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TITLE THE RENORMALIZATION GROUP AND LATTICE QCD

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SUBMITTED TO TASI 1989 REPORT IN ELEMENTARY PARTICLE PHYSICS
BOULDER, CO JUNE 12 TO 30, 1989

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THE RENORMALIZATION GROUP AND LATTICE QCD

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Lectures given at the 1989 Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder,Colorado, June 12 to June 30, (1989).

11/89

CONTENTS

1. Introduction:

The goal in the study of statistical models is to calculate the behavior of thermodynamic quantities like magnetization, susceptibility, excitation spectrum, *etc.*, in each of the regions of phase space. For that the first thing that we want to do is to map out the phase diagram. Next, we like to know the symmetries of the model (and whether they are spontaneously broken) in a given phase. These play a fundamental role in our understanding of the physics. Finally, we would like to derive expressions for how thermodynamic quantities depend on the couplings (like temperature, magnetic field, etc.).

The modern approach to the study of systems close to critical points starts with asking the following questions: (1) What are the relevant degrees of freedom? This allows one to represent complicated systems by simple models which exhibit the same long distance behavior. (2) What features of the system are important? We will find that these include the dimensionality, symmetries of the Hamiltonian/Action, and the number of components in the "spin" variables. These questions, as we will see, lead to an understanding of scaling (thermodynamic quantities are not the most general functions of all independent couplings) and universality (different physical systems show the same behavior when expressed in some scaled units).

A unified understanding of critical phenomena arose with the development of the renormalization group by K.G. Wilson. The power of the renormalization group lies in its ability to explain why (a) near a second order transition all correlations functions are dominated by a single length scale, (b)the behavior of thermodynamic quantities can be written as scaling functions, (c)singularities that occur at $T = T_c$ are power-law. Moreover, it allows us to calculate these exponents (called critical exponents) which govern the scaling behavior. MCRG is a numerical technique for implementing these ideas $--$ locating T_c and calculating the critical exponents.

The numerical solution of any given model begins by simulating the partition function $Z = \sum_{states} e^{-\beta H}$ and then calculating expectation values (correlation functions) as averages over the generated configurations. Since the possible states of a system are the same for all possible values of the couplings like temperature, phase transitions occur as an interplay between energy (Boltzmann factor $e^{-\beta H}$) and entropy S (degeneracy of a given energy state) in the free energy $F = U - TS$. This is the essence of Peierls's energy-entropy argument [1] . The calculation tool one uses (series expansions, finite size scaling, MCRG, ϵ expansion, ...) depends on the region of phase diagram one is interested in. The power of MCRG is best realized only near second order transitions. In general, the study of thermodynamic systems (systems with $N \to \infty$ degrees of freedom) can be divided into three broad categories:

(1) The high temperature limit: Thermal fluctuations randomize the system as it does

not cost much energy (compared to kT) to create excitations in the system. These excitations have a short correlation length. In terms of Peierls's energy-entropy arguments, the free energy $F = U - TS$ is dominated by the entropy term. For example the spins in a ferromagnet at high temperature are randomly aligned, there is no net magnetization, and correlation functions decay exponentially with a short correlation length. Thus, the physics in this limit can be understood by developing high temperature (weak coupling) expansions for the thermodynamic quantities. These expansions are analytic in the coupling constants and their domain of validity increases with the order to which one can calculate. Combining these high order series with Padé approximants one can estimate the critical properties of the system. Such an approach is called *series expansion*.

(2) The low temperature limit: The system exists close to its ground state as it costs a lot in energy to create excitations in the system, *i.e.*, thermal fluctuations are suppressed. Thus for $T > 0$, the behavior above the ground state can be developed as an expansion in terms of the number of excitations. These excitations have a short correlation length but often exist in a background of long range order. In the example of a ferromagnet, the ground state is one with all spins aligned, and excitations are $1, 2, \ldots$ spin-flips. In this region of phase space, the free energy, $F = U - TS$, is dominated by the energy term. The analytic tools developed to study this region are called strong coupling expansions. Again these are series expansions with a finite radius of reliability.

(3) The vicinity of phase transitions: Phase transitions, as the name implies, are the boundaries between different phases of a substance *i.e.* solid-liquid, liquid-gas, *etc.* phase boundary. The free energy in the two phases is equal along the phase boundary. If one can solve for U and S, then the critical temperature is given by $U - T_c S = 0$. At T_c, thermodynamic quantities develop singularities and standard analytic tools are not very useful for a detailed quantitative analysis. The phase transition is said to be n^{th} order if the n^{th} derivative of the free energy is singular.

Thermodynamic quantities are discontinuous across first order transitions. The generic shape of the potential for the model near a first order transitions has the profile of a Mexican hat. For example, in the simple case of an Ising ferromagnet for $T < T_c$ and $h = 0$, there are two degenerate states which are related by an overall spin flip. For $h > 0$ the degeneracy is broken and there is a unique ground state, *i.e.*, turning on h favors the ground state with the net magnetization aligned along h. So, if the system is in the ground state with $h > 0$, then as h is decreased past $h = 0$ the system will eventually tunnel to its flipped state which is now the true ground state. The magnetization M changes discontinuously at $h = 0$, and the system exhibits spontaneous magnetization ($M = M_+ - M_-$). As $T \to T_c$ from below and

along the transition line, the height of the potential barrier between the two states changes and goes to zero at T_c. At T_c the two degenerate states have coalesced and the potential has a unique minimum with zero curvature.

The simplest field theoretic model of such first order behavior is the Landau Ginsburg model with $m^2 < 0$ as shown in fig. 1. The vanishing of the second derivative of the potential implies a massless excitation ($m^2 = 0$ in the model). Massless states have infinite correlation length ($\xi \equiv 1/m$) and points with $\xi = \infty$ are called critical points. Therefore, if a line of first order transitions ends at a point inside the phase diagram, then at that end-point the theory has a second order transition. In general, multi-critical points can exist if at T_c some higher derivatives of the effective potential also vanish. Such points are not isolated critical points *i.e.* they are intersection points of lines of critical points.

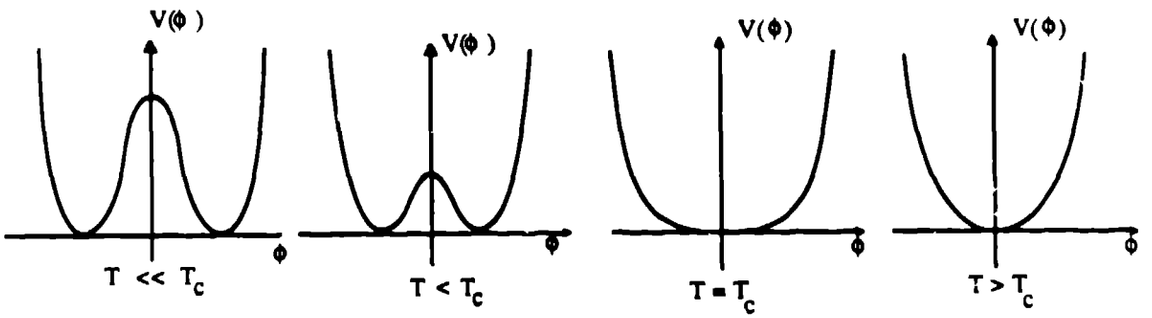

Fig. 1: The behavior of the Landau-Ginsburg effective potential for $m^2 < 0, = 0,$ *and* > 0 *or equivalently* T $\{<, =, >\}$ T_c

A large part of these lectures is devoted to developing the ideas underlying the renormalization group (RG). A brief list of references which provide excellent background and teach far more than will be covered in these lectures is [2] [3] [4] [5] [6] . I will mainly use spin models as examples. This is not a digression since there is an intimate connection between Statistical Mechanics and Quantum Mechanics or Quantum Field Theory. This connection has been reviewed in [7] and I recommend you follow up on that. The plan of these lectures is as follows: In the first two lectures I will introduce the renormalization group and the relevant concepts of statistical mechanics. The discussion will assume that the model has a simple isolated critical point. Also I will assume that the model has no hyperscaling violations. In lecture three I will discuss Monte Carlo Renormalization Group (MCRG) methods for both spin systems and 4-dimensional gauge theories. Finally in lecture 4 I will show why asymptotic freedom implies that QCD is trivial from the point of view of the renormalization group; review the status of the calculations of the non-perturbative β-function

for SU(3) gauge theory, and talk about scaling of physical observables. I will end with an outline of possible avenues of research.

2. Scaling of Thermodynamic Quantities and Critical Exponents:

Diverse physical systems show very similar behavior near a second order transition. For example, the manner in which spontaneous magnetization M in a ferromagnet vanishes as $T \to T_c$ from below is similar to how the difference in density between the liquid and vapor phase vanishes as $T \to T_c$ along the co-existence curve. In both cases, the order parameter goes to zero as a non-analytic power of the reduced temperature $t = \frac{T-T_c}{T_c}$. Other quantities like the correlation length ξ diverge. The scaling behavior of various thermodynamic quantities near T_c for a ferromagnet (the simplest and best understood example), is defined to be [8]

$$\xi \sim t^{-\nu} \tag{2.1}$$

$$C_v \sim t^{-\alpha} \tag{2.2}$$

$$\chi \sim t^{-\gamma} \tag{2.3}$$

$$M \sim h^{\frac{1}{\delta}}|_{t=0} \tag{2.4}$$

$$M \sim t^{\beta} \qquad for\ h = 0,\ T \to (T_c)_- \tag{2.5}$$

$$G_c(R) \sim \frac{1}{R^{d-2+\eta}} \quad for\ R \to \infty \tag{2.6}$$

where C_v is the specific heat at constant volume, χ is the susceptibility and G_c is the 2-point connected correlation function (the propagator familiar from field theory). Away from the critical point, G_c decays exponentially [9]

$$G_c(R) \sim \frac{e^{-R/\xi}}{R^{(d-1)/2}\,\xi^{(d-3)/2}} \quad for \quad R/\xi \to \infty \tag{2.7}$$

with a correlation length $\xi \equiv 1/m$, where m is the mass-gap. Only for $T = T_c$ does G_c develop the power-law singularity. These equations define the six exponents $\nu, \alpha, \gamma, \delta, \beta, \eta$. It will be shown that not all 6 are independent. In the simplest models (*i.e.* those that do not violate hyperscaling) there exist four scaling relations which relate these exponents. These relations can be derived from the definitions of exponents and from the scaling behavior of the free energy f_{sing}, as shown in section 3.

For $h = 0$, the Ising model has a discrete global symmetry $s \to -s$. Interactions can therefore be classified as even or odd depending on their behavior under this symmetry. For example the energy $< s_i \cdot s_j >$ is even while the magnetization $< s_i >$ is odd. Similarly the exponents are called even or odd depending on the quantity whose scaling behavior they describe. As you can guess, the two independent exponents arise

as one even and one odd. The even one is ν, the correlation length exponent. Having only one independent exponent accounts for the statement that the theory has only one length scale. From the odd sector we will determine η.

It is an empirical fact that the exponents do not take on values that are specific to each and every substance. Substances with the same value for the exponents are said to belong to the same *universality class*. On the other hand variables like T_c, which are specific to each and every substance, are non-universal. To classify substances by their universality class we have to first identify the appropriate thermodynamic variables, and then measure the critical exponents at $T = T_c$. The goal of these lectures is to present MCRG methods and to make a case that it is a reliable numerical technique for doing these calculations.

3. Scaling Relations:

Let us start by considering a theory whose free energy is a function of 3 scaling fields t, h, and u. *Scaling fields* are linear combination of the coupling constants of the theory such that under a renormalization group transformation (RGT) by a scale factor b they get multiplicatively renormalized, *i.e.*

$$t \to t' = b^{\lambda_t} t \tag{3.1a}$$

$$h \to h' = b^{\lambda_h} t \tag{3.1b}$$

$$u \to u' = b^{\lambda_u} t \tag{3.1c}$$

There is implicit assumption in this definition of the scaling fields that a linear expansion about the fixed point is sufficient. In general scaling fields are non-linear functions of the couplings. It will be shown in section 7 how such a behavior, $k \to b^{\lambda_k} k$, arises, in a linear approximation, as a consequence of flows under a RGT. The scaling behavior of the free energy under a scale change b^l is

$$f(t,h,u) \equiv \frac{1}{N} \log Z(N) = \frac{1}{b^{dl}} f'(t',\ h',u') \tag{3.2}$$

for RGT that preserve the partition function *i.e.* $Z'(N') = Z(N)$. Here N is the volume of the system and d the dimension. In these lectures I have ommitted all discussion of logarithmic corrections that can and do arise in certain models. Rewriting f' in terms of t, h, u, we get

$$f(t,h,u) = \frac{1}{b^{dl}} f(b^{\lambda_t l} t,\ b^{\lambda_h l} h, b^{\lambda_u l} u) \quad .$$

Now lets choose $b^{\lambda_t l} = \frac{\tau}{t}$ where τ is some reference temperature. Note that this is a standard trick that is used in all renormalization group analysis. Then

$$\begin{aligned} f(t,h,u) &= (\frac{t}{\tau})^{\frac{d}{\lambda_t}} \quad f(\tau,\ (\frac{\tau}{t})^{\frac{\lambda_h}{\lambda_t}} h,\ (\frac{\tau}{t})^{\frac{\lambda_u}{\lambda_t}} u) \\ &= (\frac{t}{\tau})^{2-\alpha} \quad f(\tau,\ (\frac{\tau}{t})^{\Delta} h,\ (\frac{\tau}{t})^{\phi} u) \end{aligned} \tag{3.3}$$

where the last equality is written to make the definition consistent with $C_v \sim t^{-\alpha}$ for $h = u = 0$. This requires that $(2-\alpha) = \frac{d}{\lambda_t} = d\nu$, which can be shown from the scaling of the correlation length:

$$\begin{aligned} \xi(t,0,0) &= b\,\xi(t',0,0) \\ &= (\frac{t}{\tau})^{\frac{-1}{\lambda_t}} \quad \xi(\tau,0,0) \end{aligned}$$

i.e. $1/\lambda_t = \nu$. From the definitions of M and χ we can write down further relations

$$M = \frac{\partial f}{\partial h}|_{h=0} \qquad \Rightarrow 2-\alpha-\Delta = \beta \tag{3.4}$$

$$\chi = \frac{\partial^2 f}{\partial h^2}|_{h=0} \qquad \Rightarrow 2-\alpha-2\Delta = -\gamma \tag{3.5}$$

Note that along a critical isotherm $M \sim t^{\beta} f(\frac{h}{t^{\Delta}})$. In the same limit it was postulated that $M \sim h^{\frac{1}{\delta}}$. To cancel the t dependence we require that $f(x) \sim x^{\frac{\beta}{\Delta}}$. Then $M \sim h^{\frac{\beta}{\Delta}}$ and by consistency

$$\Delta = \beta\delta \tag{3.6}$$

The desired scaling relations can now be derived from eqns (3.4), (3.5), and (3.6)

$$\begin{aligned} \alpha+\beta(\delta+1) &= 2 \\ \alpha+2\beta+\gamma &= 2 \\ \Delta = \beta\delta &= \beta+\gamma \end{aligned} \tag{3.7}$$

In fact Rushbrook and Griffiths have derived rigorous inequalities

$$\begin{aligned} \alpha+2\beta+\gamma &\geq 2 \quad (Rushbrook) \\ \alpha+\beta(\delta+1) &\geq 2 \quad (Griffiths) \end{aligned} \tag{3.8}$$

which reduce to eqn. (3.7) when one assumes scaling.

Further, the zero field susceptibility is the volume integral of the correlation function

$$\chi = \int d^d r < s(r)s(0) > \quad \sim \int d^d r \, \frac{1}{r^{d-2+\eta}} \tag{3.9}$$

from which we derive that

$$\chi = const.\ \xi^{2-\eta} \qquad \Rightarrow \gamma = \nu(2-\eta) \tag{3.10}$$

With these three scaling relations, if we regard ν, α, and η as independent exponents, then γ, δ, and β are determined in terms of them. In deriving these relations we

considered only two scaling fields by setting $u = 0$ always. This assumption is called hyperscaling, *i.e.* the theory has only one relevant length scale. Under this assumption, we expect one more relation. This can be deduced by looking at the singular part of the free energy

$$f_{sing} \sim t^{2-\alpha}$$

and also

$$f_{sing} \sim \frac{1}{L^d} \to \frac{1}{\xi^d} .$$

Thus

$$2 - \alpha = \nu d \tag{3.11}$$

This is called the hyperscaling scaling relation and is the first one to involve the dimensionality d of the system. Equivalently, the relation (3.11) using (3.10) and (3.7), is often cast as

$$2 - \eta = d\left(\frac{(\delta - 1)}{(\delta + 1)}\right) . \tag{3.12}$$

A possible source of violation of these scaling relations is called hyperscaling violation [4]. An understanding of hyperscaling violations in specific models is still very much a research topic. Even in the 3-d Ising model this issue has not been settled [10] . The reason is that if such violations are present, they are smaller than the errors in current simulations. Therefore, for the purpose of these introductory lectures I will proceed by assuming hyperscaling holds.

Before moving on to a discussion of critical phenomena in terms of the renormalization group it is instructive to review the known values of the exponents for an Ising ferromagnet. These are listed in table 1 for the Ising model in d = 1, 2, 3, and 4 dimensions. The results for the $d = 3$ Ising model are from [11] . Based on these and other such results, the accumulated wisdom about exponents is that they depend on

1) Number of spatial dimensions: this is evident even from the Ising model.
2) The number of components of the "spin" variables: For example if, instead of the Ising model, we consider $O(N)$ models (spins lying on a unit sphere in N dimensions) then the behavior depends on N.
3) Symmetries of the model.

Now we turn to the renormalization group to see how it incorporates all these features.

4. Block Spin Ideas of Kadanoff:

The first intuitive and profound steps to study critical phenomena were taken by Kadanoff. He promoted the idea that singularities in thermodynamic quantities arise due to a diverging correlation length. The basic points of his analysis are:

exponent	d=1	d=2	d=3	d=4 $\Leftrightarrow$ classical
ν		1	0.629(4)	1/2
α		log		d-4
β		1/8		1/2
η	1	1/4	0.31(5)	0
γ		7/4		1
δ		15		3
K_c	∞	$0.5\log(1+\sqrt{2})$	0.221654(6)	

Table 1: The values of the critical exponents for the Ising model for d=1, 2, 3, and 4.

1) Near the critical point the correlation length is large and spins (field variables) show co-operative behavior over macroscopic distances.
2) Spins in a local region can be replaced by an effective spin. This averaging (called blocking) defines the renormalization group transformation (RGT).
3) The interaction between these effective spins is of the same kind as between the original spins, only the strength is different. If the temperature t is the only coupling in the starting Hamiltonian, then the effective Hamiltonian is also given by one coupling, t'. In particular, $t' \equiv R(t)$ is an analytic function of t.
4) Under blocking, the correlation length changes by the scale factor of the transformation, i.e. $\xi' = \xi/b$ where b is the linear size of the region over which an average spin is defined.
5) Singularities arise due to the repetition of the blocking step an infinite number of times. If we start with a finite ξ at some finite t, then following the blocking transformations backwards gives $\xi = \infty$. The exceptions to this growth in ξ are: (a) if $t' = at + c$ with $c \neq 0$, then the critical point occurs for $T = 0$; (b) the presence of a first order transition stops the growth of the correlation length.
6) Given the analytic function $R(t)$ one can derive the critical exponent ν.

These ideas, in their fully developed form, gave rise to the modern theory of critical phenomena. The shortcoming of Kadanoff's analysis were that he assumed that $R(t)$ was expressible as the renormalization of a single coupling i.e. $t \to t'$; second, he did not specify how to calculate $R(t)$. These hurdles were surmounted by Wilson in his seminal work uniting the techniques of field theory and statistical mechanics.

I think that the most instructive way to introduce the terminology and concepts of the renormalization group is to work through an exactly solvable model. The simplest example is the 1-d Ising chain with only nearest-neighbor couplings.

5. Exact RG Solution of 1-d NN Ising Model:

The NN 1-d Ising model is defined by the Hamiltonian

$$H = -\mathcal{J}\sum_{<ij>} S_i S_j - h\sum_i S_i - C \tag{5.1}$$

where $\mathcal{J} \equiv \frac{J}{kT}$ is the exchange interaction ($\mathcal{J} > 0$ ferromagnetic, $\mathcal{J} < 0$ antiferromagnetic), h is the magnetic field (for $\mathcal{J} = 0$ the h term makes the system paramagnetic) and C is a constant. The physics of the model is governed by the couplings $[\mathcal{J}, h]$, and the exact solution of the model is known [12] . The closed form expression for the free energy $f(\mathcal{J}, h)$ is

$$f(T, h) = -\ln[\cosh h + \sqrt{\sinh^2 h + x}] \tag{5.2}$$

where

$$x = e^{-4\mathcal{J}} \tag{5.3}$$

is the Boltzmann factor for a spin flip with a maximum change in energy and turns out to be a more natural temperature variable. From the free energy we calculate the magnetization

$$M(T, h) \equiv \frac{\partial f}{\partial h} = -\frac{\sinh h}{\sinh^2 h + x} \tag{5.4}$$

and find that $M \to 0$ as $h \to 0$ for all T *i.e.* the model has no spontaneous magnetization for any value of T. The susceptibility

$$\chi(T, h) \equiv \frac{\partial^2 f}{\partial h^2} = -\frac{x\cosh h}{(\sinh^2 h + x)^{3/2}} \tag{5.5}$$

has an exponential singularity in the zero field limit as $T \to 0$

$$\chi \to \frac{1}{\sqrt{x}} = e^{2\mathcal{J}} \quad .$$

On the other hand, for $h = 0$ the free energy reduces to

$$f(T) = \ln(1 + \sqrt{x})$$

and

$$C_v(T) = \frac{\mathcal{J}^2}{\cosh^2 \mathcal{J}} \tag{5.6}$$

goes to zero as $T \to 0$. This anomalous behavior warns us that the standard definitions for the exponents do not apply for this model at its critical point $T_c = 0$.

Let us now see how the renormalization group exposes this critical behavior without using the exact solution for the free energy. The renormalization group approach is to integrate over only a fraction of the infinite number of degrees of freedom at a time and to examine the effect of this small step on the surviving degrees of freedom. For the 1-d Ising model defined by $H(\beta, h, C)$, we can integrate out every second spin and the resulting theory is described by a new $H'(\beta', h', c')$. Under this transformation, the physics of the model can be extracted from the relations $\beta' \equiv \beta'(\beta, h, C)$ *etc.*. Lets start with the partition function for the model

$$\begin{aligned} Z &= \frac{1}{2^N} Tr\, e^{-H} \\ &= \frac{1}{2^N} \sum_{s_1 \pm 1} \cdots \sum_{s_n \pm 1} e^{\beta \sum_{<ij>} s_i s_j + h \sum s_i + C} \\ &= \frac{1}{2^N} \sum_{s_1 \pm 1} \cdots \sum_{s_n \pm 1} \prod_i e^{\beta s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1}) + C} \end{aligned} \tag{5.7}$$

The last form makes it easy to see how to integrate out every other spin (perform the sum over it). To do this consider the term involving the three spins s_1, s_2, s_3

$$\sum_{s_2} e^{\beta s_1 s_2 + \frac{h}{2}(s_1 + s_2) + C}\, e^{\beta s_2 s_3 + \frac{h}{2}(s_1 + s_3) + C}$$

then the two terms that contribute to the sum over s_2 are

$$= e^{\beta(s_1 + s_3) + h(\frac{s_1 + s_3}{2} + 1) + 2C} + e^{\beta(-s_1 - s_3) + h(\frac{s_1 + s_3}{2} - 1) + 2C} \tag{5.8}$$

which seems to bear no relation to the original Hamiltonian. However, with hindsight, we can write the effective interaction between s_1 and s_3 to be

$$e^{H'(s_1, s_3)} = e^{\beta'(s_1 s_3) + h'(\frac{s_1 + s_3}{2}) + C'} \quad . \tag{5.9}$$

We now solve for β', h', and C' in terms of $[\beta, h, C]$ to get a closed form solution of the RG transformation. Since s_1 and s_3 can take on values ± 1, using eqns (5.8) and (5.9) we have 3 independent conditions to determine $[\beta',\ h',\ C']$

$$++ \qquad e^{2\beta + 2h + 2C} + e^{-2\beta + 2C} = e^{\beta' + h' + C'} \tag{5.10a}$$

$$+- \qquad e^{h + 2C} + e^{-h + 2C} = e^{-\beta' + C'} \tag{5.10b}$$

$$-+$$

$$-- \qquad e^{-2\beta + 2C} + e^{2\beta - 2h + 2C} = e^{+\beta' - h' + C'} \tag{5.10c}$$

Solving these give

$$e^{4J'} = \frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2 h} \qquad (5.11a)$$

$$e^{2h'} = e^{2h}\frac{\cosh(2J+h)}{\cosh(2J-h)} \qquad (5.11b)$$

$$e^{4C'} = e^{8C}\cosh(2J+h)\cosh(2J-h)\cosh^2 h \quad . \qquad (5.11c)$$

The relations (5.11) constitute the RG transformation

$$H'(J',\ h',\ C') = R[H(J,h,C)] \quad . \qquad (5.12)$$

The process of integrating every other spin can now be repeated an infinite number of times. Each time we get the set of equations, (5.11), that relate the coupling constants of the two theories. This RGT transformation can be interpreted graphically in terms of renormalization of the bond interaction between NN spins. Let us rewrite eqn (5.7) as

$$Z = \sum_{s_i} P(s_0 s_1)P(s_1 s_2)\ldots P(s_{n-1}s_n)$$

where

$$P(s_i s_{i+1}) = \frac{1}{2}e^{Js_i s_{i+1}+\frac{h}{2}(s_i+s_{i+1})+C}$$

is the Boltzmann factor for the bond between s_i and s_{i+1}. Then, on integrating out every other site, say s_2, we get a new Boltzmann factor between sites s_1 and s_3

$$P(s_1 s_3) \equiv \sum_{s_2} P(s_1 s_2)P(s_2 s_3)$$

where

$$P(s_1 s_3) = \frac{1}{2}\ e^{J's_1 s_3+\frac{h'}{2}(s_1+s_3)+C'}$$

defines an effective interaction for the new theory with strengths given by eqns. (5.11).

Flow equations in the 1-d Ising model

Equations (5.11a-c) are the RG equations for the couplings (also called the flow equations or recursion relations). From these we see that C depends on J and h, but does not affect their flow. So we can look for fixed points in the $[J,h]$ plane. By inspection we note that $(J=\infty,\ h=0)$ and $(J=0,\ h)$ are two fixed points of the flow equations. To examine the properties of these fixed points I consider the flow equation for J with $h=0$ (this limit still has all the important physics)

$$J' \equiv R(J) = \frac{1}{4}\ln\cosh^2 2J \quad . \qquad (5.13)$$

or equivalently

$$x' = \frac{4x}{(1+x)^2}$$

where $x \equiv \exp(-4\beta)$. Then

$$\frac{\partial x'}{\partial x} = \frac{4(1-x)}{(1+x)^3} \quad . \tag{5.14}$$

From the flow equation for h we get

$$\frac{\partial h'}{\partial h} = 1 + 0.5\,(\tanh(2\beta+h) + \tanh(2\beta-h)) \quad . \tag{5.15}$$

The structure of the fixed points can now be deduced from these two equations. For the moment let me just state some facts without much motivation in order to make the discussion of the 1-d model self-contained. These points are elaborated on later.

Ferromagnetic fixed point at $[\beta = \infty, h = 0]$**:** From eqn. (5.13), we see that the flows in β take us away from this fixed point, *i.e.* $\beta' < \beta$ and a small deviation grows under a RGT. The general flow in the β, h plane is shown in fig. 2 [13] . It shows that $\beta' < \beta$ for all $\beta > 0$, *i.e.* the system gets disordered under RGT. Such couplings are called relevant. Similarly, at the fixed point

$$\frac{\partial h'}{\partial h} \equiv \Lambda_h \rightarrow 2, \tag{5.16}$$

which shows that h is also a relevant coupling.

Thus the ferromagnetic fixed point is unstable under perturbations in both T and h. It is also the critical point of the model since $\xi = \infty$ at it. From eqn. (5.16) and (8.2) we get the exponent $\eta = 1$.

There is another fixed point at $[\beta = \infty, h = \infty]$, to which all points $[\beta = \infty, h > 0]$ converge. This point is not very interesting.

As β is decreased along the line $h = 0$, the rate of flow slows down according to eqn. (5.14) such that at $\beta = 0$ (or $x = 1$) it is zero. The flow converge to the paramagnetic fixed point $\beta = h = 0$ with zero velocity.

Paramagnetic fixed points at $[\beta = 0, h]$**:** On examining eqn. (5.15) at $\beta = 0$ we find that $\Lambda_h = 1$. This means that h has become a marginal operator. Each value of h is a fixed point, *i.e.* the model has a line of trivial fixed points at $\beta = 0$ because $\xi = 0$. Also, along this line $\eta = 0$.

Summary of results for the 1-d Ising model:

The ideas that should be extracted from this simple example are:

1) When all lengths are measured in lattice units, the theory is rescaled by the factor $b = 2$

$$R \rightarrow R' = R/2$$

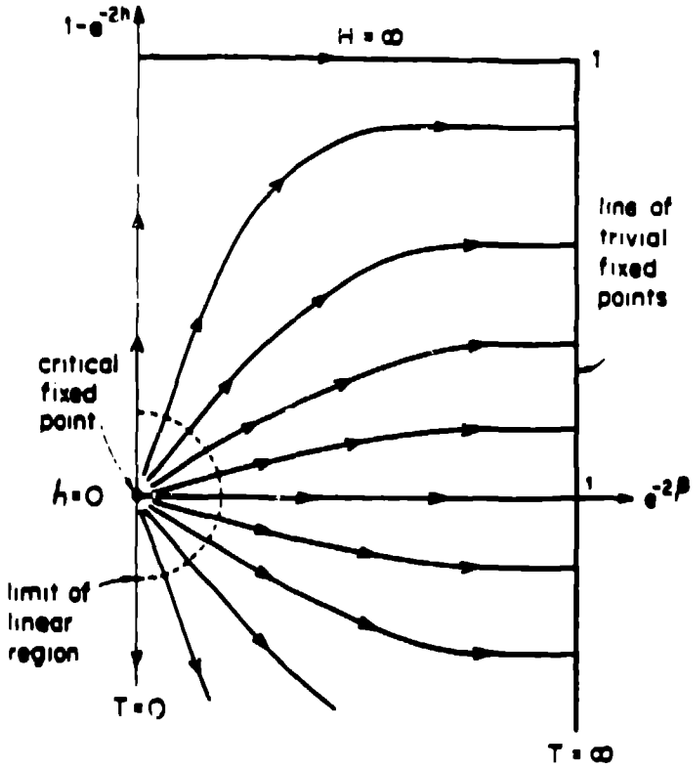

Fig. 2: *The flow in $[\beta, h]$ plane for the 1-d Ising model*

$$< s'_o s'_{R'} > \equiv < s_o s_{2R'} > \Rightarrow \xi' = \frac{\xi}{2} \tag{5.17}$$

Under a RGT the correlation length shrinks and a system is driven away from criticality for any finite β. In most models $T_c \neq 0$ and the flows are very different depending on whether the starting H is on the critical surface or not.

2) The closure property $(\beta, h, C) \rightarrow (\beta', h', C')$ is special to 1-d Ising model. Since s_2 couples only to s_1 and s_3, integrating it out can only lead to a coupling between s_1 and s_3, i.e., still NN on the blocked system.

3) The coupling constants $[\beta,\ h,\ C]$ define a point in a 3-dimensional space. Under successive RGT, consisting of integrating over ever second spin, the evolution $R[H(\beta,\ h,\ C)]$ defines a flow in this space. The line connecting points related by a RGT is called a trajectory. The generalization to arbitrary models requires the new idea that for closure under an appropriately defined RGT, the space of coupling constants has to be infinite dimensional. This is the reason why it has been extremely hard to solve for the RG behavior of models in $d > 1$.

4) The RG eqns. (5.11) have fixed points at $F_1^* = (\beta = \infty,\ h = 0)$ and $F_2^* = (\beta = 0,\ h)$. The line of trivial fixed points, F_2^* with $\xi = 0$, are the attractor of all flows except those starting at $\beta = \infty$. On the other hand the critical fixed point F_1^* is unstable in both β and h and has $\xi = \infty$ at it.

5) Under a RGT, the partition function is preserved:

$$Z_{N'}[H'] \;=\; Z_N[H] \;\;.$$

Using the definition $f[H] \;=\; \frac{1}{N}\log Z_N[H]$, we get the purported scaling relation for the free energy

$$f[H] \;=\; b^{-d} f[H']$$

6) This simple model is an example of an exponential singularity at $\beta = \infty$ (or $x = 0$ since x is a more natural variable than T). Later we will see that exponential singularities are a signature of asymptotic freedom in non-abelian gauge theories in 4 dimensions. Such theories do not conform to the standard definition of exponents.

7) In the analysis of the flows we saw that the constant C does not effect the physics, however, it takes on a definite value at the fixed points. This value is determined self-consistently using the condition that the free energy is zero at critical points.

8) From eqn. (5.11a) we see that if we start with an anti-ferromagnetic model ($\beta < 0$), then after one renormalization the theory becomes ferromagnetic. This is a simple example of how a given physical behavior can be totally lost by an inappropriate choice of the RGT.

In this RG analysis we find that the critical behavior is inferred from eqns. (5.11). In fact we only needed eqns. (5.14) and (5.15). In sections 7 and 11, I show how MCRG allows us to determine these without solving the model or knowing the flow equations.

6. Wilson's Formulation of the Renormalization Group

For the $1-d$ Ising model we derived an analytic closed form expression for the flows. This is a very special case. In general, one also generates next NN, third neighbour, 4-spin, *etc.* couplings in addition to renormalizing the NN coupling. Thus, we have to consider the infinite set, $\{K_\alpha\}$, of all possible couplings. Only then does a RGT map one point into another.

The critical points, with $\xi = \infty$, are special. Starting from H^c, a RGT produces another H^c since $\xi' = \xi/b = \infty$. Thus the set of critical points define a hypersurface in this infinite dimensional space $\{K_\alpha\}$. The RG flows on this surface can (a) meander randomly, (b) go to some limit cycle, or (c) converge to a fixed point H^*. A RGT will in general have more than one fixed points as we saw in the $1-d$ Ising model. We are interested in RGT which possess a critical fixed point *i.e.* $H^* = R(H^*)$ with $\xi = \infty$. Each such fixed point has a basin of attraction *i.e.* the set of H^c that converge to it under the RGT. This basin of attraction defines the university class since the long

distance behavior of all substances corresponding to these H^c is governed by the same fixed point.

Let me make the statement of universality more explicit with an example. The Ising model is an idealization of a ferromagnet, yet it describes the critical behavior of nickel, iron *etc.*, each of which has a different Curie point, T^c. The reason for this universality is that all these H^c lie in the basin of attraction of the Ising fixed point. For now you do have to accept on faith that the Ising model has a fixed point that describes ferromagnetic critical behavior. Unfortunately, even for the soluble $d = 2$ Ising model it has not been possible to determine H^*; the proof of it's existence is based on consistency and circumstantial evidence.

Why do H^c which lie in the basin of attraction of H^* have the same long distance physics? If we look at the behavior of these different models at the scale of each spin, then we shall see differences in the correlation functions at small separations. On the other hand, if we construct effective spins which describe the average behavior of all spins in a cell of size b^d, then the distinction in their correlation functions blurrs as the cell size b is increased. This way of looking at a substance on a coarse scale is what a RGT implements. Thus, the statement "same long distance behavior" means two things: (a) scaled correlation functions at large separation are the same and (b) correlation functions of spins averaged over a large block are the same at all scales.

I show two possible types of flows near a fixed point T^* in an idealized case of a single coupling T. In fig. 3a, the fixed point is attractive under the RGT for all starting T which lie in $0 < T < T^*$ or in $\infty > T > T^*$. On the other hand the fixed point in fig. 3b is repulsive, *i.e.* the flows go away from T^*. A coupling is called relevant if deviations in its value from the fixed point get magnified under RGT. Similarly, a coupling is called irrelevant if the deviations $\to 0$. Thus T is an irrelevant coupling in the example of fig. 3a, and relevant for the case in fig. 3b. Clearly, the space of couplings spanning the critical surface are irrelevant, while flows out of the critical surface define the relevant directions. In case of the ferromagnet, H^* is unstable with respect to variation in T and in h. For ordinary second order transitions T and h are the only two independent couplings, in accord with the scaling relations we derived earlier. In general the number of independent couplings are equal to the number of unstable directions of H^*.

For a given RGT, each of the H^c on the critical surface is the limit point of a flow followed backwards. The trajectory flowing out of H^* is special for it is the attractor for all these other trajectories. It is called the *renormalized trajectory* (RT). There is a one to one association between the H^c which lie in the basin of attraction of H^* and their corresponding trajectories which are attracted by the RT. The fixed point is unstable in the direction of the renormalized trajectory *i.e.* any deviation along it from

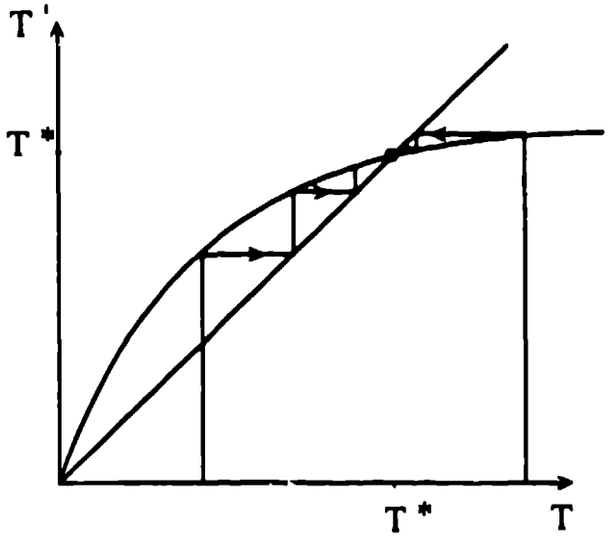

Fig. 3a: ***Example of an attractive fixed point w.r.t. coupling T.***

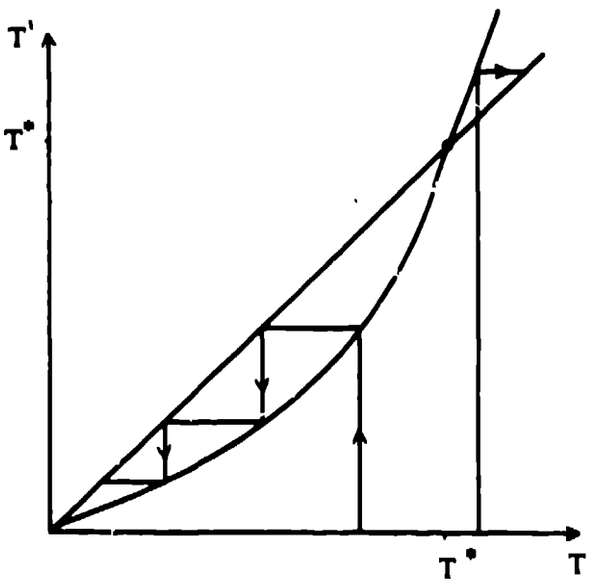

Fig. 3b: ***Example of an repulsive fixed point w.r.t. coupling T.***

H^* gets magnified under successive RGT. Yet along the RT, all irrelevant couplings are zero so there are no scaling violations. Thus the long distance physics is the same as at the fixed point and all correlation lengths in the model transform as $\xi \rightarrow \xi/b$ under a scale change b.

The flow along the RT can be parameterized by a single variable -- the distance

along it from the fixed point. This is the genesis of scaling. When thermodynamic functions are measured at different values of H corresponding to different ξ, they can all be expressed as a function of some reference H and the distance to it along the flow trajectory. For example, along the RT in the even sector, all couplings can be expressed in terms of a reference value of t and the scale change b i.e. $t' = b^{\lambda_t} t$.

I have repeatedly stressed that the H^c and the flows are a function of the RGT. Does this mean that for each valid RGT, there is a different H^c for say iron when we know that there is some unique H^c describing iron at the Curie point? In the mathematical formulation I have presented above, the answer is yes. The resolution of this paradox is that different RGT differ by ***redundant operators*** [14] . These redundant directions span a sub-space on the critical surface and correspond to a change in field variables which does not affect any physical observable. For example, in the scalar field theory a change $\phi \rightarrow \phi +$ changes $H \rightarrow H + \delta H$. Such a shift by a constant does not affect physics and the change δH defines a redundant operator. In numerical calculations, there is no a-priori way to work in a space orthogonal to these redundant operators and their presence can be a nuisance [15] . So the goal of MCRG calculations is to identify the redundant operators and then work in the sub-space orthogonal to them.

In a certain neighborhood of H^* (both on and off the critical surface), the scaling properties of the model can be analyzed in the linear approximation as discussed in section 8. It is only our ability to calculate within the linear approximation that limits us to the region near H^*. Otherwise, in principle we could work at any finite ξ along the RT and extract the critical behavior. Of course, for $\xi \approx 1$ there is no distinction between short and long distance behavior. Also, for ξ small, the RT may only be very weakly attractive and it may no longer be local. If this is the case, then a simple H (as is commonly used in simulations) will not give the critical behavior since it may not lie in the domain of attraction of the fixed point.

Having shown how universality and scaling are explained by Wilson's formulation of the renormalization group, the next question is what differentiates say nickel from iron? As mentioned earlier, even though their respective H^c are attracted by the Ising fixed point, they are not identical. This explains the difference in the Curie point for the two metals. The difference in H^c is characterized by irrelevant operators. These operators control corrections to scaling and account for the differences in the short distance properties of the substances even though they belong to the same universality class.

Let us examine these ideas in terms of the scaling of the free energy. Rewriting eqn. (3.3) with $r = 1$ and $h = 0$ gives

$$f(t, h, u) = (t)^{2-\alpha} \quad f(1, 0, t^{-\phi} u) \tag{6.1}$$

Let me first assume that f has a valid Taylor series expansion about $u = 0$ *i.e.*

$$f(1,\ 0,\ t^{-\phi}u)\ =\ f\left[1,\ 0,\ 0\right)\ (1 + c_i t^{-\phi}u + \ldots] \tag{6.2}$$

then in the limit $t \to 0$ the correction terms are well defined only if $\phi < 0$. This happens if $\lambda_h < 0$ *i.e.* u is a irrelevant field. If $\phi > 0$, then u is a second relevant field, the correction term becomes arbitrarily large and the expansion makes no sense as $t \to 0$. Fixed points with more than one relevant field are multicritical points. In that case the exponent ϕ controls the crossover behavior between the two relevant fields. The amplitude $f(1,0,0)$ in eqn. (6.2) is not universal. This is an example of a general statement: amplitudes in the scaling laws for thermodynamics functions are non-universal.

The second possibility is that f does not have a well defined Taylor expansion in u but behaves as $f(1,x) \sim f(1)/x^{\mu}$. In this case even though u is an irrelevant field, the singular behavior $(ut^{|\phi|})^{-\mu}$ will modify the hyperscaling relation to $2 - \alpha + \mu|\phi| = \nu d$. Such fields have been given the name ***dangerous irrelevant operators*** by M. Fisher and their presence is one possible source of hyperscaling violations [4].

7. Linearized Transformation Matrix and Classification of Exponents

Consider a point $\{K_\beta\}$ close to the fixed point $\{K^*_\beta\}$. Under the assumption that there are no singularities in the space of coupling constants, the transformation $K'_\alpha(K_\beta)$ can be expressed as a Taylor expansion about the fixed point

$$K'_\alpha(K_\beta)\ =\ K_\alpha(K^*_\beta)\ +\ \left.\frac{\partial K'_\alpha}{\partial K_\beta}\right|_{K^*}\ \Delta K_\beta\ +\ \ldots\ .$$

For small deviations from the fixed point one can define a "linear region" in which

$$\Delta K'_\alpha\ =\ \left.\frac{\partial K'_\alpha}{\partial K_\beta}\right|_{K^*}\ \Delta K_\beta \tag{7.1}$$

and $\Delta K_\beta = K_\beta - K^*_\beta$ is the deviation from the fixed point. The linearized transformation matrix

$$T_{\alpha\beta}\ \equiv\ \left.\frac{\partial K'_\alpha}{\partial K_\beta}\right|_{K^*} \tag{7.2}$$

determines the flows and is the starting point of MCRG analysis. The matrix $T_{\alpha\beta}$ satisfies the eigenvalue equation $T\phi = \Lambda_\phi \phi$, and is the genralization of eqns. (5.14) and (5.15). From it we can derive the critical exponents as shown in the next section. Note that $T_{\alpha\beta}$ is defined only to first order, so henceforth all statements will have the implicit assumption that they are only valid in the linear region close to H^*.

The properties of the matrix $T_{\alpha\beta}$ and its associated eigenvalues Λ_i and eigenvectors Φ_i are:

1. The eigenvalues depend on the scale factor b. Since two successive transformations by b should give the same flow as one by b^2, the general form is $\Lambda_i = b^{\lambda_i}$ where the λ_i are independent of the RGT.
2. Consider a deviations u from H^* along a given eigenvector ϕ. Then under a RGT, the flow satisfies the equation

$$u'\phi = Tu\phi = b^{\lambda_u} u\phi \quad . \tag{7.3}$$

   This is the defination of a scaling field u and justifies why we used eqns. (3.1). Note that the eigenvector ϕ embodies the linear dependence on the coupling constants $\{K_\alpha\}$.
3. Eigenvalues $\Lambda > 1$ lead to flows away from the fixed point. These eigenvalues and the corresponding eigenoperators are called relevant.
3. Eigenoperators corresponding to eigenvalues $\Lambda < 1$ die out geometrically with the number of blocking steps. These operators do not contribute to the long distance properties of the system and are consequently called irrelevant. The associated exponents λ control corrections to scaling.
4. An eigenvalue of exactly one is called marginal. To ascertain that an eigenvalue is truly marginal, one has to go beyond the linear approximation. Marginal operators do not flow under RGT. Operators that are marginal only at leading order give rise to logarithmic corrections. We will see later that asymptotically free field theories are of this kind.
5. There is an additional class of eigenoperators, called redundant operators, that are not physical. Their eigenvalues can be < 1, $= 1$, or > 1. These eigenvalues depend on the choice of the *RGT*, *i.e.* different RGT with the same scale factor will give different λ_i. A reason for this is that the amount of a redundant operator generated depends on the redefinition of fields, so different RGT give rise to different eigenvalues. In fact one way to isolate them is to repeat the calculation of $T_{\alpha\beta}$ with a different *RGT*. Then the eigenvalues that change with RGT correspond to redundant operators. Even though these eigenvalues are not physical, a RGT for which they are relevant does not converge.
6. Under certain conditions $T_{\alpha\beta}$ is block diagonal, *i.e.* the couplings break up into sub-sets that are closed under the *RGT*. For example, in the Ising model at $h = 0$, the symmetry $s \to -s$ causes interactions to break up into odd and even sectors, each with an independent eigenvalue spectrum.

For a more detailed discussion of the above points see ref [2]. I now derive the relationship between the eigenvalues of $T_{\alpha\beta}$ and the critical exponents ν and η.

8. Derivation of Exponents from the Eigenvalues of $\mathcal{T}_{\alpha\beta}$

The relation between the correlation length exponent ν and the largest even eigenvalue of $\mathcal{T}_{\alpha\beta}$ can be derived as follow: consider the correlation length ξ at two temperatures with $t_1 < t_2$

$$\begin{aligned} \xi(t_1) &\sim t_1^{-\nu} \\ \xi(t_2) &\sim t_2^{-\nu} \end{aligned}$$

Then using $\xi(t_1) = b\xi(t_2)$ with $t_2 = R_b(t_1)$ we get

$$\begin{aligned} b &= \left(\frac{t_1}{t_2}\right)^{-\nu} \\ \frac{t_2}{t_1} &= b^{\frac{1}{\nu}} \\ \Rightarrow \frac{\partial t_2}{\partial t_1} &= b^{\frac{1}{\nu}} = b^{\lambda_t} \quad . \end{aligned}$$

Therefore, if Λ_t is the largest eigenvalue of $\mathcal{T}_{\alpha\beta}$ in the even sector, then

$$\nu = \frac{1}{\lambda_t} = \frac{\log b}{\log \Lambda_t} \tag{8.1}$$

Second, the exponent η is related to the largest odd (magnetic) eigenvalue Λ_h of $\mathcal{T}_{\alpha\beta}$. This connection involves using scaling and hyperscaling relations! Lets start with the scaling behavior for h, eqns (2.4) and (2.5), then

$$\begin{aligned} h &\sim t^{\Delta} \sim t^{\beta\delta} \\ \Rightarrow h &\sim \xi^{-\frac{\beta\delta}{\nu}} \\ \Rightarrow h' &\sim \xi'^{-\frac{\beta\delta}{\nu}} = b^{\frac{\beta\delta}{\nu}} \xi^{-\frac{\beta\delta}{\nu}} = b^{\frac{\beta\delta}{\nu}} h \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial h'}{\partial h} &= b^{\frac{\beta\delta}{\nu}} = \Lambda_h \\ \Rightarrow \frac{\beta\delta}{\nu} &= \frac{\ln \Lambda_h}{\ln b} = \lambda_h \end{aligned}$$

Now using the scaling relations

$$\begin{aligned} \frac{\beta\delta}{\nu} &= \beta + \alpha \\ &= \frac{\alpha}{\nu} + \frac{2 - \alpha - \gamma}{2\nu} \\ &= \frac{\gamma + (2 - \alpha)}{2\nu} \\ &= \frac{2 - \eta}{2} + \frac{d}{2} \\ \Rightarrow \lambda_h &= \frac{d + 2 - \eta}{2} \quad . \end{aligned} \tag{8.2}$$

Thus, for models with no hyperscaling violations, MCRG allows us to calculate $T_{\alpha\beta}$ and from its relevant eigenvalues we extract the two independent exponents ν and η. Next, I want to introduce a simple field theoretic model before discussing MCRG methods.

9. Simple Field Theory: The Gaussian Model

The ideas I have discussed so far for spin models carry over unchanged to field theories. The differences are mainly of language and interpretation. For example:

a) In Euclidean field theories, the degrees of freedom are field variables which are continuous valued functions at every point in space-time. To make the problem tractable, we introduce a discrete space time grid with the lattice spacing a acting as an ultraviolet cutoff.

b) The lattice serves only as a regulator. There is no scale, like the inter-molecular distance in condensed matter problems, which provides a natural cutoff. To make contact with the continuum theory, the lattice spacing has to be taken to zero. So an important question is: Does this limit exist and if so do we recover an interacting field theory that describes the real world? In a large number of field theories the limit does exist, but the theory is trivial, *i.e.*, the renormalized coupling goes to zero as $a \to 0$. Such theories cannot be considered fundamental *i.e.* they have structure at some length scale. This length scale provides a natural cut-off for the theory. Such effective theories are mathematically well defined, interacting and can be used to describe the real world. In an effective theory the renormalized coupling depends on the cutoff. Since this coupling is known from experiments, therefore, the cutoff fixes the scale at which some new physics comes in. Below that scale the theory provides a consistent description of nature. An example of how this works is provided in the lectures by A. Hasenfratz. She showed how, in the Higgs sector of the standard model, the scalar interactions have to be considered as such an effective field theory.

c) The principle of least action governs the relative importance of a configuration *i.e.* the Boltzmann weight is $e^{-\int L}$. The interaction between the degrees of freedom is provided by the Lagrangian density L. All this means in practice is that we replace the word Hamiltonian by Action in all the previous discussion.

The formulation of Euclidean field theories has been reviewed in the lectures by Callaway and Lepage. So I will assume that you are familiar with the details. Here my aim is to show how Wilson's momentum space renormalization works in the simple case of the gaussian field theory before moving on to the application of real space renormalization to non-abelian gauge theories *i.e.* QCD.

The prototype field theory is the scalar field theory described by the action

$$A = \int d^d x \left[\frac{1}{2}(\partial_\mu\phi\partial_\mu\phi) + m^2\phi^2 + u_1(\partial_\mu\partial_\mu\phi)^2 + u_2(\partial_\mu\partial_\mu\phi\partial_\mu\partial_\mu\phi) + \lambda\phi^4 + \ldots\right] \tag{9.1}$$

Its gaussian version is the limit where only interactions that are bilinear in ϕ are retained. In terms of the fourier transformed field ϕ_k,

$$\phi(x) = \int d^d k\ e^{ikx}\phi_k \quad , \tag{9.2}$$

the action for the gaussian field theory is

$$L = \int_0^\Lambda d^d k\ \phi_k\phi_{-k}\left[\frac{k^2}{2} + m^2 + u_1(k_\mu k_\mu)^2 + u_2(k_\mu k_\mu k_\mu k_\mu) + \ldots\right] \quad . \tag{9.3}$$

The action is bilinear in ϕ_k, and the integrations over each decoupled ϕ_k in the partition function can be carried out. In eqn. (9.3), I have introduced a cutoff Λ to regularize the theory. This is necessary in order to follow the philosophy of the renormalization group *i.e.* integrate over the degrees of freedom in small steps. This is not possible if $\Lambda = \infty$ from the start since $\Lambda/b = \infty$. So we start with a finite cutoff and consider the cutoff free theory only as the limiting case. Integrating all ϕ_k for $\frac{\Lambda}{2} < k < \Lambda$ we get

$$\int_{0<k<\Lambda} D(\phi_k)e^{-L} = constant \int_{0<q<\frac{\Lambda}{2}} D(\phi_q)\ e^{\int_0^{\frac{\Lambda}{2}} d^d q\phi_q\phi_{-q}\ [\frac{q^2}{2}+\ldots]} \quad .$$

We would like to make this effective theory look like the original, just like in the 1-d Ising model. For the gaussian model this can again be done exactly. The canonical steps are: first perform a rescaling of the length scale

$$q'_\mu = 2q_\mu \tag{9.4}$$

and then rescale the field variables by

$$\phi'(q') = c\ \phi(q) \tag{9.5}$$

so that the renormalized action L' becomes

$$L' = \int_{0<q'<\Lambda} \frac{d^d q'}{2^d}\frac{\phi'_{q'}\phi'_{-q'}}{c^2}[\frac{q'^2}{2*4} + m^2 + \frac{u_1}{16}(\sum q'_\mu q'_\mu)^2 + \ldots]$$

Now choosing $c^2 = 2^{-d-2}$ and relabelling $q' \to q$ and $\phi' \to \phi$ gives

$$L' = \int_0^\Lambda d^d q\phi_q\phi_{-q}[\frac{q^2}{2} + 4m^2 + \frac{u_1}{4}(q_\mu q_\mu)^2 + \ldots]$$

which is exactly the same as the theory we started with except for a renormalization of the coupling constants

$$\begin{aligned} m^2 &\longrightarrow 4m^2 \;:\; b^2 m^2 \\ u_1 &\longrightarrow \frac{u_1}{4} \;:\; \frac{u_1}{b^2} \\ u_2 &\longrightarrow \frac{u_2}{4} \;:\; \frac{u_2}{b^2} \end{aligned} \tag{9.6}$$

where I have also written the generalization to scale factor b since the rescaling by factor 2 was arbitrary. The fixed point condition

$$L'(m'^2, u_1', u_2') \;=\; L(m^2, u_1, u_2)$$

is satisfied provided $m^2 = u_1 = u_2 = 0$. The condition $m^2 = 0$ implies that the continuum theory is massless, while $u_1 = u_2 = \ldots = 0$ means that it is free. This is the gaussian fixed point. It is unstable only with respect to the coupling m^2.

To summarize, let me stress the important points:

1) A rescaling of fields $\phi'(q') \;\rightarrow\; c\phi(q)$ was necessary to obtain a fixed point.
2) At the fixed point the theory is massless ($m^2 = 0$) and free.
3) There is no mixing between the couplings *i.e.* $m^2, u_1, u_2, \ldots$ are the scaling fields.
4) The scaling relations are simple. In fact we see that if an operator A has canonical dimension α then the conjugate couplings scale as

$$\Lambda_A \;\equiv\; b^{d-\alpha}$$

where Λ_A is the eigenvalue of the transformation. For example, in d dimensions, the ϕ^2 term has $\alpha = d - 2$, therefore m^2 scales as $b^{d-\alpha} = b^2$.

5) m^2 is a relevant operator, while all the u_i are irrelevant.
6) All eigenvalues Λ_i are independent of dimension d.
7) The exponent $\nu = 1/2$ is gotten from the scaling relation for m^2.

The next step is to generalize the model by including interactions via a term like $\lambda\phi^4$ in the action. This generalized Landau-Ginzburg model was first analyzed by Wilson using the ϵ-expansion and momentum space renormalization group [2]. The present status of results is that in $d > 4$ dimensions all such couplings renormalize to zero in the continuum limit. There do not exist any non-trivial non perturbative fixed point in the infinite dimensional coupling constant space. The only fixed point is the gaussian one. For $d = 4$, the triviality of scalar field theories has not been proven rigorously, yet all calculations point to it. In $d < 4$, the coupling λ becomes relevant. For details, I refer you to the lectures of A. Hasenfratz and to Wilson's original work where he examined these properties using the ϵ-expansion.

10. Linear Renormalization Group Transformations

In both examples (1-d Ising and the Gaussian field theory), the spin-spin correlation function under a renormalization group transformations obeys the following relation

$$G(x) = c^2 G(x') \tag{10.1}$$

In the case of the 1-d Ising model we got the identity $< s'_n s'_0 > = < s_{2n} s_0 >$ on integrating every second spin. So $c^2 = 1$. This is consistent with the required behavior of the correlation function *i.e.* $G(n) \sim \frac{1}{n^{d-2+\eta}}$ because $\eta = 1$. However, recall that we had to tune the constant C in the Hamiltonian to get the fixed point.

In the momentum space renormalization of the Gaussian model we had to rescale the field ϕ by $c = b^{-(d+2)/2}$ in order to obtain a non-trivial fixed point. Instead of integrating out all the frequencies between Λ/b and Λ, let us implement a real space blocking transformation in which the block spin is defined as the average over a block of size b^d, *i.e.* $\phi'(x') = c \sum_{x \in x'} \phi(x)$. Then the correlation function for large x in the two theories is,

$$\begin{aligned} < \phi'(x')\phi'(0) > &= c^2 b^{2d} < \phi(x)\phi(0) > \\ \Rightarrow \frac{1}{(x')^{d-2+\eta}} &= c^2 b^{2d} \frac{1}{(bx')^{d-2+\eta}} \\ \Rightarrow c &= b^{\frac{2-(d+2)}{2}} \quad . \end{aligned} \tag{10.2}$$

Since, for the Gaussian model $\eta = 0$, so the rescaling of the field is consistent between the two RGT and with the desired behavior of the correlation function.

This non-trivial rescaling of fields by a factor c to obtain a non-trivial fixed point is essential for linear renormalization group transformations. These are transformations in which the block field is a linear function of the original fields as in the examples we have considered so far. Note that in the Gaussian model $\phi'' = R_{b^2}[\phi] = R_b[R_b[\phi]]$ while in the 1-d Ising model $s' = s$. Non-linear transformations arise naturally in $SU(N)$ gauge theories and $O(N)$ non-linear sigma models. As we will see later, in those cases no such rescaling of fields is necessary to get a fixed point.

11. Numerical Methods: MCRG

Consider a magnetic system consisting of spins $\{s\}$ placed on the sites of a d dimensional lattice L and described by a Hamiltonian H. This H is to be regarded as a point in the infinite dimensional space of couplings constants $\{K_\alpha\}$. Then the physics of the model (*i.e.* all thermodynamic quantities) can be determined from the behavior of the partition function

$$Z = \sum e^{-H} = \sum e^{K_\alpha S_\alpha} \tag{11.1}$$

as a function of $\{K_\alpha\}$. Here S_α is the interaction between spins conjugate to the coupling K_α. The expectation value of any operator O is given by

$$< O > = \frac{1}{Z} \sum O e^{-H} \tag{11.2}$$

from which one can calculate the thermodynamic behavior.

In numerical simulations using Monte Carlo methods, one generates configurations of spins using one of the following methods: Metropolis [16] , heat bath [17] , molecular dynamics [18] , microcanonical [19] or the Langevin [20] [21] . Each of these algorithms gives configurations with a Boltzmann distribution $e^{-H} \equiv e^{K_\alpha S_\alpha}$. The obvious advantage of such "importance sampled" configurations over using a flat distribution is that in a thermodynamic system the probability is very highly peaked around configurations that minimize the energy and only these have any significant contribution to the partition function. The choice of which of the above methods to use depends on the efficiency of the method for a particular physics goal. These methods have been discussed in the lectures by Peter Lepage and I refer you to them for details.

Thermodynamic quantities are measured as simple statistical averages of correlation functions over these "importance sampled" configurations. By calculating the behavior of thermodynamics quantities as a function of T and h near T_c we can determine the critical properties of the model. The biggest limitation of such numerical calculations comes from the use of finite lattices. On a finite lattice the growth in ξ as $T \to T_c$ is cut off since ξ can only be $\leq L$. So all observables acquire a dependence on the lattice size L. This L dependence can be analyzed within the framework of finite size scaling. In that way we are able to extract fairly accurate values for T_c and the critical exponents.

MCRG is a numerical technique for implementing renormalization group ideas using Monte Carlo. I now show how in MCRG one does the partial integration over the degrees of freedom in small steps to generate the flows and also how to calculate $T_{c,f}$ along them.

Blocking: Numerical implementation of RGT

The implementation starts with Kadanoff's key idea to replace the spins in a small cell of size b^d by an "effective spin". The goal of this construction was to average over local fluctuations in the cell and yet preserve the physics obtained from correlation functions at large separations. We considered the example $\phi'(x') = c \sum_{x \in x'} \phi(x)$ in the gaussian model. One can intuitively see that it removes all frequencies $> \Lambda/b$, but we also need to show that it preserves the critical physics. To examine this lets write

down the constrained partition function for the renormalized theory

$$e^{-H^1(s^1)} = \sum P(s^1, s)\, e^{-H(s)} \tag{11.3}$$

where

$$P(s^1, s) \;=\; \delta\big(\varphi'(x') - c \sum_{x \in x'} \phi(x)\big) \quad . \tag{11.4}$$

for the gaussian model under the replacement $s_i \rightarrow \phi(x)$. Thus, for each configuration generated with the original H, after blocking we have another configuration on a lattice of size L/b described by the Hamiltonian H^1. All expectation values, with respect to H^1 can again be calculated as simple averages on the set of blocked configurations. Repeating this blocking n times produces a sequence of configurations distributed according to the Hamiltonians H^n.

The question then reduces to: what constraint does P have to satisfy such that the sequence of theories labelled by H^n describe the same long distance physics but on increasingly coarse lattices? Part of the answer turns out to be rather simple: the *RGT* should satisfy the Kadanoff constraint (unitarity relation)

$$\sum_{s^1} P(s^1, s) = 1 \tag{11.4}$$

independent of the state $\{s\}$. Using this in eqn (11.3)

$$\sum_{s^1} e^{-H^1(s^1)} = \sum_{s^1, s} P(s^1, s)\, e^{-H(s)} = \sum_{s} e^{-H(s)} \tag{11.5}$$

shows that the two theories H and H^1 have the same partition function. This property leads to the correct rescaling of the free energy.

It is, however, not guaranteed that all RGT satisfying Kadanoff's constraint have a fixed point; or even if they have a fixed point that it is the physically interesting one. The fixed point H^*, the *RT* and the sequence of theories, H^n, generated from a given starting H depend on the *RGT*. Since these properties are not physical, a bad RGT can miss the desired physics. For example, decimation transformations (choosing one of the b^d spins to be the block spin in the $d > 1$ Ising model) do not lead to the Ising fixed point. Such RGT should therefore not be used.

The *RGT* should incorporate the symmetry properties of the model. We saw for the anti-ferromagnetic 1-d Ising model that integrating every other spin leads to the ferromagnetic fixed point. The same is true for anti-ferromagnetic Ising models in higher dimensions when a majority rule blocking is used with block cells of size 2^d.

These constraints leave considerable freedom in the choice of the *RGT*. In fact many different *RGT* can be used to analyze a given model, however, they have different *convergence* properties. A comparative study of convergence of different RGT

can be made as follows: starting from some given T_c, one counts the number of iterations required to converge to the respective H^*. Since H^* is usually not known, this procedure is implemented indirectly by studying the convergence of exponents as a function of the number of blocking steps. The rate of convergence is controlled by how non-local is the fixed point (magnitude of the irrelevant operators), and on the presence of redundant operators with eigenvalues ≥ 1. Clearly the optimum transformation is one that converges the fastest *i.e.* whose associated fixed point is the most local. Unfortunately, in most cases the only way to test this is by detailed numerical calculations.

A short cook-book list of some general features that lead to good blocking transformations are:

a) A small scale factor b: since we are calculating the rate of flow, $T_{\alpha\beta}$, in the linear approximation, the RGT with the smallest scale factor b is desired.

b) Preserve the space-time and internal symmetries: Examples are local gauge freedom, ferromagnetic versus anti-ferromagnetic nature, *etc.*.

c) Locality: In the renormalization group method there are many places an implicit assumption of locality has been made: (a) that the fixed point Hamiltonian is local; (b) blocking corresponds to averaging fluctuations over a local region so that the 2-point correlation functions measure correlations between local block operators separated by some distance $\vec{r}$.

Methods to Calculate the Critical Exponents:

There are three MCRG methods to calculate the critical exponents. In all three methods the flows are determined from the expectation values calculated on the original and blocked configurations, and the accuracy of the calculated exponents improves when the simulations are done close to H^*. These methods, along with their relative merits and weaknesses, are discussed below.

The most popular and successful method for the calculation of the critical exponents is due to Swendsen [22] [23]. In this method, the linearized transformation matrix $T^n_{\alpha\beta}$ is calculated directly using the standard trick of the chain rule

$$T^n_{\alpha\beta} = \frac{\partial K^n_\alpha}{\partial K^{n-1}_\beta} = \frac{\partial K^n_\alpha}{\partial \langle S^n_\alpha \rangle} \frac{\partial \langle S^n_\alpha \rangle}{\partial K^{n-1}_\beta} \quad . \tag{11.6}$$

Each of the two terms on the right is a connected 2 point correlation matrix

$$U^n_{\alpha\beta} = \frac{\partial \langle S^n_\alpha \rangle}{\partial K^{n-1}_\beta} = \langle S^n_\alpha S^{n-1}_\beta \rangle - \langle S^n_\alpha \rangle \langle S^{n-1}_\beta \rangle \tag{11.7}$$

and

$$D^n_{\sigma\beta} \equiv \frac{\partial \langle S^n_\sigma \rangle}{\partial K^n_\beta} = \langle S^n_\sigma S^n_\beta \rangle - \langle S^n_\sigma \rangle \langle S^n_\beta \rangle \quad . \tag{11.8}$$

Here $\langle S^n_\sigma \rangle$ are the expectation values on the n^{th} renormalized lattice and K^n_σ are the corresponding couplings. From the correlation functions, it is easy to see that if the Hamiltonian is symmetric under $s \rightarrow -s$, then $T^n_{\alpha\beta}$ will factor into two block diagonal matrices -- those involving even-even interactions and those with odd-odd.

The correlation length exponent ν is obtained from the leading thermal eigenvalue Λ_t of the even part of $T^n_{\alpha\beta}$ (as discussed in section 8)

$$\nu = \frac{\ln b}{\ln \Lambda_t} \tag{11.9}$$

where b is the scale factor of the *RGT*. Similarly, the correlation function exponent η is given by

$$\eta = d + 2 - 2\,\frac{\ln b}{\ln \Lambda_h} \tag{11.10}$$

where Λ_h is the largest eigenvalue in the odd sector of $T^n_{\alpha\beta}$.

I have restricted the discussion to the special case of one relevant even and one odd eigenvalue. In spin models, these correspond to t and h being the two relevant couplings. In general, however, systems can have multi-critical points *i.e.* there can be more than one relevant interaction. This shows up in numerical simulations when there exists more than one eigenvalue that is greater than one in a given sector of $T_{\alpha\beta}$.

The second method to calculate the leading relevant exponent is due to Wilson [24]. Consider once again the 2-point connected correlation function $\langle S^1_\alpha S^j_\beta \rangle_c$ with $j > i$ and expand $\langle S^1_\gamma \rangle$ in terms of the eigenoperators O^1_μ of the *RGT*

$$\langle S^1_\alpha S^j_\beta \rangle_c = \sum_\mu c^1_{\alpha,\mu} \langle O^1_\mu S^j_\beta \rangle = \sum_\mu \Lambda_\mu^{j-1}\, c^1_{\alpha,\mu} \langle O^j_\mu S^j_\beta \rangle \tag{11.11}$$

where Λ_μ is the eigenvalue corresponding to the eigenoperator O_μ. If close to H^* we make the following approximations: (a) the blocking level dependence in O^1_μ and in the expansion coefficients $c^1_{\alpha\beta}$ can be neglected, and (b) for $j - i$ large, the sum can be approximated by the leading term (dominance of the leading eigenvalue Λ_t) then

$$\langle S^1_\alpha S^j_\beta \rangle_c \sim \Lambda_t^{j-1}\, c_{\alpha,t} \langle O_t S^j_\beta \rangle \quad . \tag{11.12}$$

Consequently, an estimate for the leading eigenvalue Λ_t is then obtained for each α and β as

$$\Lambda_t = \frac{\langle S^1_\alpha S^j_\beta \rangle}{\langle S^{j+1}_\alpha S^j_\beta \rangle} \quad . \tag{11.13}$$

Note that the many different operators $\langle S_\alpha^i S_\beta^j \rangle$ are highly correlated, and these correlations have to be taken into account in the error analysis. Also, the corrections to eqn. (11.12) are suppressed only as $(\frac{\lambda_2}{\lambda_1})^{j-i}$. So for accurate results one should take $j-i$ large, while making sure that flows stay in the linear region.

The third method uses the 2-lattice method due to Wilson to estimate the thermal exponent ν. Let the RT be parameterized by the relevant scaling field K, then under a RGT

$$(K^2 - K^{c*}) = b^{\frac{1}{\nu}} (K^1 - K^{c*}) \tag{11.14}$$

where the flow is from K^1 to K^2. From this flow one can calculate ν and K^{c*}. In practice we do not know the RT, but we can proceed as follows: block expectation values are calculated on n and $n-1$ levels on 2 lattices of size L and L/b with simulation temperature T^1 and T^2 respectively. These block expectation values are compared at i and $i-1$ blocking steps respectively for the two simulations. The starting temperature T^2 is adjusted until the compared block expectation values match for i greater than some number of blocking steps. Then, under the assumption that flows after $i < n$ steps have converged to the RT, eqn (11.14) can be recast as

$$(T^2 - T^c) = b^{\frac{1}{\nu}} (T^1 - T^c) \quad . \tag{11.15}$$

This general flow diagram for the 2-lattice method is shown in fig. 4. From a sequence of such calculations that give matching couplings one can determine ν and T^c using eqn. (11.15). The advantage of this method is that only expectation values $\langle S_\alpha^i \rangle$ enter into the calculation, and these can be calculated with small statistical errors. The disadvantage is that it is hard to judge whether the flows have converged to the RT. One usually needs a large starting lattice and many blocking steps to make a careful check.

The chief drawback of Swendsen's method is that the matrix $T_{\alpha\beta}$ is calculated in a truncated space of interactions S_α. Since, for a given model it is not known a-priori how many couplings are needed to get stable results, one has to test for truncation errors. In practice, the method works much better than one would naively expect due to a fortuitous cancellation of truncation errors as discussed in [25] .

When simulations are done at H^c, as in Swendsen's method, one would naively expect large errors due to critical slowing down and finite size effects. Empirically it is seen that even though individual correlation functions suffer severly from critical slowing down and finite volume effects, there is a remarkable cancellation in the matrix $T_{\alpha\beta}$. Also, the variation in the elements of $T_{\alpha\beta}$ with lattice size is remarkably small. Thus results converge to the correct values of the exponents on moderately large lattices and with present day statistics.

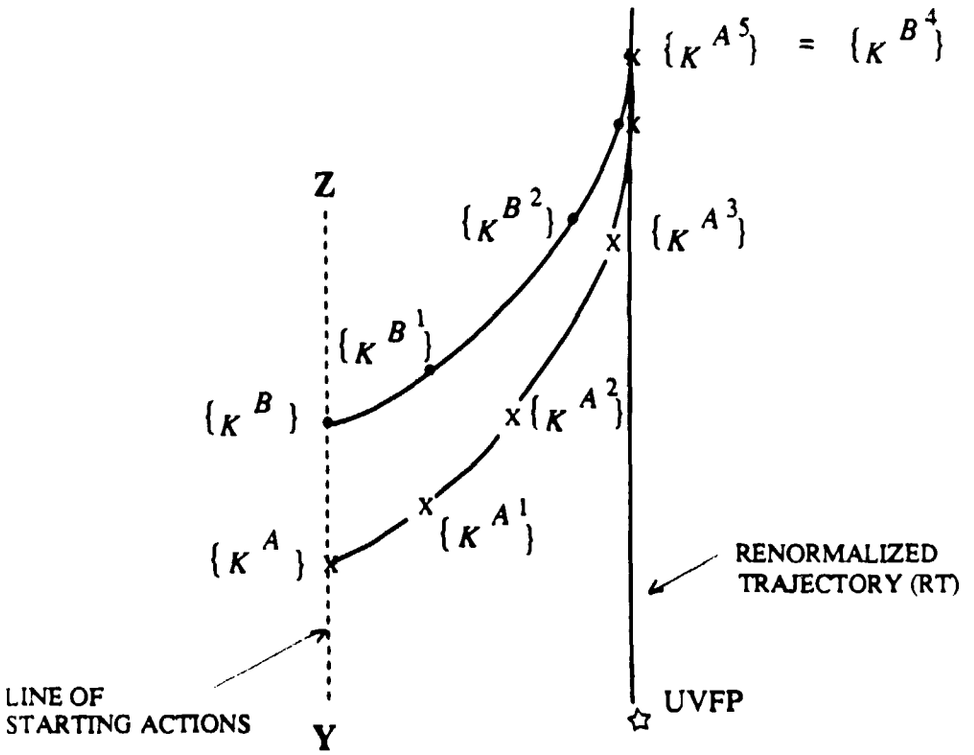

Fig. 4: *Flows for the two simulations in Wilson's 2-lattice method for the case when block actions match.*

For simulations starting at $H \neq H^c$, the eigenvalues initially converge to their fixed point value as flows kill the irrelevant operators, but eventually become worse once the flows go past the linear region of H^*. So the convergence in the eigenvalues as a function of the blocking steps has to evaluated carefully. Also, in this case one expects the three methods to be competitive, though no careful tests have been done so far.

In all three methods described above there is an implicit assumption that the flows converge to H^* or the RT. This can be achieved by either blocking the lattice a sufficient (usually large) number of times or by using a more elaborate Hamiltonian $\{K''\}$ which lies closer to the RT. The main problem with the second approach is that such a Hamiltonian may contain a large number of non-local terms, which makes it slow to simulate. On the other hand if we believe in the central assumption of the renormalization group approach that the couplings fall off exponentially with the range so that the RT and H^* can be approximated by a small number of short range couplings, then do we expect the use of "improved actions" to bare fruit. This avenue is discussed further in section 15.

Why do we expect MCRG results obtained on small lattices with restricted number of blocking steps to be reliable? Since the basis of renormalization group methods is the calculation of the rate of change of couplings along the flows, one believes that the relevant variable that dictates finite size effects is the range of couplings and not the correlation length. If we couple this with the assumption that the strength of couplings fall off exponentially with a small range, then all we need is that the lattice should be large enough to accommodate all important couplings. Present calculations on the 3-d Ising model suggest that one can block down to 8^3 lattices and still get reliable results vis-a-vis finite size effects. So the central assumption of the renormalization group seems to be borne out, even though we don't have a solid understanding of why.

The first step in getting accurate results from MCRG calculations is to know T_c. This too has to be done numerically since it is not known for most models. So I now describe Wilson's 2-lattice method with which one can obtain a successively improved estimate for H^c.

Method to Calculate the Critical Temperature:

The critical temperature is not known for most models. So one starts with a rough determination of the phase boundary. This is done by scanning the coupling constant space using numerical simulations and looking for either discontinuities or divergences in thermodynamic quantities. The order of the transition has to be determined carefully since on finite lattices there are no singularities, and discontinuities at first order transition get rounded. At a second order transition, C_v, χ, *etc.*, have a peak which becomes sharper and diverges only as $V \to \infty$. Thus, only by taking data on lattices of different sizes and using finite size scaling can one systematically improve the estimate of T_c. Wilson's 2-lattice MCRG method provides an accurate way to improve the infinite volume estimate of T_c.

Consider two $MCRG$ simulations L and S with the same starting couplings $\{K_\alpha^0\}$ but on lattices of size $L = b^n$ and $S = b^{n-1}$. Measure expectation values on the original and all block lattices. The key observation about flows is that if the starting $\{K_\alpha^0\}$ is critical then after some number of blockings the renormalized theories will converge to H^*. This means that all block correlation functions will attain their fixed point values *i.e.* $\langle L_\alpha^m \rangle = \langle S_\alpha^{m-1} \rangle$ for m sufficiently large. If, on the other hand, the starting H is non critical, then we can expand block correlation functions about their value at H^* in the linear approximation as

$$\begin{aligned} \langle L_\alpha^m \rangle - \langle S_\alpha^{m-1} \rangle &= \frac{\partial}{\partial K_\beta^0} \left\{ \langle L_\alpha^m \rangle - \langle S_\alpha^{m-1} \rangle \right\} \Delta K_\beta^0 \\ &= \left\{ \langle L_\alpha^m L_\beta^0 \rangle_c - \langle S_\alpha^{m-1} S_\beta^0 \rangle_c \right\} \Delta K_\beta^0 \end{aligned} \tag{11.16}$$

from which one can determine ΔK_α^0 [11]. The critical coupling is then given by

$$K_\alpha^c = K_\alpha^0 - \Delta K_\alpha^0 \ . \tag{11.17}$$

This estimate can and should be improved iteratively. Note that in this 2-lattice method the individual expectation values have finite size errors, however, by comparing expectation values that have been measured on the same size lattices, these errors are vastly reduced.

This completes my intended introduction to RG and MCRG methods. I now turn, posthaste, to their application to QCD.

12. Block Transformations for 4-d SU(N) LGT

The blocking transformations for gauge theories in 4-d are to some extent non-intuitive. The degrees of freedom are SU(N) matrices which are associated with links on the lattice rather than on the sites. Nevertheless the steps one takes are similar to spin models. First we define the block lattice; the simplest example is $b = 2$ for which the basic cell is a 2^4 hypercube. Second the block link should join two adjacent block sites and represent the average value of the gauge field A_μ in the cell. The goal is to construct the block link as an average over the maximum number of the original degrees of freedom in a cell in a gauge covariant way.

The simplest way to construct the block link in a gauge-covariant way is to average the SU(N) matrices that represent the path ordered product of links between the block sites. This leaves intact the local gauge freedom at the block sites, which is the desired gauge freedom on the block lattice. It is somewhat arbitrary what paths to choose except that we like to keep the transformation local and at the same time use the maximum number of degrees of freedom. These features improve the convergence to the RT. It is worth noting that the requirement of gauge covariance is not essential provided the effects of the non-covariant operators generated under blocking are small *i.e.* they are irrelevant operators with a small eigenvalue.

A complication with SU(N) gauge theories is that the sum of SU(N) matrices is not proportional to a SU(N) matrix (SU(2) is an exception). So in order to average the fields, the common practice is to project the sum of paths Σ (which is in general a 3×3 complex matrix) back on to SU(3). This is done by finding the SU(3) matrix U that maximizes the $Tr\ (\Sigma^\dagger U)$ or by generating a matrix with probability $e^{-p\ Tr\Sigma^\dagger U}$ with p a free parameter chosen to optimize convergence. In either case we are losing some dynamic information in the projection, so it needs to be ascertained whether this construction throws out some essential physics! Again note that it is not essential that the blocked links be group elements, though this projection makes calculations on block lattices simple and one does not have to worry about normalizations.

The two methods described below construct the block link in a gauge covariant manner and differ primarily in the definition of the unit cell.

1) $b = 2$ **by Swendsen [26] :** The transformation, in its generalized form, is shown in Fig. 5 where the α_i are tunable parameters. One can in principle include all possible paths that start and end at the block site as that ensures that no gauge fixing is necessary. On the other hand to keep the construction local, all present calculations have used only up to 4 link paths. These calculations show that the convergence of the RGT is improved considerably by optimizing the parameters α_i. I refer you to the original literature for a discussion on this tuning.

Fig. 5: *Swendsen's blocking transformation for gauge theories in 4-d with scale factor* $b = 2$.

2) $b = \sqrt{3}$ **by Cordery, Gupta and Novotny [27] :** This transformation is specific to gauge theories in 4-dimensions. It uses the fact that there are 4 positive 3-cubes associated with each site. The body diagonals of these cubes are orthogonal and of length $\sqrt{3}$ as shown in Fig. 6. The block cell consists of the block site and its 8 nearest neighbors. By performing a local gauge transformation on the 8 NN sites, all 8 links coming out of the block site can be set to the identity. In fig. 6, the unshaded links, say U_a and U_b in the path U_1 for example, can be gauge fixed to the identity, leaving the gauge freedom only at the block sites. The unfixed links connect different block cells; note that their value is identical to the original gauge-covariant path. The total number of free links per block cell is 28 (the figure shows the seven associated with one of the 3-cubes). The block link is constructed as the average of the 6 paths $U_1 \cdots U_6$ which are of equal length. Thus, the transformation uses $\frac{24}{28}$ degrees of freedom at each step, since only the path U_7 is ignored.

It is easy and natural to include scalar matter fields at sites and couple them to the gauge degrees of freedom as one would like to do for the $SU(2) \times U(1)_Y$ theory. Another advantage of this $b = \sqrt{3}$ transformation over the $b = 2$ version is made clear in the block diagonalization process of Mütter and Schilling [28] for defining the renormalized Dirac operator. There one finds that on the block lattice the 9 original modes split into a single light mode and 8 heavy ones.

The only nuisance working with this transformation is that under the first *RGT*,

the new hypercubic lattice is rotated with respect to the old basis leading to a jagged boundary for the box. This can be undone by a second application of the *RGT* with different basis vectors. So the original box geometry is recovered after a scale change by a factor of 3.

Having defined a RGT, all the MCRG methods discussed earlier can be used on non-abelian gauge theories. So far, the main application of MCRG to Lattice QCD has been to calculate the non-perturbative β-function for both SU(2) and SU(3) pure-gauge theories. Present results show that the $b = \sqrt{3}$ *RGT* transformation has better convergence behavior both at strong and at weak coupling than the $b = 2$ transformation as discussed in [29] . I present the status of these results in section 14.

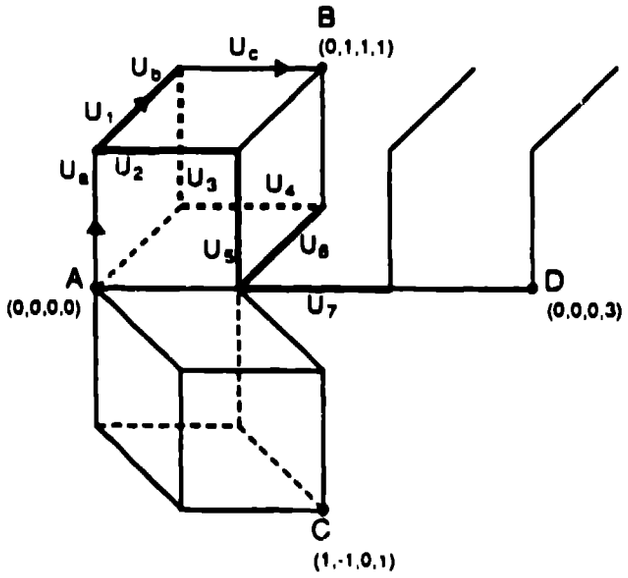

Fig. 6: $b = \sqrt{3}$ *blocking transformation for gauge theories in 4 dimensions*

13. Asymptotic Freedom makes QCD Simple

To analyze QCD, the first thing that we want to determine is the relevant fixed point. Since pure gauge SU(3) has only one coupling g, we should be able to locate the critical point by just tuning g. Politzer and Gross and Wilczek showed that the β-function of QCD is negative [30] . They used perturbation theory to calculate the β function (differential flow equation for g) with respect to a change in momentum scale i.e. $\frac{\partial g}{\partial \log \mu}$. This corresponds to following the flow along the RT backwards i.e. making the lattice spacing smaller. A negative β-function implies that the coupling $g \to 0$ as the cut-off $\Lambda \to \infty$. Thus the prediction from perturbation theory is that the

fixed point is at zero bare coupling. Further, it means that all scaling behavior can be calculated in a perturbation expansion about $g = 0$ *i.e.* the 2-loop β-function (which is gauge and regularization scheme invariant) gives all the scaling laws. For example all physical quantities with the dimensions of mass scale as

$$ma = cf(g) \equiv c\,(\frac{1}{\beta_0 g^2})^{-\frac{\beta_1}{2\beta_0^2}}\ exp[-\frac{1}{2\beta_0 g^2}] \qquad (13.1)$$

for g close to zero. Here $f(g)$ is a universal scaling function.

Asymptotic freedom has made QCD trivial from the point of all the RG machinery we have developed. On the other hand non-abelian gauge theories are not trivial, for even though the fixed point is at zero bare coupling, the renormalized charge is not zero and one gets a well defined interacting theory in the continuum limit. This can be seen in the following way: eqn. (13.1) states that the lattice mass ma of all states goes to zero in the continuum limit because $a \to 0$ as the prescribed function $f(g)$. The limit is taken keeping the physical mass m constant, consequently all ratios of masses have a finite limit. So, if the theory has a mass gap, which we can use as the basic scale, then the spectrum of QCD is predicted in terms of it. From eqn. (13.1) one sees that $g \to 0$ as $a \to 0$, and the two are related through the invariant scale of the theory, c/m. Yet, how does this scale get generated since QCD has only one parameter, the dimensionless coupling g? I remind you that you have been introduced to this phenomenon in field theories which posses no explicit scale under the name dimensional transmutation.

Let me now give an intuitive picture of the statement that the renormalized charge does not go to zero as $a \to 0$. At strong coupling (g large), Wilson (and Wegner for Z_2) showed that all gauge theories in 4 dimensions have an area law *i.e.* the expectation value of large Wilson loops is dominated by the area term, $\langle loop \rangle \sim \exp(-\sigma\ area)$. The coefficient σ is the string tension, so an area law implies that the potential has a linear confining piece. Further, if the SU(3) theory does not have a phase transition separating the weak coupling phase from the strong coupling phase, then, in the continuum limit also, σ is a non-zero constant of the theory and σa^2 scales like $(ma)^2$. The occurrence of a non-zero σ at the fixed point $g = 0$ implies that linear confinement and asymptotic freedom occur simultaneously in non-abelian gauge theories.

QCD, through Eqn. (13.1), requires that there exist a mass gap for a consistent description. Let me state precisely the set of assumptions under which this is valid. If QCD has only one relevant coupling (*i.e.* a single universal scaling function defined in eqn. (13.10)), and no zero mass state at even one given value of the lattice spacing a, then in the continuum limit all states are either massless or there has to be a mass-gap.

The only caveat to this deduction is that there should be no phase boundary separating the the phase with the choosen point with spacing $\hat{a}$ and the continuum limit (*i.e.* they are analytically connected). A continuum theory with all states having zero mass would be a trivial theory and present lattice calculations do not show evidence for a mass-less state in the pure gauge sector. So barring the presence of a phase transition line at coupling weaker than that used in Monte Carlo calculations, pure gauge QCD has a mass-gap. Next, I address the question of mass-less pions in the limit of zero mass quarks. Note that for each flavor, we have to tune the quark mass to get the correct physical spectrum. The tuning can alternately be done by using the associated "pion". Therefore the "pion" mass is an independent parameter and not a prediction of the theory. For example, with SU(3) of flavor, we have to tune either the mass of the u, d, s quarks or that of the π, η and η' mesons to reach the physical point. The fact that the Goldstone theorem, associated with the spontaneous breaking of chiral symmetry, states that pions are massless in the limit of zero mass quarks does not invalidate our previous conclusion. All it says is that to define the physical world at any give lattice scale we have to tune the "pion" mass. If this value is zero, then it is tuned to zero for all a. To conclude, since the two assumptions listed above are rather mild and borne out by present data, therefore, if QCD is the theory of strong interactions, then it predicts that the lightest glueball state is massive.

Given the scaling law, eqn. (13.1), the only unknowns in the theory are the constants c, one for each state. Their determination turns out to be very hard since the c_i are intrinsically non-perturbative. The only first principle way we know at present of determining them is numerical calculations. You have already heard a lot about these simulations from other lecturers, so in the next section I will only talk about the status of their reliability in light of MCRG calculations.

14. Non-Perturbative β-function and Scaling:

Most non-perturbative lattice calculations have been done at $\beta \equiv 6/g^2 \approx 6.0$ where $\xi < 10$. So the big question that arises in evaluating numerical results is how do we know that these calculations represent continuum physics. As discussed in the last section, asymptotic scaling tells us that all physical quantities with dimensions of mass should scale according to eqn. (13.1) near $g = 0$ *i.e.* the change in an individual mass should agree with the function $f(g)$ calculated from the 2 loop perturbative β function. At $\beta = 6.0$, there is no reason for perturbation theory to hold and the check on whether lattice calculations reproduce continuum physics is to show that mass-ratios are constant as a function of β. A sufficient condition that mass ratios remain constant is the existence of a universal non-perturbative function which tends to $f(g)$ as $g \to 0$. This non-perturbative function can be calculated using Wilson's 2-lattice

MCRG method [6].

In pure gauge theory there are three physical quantities that have been measured in many lattice simulations -- the string tension σ, the glueball spectrum and the deconfinement transition temperature T_c. So we can check whether scaling exists for these three observables. The second approach is to make detailed comparisons with the non-perturbative β-function to estimate the minimum value of β beyond which we expect scaling to within some accuracy.

Wilson's 2-lattice method calculates the change in β for a scale change b i.e. the integral of the β-function for a scale change b. This $\Delta\beta$ can be calculated provided we can follow flows along the RT. If K'^2 is the renormalized theory obtained from K'^1 under a scale change b, then by definition $\Delta\beta(\frac{K'^1+K'^2}{2}) = K'^1 - K'^2$ evaluated at the midpoint of the interval. In general, we do not know the RT and numerical simulations are done using simple local actions, for example, β is the only coupling. Wilson provided the link between the goal, to measure flows along the RT, and simulations done by varying just β.

The outline of the method is as follows: first simulate a system of size $L = (b^n)^d$ with couplings K'^A_α and calculate the expectation values of Wilson loops on the original lattice and the n block lattices. Next simulate a second system of size $S = (b^{n-1})^d$ with couplings K'^B_α chosen judiciously, and again calculate the expectation values on the n levels. Adjust the couplings K'^B_α (which requires a new simulation each time) until the expectation values from the two simulations match on the same size lattices, i.e. compare the m^{th} blocked level on the larger starting lattice L with the $(m-1)^{th}$ on the smaller lattice S. The test of convergence of the two theories L^m and S^{m-1} is that the expectations values should match simultaneously at the last few levels. This ideal situation was shown in a two coupling constant space in Fig. 4. At matching, the correlation length on L (starting couplings K'^A_α) is larger than on S (K'^B_α) by the scale factor b since L has been blocked one more time. Thus $\Delta\beta$ for a scale change b is $(K'^A - K'^B)$. If the starting trajectory is taken to be the Wilson axis (or any 1 parameter line specified by β) then $\Delta\beta = (\beta^A - \beta^B)$.

Comparing expectation values is equivalent to matching the action. There is a one to one correspondence between the value of the couplings and the expectation values of Wilson loops. Under the assumption that the fixed point action is local (i.e. at any scale a few short range couplings are sufficient to characterize the action), matching the expectation values of a few small Wilson loops is sufficient to guarantee that the two actions are equal. For this assumption to hold, the matching lattices should be large enough to accommodate all the important couplings and all the corresponding loops should be measured and matched. The key point is that while loop expectation values will have finite size effects, these effects are the same for the two simulations

when matching occurs since the comparison is on approximately the same physical size lattices.

In practice it is sufficient to do two simulations S_1 and S_2 which bracket L and then use interpolation. Once matching takes place on lattices which are large enough to accommodate the important couplings, thereafter, the check that the two flows move together can be made on 1^4 lattices too! Finite size effects in $MCRG$ are controlled by the range of the couplings and not the correlation length. Let me reiterate: the reason that $MCRG$ has good control over finite size effects and is a powerful method is because the range of interactions falls off exponentially even on the critical surface for "good" RGT.

Finally some numbers:

Asymptotic scaling for the simple plaquette action in pure gauge SU(3), ($\beta \equiv \frac{6}{g^2}$), is defined by the 2-loop perturbative β-function,

$$\frac{\partial(\beta)}{\partial(lna)} = -\frac{33}{4\pi^2} - \frac{459}{16\pi^4}\beta + \cdots \quad . \tag{14.1}$$

Since $\frac{\partial\beta}{\partial(lna)}$ is essentially constant over some small scale change b, one can integrate it trivially to get the quantity which can be compared against the non-perturbative β-function calculated using $MCRG$,

$$\Delta\beta = -\frac{\partial\beta}{\partial(lna)}\log b \quad . \tag{4.2}$$

Here also $\Delta\beta$ is defined at the midpoint of the interval spanning a scale change b.

In fig. 7a, I show the $\Delta\beta$ calculated using the $b = 2$ and $b = \sqrt{3}$ transformations. The main features are (a) there is a very large dip in the non-perturbative β-function below $\beta = 6$, and (b) between $6 < \beta < 7$, $\Delta\beta$ lies below the 2-loop perturbative value. The deviation is $\approx 30\%$. The conclusions from this data are that there is no perturbative scaling below $\beta = 6.0$ and furthermore we should not expect asymptotic scaling to better than 10% below $\beta \approx 7.0$.

I would like to bring up a technical point since it serves as a word of caution. These MCRG calculations have been done on rather small lattices. For example the $b = \sqrt{3}$ calculation uses $L = 9^4$ lattices. Therefore, at say $\beta = 7.5$, the box is effectively at very high temperature. The question that arises then is that: even though at matching the two compared systems are at the same physical temperature, does the measured $\Delta\beta$ approach the 2 loop value faster because of the high temperature? The only way to check this is to work on larger lattices. There is some evidence of such finite size effects from the results of a recent calculation in SU(2) [31]. Decker and deForcrand find

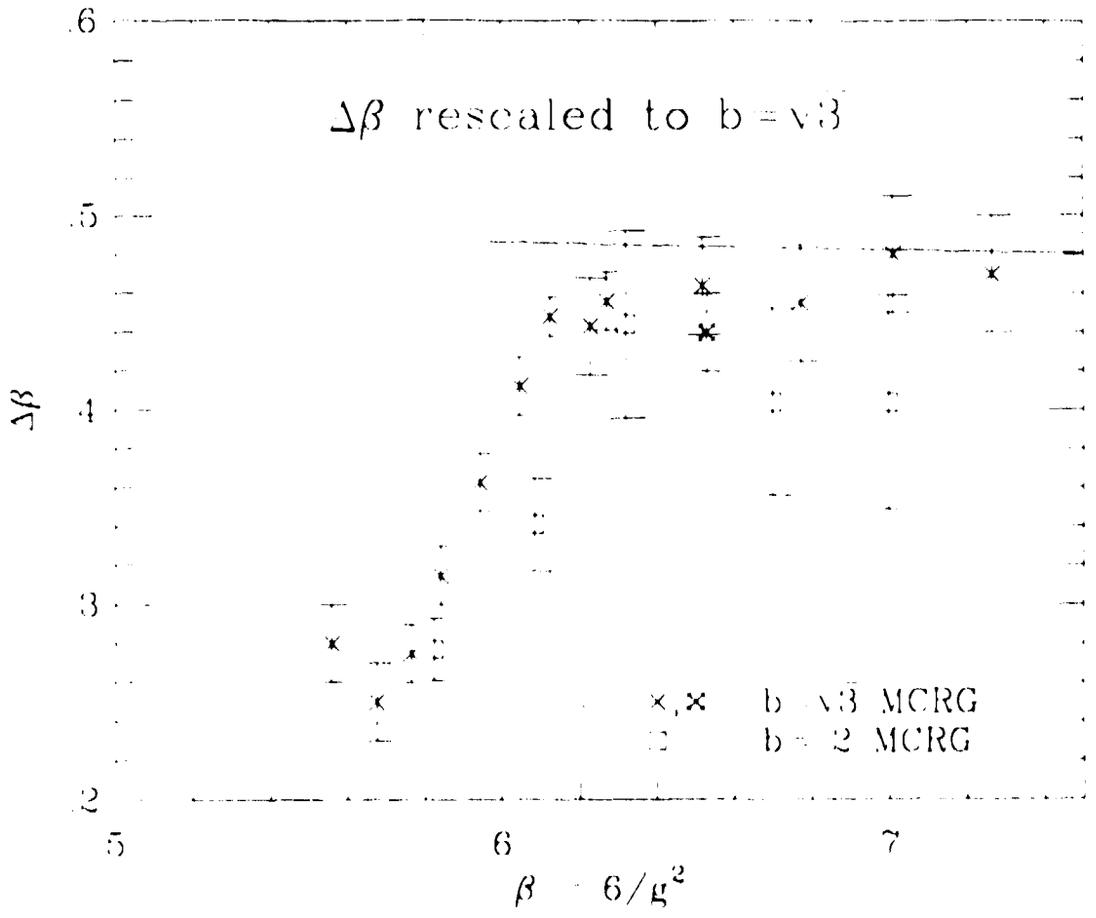

Fig. 7a: *MCRG results for the non-perturbative $\Delta\beta$. The $b = 2$ results have been rescaled to $b = \sqrt{3}$.*

that deviations in $\Delta\beta$ from the 2-loop result are larger for 32^4 lattices with $b = 2$ RGT than those gotten using a 18^4 lattice and $b = \sqrt{3}$ RGT. Since the MCRG calculation is a small overhead on lattice generation, this brute force test will certainly be done when one simulates larger quenched lattices to measure zero temperature observables.

Next we test scaling using the world data for σ, T_c and $M_{0^{++}}$[7]. To calculate $\Delta\beta$ we use the dimensionless lattice value, say ma, calculated at two values of the coupling, β_1 and β_2. Then $\Delta\beta = \beta_2 - \beta_1$ for a scale change $\frac{ma_1}{ma_2}$ holding m constant. To compare with the MCRG result I am forced to rescale $\Delta\beta$ since the measurements of observables have not been made for a constant scale change, so in fig. 7b all the $\Delta\beta$ data from observables is rescaled to $b = \sqrt{3}$. This rescaling does not introduce a significant error, however, as long as one only chooses data with $1.5 \leq b \leq 2.0$.

We find that the data for σ and T_c agree with MCRG results, however, $M_{0^{++}}$ data show large deviations for $\beta < 6.0$. The $\Delta\beta$ from all observables seems to come together at $\beta \approx 6.0$. Unfortunately, it has not been possible to really test scaling for

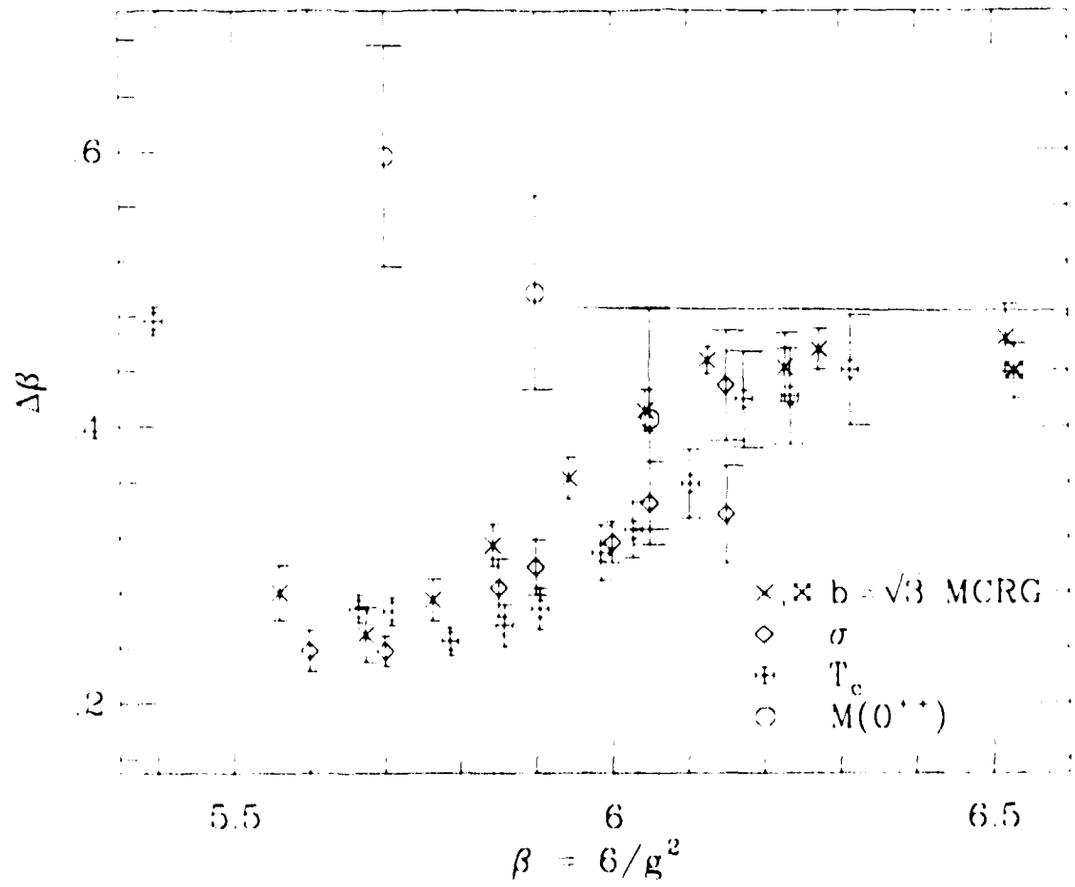

Fig. 7b: *Comparison of $\Delta\beta$ obtained from observables σ, T_c and $M_{0^{++}}$ with the MCRG results.*

$\beta > 6.1$, *i.e.* whether the $\Delta\beta$ continue together, since we have run out of data. So for present we assume scaling to within 10% on faith.

To understand the cause of the dip in the β-function and the lack of scaling for $\beta < 6.0$ we have to examine the phase diagram in the two coupling plane -- fundamental and adjoint representation of the plaquette action. There one finds a line of first order transitions ending in a spurious critical point above the Wilson axis [32]. If one were to extend the transition line it would intersect the Wilson axis at $\beta \approx 5.7$, which coincides with the location of the minimum of the dip. At this spurious critical point the 0^{++} glueball mass vanishes while lines of constant σ and T_c get bunched up. This is why $\Delta\beta$ calculated from σ and T_c show a dip while that from 0^{++} glueball shows a rise. Therefore, even though this spurious singularity lies off the Wilson axis it has changed the scaling behavior in its vicinity.

To conclude, given that there is no scaling for $\beta < 6.0$ and possibly large deviations up to $\beta \approx 7.0$, the only way we can one improve the lattice results (improve scaling)

is to find an action closer to the RT. This hope and desire is discussed next.

15. The Holy Grail: The Renormalized Trajectory

The hope of lattice practitioners is that we can extract continuum physics from simulations with $\xi \approx 10$, and that at these values of couplings one can reliably relate continuum perturbation theory with that on the lattice. The latter is necessary in all calculations where it is essential to fix the normalization of operators, for example in matrix elements calculations as discussed by C. Bernard.

If we knew the location of the renormalized trajectory, then any simulation along it that satisfied the conditions $L >> \xi >> 1$ would give the correct mass ratios and the desired continuum physics. This ideal situation will be hard to achieve in practice. The best we can hope for is to find an action that lies closer to the renormalized trajectory. MCRG methods are essential to determine and evaluate such improved actions. The status of this approach is contained in [33] and [34] , and it is obvious that far more work needs to be done to find and demonstrate that one has derived an action that significantly improves scaling.

Another avenue for research is to develop MCRG methods to map the flows for the Dirac operator. This would be very useful for developing multigrid ideas for inverting the fermion matrix. A small step in this direction for the $b = \sqrt{3}$ transformation is described in [34].

In the calculation of matrix elements it is necessary to fix the normalization of lattice operators with respect to their value in some continuum regularization scheme at the same physical scale. The lattice coefficients when analyzed in perturbation theory have a dependence and in some cases even diverge as $1/a$. These lattice artifacts need to be removed. One way to handle this is to add operators to the lattice action that kill this unwanted behavior. The Rome group (ELC) is at present trying to implement this program using perturbation theory. Since this is a very new development let me end with citing a reference [35] .

Acknowledgements:

It is a pleasure to thank T. DeGrand and D. Toussaint for inviting me to give these lectures. While MCRG methods have not provided any hard physical numbers in Lattice QCD, the philosophy prevades the field and justifies many of the calculations. I hope that I have managed to at least show this unifying structure in these lectures.

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