Conf-820737--1

CONF-820737--1

DE82 [illegible]

MOMENT SERIES FOR MOMENT ESTIMATORS OF THE PARAMETERS OF A WEIBULL DENSITY

K. O. Bowman*

Mathematics and Statistics Research Department
Computer Sciences Division
Union Carbide Corporation, Nuclear Division
Oak Ridge, Tennessee 37830

L. R. Shenton

Office of Computing and Information Service
Boyd Graduate Studies Building
University of Georgia
Athens, Georgia 30602

*Research sponsored by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Department of Energy under contract W-7405-eng-26 with the Union Carbide Corporation

DISCLAIMER

ABSTRACT

Taylor series for the first four moments of the coefficients of variation in sampling from a 2-parameter Weibull density are given; they are taken as far as the coefficient of n^{-24}. From these a four moment approximating distribution is set up using summatory techniques on the series. The shape parameter is treated in a similar way, but here the moment equations are no longer explicit estimators, and terms only as far as those in n^{-12} are given. The validity of assessed moments and percentiles of the approximating distributions is studied. Consideration is also given to properties of the moment estimator for 1/c.

1. INTRODUCTION

The 3-parameter distribution function is

$$F(t) = 1-\exp\{-(t-s)^c/b^c\} \quad (t \geq s, c, b>0) \qquad (1)$$

where s is the origin, b, and c the scale and shape parameters respectively.

In one form or another (the parameter 1/b is sometimes used) the density has widespread application, the precise reason for its use not being clear always (it has a slight advantage in the simplicity of its distribution function, but this is a minor point in the face of computers of one sort or another). However, it seems well suited to situations involving breaking strengths (Barlow et al., 1979; Cain and Knight, 1981, for example), survival times (Peto and Lee, 1973, for example), etc. It has been used as a model for wind speed (Stewart and Essenwanger, 1978), a main attraction here being the interest in wind power, which is proportional to the cube of wind speed; this translates into changing the value of c in the Weibull model.

Our interest is in the nature of series for the mean, variance, etc., of moment estimators for the parameters in the 3-parameter case. However, the complications here are such as to confine attention more or less to basic asymptotics (a partial study of the situation has so far produced the first 12 terms in the moments of the skewness). Although it is quite likely that properties of estimators in the 3-parameter case will differ considerably from those in the 2-parameter case, the study of the latter should bring out some of the difficulties. The only previous study of the series in this case (Newby, 1980) goes no further than basic asymptitics, and these were not free from error.

1.1 What do we expect from a study of estimators?

Of course, a study of estimation problems should at least have a better than fuzzy aim. Basically estimates of parameters survive only if they lead to passing a satisfactory goodness of fit test. It seems reasonable to assume that refinements predicated on methods of estimation pale in comparison to model selection. Again it is always tempting to base decisions on narrow choices of criteria. Since sample size plays only a minor role in basic asymptotic assessments of variances (and biases), we are deceiving ourselves when decisions are based on asymptotic comparisons. In addition, it is all too easy to introduce a caveat invoking "a large enough sample," a transparent circularity digression.

When we pay some attention to what comes after the first order asymptotic in means and variances, for example, we find a change of attitude to the asymptote, for we may be confronted with a few decreasing terms followed by one or more surges and variegated sign patterns. It is surely time we became aware of the existence of higher order terms and studied ways of using the information they contain. The statistics community seems to be half a century behind the times in this respect and completely unaware of advances and studies due to a school of theoretical physicists (see for example, the preface to Baker and Gammel, 1970).

1.2 Problems with the 3-parameter Weibull and aims of this study. Moments of the maximum likelihood estimators in this case probably do not exist and other procedures are needed. Fitting by moments is quite straightforward, using the skewness to fix the shape parameter c, then the variance to fix b, and lastly, the mean to fix the start s. But properties of the distribution of these estimatorss presents formidable mathematical difficulties, although a computer assisted approach is feasible. The 2-parameter case has already been studied; Bowman and Shenton (1981) have given details for the first four moments of the coefficient of variation, using summatory algorithms on the series (carried out to terms of order n^{-24}, n being the sample size). Here we discuss characteristics of these series and series for the first four moments of the moment estimator (c*). Having the four moments for ν* and c* we can compare the percentage points of the one against the other using a four moment approximating distribution. Questions of validity are considered. Lastly, some general comments are added concerning the information in (what appear to be) divergent series.

2. LEVIN'S ALGORITHM AND ν* MOMENTS WHEN c = 1.5

2.1 The series (Table 1) alternate in sign and diverge faster than the single factorial series $(1-1!/n+2!/n^2-\ldots)$ but not as fast as the double factorial series $(1-2!/n+4!n^2-\ldots)$. We think the Levin algorithm (Levin, 1973) using

[TABLE 1 about here]

$$\alpha_r = \frac{\sum_{j=0}^{r} (-1)^r \binom{r}{j} \left(\frac{j+1}{r+1}\right)^{r-1} \frac{A_{j+1}}{a_j}}{\sum_{j=0}^{r} (-1)^r \binom{r}{j} \left(\frac{j+1}{r+1}\right)^{r-1} \frac{1}{a_j}} \qquad (2.1)$$

where for the series $a_0 + a_1/n + \ldots$,

$$A_{j+1} = a_0 + a_1/n + \ldots + a_j/n^j, \quad (j = 0,1,\ldots)$$

applied to certain divergent series is divergent itself, but there exists a best member of the sequence (or stopping point). Now a peculiar aspect of series for statistical moments is that, for small n, we can often derive exact results using dimension reducing transformations or quadrature; the latter poses problems when n exceeds five or so. In the present case exact results have been found for n=2, and 4 using quadrature. Details for the first four moments for these values of n are given in Table 2.

TABLE 2. Levin's t-algorithm and the moments series for ν^*, n=2,3,4. (Entries are α_{r-1}; c=1.5)

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=2	6	.40486185	.06017495	-.01056879	.04307018
	7	.39979784	.06217180	-.00907002	.03302390
	8	.39479750	.06467342	-.00534255	.02057064
	9	.39044492	.06727881	-.00085487	.01098028
	10	.38694110	.06965979	.00356697	.00954210
	11	.38417228	.07149851	.00729993	.01515748
	12	.38196432	.07259348	.01022352	.02266028
	13	.38020300	.07280097	.01228739	.02792580
	14	.37885619	.07204075	.01412681	.02847297
	15	.37795604	.07018375	.01586526	.02370270
	16	.37759054	.06714841	.01863328	.01337855
	17	.37788403	.06275753	.02126346	-.00030514
	18	.37899629	.05713638	.01761667	-.01086268
	19	.38106947	.05199600	-.02680927	.00972894
	20	.38395273	.05679861	-.15955086	.09882317
	True	.382657	.06360	.00649	.008786

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=3	6	.49557302	.05132049	.00150823	.01928206
	7	.49355822	.05191633	.00191748	.01525018
	8	.49189084	.05253079	.00265513	.01097594
	9	.49068668	.05309256	.00340504	.00837420
	10	.48986117	.05353657	.00403855	.00793691
	11	.48928991	.05382896	.00449792	.00862812
	12	.48888579	.05397361	.00481116	.00943739
	13	.48859966	.05398413	.00500293	.00990738
	14	.48840697	.05388140	.00516034	.00989262
	15	.48829599	.05367738	.00529306	.00946541
	16	.48826233	.05338954	.00548973	.00870570
	17	.48830396	.05302404	.00564607	.00784777
	18	.48842022	.05261345	.00540456	.00730647
	19	.48860605	.05229469	.00304165	.00850647
	True	.488905	.052842	.0047611	.008096

TABLE 1. Moments of ν^*; $c = 1.5$.

S	$\mu_1'(\nu^*)$		$\mu_2(\nu^*)$		$\mu_3(\nu^*)$		$\mu_4(\nu^*)$	
0	6.7896869309735	-01						
	-5.6132013492859	-01	2.6772673984050	-01				
	1.6557176844808	-01	-6.2527413386551	-01	3.2420881596408	-01	2.1503282167687	-01
	-1.8474620536721	00	2.8054197020753	00	-2.8812804441747	00	1.0502505615668	-01
	2.1922586200894	01	-3.1740301673027	01	3.0748070794268	01	-1.5448458389139	01
5	-4.3580645232247	02	6.1698213591867	02	-5.8538726736417	02	3.6415327254464	02
	1.2140053990087	04	-1.6985660574568	04	1.6234033676654	04	-1.1056445045515	04
	-4.4319272510179	05	6.1568194941015	05	-5.9254772698640	05	4.2823776889988	05
	2.0203579868046	07	-2.7938852616884	07	2.7053438277634	07	-2.0334982988104	07
	-1.1120346826810	09	1.5329655502103	09	-1.4920295848592	09	1.1530923018020	09
10	7.2096574734542	10	-9.9160099223580	10	9.6926666881294	10	-7.6471743602422	10
	-5.4024459971249	12	7.4175948216140	12	-7.2765971642972	12	5.8332256542299	12
	4.6095483370065	14	-6.3204220814912	14	6.2191074550240	14	-5.0490312473491	14
	-4.4246256616307	16	6.0603265024852	16	-5.9786222434405	16	4.9040292832800	16
	4.7306820232839	18	-6.4738200315179	18	6.4007815420352	18	-5.2953093012828	18
15	-5.5869875796767	20	7.6400548503158	20	-7.5684964011259	20	6.30650928[illegible]3133	20
	7.2369947493057	22	-9.8902833121783	22	9.8142882027352	22	-8.2281468998640	22
	-1.0219131662144	25	1.3958391428942	25	-1.3871913557010	25	1.1691746931915	25
	1.5647248801081	27	-2.1362971186825	27	2.1258940610359	27	-1.8000727655091	27
	-2.5858544423711	29	3.5290314664515	29	-3.5160392107693	29	2.9892767600516	29
20	4.5932258200979	31	-6.2663989110185	31	6.2500388332982	31	-5.3328414511363	31
	-8.7373158431193	33	1.1916385088463	34	-1.1896832619930	34	1.0183630563144	34
	1.7739714027278	36	-2.4187672237888	36	2.4169335423533	36	-2.0748600181908	36
	-3.8328687301095	38	5.2247417831321	38	-5.2249959528916	38	4.4971880275419	38
24	8.7887933193700	40	-1.1977722473854	41	1.1987171388092	41	-1.0341841311369	41

($\nu^* = \sqrt{m_2}/m_1'$, where m_2 is the second central moment of the sample, and m_1' the mean.)

TABLE 2--(Continued)

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=4	6	.54117667	.04353539	.00327633	.01095807
	7	.54021557	.04376028	.00341579	.00900337
	8	.53952911	.04396895	.00363020	.00727274
	9	.53909662	.04414201	.00382233	.00643290
	10	.53883094	.04426374	.00396656	.00631124
	11	.53866334	.04433463	.00405987	.00645609
	12	.53855483	.04436501	.00411736	.00660624
	13	.53848469	.04436465	.00414895	.00668134
	14	.53844185	.04434288	.00417348	.00667184
	15	.53841989	.04430541	.00419236	.00660458
	16	.53841484	.04425801	.00421892	.00650020
	17	.53842363	.04420338	.00423718	.00639565
	18	.53844390	.04414778	.00420655	.00633948
	19	.53847285	.04410983	.00395035	.00647756
	20	.53850428	.04415703	.00336182	.00689265
	True	.5385352	.0441541	.0041379	.0064037

(True given in an appendix on small sample results.)

The "boxed" entries are those closest to the true value. In the case of the variance there is not much to choose between r=8 and r=17. The consistency of the stopping point as n increases is noteworthy. In Table 3 we show the sequences for n=5 and n=10.

TABLE 3. Levin's t-algorithm and the moments series for ν^*, n=5,10; c=1.5

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=5	7	.56809277	.03766328	.00331170	.00605289
	8	.56776951	.03774945	.00339003	.00527773
	9	.56758617	.03781515	.00345310	.00496769
	10	.56748250	.03785700	.00349608	.00492982
	11	.56742171	.03787906	.00352147	.00497093
	12	.56738511	.03788745	.00353590	.00500903
	13	.56736316	.03788679	.00354316	.00502583
	14	.56735080	.03788075	.00354857	.00502246
	15	.56734505	.03787145	.00355242	.00500770
	16	.56734405	.03786066	.00355762	.00498707
	17	.56734644	.03784916	.00356077	.00496833

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=10	7	.62353379	.02206202	.00161832	.00179863
	8	.62351206	.02206626	.00162092	.00176415
	9	.62350304	.02206853	.00162229	.00175653
	10	.62349912	.02206957	.00162297	.00175610
	11	.62349733	.02206996	.00162328	.00175668
	12	.62349649	.02207006	.00162341	.00175704
	13	.62349610	.02207003	.00162346	.00175714
	14	.62349594	.02206996	.00162349	.00175709
	15	.62349589	.02206988	.00162351	.00175701
	16	.62349589	.02206981	.00162353	.00175693
	17	.62349591	.02206976	.00162353	.00175688

2.2 Another algorithm (a modified Borel-Padé described in Shenton and Bowman, 1977a) basically considers

$$S(n) \sim e_0 + e_1/n + \ldots \qquad (2.2)$$

$$\sim \int_0^\infty e^{-t} t^{a-1} \{k_0 + k_1(h/n)t + k_2(h/n)^2 t^2 + \ldots\} dt \qquad (a>0; h>0)$$

leading to the summation formula (to be referred to as 2cB)

$$F_r(n;a,h) = N \sum_{s=0}^{r-1} K_s(a,h)\, \Phi_s(N;a) \quad (N = n/h) \qquad (2.3)$$

where

$$K_s(a,h) = \sum_{r=0}^{s} \binom{s}{r} \frac{e_r}{h^2 \Gamma(a+2r)},$$

$$\Phi_s(N;a) = \int_0^\infty \frac{e^{-t} t^{a+2s-1} dt}{(N+t^2)^{s+1}}.$$

$\Phi_s(\cdot)$ can either be calculated by quadrature or using the recurrence

$$4s(s+1)\Phi_{s+1}(N;a) = \sum_{r=0}^{2} G_r \Phi_{s-r}(N;a), \quad (s=2,3,\ldots) \qquad (2.4)$$

with

$$G_0 = 2s(6s+2a-3),$$
$$G_1 = -\{12s^2+8s(a-3)+a^2-7a+12+N\},$$
$$G_2 = (2s+a-3)\,(2s+a-4);$$

and

$$\Phi_0(N;a) = \int_0^\infty e^{-t} t^{a-1} (N+t^2)^{-1} dt,$$

$$\Phi_1(N;a) = \frac{1}{2} a\, \Phi_0(N;a) - \frac{1}{2} \Phi_0(N;a+1),$$

$$\Phi_2(N;a) = \{(4a+6)\Phi_1(N;a) - (a^2+a+N)\Phi_0(N;a) + \Gamma(a)\}/8.$$

Actually these indicate that $\Phi_s(\cdot\cdot)$ is a linear function of the basic functions $\Phi_0(\cdot\cdot)$, and $\Phi_1(\cdot\cdot)$ and indeed

$$\Phi_s(N;a) = N\, \Pi^{(0)}_{s_2}(N) + \Pi^{(1)}_{s_0}(N)\Phi_0(N;a) + \Pi^{(2)}_{s_2}(N)\Phi_1(N;a)$$

where $s_i = [s-i-1)/2]$, $i = 1,2,3^{(s\geq 3)}$ (2.5)

and $\Pi^{(\ )}_{s_i}(\cdot)$ are real polynomials.

Results are given in Table 4, and whereas there are slight discrepancies for n=5 and the Levin values (Table 3), the agreement is quite satisfactory. One should notice the reduction in the sizes of the first differences for each moment as n increases. This characteristic also applies to the Levin sequences provided the differences relate to a neighborhood of the best stopping point. It could be that the Borel sequences converge, whereras as noted earlier those for Levin do not.

Our preferred values for the four moments are:

n	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
5	0.568	0.0378	0.0035	0.0049
10	0.6236	0.02207	0.00162	0.00175

TABLE 4. Borel-Padé Sequences for the moments series for ν^* when c = 1.5, n = 5,10

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=5	21	.573267	.037860	.003306	.004785
	22	.572835	.037874	.003312	.004771
	23	.572446	.037886	.003318	.004760
	24	.572094	.037894	.003325	.004752
	25	.571774	.037900	.003332	.004746
	S_3	.568643	.037912	.003016	.004731
	S_5	.567936	.037896	.003470	.005021

	r	$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
n=10	21	.624125	.022069	.001618	.001754
	22	.624057	.022070	.001618	.001754
	23	.623999	.022070	.001619	.001753
	24	.623947	.022070	.001619	.001753
	25	.623903	.022070	.001619	.001753
	S_3	.623586	.022070	.001635	.001752
	S_5	.623529	.022070	.001622	.001760

(Entries are $F_r(n;1,1)$; see (2.3). S_3, S_5 refer to Shanks' (1955) extrapolate on F_{23}, F_{24}, F_{25} and F_{21}, F_{22}, F_{23}, F_{24}, F_{25}. Indeed

$S \equiv (F_{23}, F_{25} - F_{24}^2)/\Delta^2 F_{24}$, and

$$S_5 = \begin{vmatrix} F_{21} & F_{22} & F_{23} \\ \Delta F_{21} & \Delta F_{22} & \Delta F_{23} \\ \Delta^2 F_{21} & \Delta^2 F_{22} & \Delta^2 F_{23} \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ \Delta F_{21} & \Delta F_{22} & \Delta F_{23} \\ \Delta^2 F_{21} & \Delta^2 F_{22} & \Delta^2 F_{23} \end{vmatrix} .$$

These extrapolates are to be used with caution; sometimes they reverse a trend throwing suspicion on the process.)

2.3 We now assume that for higher values of n the stopping rule for the Levin sequences holds. It is possible, in view of the conjectured divergency, that each n has its own best stopping point for each moment. However, the sequence values become lightly packed for larger n; for example for c=1.5, n=50, we have for var(ν^*),

α_{10} = 5.1231738-03	α_{16} = 5.1231117-03
α_{11} = 5.1231473-03	α_{17} = 5.1231104-03
α_{12} = 5.1231320-03	α_{18} = 5.1231096-03
α_{13} = 5.1231228-03	α_{19} = 5.1231090-03
α_{14} = 5.1231173-03	α_{20} = 5.1231087-03
α_{15} = 5.1231138-03	α_{21} = 5.1231085-03

and α_{16} is the best value flagged from earlier cases. It seems reasonable to take 5.12311-03 as the preferred value. Of course, increasing n still further and choosing a best value would have to take into account the basic accuracy of the moment series coefficients.

2.4 A further set of comparisons for c=2.0 (Table 5) shows that conclusions similar to those drawn for c=1.5 hold.

2.5 A summary of the characteristics of the moment series for ν^* is given in Table 6. It will be noticed that divergency is pronounced for small c, corresponding to marked skewness (and long-tailed) in the Weibull density. The divergency becomes less severe as c increases, but now there is a disruption in the alternating sign pattern,

TABLE 5 ~~Moment Assessments for ν^*, c=2.0~~
Levin and Borel (modified)

n		$\mu_1'(\nu^*)$	$\mu_2(\nu^*)$	$\mu_3(\nu^*)$	$\mu_4(\nu^*)$
2	L	.306971 11	.047562 9	.006066 7	.006613 5
	T	.306853	.047434	.006883	.006036
	DS	.312	.0573	.0120	.00183
3	L	.388759 11	.037496 9	.003423 7	.004547 5
	T	.388807	.037349	.003626	.004275
	DS	.3848	.0395	.00453	.00635
4	L	.425847 11	.029815	.002288 7	.002972 5
	T	.425863	.029785	.002340	.00288
	DS	.4262	.0293	.00263	.00338
5	L	.446871 11	.024596 9	.001640 7	.002040 5
	2cB*	.446913	.024645	.001608	.002047
	DS	.44702	.02441	.00177	.002170
10	L	.486309 11	.013020 9	.000519 7	.000559 5
	2cB*	.486309	.013021	.000518	.000559
	DS	.486299	.013030	.000510	.000563
20	L	.5049	.006696	.000146	.000143
	S_1	.5048	.006708	.000152	.000144
	S_2	.5044	.006675	.000140	.000141

(L is the Levin t-algorithm, the parenthetic entry referring to the best approximant α_r. T refers to values computed by quadrature. 2cB refers to the Borel-Padé algorithm effectively using all the available coefficients (see expression (2.3)); for μ_1' we have used a=2, h=1 with terms up to n^{-4} truncated, and for μ_2, μ_3, and μ_4 we have used a=1, h=1 with the n^{-1} term omitted for μ_2 and no truncations for μ_3 and μ_4. DS refers to the direct sum of the series stopping at the first numerically smallest term. S_1 and S_2 refer to simulations of 10^5 cycles each, the s.d. of the mean being 0.0003 approx.)

TABLE 6. Magnitudes and sign patterns occurring in the first four moment series for the coefficient of variation ν*, as affected by the shape parameter c.

Moment	$\mu_1'(\nu^*)$		$\mu_2(\nu^*)$		$\mu_3(\nu^*)$		$\mu_4(\nu^*)$	
c	$\lvert v_{24}^{(1)}/v_0^{(1)}\rvert$		$\lvert v_{24}^{(2)}/v_1^{(2)}\rvert$		$\lvert v_{24}^{(3)}/v_2^{(3)}\rvert$		$\lvert v_{24}^{(4)}/v_2^{(4)}\rvert$	
0.9	2.9	67	4.6	67	6.3	66	2.3	67A
1.0	8.6	60	1.7	61	3.6	60	1.1	61A
1.1	4.2	55	1.0	56	3.0	55	7.4	55A
1.2	1.5	51	4.2	51	1.7	51	3.5	51A
1.3	2.5	47	7.4	47	4.0	47	6.9	47A
1.4	1.2	44	3.8	44	2.6	44	3.9	44A
1.5	1.3	41	4.5	41	3.7	41	4.9	41A
1.6	2.6	38	9.4	38	9.2	38	1.1	39A
1.7	7.9	35T	3.0	36	3.4	36	3.5	36A
1.8	3.1	33T	1.2	34	1.6	34	1.4	34A
1.9	1.3	31T	5.0	31	7.4	31	6.1	31A
2.0	3.7	28T	1.5	29	2.4	29	1.9	29A
2.1	2.4	26	9.6	26L	1.7	27L	1.2	27L
2.2	5.7	25B	2.5	26B	5.4	26B	3.5	26B
2.3	1.6	25B	6.7	25B	1.4	26B	8.8	25B
2.4	3.2	26B	1.3	27B	2.3	27B	1.6	27B
2.5	1.2	26B	4.7	26B	1.0	27B	5.9	26B
2.6	5.6	25B	2.3	26B	5.3	26B	2.9	26B
2.8	1.2	25B	4.9	25B	1.2	26B	6.1	25B
3.0	6.0	25B	2.4	26B	6.0	26B	3.0	26B
3.2	1.8	26B	7.3	26B	1.9	27B	8.9	26B
3.5	3.4	26B	1.3	27B	3.4	27B	1.6	27B
3.8	7.3	26B	2.7	27B	6.7	27B	3.2	27B
4.0	2.6	27B	9.6	27B	2.4	28B	1.1	28B

(Introduction of letters of T, L, and B indicate disruption of alternating sign pattern. Disruption occurs at the top (T) of the series, bottom (L), and both top and bottom (B). A refers to a sign pattern with alternation except for the first two terms. In the moment columns each second column refers to the power of ten used as a multiplier. $v_{24}^{(i)}/v_s^{(i)}$, i = 1,2,3,4 refer to coefficients in the series.)

anomolies creeping in for both the initial terms and those for the highest coefficients (n^{-24}, n^{-23}, etc.). It will be recalled that the density itself tends towards symmetry with c slightly larger than three and thereafter achieves negative skewness. We have not carried out extensive studies of the series for c > 4.0.

2.6 Moment assessments for $c = 0.8\ (0.1),\ 2.6\ (0.2),\ 3.2,\ 3.5,\ 3.8,\ 4.0$ and $n = 13(1)50(5)100$ have been tabulated; for $\mu_1'(\nu^*)$, $\mu_2(\nu^*)$, $\mu_3(\nu^*)$, $\mu_4(\nu^*)$ we used 13, 18, 12, and 11 coefficients of the corresponding series. A selection is given in Table 7. From these it will be seen that for a sample of n known to come from a Weibull density with parameter c (which fixes ν) the estimate ν* of ν is expected to underestimate the true ν. Using a least squares procedure on the tabulated values, we have derived the unbiased c, given ν* and n, namely

[TABLE 7 about here]

$$\bar{c} \sim \{1+b_{01}/n+b_{02}/n^2+\nu^*(b_{10}+b_{11}/n)+\nu^{*2}b_{20}\}/\{a_{00}+a_{01}/n+a_{02}/n^2+\nu^*(a_{10}+a_{11}/n)+\nu^*a_{20}\} \quad (0<\nu^*<1) \tag{2.6}$$

where

a_{00} =	-0.02586359654	b_{00} =	1.0
a_{01} =	0.1368685508	b_{01} =	-0.03224982824
a_{02} =	0.1941632032	b_{02} =	-0.05540306547
a_{10} =	0.9555907114	b_{10} =	-0.4528195583
a_{11} =	-0.2097962308	b_{11} =	-0.9542316494
a_{20} =	-0.2918002479	b_{20} =	0.08948942647,

the errors being numerically less than 0.5%. The grid of values used was $n=10(1)20,22,25(5)50(10)100$ and $0.8 \le c \le 4$ involving 575 points.

In a similar way if we need a quick, fairly accurate solution to the equation

$$\frac{\Gamma(1+2/c)}{\Gamma^2(1+1/c)} = 1 + \nu^2, \tag{2.7}$$

then

$$1/c = \frac{c_1\nu + c_2\nu^2 + c_3\nu^3 + c_4\nu^4}{1 + d_1\nu + d_2\nu^2 + d_3\nu^3} \tag{2.8}$$

where

c_1 =	0.779960622	d_1 =	0.188028602
c_2 =	0.587095391	d_2 =	0.609555293
c_3 =	0.471569800	d_3 =	0.00282363508
c_4 =	-0.0382146209		

The error in the approximation to c is 0.0004% or less for $0.6 \le c \le 6.6$ $(0.178 \le \nu \le 1.758)$.

If we replace ν by ν* then the moment estimator of c is the real solution of

$$\Gamma(1+2/c^*)/\Gamma^2(1+1/c^*) = 1 + \nu^{*2} \tag{2.8}$$

showing that the distribution of c* is a function of c only, ν* being scale free (this is also evident from the tabulated moments of ν* which do not involve b).

3. THE DISTRIBUTION OF c*

3.1 Moment series. It will be evident from the equation for c* that the Taylor series for its moments will be more complicated. However, we use a two-stage process, expressing c* in terms of ν*, and ν* in terms of the moments m_1' and m_2'. Thus we set

TABLE 7. Moments of ν^* using the Levin sequences.

n	c	1.0	1.5	2.0	2.5	3.0	3.5	4.0
	v	1.000	0.679	0.523	0.428	0.363	0.316	0.281
15	μ_1'	0.914	0.642	0.499	0.409	0.348	0.303	0.268
	σ	0.200	0.125	0.094	0.077	0.067	0.059	0.053
	$\sqrt{\beta_1}$	0.780	0.457	0.300	0.236	0.217	0.219	0.231
	β_2	4.278	3.533	3.227	3.110	3.058	3.036	3.026
20	μ_1'	0.934	0.651	0.505	0.414	0.352	0.306	0.272
	σ	0.182	0.110	0.082	0.067	0.058	0.051	0.046
	$\sqrt{\beta_1}$	0.779	0.421	0.266	0.205	0.186	0.186	0.195
	β_2	4.335	3.460	3.181	3.086	3.046	3.026	3.019
25	μ_1'	0.946	0.657	0.508	0.417	0.354	0.308	0.273
	σ	0.168	0.099	0.073	0.060	0.052	0.046	0.041
	$\sqrt{\beta_1}$	0.766	0.392	0.241	0.184	0.165	0.164	0.172
	β_2	4.332	3.402	3.150	3.070	3.037	3.021	3.015
30	μ_1'	0.954	0.660	0.511	0.419	0.356	0.310	0.275
	σ	0.157	0.091	0.067	0.055	0.047	0.042	0.038
	$\sqrt{\beta_1}$	0.750	0.367	0.222	0.168	0.150	0.149	0.156
	β_2	4.304	3.356	3.128	3.059	3.031	3.018	3.012
35	μ_1'	0.960	0.663	0.513	0.420	0.357	0.311	0.275
	σ	0.148	0.085	0.062	0.051	0.044	0.039	0.035
	$\sqrt{\beta_1}$	0.732	0.347	0.207	0.156	0.139	0.137	0.143
	β_2	4.266	3.319	3.112	3.051	3.027	3.016	3.010
40	μ_1'	0.965	0.665	0.514	0.421	0.358	0.312	0.276
	σ	0.140	0.080	0.058	0.047	0.041	0.036	0.033
	$\sqrt{\beta_1}$	0.714	0.329	0.195	0.146	0.129	0.128	0.134
	β_2	4.224	3.289	3.099	3.045	3.024	3.014	3.009
45	μ_1'	0.969	0.667	0.515	0.422	0.358	0.312	0.277
	σ	0.134	0.075	0.055	0.045	0.038	0.034	0.031
	$\sqrt{\beta_1}$	0.697	0.314	0.185	0.137	0.122	0.120	0.125
	β_2	4.182	3.265	3.089	3.040	3.021	3.012	3.008
50	μ_1'	0.972	0.668	0.516	0.422	0.359	0.313	0.277
	σ	0.128	0.072	0.052	0.042	0.036	0.032	0.029
	$\sqrt{\beta_1}$	0.681	0.301	0.176	0.131	0.115	0.114	0.119
	β_2	4.140	3.244	3.080	3.037	3.019	3.011	3.007
70	μ_1'	0.980	0.671	0.518	0.424	0.360	0.314	0.278
	σ	0.111	0.061	0.044	0.036	0.031	0.027	0.025
	$\sqrt{\beta_1}$	0.624	0.261	0.150	0.110	0.097	0.096	0.100
	β_2	3.992	3.185	3.058	3.026	3.014	3.008	3.005
100	μ_1'	0.986	0.673	0.519	0.425	0.361	0.314	0.279
	σ	0.095	0.051	0.037	0.030	0.026	0.023	0.021
	$\sqrt{\beta_1}$	0.560	0.223	0.126	0.092	0.081	0.080	0.083
	β_2	3.826	3.136	3.042	3.019	3.010	3.006	3.004

(For c=1, there is the special property that ν^* is distributed independently of the mean (see Bowman and Shenton, 1981) so that this special property is used along with Levin's algorithm. For $c > 1$, we have used Levin's α_{16} for $\mu_1'(\nu^*)$, α_{10} for $\mu_2(\nu^*)$, α_{13} for $\mu_3(\nu^*)$, and α_{10} for $\mu_4(\nu^*)$. If a 4-moment Pearson distribution is now fitted, assuming the value of c is known, our guess is that the middle percentage points (1, 5, 10, 90, 95, and 99) are in error by not more than 5% and very likely less for samples exceeding 30 or so.)

$$c^* = c + c_1(\nu^*-\nu) + c_2(\nu^*-\nu)^2/2! + \ldots$$

with $$c_s = \frac{d^s c^*}{d\nu^{*s}} \qquad (3.1)$$

If we wish to carry the series (3.1) and similar ones for higher moments so that in expectation all terms are included contributing to, say, n^{-12}, then we need all derivatives up to c_{24}. These can be found using Faa di Bruno's formula for a derivative of a function of a function (see for example Shenton and Bowman, 1977b, pp. 14, 130, 169; for several generalizations see Good, 1961).

3.2 Derivatives of c^* with respect to ν^*. From (1.1)

$$\Gamma(1+2/c^*)/\Gamma^2(1+1/c^*) = 1 + \nu^{*2} \qquad (3.2)$$

so that taking logarithmic derivatives

$$\frac{1}{c^{*2}}\left\{\psi\left(1+\frac{1}{c^*}\right) - \psi\left(1+\frac{2}{c^*}\right)\right\}\frac{\partial c^*}{\partial \nu^*} = \frac{1}{2}\left\{\frac{1}{\nu^*+i} + \frac{1}{\nu^*-i}\right\} \qquad (3.3)$$

where $\psi(x) = d\ell n\,\Gamma(x)/dx$, $i = \sqrt{(-1)}$. Clearly we can drop the asterisks and replace them when necessary. We write c_r for $\partial^r c/\partial\nu^r$, the modified (3.3) in the form

$$c_1 J(c) = \nu_0. \qquad (3.4)$$

Using the formula of Leibniz for the s-th derivative of a product,

$$\frac{\partial^s J(c)}{\partial c^s} = J^{(s)}(c)$$

$$= \sum_{r=0}^{s} \binom{s}{r} \frac{(-1)^r}{c^{r+2}} (r+1)!\, H^{(s-r)}(c) \qquad (3.5)$$

where

$$H^{(0)}(c)=H(c)=\psi\left(1+\frac{1}{c}\right)-\psi\left(1+\frac{2}{c}\right),\quad H^{(m)}(c)=d^m H(c)/dc^m.$$

From (3.4)

$$Jc_2 + J^{(1)}c_1^2 = \nu_1 = \partial\nu_0/\partial\nu,$$

$$Jc_3 + 3J^{(1)}c_1c_2 + J^{(2)}c_1^3 = \nu_2, \qquad (3.6)$$

$$Jc_4 + 4J^{(1)}c_1c_3 + 3J^{(1)}c_2^2 + 6J^{(2)}c_1^2c_2 + J^{(3)}c_1 = \nu_3$$

and so on, where

$$\nu_s = \frac{\partial^s\nu_0}{\partial\nu^s} = \frac{(-1)^s}{2}\, s!\left(\frac{1}{(\nu+1)^{s+1}} + \frac{1}{(\nu-i)^{s+1}}\right).$$

Now the structure of these formulas is the same (Luckacs, 1955) as occurs in the expression of noncentral moments (μ_r') in terms of cumulants.

For example,

$$\kappa_2 + \kappa_1^2 = \mu_2',$$

$$\kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 = \mu_3', \qquad (3.7)$$

$$\kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 = \mu_4'.$$

But these formulas are equivalent to

$$\mu_r' = \sum_{s=0}^{r-1}\binom{r-1}{s}\kappa_{r-s}\mu_s' \quad (r = 2,3,\ldots) \qquad (3.8)$$

giving μ_r' in terms of $\mu_{r-1}', \mu_{r-2}', \ldots, \mu_0'$ (note

$\kappa_1 = \mu_1'$, $\mu_0' = 1$). Hence the left side members of (3.7) and the generalization can be set up recursively from previous members and awkward combinatorial problems avoided, a distinct advantage in digital implementation.

3.3 Moment series for c*. A tabulation is given in Table 8 for the first four moments for a selection of values from c = 0.8(0.1)2.6(0.2)3.2, 3.5, 3.8, 4.0. The sign pattern for c=1 (apart from one anomoly in each of $\mu_1'(c^*)$ and $\mu_4(c^*)$) is alternating. As c increases this regular pattern is disrupted and the plus signs start to predominate, especially for the higher moments. As for magnitude, very approximately, the coefficient of n^{-12} decreases from (24)!, for c=1, towards (12)! for c=3 for $\mu_1'(c^*)$, with slight increases for the higher moments. See Table 9 for further details.

[Table 8 about here]

TABLE 9. Magnitude of coefficients for c* moment series

Moment	$\mu_1'(c^*)$		$\mu_2(c^*)$		$\mu_3(c^*)$		$\mu_4(c^*)$	
c	$\lvert C_{12}^{(1)}/C_0^{(1)} \rvert$		$\lvert C_{12}^{(2)}/C_1^{(2)} \rvert$		$\lvert C_{12}^{(3)}/C_2^{(3)} \rvert$		$\lvert C_{12}^{(4)}/C_2^{(4)} \rvert$	
0.8	1.1	31	3.4	31	1.2	31	2.5	31
0.9	4.3	27	1.7	28	1.8	28	1.6	28
1.0	7.9	24	3.7	25	6.7	26	4.4	25
1.1	4.4	22	2.3	23	6.1	23	3.2	23
1.2	5.3	20	3.1	21	5.0	21	4.7	21
1.3	1.2	19	7.3	19	9.3	19	1.2	20
1.4	3.9	17	2.6	18	2.9	18	4.3	18
1.5	1.8	16	1.2	17	1.3	17	2.1	17
1.6	1.0	15	7.2	15	7.1	15	1.2	16
1.7	7.2	13	5.0	14	4.6	14	8.2	14
1.8	5.3	12	3.7	13	3.3	13	5.9	13
1.9	4.0	11	2.8	12	2.3	12	4.2	12
2.0	2.6	10	1.8	11	1.4	11	2.5	11
2.1	1.0	09	6.7	09	5.1	09	8.4	09
2.2	4.1	07	3.9	08	4.4	08	1.3	09
2.3	5.3	06	1.9	08	3.9	08	1.8	09
2.4	3.6	06	1.4	08	4.2	08	2.4	09
2.5	1.0	07	7.9	07	4.2	08	2.9	09
2.6	2.2	07	4.3	07	3.3	08	3.2	09
2.8	4.7	07	3.8	08	5.5	07	2.7	09
3.0	5.5	07	6.9	08	6.3	08	6.8	08
3.2	2.9	07	7.6	08	1.1	09	2.4	09
3.5	9.5	07	6.5	07	1.0	09	6.0	09
3.8	2.6	08	1.9	09	7.6	08	3.7	09
4.0	3.0	08	3.1	09	2.6	09	2.3	09

(In the moment columns each second column refers to the power of ten used as a multiplier.)

These properties suggest that E(c*) will exceed c, and Var(c*) will exceed the asymptotic variance ($\mathrm{Var}_1(c^*)$) for c in the region of 1.5 or more. Numerical evidence for 8.0 < c < 4.0 and 10 < n < 100 suggests E(c*-c) > 0, and Var(c*)/$\mathrm{Var}_1(c^*)$ > 1. For example, when c=1.9, n=10, E(c*-c) = 0.2, and the variance ratio is 1.4; similarly, when n=10, c=1.5, E(c*-c) = 0.8, and the variance ratio is 1.7.

(11a)

TABLE 8. Moment series for c*, where $\Gamma(1+2/c^*)/\Gamma^2(1+1/c^*) = 1 + \nu^{*2}$ ($\nu^* = \sqrt{m_2}/m_1'$, the coefficient of variation).

C	S	$\mu_1'(c^*)$	$\mu_2(c^*)$	$\mu_3(c^*)$	$\mu_4(c^*)$
1.0	0	1.00000000000 +00			
		2.64493406685 +00	1.00000000000 +00		
		-1.71123763218 +01	-1.16217622550 +01	-1.30395598911 -01	3.00000000000 +00
		8.53180565368 +02	1.04699772400 +03	4.04537576826 +02	-1.94936949445 +01
		-7.08236389752 +04	-1.18682984508 +05	-7.42938033595 +04	-1.26370978873 +04
	5	9.16919113987 +06	1.93912796742 +07	1.69161175037 +07	6.47275757861 +06
		-1.67242143328 +09	-4.23691027771 +09	-4.70621073393 +09	-2.68301982152 +09
		4.04878601561 +11	1.18709573920 +12	1.59183254710 +12	1.18649611510 +12
		-1.24957512007 +14	-4.13831172021 +14	-6.46747649882 +14	-5.89933637320 +14
		4.77569676051 +16	1.75476767742 +17	3.11777660021 +17	3.34303106097 +17
		-2.21096768650 +19	-8.89115426850 +19	-1.76327643511 +20	-2.16292996989 +20
		1.21859738849 +22	5.30619049143 +22	1.15816021472 +23	1.59366933885 +23
	12	-7.88382598313 +24	-3.68537991717 +25	-8.75534174063 +25	-1.33158479075 +26
1.5	0	1.50000000000 +00			
		2.39460463705 +00	1.53845275798 +00		
		1.06599601622 +00	4.35186733758 +00	5.21738040174 +00	7.10051066563 +00
		2.69713017770 +01	6.48215451578 +01	7.03747501861 +01	8.37131826997 +01
		-3.53721215742 +02	-8.06447688276 +02	-3.95658737729 +02	1.08724472751 +03
	5	9.59416683127 +03	3.05879721603 +04	3.93587343555 +04	2.11853890204 +04
		-3.29037238977 +05	-1.24215369477 +06	-1.97061328158 +06	-1.25192390674 +06
		1.43244108906 +07	6.26518882532 +07	1.22282944586 +08	1.18557479543 +08
		-7.58422614574 +08	-3.74361577598 +09	-8.57722170066 +09	-1.06473968110 +10
		4.75599220101 +10	2.60218037332 +11	6.81247904037 +11	1.02287789404 +12
		-3.46116462749 +12	-2.07055762499 +13	-6.07053476093 +13	-1.06134168661 +14
		2.87708828012 +14	1.86190755390 +15	6.02121347100 +15	1.19499508294 +16
	12	-2.69712794353 +16	-1.87220275839 +17	-6.59986794137 +17	-1.45975868166 +18
2.0	0	2.00000000000 +00			
		2.70621619094 +00	2.49178668783 +00		
		4.09622030667 +00	1.21358171725 +01	1.30938563578 +01	1.86270026930 +01
		1.07750575800 +01	5.87385616295 +01	1.41836390122 +02	3.20518319944 +02
		1.73139719173 +01	2.18087615386 +02	1.04488271269 +03	3.70018540429 +03
	5	1.22861199259 +02	1.22874693140 +03	7.26966573635 +03	3.40537720008 +04
		-6.09643673836 +02	3.01032157428 +02	3.43172490855 +04	2.75931393146 +05
		9.96309999457 +03	6.47799658547 +04	3.33913330334 +05	2.15257142591 +06
		-1.73642702591 +05	-1.03383596766 +06	-1.92966573241 +06	1.01060605221 +07
		3.31904521127 +06	2.21690539839 +07	7.15502179883 +07	1.83808487647 +08
		-7.44175186773 +07	-5.47919534477 +08	-1.91843636327 +09	-3.29254172381 +09
		1.86962029389 +09	1.48340632029 +10	5.72663115649 +10	1.26472136972 +11
	12	-5.20720028160 +10	-4.43006816905 +11	-1.87260123534 +12	-4.66556464359 +12
2.5	0	2.50000000000 +00			
		3.21080683541 +00	3.85125310096 +00		
		5.91359408349 +00	2.11260614720 +01	2.68109244124 +01	4.44964513430 +01
		1.31497936214 +01	9.69408672744 +01	3.02659719099 +02	8.54706236178 +02
		3.19036593715 +01	4.11558422187 +02	2.33021878154 +03	1.01185924436 +04
	5	5.94139796099 +01	1.54260294167 +03	1.48074310784 +04	9.46841021346 +04
		7.69255201951 +00	4.58954031641 +03	8.00056858598 +04	7.53142273523 +05
		-6.67092260119 +02	5.77412371479 +03	3.49142049167 +05	5.16917882133 +06
		-1.90368917783 +03	-3.00588611116 +04	1.05894424669 +06	3.00161450742 +07
		4.02756907485 +04	1.55104573000 +05	2.56936352297 +06	1.45177698170 +08
		5.34231938856 +05	7.52885601116 +06	4.71401051553 +07	7.42259369228 +08
		2.95457913454 +06	9.34343339404 +07	1.03071583918 +09	8.35049082213 +09
	12	-2.61458998108 +07	3.05571651390 +08	1.12854524846 +10	1.30799503718 +11
3.0	0	3.00000000000 +00			
		3.82132187478 +00	5.63714982025 +00		
		7.55069084600 +00	3.24471750061 +01	4.88600986935 +01	9.53323742878 +01
		1.66677744055 +01	1.49609515936 +02	5.64826632612 +02	1.91877256339 +03
		3.53337568366 +01	6.05430408881 +02	4.31019115087 +03	2.30455776023 +04
	5	4.61293354235 +01	2.03314581679 +03	2.61190739435 +04	2.12245487874 +05
		-8.30546942913 +01	4.67397937133 +03	1.28212520555 +05	1.61065206363 +06
		-9.58412196960 +01	6.35594863897 +03	5.05747305909 +05	1.02827188568 +07
		1.07237121181 +04	1.11691543390 +05	2.17146313306 +06	5.77578233599 +07
		1.08525524240 +05	2.06596344138 +06	2.22554610008 +07	3.60852002325 +08
		-5.32138022464 +03	1.39651311731 +07	2.64925361942 +08	3.41652787533 +09
		-1.35479035481 +07	-1.18970429170 +08	1.08065423512 +09	3.30646818568 +10
	12	-1.66127616752 +08	-3.89968568664 +09	-3.08174526253 +10	6.47214241681 +10

(Notice the sign pattern irregularities as c increases and the decrease in magnitude in the higher coefficients.)

(11-B)

TABLE 10. Levin and Padé approximants for c* moments.

c	r	N=25 μ_1	N=25 μ_2	N=25 μ_3	N=25 μ_4	N=30 μ_1	N=30 μ_2	N=30 μ_3	N=30 μ_4
1.0	9	1.0949728	0.0372344	0.0033566	0.0347445	1.0798484	0.0307982	0.0023006	-0.0007795
	10	1.0950276	0.0373526	0.0037434	0.0033054	1.0798738	0.0308527	0.0024777	0.0025505
	11	1.0950562	0.0374226	0.0039414	0.0043164	1.0798864	0.0308835	0.0025660	0.0029895
	12	1.0950702	0.0374535	0.0039704	0.0046815	1.0798921	0.0308958	0.0025749	0.0031664
	13	1.0950773	0.0374746	0.0040353	0.0049112	1.0798948	0.0309041	0.0026012	0.0032735
	S3	1.0950844	0.0375207	0.0039180	0.0053006	1.0798972	0.0309208	0.0025614	0.0034381
	S5	1.0950843	0.0374937	0.0040316	0.0052856	1.0798971	0.0309104	0.0025983	0.0034648
	P10	1.0955000	0.0370200	0.0042100	0.0102000	1.0801000	0.0306700	0.0026870	0.0047570
	P11	1.0947000	0.0346600	0.0035200	0.0041500	1.0797000	0.0289000	0.0023350	0.0028970
1.5	9	1.5987553	0.0718490	0.0131371	0.0205066	1.5817627	0.0580862	0.0085160	0.0127585
	10	1.5987578	0.0718752	0.0132863	0.0205384	1.5817636	0.0580953	0.0085696	0.0127741
	11	1.5987592	0.0718882	0.0133660	0.0206300	1.5817640	0.0580993	0.0085947	0.0128070
	12	1.5987598	0.0718943	0.0134040	0.0207284	1.5817642	0.0581010	0.0086054	0.0128385
	13	1.5987601	0.0718971	0.0134218	0.0208095	1.5817643	0.0581017	0.0086099	0.0128618
	S3	1.5987603	0.0718996	0.0134374	0.0211908	1.5817643	0.0581023	0.0086132	0.0129282
	S5	1.5987603	0.0718996	0.0134373	0.0209189	1.5817643	0.0581023	0.0086132	0.0128849
	P10	1.5987700	0.0719040	0.0134770	0.0214170	1.5817670	0.0581040	0.0086240	0.0130110
	P11	1.5987600	0.0719010	0.0134530	0.0210040	1.5817640	0.0581030	0.0086170	0.0129050
2.0	9	2.1155475	0.1235319	0.0336408	0.0648167	2.0951835	0.0990390	0.0214528	0.0390282
	10	2.1155476	0.1235278	0.0336363	0.0647768	2.0951835	0.0990363	0.0214519	0.0390159
	11	2.1155476	0.1235657	0.0336364	0.0649225	2.0951835	0.0990437	0.0214519	0.0390441
	12	2.1155476	0.1235426	0.0336372	0.0648706	2.0951835	0.0990419	0.0214521	0.0390380
	13	2.1155476	0.1235408	0.0336380	0.0648623	2.0951835	0.0990416	0.0214523	0.0390368
	S3	2.1155476	0.1235406	0.0336559	0.0648608	2.0951835	0.0990416	0.0214531	0.0390365
	S5	2.1155476	0.1235467	0.0336403	0.0648720	2.0951835	0.0990418	0.0214526	0.0390376
	P10	2.1155476	0.1235397	0.0336397	0.0648605	2.0951835	0.0990416	0.0214526	0.0390366
	P11	2.1155475	0.1235402	0.0336390	0.0648594	2.0951835	0.0990416	0.0214525	0.0390364
2.5	9	2.6388235	0.1952872	0.0701416	0.1656667	2.6141264	0.1560170	0.0446131	0.0988083
	10	2.6388232	0.1952872	0.0701420	0.1656683	2.6141263	0.1560170	0.0446132	0.0988085
	11	2.6388233	0.1952872	0.0701416	0.1656727	2.6141263	0.1560170	0.0446131	0.0988091
	12	2.6388233	0.1952872	0.0701413	0.1656633	2.6141263	0.1560170	0.0446131	0.0988065
	13	2.6388233	0.1952869	0.0701469	0.1656434	2.6141264	0.1560171	0.0446134	0.0988021
	S3	2.6388233	0.1952872	0.0701415	0.1656809	2.6141264	0.1560170	0.0446131	0.0988128
	S5	2.6388233	0.1952872	0.0701417	0.1656677	2.6141264	0.1560170	0.0446131	0.0988084
	P10	2.6388233	0.1952873	0.0701419	0.1656175	2.6141264	0.1560170	0.0446132	0.0987839
	P11	2.6388233	0.1952872	0.0701428	0.1656876	2.6141264	0.1560170	0.0446133	0.0988104
3.0	9	3.1660955	0.2887541	0.1286604	0.3648222	3.1364297	0.2303361	0.0818077	0.2169620
	10	3.1660955	0.2887541	0.1285770	0.3648729	3.1364297	0.2303361	0.0818150	0.2169699
	11	3.1660956	0.2887567	0.1286557	0.3650496	3.1364298	0.2303367	0.0818077	0.2169788
	12	3.1660956	0.2887562	0.1286650	0.3648476	3.1364298	0.2303366	0.0818085	0.2169681
	13	3.1660956	0.2887562	0.1286649	0.3648695	3.1364298	0.2303366	0.0818085	0.2169699
	S3	3.1660956	0.2887562	0.1286649	0.3648673	3.1364298	0.2303366	0.0818085	0.2169697
	S5	3.1660956	0.2887562	0.1286648	0.3649202	3.1364298	0.2303366	0.0818085	0.2169717
	P10	3.1660956	0.2887558	0.1286596	0.3649093	3.1364298	0.2303367	0.0818079	0.2169744
	P11	3.1660957	0.2887560	0.1286639	0.3649262	3.1364298	0.2303366	0.0818084	0.2169767

(Notes on Table 10. Sequences for the four moments are those for Levin's t-algorithm (2.1) and refer to α_{r+1} for each moment. S_3 and S_5 refer to the Shank's extrapolates (see footnote to Table 4) based on the last 3 and 5 sequence values respectively. If either of these extrapolates reverses the sequence trend, it should be ignored; generally we look for monotonicity in the sequences. Caution is needed in interpreting the Shanks' extrapolates.

For the Padé fractions we have used the Stieltjes continued fraction forms; for example, for the mean we use

$$\mu_1'(c^*) \sim \frac{nc}{n} + \frac{p_1^{(1)}}{1} + \frac{q_1^{(1)}}{n} + \ldots + \frac{p_5^{(1)}}{1} + \frac{q_5^{(1)}}{n}$$

and P_{11}, P_{12} refer to the approximants stopping at the partial numerators p_5, q_5 respectively. For the variance we use a similar expression, the first partial numerator now being $q_0^{(2)}$. Similarly, $\mu_3(c^*)$ and $\mu_4(c^*)$ have $q_0^{(3)}/n^2$, and $q_0^{(4)}/n^2$ as first partial numerators.

Generally, there is good agreement in the two types of approximants for $c \geq 2$--five or six decimal place agreement seems to be common. There is a deterioration for smaller c and especially for the 3-rd and 4-th central moments. Thus for c=1, n=25, our preferences would be $\mu_3(c^*) \sim 0.0060$, and $\mu_4(c^*) \sim 0.005$ with some doubt; the situation for n=30 is only slightly improved. Even so, the effect on the percentiles is surprisingly small (see Tables 13a, 13b).

The reason for the deterioration in the summation algorithms for $c < 2$ doubtless lies in the largeness of the higher coefficients in the series, together with a "bumpiness" in the early terms especially for μ_3 and μ_4.)

3.4 Summation of the c* series. The diversified structure of the series' coefficients arising from the 100 cases tabulated (25 values of c, for four moments) makes it imperative to diminish the labor involved in a detailed study; so we confine attention generally to samples in the region of 25 or more. This makes less stringent demands on the summatory algorithms chosen.

Again, since magnitudes decrease and sign patterns become irregular as c increases, summatory algorithms successful for small c may fail for large c ($1.6 \le c < 4$).

For c in the vicinity of unity, we use Levin's t-algorithm or its truncated versions; some illustrations are given in Table 10 and an appendix. We look first of all for monotonicity, and secondly, smallness of first differences.

A word on notation--we use S_3 and S_5 to denote the Shanks' approximant based on **the last three, and last five values computed. (See the footnote to Table 4.) For c*, terms up to the coefficient of n^{-12} are always used.**

In addition $L(tr=s,S_r)$ means a Levin algorithm with S initial terms truncated, with a Shanks' smoothing formula applied to the last r (3 or 5) terms. Similarly, 2cB ($a=\alpha$, $tr=s,S_r$) and 1cB ($a=\alpha$, $tr=s,S_r$) refer to the Borel-Padé type algorithm described in paragraph 2.2.

We should warn that truncation of a series does not relate linearly to summation algorithms in general. For example, different diagonals of a Padé table are generally distinct, and removing the first term of a series or adding a term at the beginning can change drastically the continued fraction representation.

3.5 Detailed illustrations.

3.5.1. There is undoubtedly a summation problem for small n, so we confine attention for the most part to $n \ge 20$; the difficulties stem from the variety of patterns which emerge for the series, so that no one approach works for small samples over the parameter space of the shape parameter. We must point out that the study, as it is, involves some 200 series so that detailed individual cases cannot be undertaken.

Table 10 gives a general view of the usefulness of the Levin algorithm. Table 11 treats the four moments of c* and several summation algorithms. The series are noteworthy for the preponderance of positive coefficients and divergence at about the rate of the single factorial series. Higher moments are less easy to sum than lower. Another characteristic to notice is the bumpiness of the coefficients, in contrast to the series for ν*. As to potential error in the c* - series higher coefficients, we can only say that we have used double-double precision arithmetic on an IBM computer, amounting to the retention of about 30 decimal deigits.

Assessment of moments of c*, c = 2.5, n = 20

TABLE 11a. $\mu_1'(c^*)$

	2cB(a=1,tr=1)		2cB(a=1,tr=0)		2cB(a=2,tr=0)	
r	Fr	Δ	Fr	Δ	Fr	Δ
9	2.676077	235	2.668530	1766	2.668901	3744
10	2.676312	172	2.670296	1314	2.662645	2785
11	2.676484	129	2.671610	999	1.665430	2117
12	2.676613	100	2.672609	775	2.667547	1641
13	2.676713		2.673384		2.669188	
S_5	2.67712		2.67864		2.67606	

	Direct Sum
r	term
0	2.5
1	0.1605403
2	0.0147840
3	0.0016437
4	0.0001994
5	0.0000186
6	0.0000001
7	-0.0000005
DS(6)	2.677187

The preferred value is 2.67712 because of the small differences in 2cB(a=1,tr=1); this agrees with the direct sum to the n^{-6} term. A simulation of 10^5 cycles gave 2.6777. Levin without truncation gave 2.6772.

TABLE 11b. $\mu_2(c^*)$

	2cB(a=2,tr=0)		2cB(a=1,tr=1)	
r	Fr	Δ	Fr	Δ
9	0.2493811	17514	0.2587758	3567
10	0.2511325	14470	0.2591325	2663
11	0.2525795	11530	0.2593988	2042
12	0.2537325	9340	0.2596630	1600
13	0.2546665		0.2597630	
S_5	0.259590		0.260476	
σ	0.5095		0.5104	

Preferred 2cB is 0.5104. A simulation gave $\sigma_s \sim 0.5105$. The direct sum gave DS(9) = 0.260625 with the n^{-10} coefficient 0.0000007, with $\sigma \sim 0.5104$. The Levin assessment gave 0.5105, our final choice.

TABLE 11c. $\mu_3(c^*)$

	2cB(a=1,tr=0)		2cB(a=1,tr=1)	
r	Fr	Δ	Fr	Δ
9	0.1140509	17471	0.1214065	7610
10	0.1157980	13752	0.1221675	5711
11	0.171732	11058	0.1227386	4420
12	0.1182790	9049	0.1231806	3496
13	0.1191839		0.1235302	
S_5	0.124287		0.125189	
$\sqrt{\beta_1}$	0.9347		0.9415	

Comparisons are:

2cB $\sqrt{\beta_1} \sim 0.9415$

Simulation $\sqrt{\beta_1} \sim 0.9512$

Levin $\sqrt{\beta_1} \sim 0.9440$

DS(10) 0.125625 (for $\mu_3(c^*)$)

$\sqrt{\beta_1} \sim 0.9442$

Preferred value $\sqrt{\beta_1}$ = 0.9442

TABLE 11d. $\mu_4(c^*)$

	2cB(a=1,tr=0)		2cB(a=1,tr=1)	
r	Fr	Δ	Fr	Δ
9	0.2630947	76628	0.2966641	45608
10	0.2707575	62896	0.3012249	35777
11	0.2770471	52531	0.3048026	28778
12	0.2823002	44498	0.3070804	23608
13	0.2867500		0.3100412	
S_5	0.318118		0.323717	
β_2	4.684		4.766	

Levin $\beta_2 \sim 4.831$

{Ds(12) 0.3283169 (n^{-12} term 0.00003)

$\beta_2 \sim 4.839$

Final choice $\beta_2 \sim 4.84$. A small sample case (c=1.5, n=10) is given in the appendix.

3.5.2 Simulation comparisons. A check on the summation algorithm for several values of the shape parameter c with samples of n=20 is shown in Table 12. Agreement with the simulation assessments improves as the skewness of the population decreases (c increases from I towards 3--the Weibull density has zero skewness when c=3.6 approx.).

TABLE 12. Moments of c* by series (Levin) and simulation (10^5 runs)

c	n		$\mu_1'(c^*)$	$\sigma(c^*)$	$\sqrt{\beta_1}(c^*)$	$\beta_2(c^*)$
1.0	20	L	1.1177	0.2188	0.6645	3.5796
		S	1.1174	0.2183	0.6751	4.1027
1.5	20	L	1.6248	0.3072	0.8166	4.3727
		S	1.6247	0.3068	0.8240	4.6079
2.0	20	L	2.1470	0.4050	0.9035	4.7211
		S	2.1472	0.4048	0.9106	4.8945
	25	L	2.1155	0.3515	0.7747	4.2498
		S	2.1156	0.3511	0.7705	4.2502
2.5	20	L	2.6772	0.5105	0.9440	4.8307
		S	2.6777	0.5105	0.9512	5.0155
3.0	20	L	3.2123	0.6214	0.9601	4.8627
		S	3.2130	0.6215	0.9669	5.0429

L ≡ Levin's t-algorithm using all series coefficients to n^{-12}. S ≡ simulation of 10^5 cycles.)

3.6 Percentage points comparisons. Using the moment series for ν* and c* along with the mapping in (2.7) we compare standard percentile levels derived from the 4-moment Pearson density approximants (Tables 13a, 13b) for samples of 15, 20, and 25 at five values of c. There are also simulation comparisons for samples of 20. The reader may agree that the results are satisfactory.

Another check on the summatory algorithms arises from a study of Pearson and Tukey (1965) on the relation between distances between percentage points (for Pearson curves) and the mean and standard deviation. For a region of the (β_1, β_2) plane ($\beta_1 < 4$, $\beta_2 < 11$, approximately), we may approximate the mean by $\hat{\mu} = [50\%] + 0.185\Delta$ where $\Delta = [95\%] + [5\%] - 2[50\%]$; here [50%], for example, refers to the median. For the standard deviation Pearson and Tukey give the equations,

$$\hat{\sigma}'_{0.05} = \frac{[95\%] - [5\%]}{\max\{3.29 - 0.1(\Delta/\hat{\sigma}'_{0.05})^2,\ 3.08\}},$$

$$\hat{\sigma}'_{0.025} = \frac{[97.5\%] - [2.5\%]}{\max\{3.98 - 0.138(\Delta/\hat{\sigma}'_{0.025})^2,\ 3.66\}}$$

and the final assessment $\sigma \sim \max\{\hat{\sigma}'_{0.05}, \hat{\sigma}'_{0.025}\}$.

We consider these values for the mean and standard deviation of c* using the percentiles of c* derived from the percentiles of ν* under the mapping (2.7). The point to notice is our concern for c* moments derived by a complicated numerical process and how these compare with assessments derived from more stable and [illegible] series for ν* (up to the n^{-24} coefficients). The agreement for samples of 15 or more is quite remarkable (Table 14).

TABLE 13a. Percentage points of V* from V* moments (direct) and C* moments (indirect)

%	N=15 Direct	N=15 I	N=20 Direct	N=20 I	N=20 M C	N=25 Direct	N=25 I
C=1.0							
1	0.5454	0.5475	0.5956	0.5964	0.595	0.6309	0.6315
5	0.6312	0.6249	0.6757	0.6726	0.677	0.7067	0.7048
10	0.6813	0.6743	0.7220	0.7186	0.723	0.7500	0.7481
90	1.1747	1.1736	1.1705	1.1711	1.170	1.1645	1.1649
95	1.2744	1.2479	1.2613	1.2525	1.261	1.2481	1.2441
99	1.4921	1.3651	1.4606	1.4107	1.465	1.4319	1.4099
C=1.5							
1	0.3864	0.3869	0.4243	0.4245	0.425	0.4502	0.4504
5	0.4532	0.4515	0.4840	0.4834	0.484	0.5049	0.5046
10	0.4902	0.4886	0.5171	0.5165	0.518	0.5352	0.5349
90	0.8046	0.8042	0.7942	0.7940	0.794	0.7858	0.7856
95	0.8605	0.8565	0.8427	0.8416	0.842	0.8291	0.8286
99	0.9769	0.9591	0.9429	0.9393	0.943	0.9179	0.9170
C=2.0							
1	0.2985	0.2986	0.3290	0.3290	0.330	0.3496	0.3496
5	0.3523	0.3520	0.3766	0.3765	0.376	0.3929	0.3928
10	0.3821	0.3819	0.4029	0.4028	0.403	0.4168	0.4167
90	0.6212	0.6209	0.6113	0.6110	0.611	0.6039	0.6037
95	0.6606	0.6604	0.6450	0.6448	0.645	0.6338	0.6337
99	0.7394	0.7411	0.7121	0.7127	0.714	0.6930	0.6932
C=2.5							
1	0.2424	0.2425	0.2679	0.2679	0.269	0.2852	0.2852
5	0.2875	0.2873	0.3079	0.3077	0.308	0.3215	0.3214
10	0.3125	0.3123	0.3299	0.3298	0.330	0.3414	0.3413
90	0.5102	0.5100	0.5014	0.5012	0.501	0.4949	0.4948
95	0.5416	0.5412	0.5282	0.5281	0.528	0.5187	0.5186
99	0.6032	0.6026	0.5806	0.5806	0.582	0.5649	0.5650
C=3.0							
1	0.2037	0.2038	0.2256	0.2257	0.226	0.2405	0.2405
5	0.2425	0.2423	0.2601	0.2600	0.260	0.2718	0.2718
10	0.2641	0.2640	0.2792	0.2791	0.279	0.2891	0.2890
90	0.4350	0.4349	0.4271	0.4270	0.427	0.4214	0.4213
95	0.4617	0.4615	0.4499	0.4498	0.450	0.4416	0.4416
99	0.5136	0.5133	0.4941	0.4940	0.495	0.4807	0.4806

M. C. is 10^5 simulation, I is indirect.

TABLE 13b. Percentage points of C* from C* moments (direct) and V* moments (indirect)

%	N=15 Direct	N=15 I	N=20 Direct	N=20 I	N=20 M C	N=25 Direct	N=25 I
C=1.0							
1	0.7439	0.6880	0.7225	0.7008	0.699	0.7229	0.7131
5	0.8075	0.7919	0.8047	0.7996	0.800	0.8098	0.8074
10	0.8554	0.8546	0.8571	0.8576	0.858	0.8615	0.8618
90	1.5112	1.4945	1.4108	1.4037	1.401	1.3512	1.3475
95	1.6420	1.6242	1.5156	1.5078	1.505	1.4406	1.4365
99	1.9000	1.9078	1.7285	1.7311	1.696	1.6233	1.6250
C=1.5							
1	1.0429	1.0237	1.0652	1.0611	1.061	1.0917	1.0906
5	1.1713	1.1657	1.1930	1.1911	1.192	1.2125	1.2118
10	1.2513	1.2507	1.2684	1.2680	1.268	1.2826	1.2823
90	2.1558	2.1480	2.0267	2.0242	2.022	1.9497	1.9484
95	2.3543	2.3449	2.1819	2.1788	2.181	2.0802	2.0786
99	2.7980	2.8015	2.5221	2.5238	2.519	2.3610	2.3621
C=2.0							
1	1.3648	1.3682	1.4234	1.4247	1.420	1.4666	1.4672
5	1.5459	1.5455	1.5867	1.5861	1.587	1.6172	1.6168
10	1.6538	1.6529	1.6830	1.6823	1.684	1.7055	1.7050
90	2.8389	2.8372	2.6747	2.6739	2.673	2.5754	2.5748
95	3.1086	3.1056	2.8844	2.8829	2.885	2.7507	2.7498
99	3.7336	3.7341	3.3524	3.3524	3.344	3.1327	3.1326
C=2.5							
1	1.7089	1.7072	1.7804	1.7805	1.766	1.8349	1.8351
5	1.9242	1.9227	1.9776	1.9769	1.976	2.0177	2.0172
10	2.0555	2.0547	2.0956	2.0950	2.096	2.1259	2.1255
90	3.5518	3.5487	3.3430	3.3417	3.343	3.2175	3.2168
95	3.8968	3.8925	3.6103	3.6085	3.606	3.4405	3.4394
99	4.6964	4.6992	4.2086	4.2092	4.197	3.9281	3.9282
C=3.0							
1	2.0409	2.0395	2.1299	2.1293	2.125	2.1960	2.1958
5	2.2976	2.2964	2.3644	2.3637	2.363	2.4138	2.4134
10	2.4552	2.4547	2.5058	2.5054	2.507	2.5437	2.5434
90	4.2785	4.2753	4.0235	4.0220	4.025	3.8703	3.8695
95	4.7006	4.6967	4.3505	4.3486	4.346	4.1428	4.1417
99	5.6789	5.6835	5.0821	5.0837	5.072	4.7391	4.7398

M. C. is 10^5 simulation. (Based on 4-moment Pearson density approximation, the series summed by Levin's t-algorithm. For ν^* see 2.6. For c* all coefficients were used.)

TABLE 15. Mean and standard deviation of c* computed directly from c* series compared to Pearson-Tukey approximants based on percentiles of c* derived from those of ν*

		N=15		N=20		N=25	
C=1.0	μ_1'	1.1556	1.1558	1.1177	1.1178	1.0951	1.0951
	σ	0.2589	0.2593	0.2188	0.2186	0.1936	0.1932
C=1.5	μ_1'	1.6697	1.6696	1.6248	1.6247	1.5988	1.5987
	σ	0.3711	0.3696	0.3072	0.3057	0.2681	0.2664
C=2.0	μ_1'	2.2023	2.2018	2.1470	2.1468	2.1155	2.1154
	σ	0.4937	0.4928	0.4050	0.4035	0.3515	0.3496
C=2.5	μ_1'	2.7449	2.7443	2.6772	2.6769	2.6388	2.6387
	σ	0.6245	0.6232	0.5105	0.5086	0.4419	0.4396
C=3.0	μ_1'	3.2940	3.2933	3.2123	3.2119	3.1661	3.1659
	σ	0.7613	0.7597	0.6214	0.6191	0.5374	0.5345

(*Levin's t-algorithm, using all available coefficients in the series for $\mu_1'(c^*)$ and $\mu_2(c^*)$.
**Pearson-Tukey values derived from their $\max\{\hat{\sigma}^1_{0.05}, \hat{\sigma}^1_{0.025}\}$ based on percentiles of c* derived from ν* moment series.)

4. THE MOMENT ESTIMATOR FOR 1/c

The equation for the estimator d* of d(=1/c) is

$$\Gamma(1+2d^*)/\Gamma^2(1+d^*) = 1 + \nu^{*2}. \qquad (4.1)$$

A quick approximate solution (see 2.8 for a comparison) is

$$d^* \sim 0.908919\nu^* + 0.91081\nu^{*2} \qquad (4.2)$$
$$(0 < \nu < 2)$$

hinting that the distribution of d* will be done (in some sense) to that of ν*. A modification of the approach of ¶3 now leads to series developments for the moments of d* up to terms in n^{-12}. From tabulations for the same parameter space as was used for c*, we find the series for d* in general have the same sign and magnitude patterns (however, we are limited to fewer coefficients). Thus development provides yet another check on the validity of moment assessments. As an example, the assessments of moments when n=10, c=1.5 are:

	$\mu_1'(a^*)$	$\sigma(d^*)$	$\sqrt{\beta_1}(d^*)$	$\beta_2(d^*)$
(i) Levin:	0.607184	0.156519	0.4633	3.4253
(ii) Padé:	0.607190	0.156542	0.4634	3.4122

The 4-moment percentiles of d* using (1) are

	1%	5%	10%	90%	95%	99%
d*	0.2924	0.3714	0.4165	0.8125	0.8825	1.0256
Derived ν*	0.3231	0.4004	0.4437	0.8170	0.8843	1.0257
Direct ν*	0.3230	0.4006	0.4439	0.8171	0.8846	1.0260

along with a comparison for ν*. In addition we have from the d* percentiles, the c* values 0.975, 1.133, 1.231, 2.401, 2.693, and 3.420 which can be compared with the less reliable (because of the bumpiness of the higher moments) results in Table 12.

Further comparisons of moments and percentiles (not reported here) give grounds for confidence in $1 \leq c \leq 4$ and $n > 15$ approximately. Actually there is reason to believe that if percentage points of c* are needed, it is better to proceed via d*.

5. CONCLUDING REMARKS

(i) The series for the moments of ν^* taken as far as the n^{-24} term appear to be divergent. As c, the shape parameter varies from 1 to about 4, the regular alternating sign pattern is increasingly disrupted (especially for the higher moments), whereas the magnitude pattern is diluted (the n^{-24} coefficient decreases from about 10^{60} to 10^{25}). The Levin t-algorithm, with stopping point signalled from exact small sample results, works well.

(ii) Series for the shape parameter c* (take as far as n^{-12}) are more difficult to sum because of irregular sign and magnitude patterns.

(iii) Series for d* (estimating d=1/c) are similar to those for the coefficient of variation.

(iv) Validation is by numerical investigation--error bounds for moments and percentiles are out of the question. We use several summation algorithms (Levin, Levin with truncation, Padé, simulation) and in them study consistency. There can be difficulties here--for example, adjacent close approximants may still be in error. We have described some highly successful cases and some problematical cases--for example $\mu_4(c^*)$ when c=1.5 or so and n is small.

There are outstanding problems, such as:
(a) the response of an algorithm to slight errors in series coefficients for low orders of n^{-1} and large errors in coefficents for high orders of n^{-1}; (b) the choice of algorithm for summation purposes. Levin's t-algorithm works well for alternating series and magnitudes lying between the single and double factorial series. The Padé approach behaves similarly and very likely has wide application (see the Baker-Gammel-Wills conjecture (Baker, 1975)); (c) the construction of algorithms which relate specifically to moments of statistics expressable as multiple integrals.

Finally, it should be eminently clear that low order asymptotics to measures such as means, covariances, etc., should be viewed with great caution. Even if the first few coefficients are seductively small, there may be rude awakenings round the corner; for example, an n^{-1} term in a variance may exist but all higher order terms may not.

To those not well acquainted with summation problems reference may be made to:
(i) Baker and Gammel (1970, Baker (1975), Graves and Morris (1973), and Brezinski (1980) for modern studies on Padé methods; (ii) Wall (1948), Perron (1950), Stieltjes (1918), Borel (1928) for classical studies; (iii) Van Dyke (1974, 1975) for general remarks on divergent series; (iv) Shohat and Tamarken (1963) for the moment problem.

APPENDIX

A small sample case. To illustrate problems which arise for small samples, we take c=1.5 and n=10. In particular the fourth central moment has the successive coefficients (approximated for convenience)

$$\mu_4(c^*) \sim 0.07 + 0.08 + 0.11 + 0.21 - 1.25 + 11.86 - 106.5 + 1022.9 - 10613.4 + \ldots;$$

it should be noted that the first coefficient is merely three times the square of the variance asymptote and provides no unexpected information. Also note the disrupted sign pattern.

We try the Levin algorithm.

r	$\mu_1'(c^*)$	$\mu_2(c^*)$	$\mu_3(c^*)$	$\mu_4(c^*)$ Truncate & Start at n^{-4} term
		α_r		
2	1.7502	0.1279	0.1080	0.2335
3	1.7418	0.2656	0.1097	0.1679
4	1.6397	0.2403	0.1163	0.1722
5	1.7749	0.2398	0.1244	0.1957
6	1.7675	0.2426	0.1345	0.2229
7	1.7663	0.2457	0.1464	0.1979
8	1.7663	0.2485	0.1585	

We base $\mu_4(c^*)$ on α_6 yielding the value 0.3776. Our preferred assessments are:

$\mu_1' = 1.7663$, $\mu_2 = 0.2485$, $\sqrt{\beta_1} = 1.2789(?)$, $\beta_2 = 6.1132(?)$ with rather low confidence in β_2. If we take α_7 instead of α_6, our alternative for the kurtosis is $\beta_2^{(a)} = 5.7102$. We not have the Pearson 4-moment fits for c* (Table A1). Thus the change in β_2 does affect the c* percentiles but this change is damped out in the ν^* derived values.

TABLE A1. Percentiles of ν^* derived from c* compared to direct values n=10, c=1.5

%		1	5	10	90	95	99
c*	(a)	0.995	1.134	1.226	2.413	2.628	3.355
	(b)	1.038	1.149	1.230	2.426	2.713	3.360
Derived	(a)	1.005	0.884	0.820	0.442	0.400	0.329
ν^*	(b)	0.964	0.873	0.818	0.440	0.400	0.328
ν^* direct		1.026	0.885	0.817	0.444	0.401	0.323

((a) uses the moments with kurtosis β_2, and (b) with kurtosis $\beta_2^{(a)}$)

REFERENCES

Baker, G. A., Jr. (1975). Essentials of Padé approximants, Academic Press, New York.

Baker, G. A., Jr. and Gammel, J. L. (1970). The Padé approximant in theoretical physics, Academic Press, New York.

Barlow, R. E., Toland, R. H., and Freeman, T. (1979). Stress-rupture life of Kevlar/epoxy sperical pressure vessels, Lawrence Livermore Laboratory Report UCID-17755, Part 3.

Borel, E. (1920) Lecons sur tes series divergentes, Gauthier-Vaillars, Paris (translated by Charles L. Critchfield and Ann Vakar (1975), Los Alamos Scientific Laboratory).

Bowman, K. O. and Shenton, L. R. (1981). "Moment series for the coefficient of variation in Weibull sampling," Proc. Statis. Comput. Sect. Amer. Statis. Assoc., 148-153.

Bowman, K. O. and Shenton, L. R. (1981). Estimation Problems associated with the Weibull distribution, Report ORNL/CSD-79, Union Carbide Corporation.

Brezinski, C. (1980). Padé-Type approximation and general orthogonal polynomials, Birkhausen Verlag, Basel-Boston-Stuttgart.

Good, I. J. (1961). "The multivariate saddlepoint method and chi-squared for the multinomial distribution," Annals of Math. Statis., 32, 535-548.

Levin, D. (1973). "Development of non-linear transformations for improving convergence of sequences," Internat. J. Computer Math., B3, 371-388.

Lukacs, E. (1955). "Applications of Faa di Bruno's formula in mathematical statistics," Amer. Math. Monthly, 62, 340-348.

Graves-Morris, P. R., ed. (1973). Padé approximants and their applications, Academic Press, New York.

Newby, M. J. (1980). "The properties of moment estimators for the Weibull distribution based on the sample coefficient of variation," Technometrics, 22, 2, 187-194.

Peto, R. and Lee, P. (1973). "Weibull distributions for continuous-carcinogenesis experiments," Biometrics, 29, 457-470.

Pearson, E. S. and Tukey, J. W. (1965). "Approximate means and standard deviations based on distances between percentage points of frequency curves," Biometrika, 52, 533-546.

Shanks, Daniel (1955). "Non-linear transformations of divergent and slowly convergent sequences," J. Math. and Physics, 34, 1-42.

S[illegible] and Bowman, K. O. (1977b). "A new algorithm for summing divergent series: Part 3, applications," J. Computa. and Appl. Math., 3, 35-51.

Shenton, L. R. and Bowman, K. O. (1977a). Maximum likelihood estimation in small samples, Griffin's Statistical Monograph and Courses No. 38, Macmillan Publishing Co., Inc., New York.

Stewart, D. A. and Essenwanger, A. M. (1978). "Frequency distribution of wind speed near the surface," J. Appl. Meteorology, 17, 1633-1642.

Shohat, J. A. and Tamarkin, J. D. (1963). The problem of moments, Amer. Math. Soc., Providence, Rhode Island.

Stieltjes, T. J. (1918). Oeuvres complétes, Vol. 2, P. Noordhoff, Groningen.

Van Dyke, M. (1974). "Analysis and improvement of perturbation series," Quan. 7 J. Mech. Appl. Math. 27, 423-450.

Van Dyke, M. (1975). "Computer extension of perturbation series in fluid mechanics. SIAM J. Appl. Math., 28, 3, 720-734.

Wall, H. S. (1948). Analytic theory of continued fractions, Van Nostrand-Reinhold, Princeton, New Jersey.