption regarding the magnitude of Δ', we may repeat the manipulations leading to Eq. (37), but now using Eq. (39) instead of Eq. (36). Then Eq. (33) implies

$$\gamma \sim \frac{a^2 \Delta'}{\tau_s \lambda} , \tag{40}$$

while Eq. (32) yields $k_\parallel B_o \Delta' \psi/\lambda \sim 4\pi\rho\gamma^2 \xi_r'' \sim (B_o/v_A)^2 \gamma^2 \xi_r/\lambda^2$. Assuming that the two terms on the left-hand side of Eq. (34) contribute comparably to $E_\parallel$, we obtain

$$\gamma \sim k_\parallel v_A (\Delta' \lambda)^{\frac{1}{2}} \tag{41}$$

We can solve Eqs. (40) and (41) for γ and λ in the two cases of interest:
(i) If $\Delta' \sim a^{-1}$, we find

$$\gamma^5 \sim (k_\parallel' a^2)^2 (\Delta' a)^4 \tau_s^{-3} \tau_A^{-2} , \tag{42}$$

where instability occurs only if $\Delta' > 0$. The width is given by

$$(\lambda/a)^3 = (\Delta' a)(k_\parallel' a^2)^{-2} (\tau_A/\tau_s)^2 \tag{43}$$

A typical solution $\psi(r)$ for this case is depicted in Fig. 5. The growth rate of Eq. (42) is typically smaller than that of Eq. (37); yet it also displays the shear-driven, hybrid features which were noted previously.
(ii) If $\Delta' \sim \lambda^{-1}$, we obviously recover Eq. (37) for γ. The width in this case satisfies

$$(\lambda/a)^3 \sim (k_\parallel' a^2)(\tau_s/\tau_A) \tag{44}$$

Two quite different radial mode structures can produce $\Delta' \sim \lambda^{-1}$, and thus Eqs. (37) and (44). First, $\psi(r_s)$ might be quite small, of order λ/a compared to its maximum value outside the layer. This circumstance is depicted in Fig. 6a; we shall see that it commonly applies to modes with $m=n=1$. Second, it could happen that ψ' is large, proportional to λ^{-1}, near r_s. Then ψ would appear nearly discontinuous, as shown in Fig. 6b. This possibility is rarely considered in the literature, although both analytical[40] and numerical[23] studies have indicated its possible importance.

It should now be clear that analytical solution of the tearing layer equations is not possible without some a priori knowledge of the behavior of ψ outside the tearing layer: Eqs. (21), (32) and (33) must in general be solved simultaneously. In most of the literature, this task is avoided by assuming ψ to be continuous across the layer, as in case (i). It is then reasonable to consider the tearing mode problem solved, once γ has been evaluated in terms of Δ'.

As noted previously, case (ii) is of importance mainly for $m = n = 1$ modes. The distinguishing feature of such modes is apparent from Eqs. (22) and (25), which show that

$$g \sim (nr/R)^2 \ll 1 \text{ , for } m = 1.$$

Since toroidal curvature terms, which Eq. (21) omits, are also of order (r^2/R^2), the use of a cylindrical model for $m = 1$ modes is questionable.[11] But even in a torus, a lowest order approximation for the $m = 1$ internal kink mode presumably would be obtained by neglecting g, so that Eq. (21) is approximated by

$$k_{\parallel} B_o \, (r \, \psi'/k^2)' \simeq [r \, (k_{\parallel} B_o)' / k^2] \, \psi .$$

or

$$[r \, (k_{\parallel} B_o)^2 (\psi/k_{\parallel} B_o)' / k^2]' \cong 0$$

Thus the exterior solution is characterized by[12]

$$\psi/k_{\parallel} B_o \simeq \xi = \text{const.}, \text{ for } r < r_s$$

and $\psi/k_{\parallel} B_0 = 0$ for $r > r_s$, to satisfy the boundary condition at $r = a$ ("fixed-boundary" kink mode). It is clear that this form of the exterior solution corresponds to Fig. 6a: $\psi \approx$ const. $k_{\parallel}\lambda$, for $r \sim r_s$.

The preceding argument has sometimes been taken to imply a strict correspondence between $m = 1$ modes and Eq. (37), on the one hand, and between $m \geq 2$ modes and Eq. (42), on the other. In fact the correspondence is far from clear-cut; in particular, it is contradicted by $m \geq 2$ modes having the discontinuous form depicted in Fig. 6b. Furthermore, $m = 1$ modes have been found whose growth rates resemble Eq. (42).[39]

Once the influence of the form of the exterior solutions is understood, the analytical solution of Eqs. (32) and (33) is relatively straightforward. An especially elegant procedure is presented in Ref. 39. We do not review the detailed solution here.

D. Summary

To recapitulate: tearing instabilities are analyzed by solving a boundary layer problem for the mode-rational flux surface. The kink-like excitation of the exterior region is sufficiently slow to be treated as a neighboring equilibrium. With the neglect of stress anisotropy, this description corresponds to the marginal stability condition of ideal MHD (Newcomb equation). In the interior region, $B_r''(r)$ becomes large, and the description hinges upon coupling B_r'' to the parallel electric field of Eq. (34). Various nominally small effects, of which only resistivity has been considered here, can provide the coupling. The resulting tearing layer equation is generally (equivalent to) a fourth order radial differential equation for B_r. The form of the final dispersion relation depends not only upon the $B_r''-E_{\parallel}$ coupling, but also upon gross features (e.g., continuity across the layer) of the exterior solutions. Similarly, the precise value and sign of the growth rate can depend upon critical details (e.g., Δ') of the exterior solutions.

The coupling between B_r'' and $E_{\parallel}$ becomes particularly delicate at low frequency and small resistivity. When $\omega \sim \omega_*$ as for the observed modes, a host of higher order terms, in both the equation of motion and the (generalized) Ohm's law, become potentially important. This explains why, fourteen years after the lengthy FKR study, the linear theory of tearing instabilities remains of interest. It also explains how the analysis outlined above can yield a remarkably disparate collection of dispersion relations, as shown in the following section.

V. Tearing Mode Dispersion Relations

Finally we display some tearing mode growth rates. What follows is at best a representative sampling, and far from complete. Our purpose is not only to display the wide variety of results which can be obtained, but also to touch upon some topics of recent interest.

Future research on tearing modes may well be concentrated on their non-linear properties. Yet a characteristic feature of the modes - their sensitive dependence on small details - suggests a long-term interest in the linear theory.

A. Classical tearing modes

The FKR tearing mode growth rate, derived from a slab model with $\nu >> \gamma >> \omega_*$, is given by[2]

$$\gamma = \gamma_C \equiv \left\{ \left[\frac{\Delta' a \; \Gamma(\frac{1}{4})}{2\pi \; \Gamma(3/4)} \right]^4 \frac{(k_\parallel' a^2)^2}{\tau_s^3 \; \tau_A^2} \right\}^{1/5} \tag{45}$$

for $\Delta' a \sim 1$, as in case (i) of Sec. IV. Here, Δ' is given by Eq. (38), τ_s is the resistive skin time of Eq. (13) and $\tau_A = a/v_A$ is the Alfven time. Of course, this result is just the exact version of Eq. (42). We recall that instability occurs only when $\Delta' > 0$.

Equation (45) describes a resistive tearing mode: the decoupling of plasma and magnetic field which allows $B_r(r_s) \neq 0$ is provided by resistivity. As noted previously, other mechanisms can be similarly effective. In the low collision frequency limit, the inertial term, $4\pi\omega_p^{-2}\, \partial \underset{\sim}{J}/\partial t$, takes the place of $\eta \underset{\sim}{J}$, in Ohm's law. The resulting growth rate,[3,4]

$$\gamma = (8\gamma_C^5 \nu^{-3})^{\frac{1}{2}} \tag{46}$$

may be obtained directly from Eq. (45), by the substitution $\nu \rightarrow 2\gamma$ (the factor of 2 comes from the Spitzer-Harm[41] resistivity for unit ionic change).

Equation (46) describes the classical collionless tearing mode. A distinct type of collisionless tearing mode, in which electron inertia is similarly responsible for field-fluid decoupling, is considered below.

The classical tearing mode instability criterion, $\Delta' > 0$, strictly pertains only in cylindrical or slab geometry. The same resistive fluid model which yields Eq. (45) has recently been analyzed in toroidal geometry.[16] It is found that Δ' should be replaced by a much more complicated quantity, Δ'_* (our notation), which includes in particular the effects of toroidal curvature. More surprising is the discovery that, under certain circumstances, the difference between Δ' and Δ'_* can be significantly large.

B. Drift-tearing modes

"Drift-tearing" modes have frequencies comparable to the diamagnetic frequency, ω_*. In this case, the ion diamagnetic drift enters the equation of motion, through perturbation of the ion pressure (magnetoviscous contributions are cancelled by the convective inertial term).[6,42] One finds that Eq. (32) should be replaced by

$$4\pi \rho \omega (\omega - \omega_{i*}) \xi'' - k_{\parallel} B_o \psi'' = 0. \qquad (47)$$

where $\omega_{i*} = (cm/eB_o r) p_i'/n_i$, $m = 1, 2, \ldots$. Similarly, perturbation of p_e yields important $\omega_{e*} = -(cm/e B_o r) p_e'/n_e$ terms in the Ohm's law of Eq. (33). The simplest version of the resulting dispersion relation may be written as[4,13]

$$\omega (\omega - \omega_{i*}) (\omega - \omega_{e*})^3 = i \gamma_c^5 . \qquad (48)$$

Equation (47) pertains to an isothermal plasma, and, strictly speaking, requires $\nu >> \omega$. The point is that the static, or "dc" resistivity in Ohm's law cannot describe perturbations with $\omega \sim \nu$. When an "ac" Ohm's law is used, and when electron temperature gradient effects are retained; one obtains the dispersion relation[14]

$$\omega\,(\omega - \omega_{i*})\,(\omega - \omega_{e*} - \alpha_2\,\omega_{T*})^3\,\alpha_1^{\,3} = i\,\gamma_c^{\,5} \tag{49}$$

where the α's are complex-valued functions of ω and ν, and $\omega_{T*} \equiv -(cm/eB_o r)\,T_e'$. A variational solution to the guiding-center Fokker-Planck equation yields

$$\alpha_1 = 0.98\,(1-0.54\,i\omega\tau)\,(1-2.97\,i\omega\tau - 1.04\,\omega^2\tau^2)^{-1}$$

$$\alpha_2 = 0.80\,(1-0.54\,i\omega\tau)^{-1}$$

where $\tau \sim \nu^{-1}$ is the electron collision time of Braginskii,[43] for unit ionic charge:

$$\tau = \frac{3}{16\pi^{\frac{1}{2}}}\;\frac{m_e^2\,v_{Te}^{\,3}}{e^4 Z^2\,n_e\,\ell n\,\Lambda}$$

Equation (49) can be shown to include Eqs. (45)-(48) as limiting cases. It also predicts a new, "thermo-electric" tearing instability, which is driven, not by magnetic shear, but by the electron temperature gradient:[14,15]

$$\gamma \simeq 0.43\,(\omega_{e*} + 0.8\,\omega_{T*})\,\omega_{T*}\tau\,, \quad \text{for } \omega_*\tau < 1\,. \tag{50}$$

A similar T_e'-driven mode occurs at a low collision frequency in toroidal geometry, where magnetic trapping of electrons is important.[17]

C. Current channel modes

In the derivation of Eq. (48) or (49), terms of order $k_\parallel v_{Te}$ $[v_{Te} \equiv (2T_c/m_e)^{\frac{1}{2}}]$ are presumed small compared to ω or ν. Hence at small collision frequency, $\omega \gg \nu$, the relation between drift-tearing modes and electrostatic drift waves ($\omega < k_\parallel v_{Te}$) is obscured. But the essential effects of finite $k_\parallel v_{Te}/\omega$ can be understood from a simple, isothermal electron fluid model. By retaining both the electron pressure gradient term and the inertial term in Ohm's law, we find that Eq. (33) should be replaced by[6,13]

$$\frac{a^2}{\tau_s}\psi'' + i\left[\frac{\nu}{\nu - i\omega + ik_\parallel^{\,2} v_{Te}^{\,2}/\omega}\right](\omega - \omega_{e*})\,(\psi - k_\parallel B_o\,\xi) = 0 \qquad (51)$$

Here some numerical factors are omitted for simplicity, and $\nu = (\omega_p^{\,2} a^2/c^2)\tau_s^{-1}$. The familiar differential equation for drift waves is now obtained by using Eq. (47) to eliminate ψ'' from Eq. (51), and then considering the limits $\omega \ll k_\parallel v_{Te}$ (thus excluding the immediate vicinity of r_s), $\omega \gg \nu$ and $k_\parallel B_o \xi \gg \psi$ (thus excluding coupling to an exterior, kinked region). Note that in this case ξ cannot be interpreted as the plasma displacement; it strictly measures the electrostatic $\underset{\sim}{E} \times \underset{\sim}{B}$ drift.

More interestingly, Eq. (51) shows that for sufficiently small ω and ν, two nested singular regions can appear. Recall that the usual tearing layer is defined by the presence of an appreciable parallel electric field ($\psi - k_\parallel B_o \xi \neq 0$). The width, λ, of this region may be estimated, for a given ω, from Eq. (41). But the square-bracketted factor in Eq. (51) can make ψ'' become small, as one proceeds away from r_s, before the tearing layer boundary is reached. The resulting inner singular region evidently corresponds to a current channel (since $\psi'' \propto J_\parallel$);[15] denoting the current channel half-width by λ_c, we have, from Eq. (51),

$$k_\parallel^{\,2}\, v_{Te}^{\,2} \sim (k_\parallel{}')^2\, v_{Te}^{\,2}\, \lambda_c^{\,2} \sim |\omega(\nu - i\omega)| \qquad (52)$$

Our discussion in Sec. IV, and also Eqs. (45)-(49), assume $\lambda_c \geq \lambda$, so that the current channel is not distinguishable. In the opposite case,

$$\lambda_c \ll \lambda \qquad (53)$$

a somewhat different analysis pertains: in general, one must solve a double boundary layer problem, matching solutions for the current channel to those for the ordinary tearing layer, before matching the tearing layer solution to kink modes in exterior regions.[17] We do not review solution to the double boundary layer problem here, beyond remarking that the current channel analysis itself is straightforward, because the ion motion, as in Eq. (32), is relatively unimportant.

In the simplest situation, the current channel solution can be matched directly to the exterior kink, without regard to the outer tearing layer. Notice that in this case, the quantity Δ' of Eq. (38) must be defined with respect to λ_c, rather than λ. When $\nu << \omega$, so that Eq. (52) becomes

$$\lambda_c \sim (\omega / k_\parallel' \, v_{Te}),$$

this procedure yields a collisionless tearing mode, alluded to previously, which is quite different from that of Eq. (46). The growth rate of this "current-channel collisionless tearing mode" is given by[5]

$$\gamma = \gamma_{cc} \equiv \frac{\Delta' k_\parallel'}{2\pi^{\frac{1}{2}}} \left(\frac{c}{\omega_{pe}} \right)^2 v_{Te} \tag{54}$$

For tokamak parameters, Eq. (54) indeed yields $\lambda_c < \lambda$; Eq. (46) is not similarly consistent.[15]

More recently, the same matching procedure has been applied to a collision dominated current channel.[15] This will occur if $\nu > \omega$ but

$$\lambda_c \sim (k_\parallel' \, v_{Te})^{-1} (\omega \nu)^{\frac{1}{2}} << \lambda ,$$

as can be seen from Eq. (52). The resulting "semi-collisional" tearing mode has the growth rate

$$\gamma = [\, 3\pi^{\frac{1}{4}} / 4\Gamma\,(11/4)\,]\, \gamma_{cc}^{2/3} \, \nu^{\frac{1}{3}} \tag{55}$$

for $\gamma \gg \omega_*$, and a more complicated expression for the case $\gamma \ll \omega_*$.

A more transparent version of Eq. (53), the criterion for formation of a current channel, is difficult to construct. The problem is that λ_c/λ depends not only upon ω, which is usually not known in advance, but also, rather delicately, upon the shear, the plasma β, and so on. Thus the possible appearance of a current channel is best examined in individual cases.

D. m = 1 modes

The classical m = 1 tearing mode corresponds to case (ii) of Sec. IV B; that is, $\Delta' \lambda \sim 1$, because $\psi(r_s)$ is small. In this case our previous growth rate estimate, Eq. (37), is fortuitously exact;

$$\gamma = \gamma_{cl} \equiv (k_{\parallel}' a^2)^2 / (\tau_s \tau_A^2) \tag{56}$$

More recently the relation between tearing modes and ideal MHD kink modes, at m = 1, has been examined.[39] It is found that Eq. (37) pertains when the kink is marginally stable, and that when the kink is actually stable, $\gamma_{MHD} < 0$, a new type of tearing instability can appear. Disregarding diamagnetic effects, one obtains for this mode the growth rate

$$\gamma = \left\{ \left[\frac{\Gamma(5/4)\, r_s}{\pi \Gamma(-\frac{1}{4})\, a G} \right]^4 \frac{(k_{\parallel}' a^2)^6}{\tau_A^2 \tau_s^3} \right\}^{1/5} \tag{57}$$

Here,

$$G \equiv \int_0^{r_s} r\, g(r)\, dr,$$

where g is defined in Eq. (22). A noteworthy feature of Eq. (57) is its close resemblance to Eq. (45), describing a mode with $\Delta' a \sim 1$. Thus, as noted in Sec. IV C, the classical growth rate scaling, $\tau_s^{-3/5} \tau_A^{-2/5}$, is not restricted to modes with $m \geq 2$.

Another modification of the classical m=1 tearing mode is obtained by using the ac Ohm's law which leads to Eq. (49), but allowing $\Delta' \lambda \sim 1$ (which can also pertain for $m > 1$). This yields the dispersion relation[40,44]

$$i\,\alpha_1\ \omega(\omega - \omega_{i*})\,(\omega - \omega_{e*} - \alpha_2\,\omega_{T*}) = \gamma_{cl}^{\ 3}, \qquad (58)$$

where γ_{cl} is defined in Eq. (56). Equation (58) allows extension of the classical result to regimes in which $\omega \sim \omega_* \sim \nu$.

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Figure Captions

Figure 1. Perturbed flux surfaces are indicated by solid curves. The dashed curves are unperturbed flux surfaces. In (1a), the safety factor is nonresonant and the deformation, exaggerated for clarity, is relatively harmless. In the resonant case of (1b), the entire shaded region is filled by a single field line, and the flux surfaces have been locally destroyed.

Figure 2. Magnetic field lines of the field component $\underset{\sim}{B}_n$, near the radius r_*, at which the sheared equilibrium field $\underset{\sim}{B}_o$ is perpendicular to the plane of the figure. The field strength, $|\underset{\sim}{B}_n|$, is indicated by the thickness of the lines.

Figure 3. A flux tube, distorted by a radial displacement, $\xi_r \propto \exp(i\,\underset{\sim}{k}\cdot\underset{\sim}{x})$, of the plasma "frozen" inside it. Dashed lines show the unperturbed flux tube. It is evident that the radial field perturbation would vanish if the displacement were constant along the direction of $\underset{\sim}{B}_o$. In fact the indicated relation between B_r and ξ_r can be inferred from the figure.

Figure 4. Perturbed version of Fig. 2. The geometry of the distortion is such that annihilation and reconnection takes place even for an infinitesimal perturbation.

Figure 5. Radial structure of the radial field perturbation $B_r = i\,\psi$, associated with a kink-tearing mode having $\Delta' \sim a^{-1}$. Dashed lines indicate the approximate boundaries of the tearing layer.

Figure 6. Possible forms of $\psi(r)$, near and within the tearing layer, which yield $\Delta' \sim \lambda^{-1}$. In case (6a), $\psi(r_s)$ is quite small; in case (6b), ψ changes sharply within the tearing layer.

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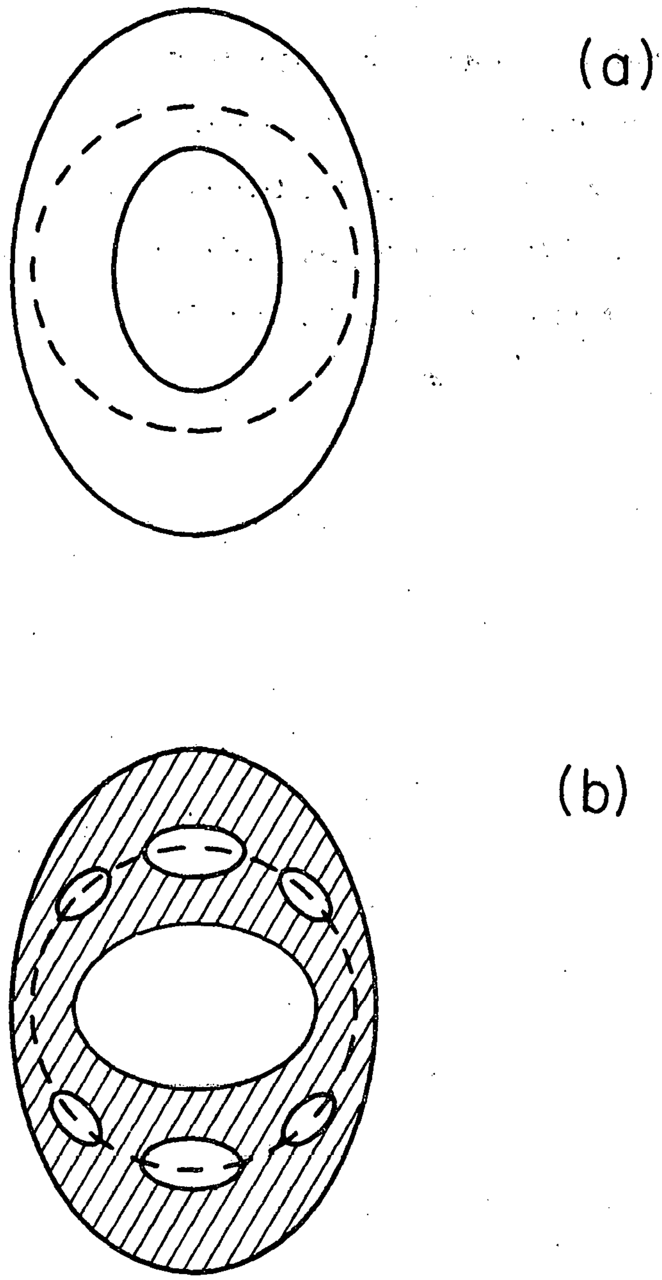

FIGURE 1

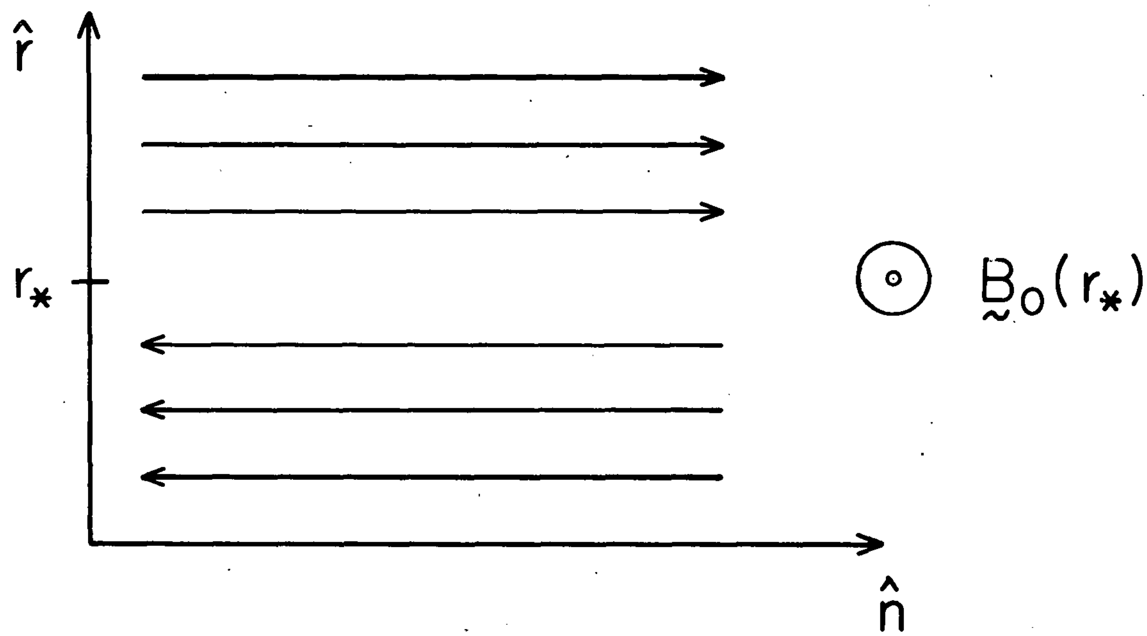

FIGURE 2

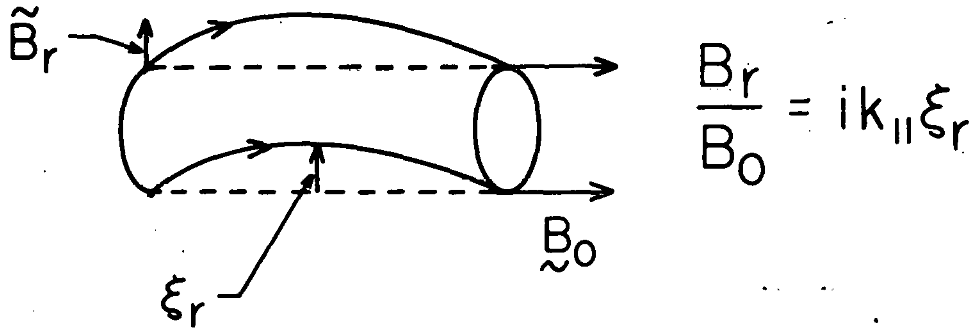

FIGURE 3

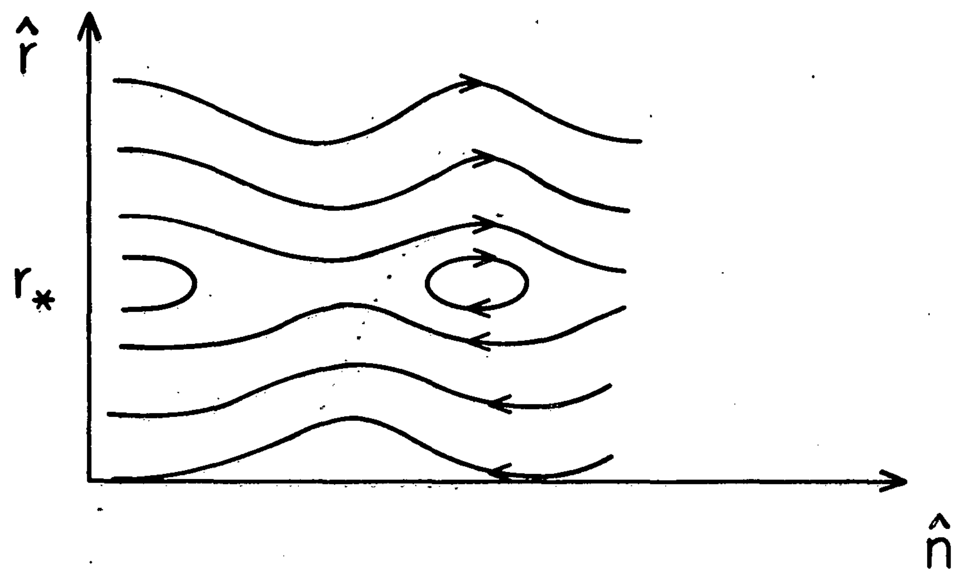

FIGURE 4

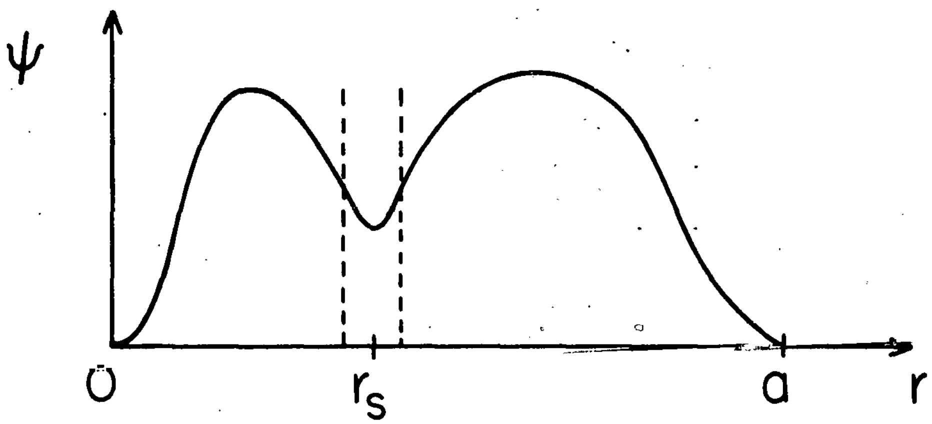

FIGURE 5

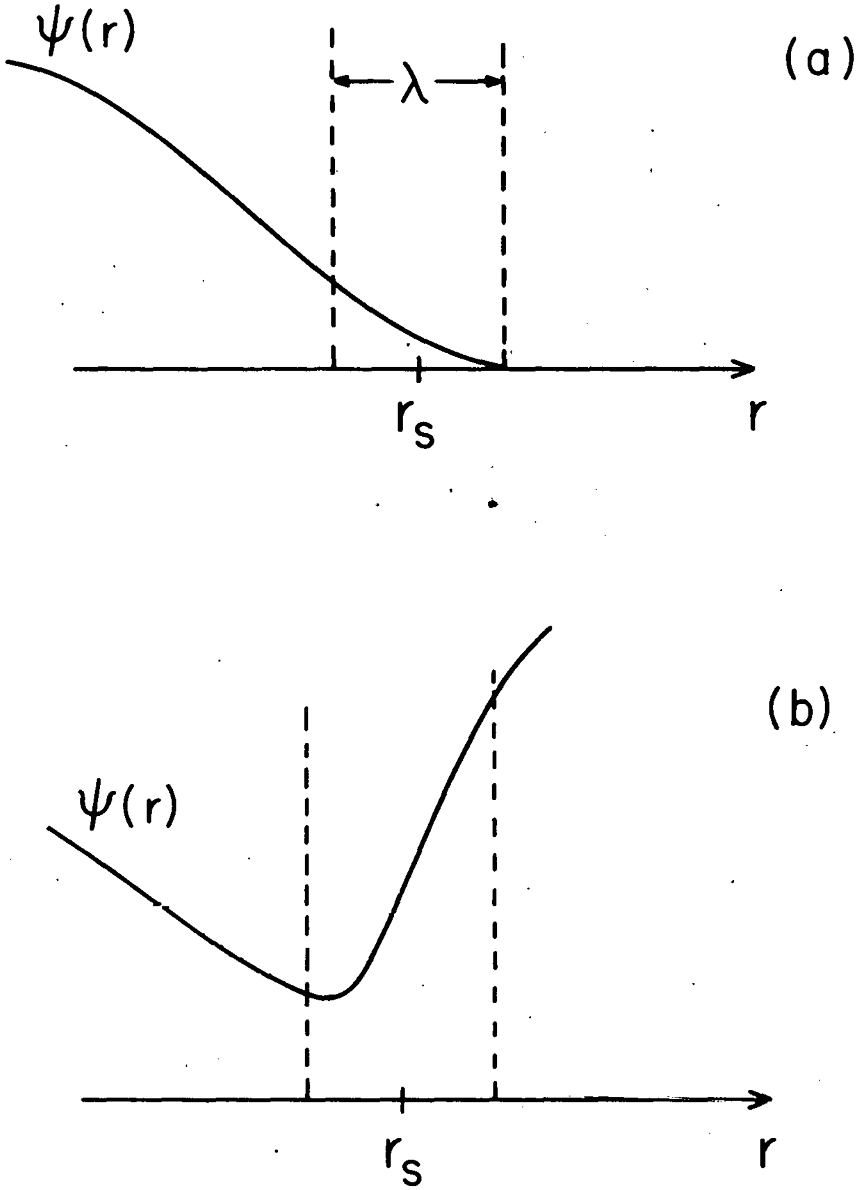

FIGURE 6

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