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CHAOTIC TRANSIENTS, HIGHER DIMENSIONAL PHENOMENA, AND COUPLED ORDINARY DIFFERENTIAL EQUATIONS

Progress Report

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I. SUMMARY OF PAST PROGRESS

From April 1982 we have had DOE funding for work on chaos in nonlinear dissipative dynamical systems. The present proposal is essentially for a continuation of the previous line of work. Therefore, in this section we briefly review our past progress under the **previous DOE funding.**

A. Crises and Chaotic Transients

Crises – sudden macroscopic changes in chaotic attractors due to their collision with unstable orbits – and the associated chaotic transients have been extensively studied by us in a series of papers listed below.

1. "Chaotic Attractors in Crisis", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **48**, 1507 (1982).
2. "Crises, Sudden Changes in Chaotic Attractors, and Transient Chaos", C. Grebogi, E. Ott, and J. A. Yorke, Physica **7D**, 181 (1983).
3. "Fractal Basins Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **50**, 935 (1983).
4. "Structure and Crises of Fractal Basin Boundaries", S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. **107A**, 51 (1985).
5. "Super Persistent Chaotic Transients", C. Grebogi, E. Ott, and J. A. Yorke, Ergodic Theory and Dynamical Systems **5**, 341 (1985).
6. "Critical Exponent of Chaotic Transients in Nonlinear Dynamical Systems", Phys. Rev. Lett. **57**, 1284 (1986).
7. "Critical Exponents for Crisis-Induced Intermittency", C. Grebogi, E. Ott, F. Romeiras, and J. A. Yorke, Phys. Rev. A **36**, 5365 (1987).

B. The Dimension of Chaotic Attractors

In this area we have investigated the various definitions of attractor dimension, their possible relationship to Lyapunov numbers, their implications for the effect of noise on attractors, and numerical techniques for their calculation. Papers on this topic are listed below.

8. "The Dimension of Chaotic Attractors", J. D. Farmer, E. Ott, and J. A. Yorke, Physica **7D**, 153 (1983).

9. "Is the Dimension of Chaotic Attractors Invariant Under Coordinate Changes?", E. Ott, W. D. Withers, and J. A. Yorke, J. Stat. Phys. **36**, 659 (1984).

10. "A Scaling Law: How an Attractor's Volume Depends on Noise Level", E. Ott, E. D. Yorke, and J. A. Yorke, Physica **16D**, 62 (1985).

11. "Lorenz Cross-Sections and the Structure and Dimension of Higher Dimensional Attractors", E. Kostelich and J. A. Yorke, Physica **24D**, 263 (1987).

12. "Unstable Periodic Orbits and the Dimension of Chaotic Attractors", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **36**, 3522 (1987).

13. "Unstable Periodic Orbits and the Dimensions of Multifractal Chaotic Attractors", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).

14. "Strange Saddles and the Dimensions of Their Invariant Manifolds", G. - H. Hsu, E. Ott, and C. Grebogi, Phys. Lett. **127A**, 199 (1988).

15. "Dimensions of Strange Nonchaotic Attractors", M. Ding, C. Grebogi, and E. Ott, Phys. Lett. **137A**, 167 (1989).

16. "Pointwise Dimension and Unstable Periodic Orbits", C. Grebogi, E. Ott, and J. A. Yorke, in *Essays on Classical and Quantum Dynamics*, Ed. H. Uberall (Gordon and Breach, 1989), 10 pages, in print.

17. "Lyapunov Partition Functions for the Dimensions of Chaotic Sets", E. Ott, T. Sauer, and J. A. Yorke, Phys. Rev. A **39**, 4212 (1989).

C. Fractal Basin Boundaries

In this area we have been the first to quantitatively assess and study the extent to which the existence of fractal basin boundaries pose practical difficulties in predicting outcome from imperfect initial data.

18. "Final State Sensitivity: An Obstruction to Predictability", C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. **99A**, 415 (1983).

19. "Fractal Basin Boundaries in Nonlinear Dynamical Systems", C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, in *Statistical Physics and Chaos in Fusion Plasmas*, Ed. C. W. Horton and L. E. Reichl (Wiley, New York, 1984).

20. "An Obstruction to Predictability", S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, in Proc. XIIIth Intn'l. Colloq. on Group Theor. Methods in Physics (World Scientific publishing Co., Singapore, 1984).

21. "Fractal Basin Boundaries", S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica **17D**, 125 (1985).

22. "Metamorphoses of Basin Boundaries", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **56**, 1011 (1986).

23. "Basin Boundary Metamorphoses: Changes in Accessible Boundary Orbits", C. Grebogi, E. Ott, and J. A. Yorke, Physica **24D**, 263 (1987).

24. "Multi-dimensional Intertwined Basin Boundaries and the Kicked Double Rotor", E. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Lett. **118A**, 448 (1986).

25. "Multi-dimensional Intertwined Basin Boundaries: Basin Structure of the Kicked Double Rotor", E. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, Physica **25D**, 347 (1987).

26. "Fractal Basin Boundaries with Unique Dimension," C. Grebogi, E. Ott, J. A. Yorke, and H. E. Nusse, Ann. N.Y. Acad. Sci. **497**, 117 (1987).

27. "Basic Sets: Sets that Determine the Dimensions of Basin Boundaries", C. Grebogi, H. E. Nusse, E. Ott, and J. A. Yorke, *Springer Lecture Notes in Mathematics*, Vol. 1342 (Dynamical Systems)(Springer-Verlag, 1988), p. 220.

28. "Fractal Boundaries for Exit in Hamiltonian Dynamics", S. Bleher, C. Grebogi, E. Ott and R. Brown, Phys. Rev. A **38**, 930 (1988).

29. "Scaling of Fractal Basins Boundaries Near Intermittency Transitions to Chaos", B. -S. Park, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **40**, 1576 (1989).

D. Other Research

Other research papers not falling in the above three categories are the following. These cover work in quasiperiodicity, Hamiltonian systems, windows and their scaling, fat fractals, shadowing, etc.

30. "Are Three-Frequency Quasiperiodic Orbits to Be Expected in Typical Nonlinear Dynamical Systems?", C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **51**, 339 (1983).

31. "Correlations of Periodic Area-Preserving Maps", J. D. Meiss, J. R. Cary, C. Grebogi, J. D. Crawford, and H. D. I. Abarbanel, Physica **6D**, 375 (1983).

32. "Relativistic Ponderomotive Hamiltonian", C. Grebogi and R. G. Littlejohn, Phys. Fluids **27**, 1996 (1984).

33. "Strange Attractors That Are Not Chaotic", C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica **13D**, 261 (1984).

34. "Quasiperiodicity and Chaos", C. Grebogi, E. Ott, and J. A. Yorke, in Proc. XIIIth Intn'l. Colloq. on Group Theor. Methods in Physics (World Scientific Publishing Co., Singapore, 1984).

35. "Attractors on an N-Torus: Quasiperiodicity Versus Chaos", C. Grebogi, E. Ott, and J. A. Yorke, Physica **15D**, 354 (1985).

36. "Nonlinear Dynamics", C. Grebogi, in *The Role of Supercomputers in Energy Research Program*, U.S. Department of Energy, Office of Scientific Computing (1984).

37. "The Exterior Dimension of Fat Fractals", C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. **110A**, 1 (1985).

38. "Scaling Behavior of Windows in Dissipative Dynamical Systems", J. A. Yorke, C. Grebogi, E. Ott, and L. Tedeschini-Lalli, Phys. Rev. Lett. **54**, 1095 (1985).

39. "Effect of Noise on Time-Dependent Quantum Chaos", E. Ott, J. D. Hanson, and T. M. Antonsen, Phys. Rev. Lett. **53**, 2187 (1984).

40. "Quasiperiodically Forced Damped Pendulà and Schrödinger Equations with Quasiperiodic Potentials: Implications of Their Equivalence", A. Bondeson, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **55**, 2103 (1985).

41. "How Often Do Simple Dynamical Processes Have Infinitely Many Coexisting Sinks?", L. Tedeschini-Lalli and J. A. Yorke, Comm. Math. Phys. **106**, 635 (1986).

42. "Markov Tree Model of Transport in Hamiltonian Systems", J. D. Meiss and E. Ott, Phys. Rev. Lett. **55**, 2741 (1985).

43. "Broadening of Spectral Peaks at the Merging of Chaotic Bands in Period-Doubling Systems", R. Brown, C. Grebogi, and E. Ott, Phys. Rev. A **34**, 2248 (1986).

44. "Markov Tree Model of Transport in Area-Preserving Maps", J. D. Meiss and E. Ott, Physics **20D**, 387 (1986).

45. "Lorenz-like Chaos in a Partial Differential Equation for a Heated Fluid Loop", J. A. Yorke, E. D. Yorke, and J. Mallet-Paret, Physica **24D**, 279 (1987).

46. "Chaos, Strange Attractors, and Fractal Basin Boundaries in Nonlinear Dynamics", C. Grebogi, E. Ott, and J. A. Yorke, Science **238**, 632 (1987).

47. "Strange Nonchaotic Attractors of the Damped Pendulum with Quasiperiodic Forcing", F. J. Romeiras and E. Ott, Phys. Rev. A **35**, 4404 (1987).

48. "Quasiperiodically Forced Dynamical Systems with Strange Nonchaotic Attractors", F. J. Romeiras, A. Bondeson, E. Ott, T. M. Antonsen, and C. Grebogi, Physica **26D**, 277 (1987).

49. "Chaotic Fluid Convection and the Fractal Nature of Passive Scalar Gradients", E. Ott and T. M. Antonsen, Phys. Rev. Lett. **61**, 2839 (1988).

50. "Critical Exponents for Power-Spectra Scaling at Mergings of Chaotic Bands", F. Romeiras, C. Grebogi, and E. Ott, Phys. Rev. A **38**, 463 (1988).

51. "Numerical Orbits of Chaotic Processes Represent True Orbits", S. M. Hammel, J. A. Yorke, and C. Grebogi, Bull. Am. Math. Soc. **19**, 465 (1988).

52. "Chaotic Flows and Magnetic Dynamos", J. M. Finn and E. Ott, Phys. Rev. Lett. **60**, 760 (1988).

53. "Chaotic Flows and Fast Magnetic Dynamos", J. M. Finn and E. Ott, Phys. Fluids **31**, 2992 (1988).

54. "Evolution of Attractors in Quasiperiodically Forced Systems: From Quasiperiodic to Strange Nonchaotic to Chaotic", M. Ding, C. Grebogi, and E. Ott, Phys. Rev. A **39**, 2593 (1989).

55. "Spatio-temporal Dynamics in a Dispersively Coupled Chain of Nonlinear Oscillators", D. K. Umberger, C. Grebogi, E. Ott, and B. Afeyan, Phys. Rev. A **39**, 4835 (1989).

56. "Theory of First Order Phase Transitions for Chaotic Attractors of Nonlinear Dynamical Systems", E. Ott, C. Grebogi, and J. A. Yorke, Phys. Lett. **135A**, 343 (1989).

57. "Routes to Chaotic Scattering", S. Bleher, E. Ott, and C. Grebogi, Phys. Rev. Lett. **63**, 919 (1989).

58. "Quasiperiodic Forcing and the Observability of Strange Nonchaotic Attractors", F. J. Romeiras, A. Bondeson, E. Ott, T. M. Antonsen, and C. Grebogi, Physica Scripta **40**, 442 (1989).

59. "Chaos, Strange Attractors, and Fractal Basin Boundaries", C. Grebogi, Trans. Am. Nucl. Soc. **60**, 346 (1989).

60. "Strange Saddles in Scattering Hamiltonian Systems", G. H. Hsu, S. Bleher, C. Grebogi, and E. Ott, in *Essays on Classical and Quantum Dynamics*, Ed. H. Uberall (Gordon and Breach, 1989), 14 pages, in print.

61. "Fractal Structure in Physical Space in the Dispersal of Particles in Fluids", L. Yu, C. Grebogi, and E. Ott, in Proc. Conf. on Nonlinear Structures in Physical Systems - Pattern Formation, Chaos, and Waves (Springer-Verlag, 1990).

62. "A Procedure for Finding Numerical Trajectories on Chaotic Saddles", H. E. Nusse and J. A. Yorke, Physica **36D**, 137 (1989).

II. CONTROL OF CHAOTIC PROCESSES

The following is a brief discussion of our recent progress on the problem of controlling chaotic processes using unstable periodic orbits (discussed in Sec. I.B.1).

To simplify the analysis we consider continuous time dynamical systems which are *three dimensional* and depend on one system parameter which we denote p. (For example, $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}, p)$, where $\mathbf{x}$ is three dimensional.) We assume that the parameter p is available for external adjustment, and we wish to temporally program our adjustments of p so as to achieve improved performance. We emphasize that our restriction to a three dimensional system is mainly for ease of presentation, and we believe that the case of higher dimensional (including infinite dimensional) systems can be treated by similar methods.

We imagine that the dynamical equations describing the system are not known, but that experimental time series of some scalar dependent variable $z(t)$ can be measured. Using delay coordinates [PCFS,T] with delay T one can form a delay coordinate vector, $\mathbf{X}(t) = [z(t), z(t-T), z(t-2T), \ldots, z(t-MT)]$. We are interested in periodic orbits and their stability properties, and we shall use $\mathbf{X}$ to obtain a surface of section for this purpose. In the surface of section, a continuous-time periodic orbit appears as a discrete time orbit cycling through a finite set of points. We require the dynamical behavior of the surface of section map in *neighborhoods* of these points in order to study the stability of the periodic orbits. To embed a small neighborhood of a point from $\mathbf{x}$ into $\mathbf{X}$, we typically only require as many dimensions as there are coordinates of the point. Thus, for our purposes, $M = D - 1$ is generally sufficient. (This is in contrast with $M + 1 = 2D + 1$, typically required for global embedding.) Hence, for the case considered ($D = 3$), our surface of section is two-dimensional.

We suppose that the parameter p can be varied in a small range about some nominal value p_0. Henceforth, without loss of generality, we set $p_0 \equiv 0$. Let the range in which we are allowed to vary p be $p_* > p > -p_*$.

Using an experimental surface of section for the embedding vector $\mathbf{X}$, we imagine that we obtain many experimental points in the surface of section for $p = 0$. We denote these points $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$, $\ldots$, $\boldsymbol{\xi}_k$, where $\boldsymbol{\xi}_n$ denotes the coordinates in the surface of section at the nth piercing of the surface of section by the orbit $\mathbf{X}(t)$. For example, a common choice of the surface of section would be $z(t - MT)$ equals a constant, and $\boldsymbol{\xi}_n = [z(t_n), \ldots, z(t_n - (M-1)T)]$, where $t = t_n$ denotes the time at the nth piercing. From such experimentally determined sequences it has been demonstrated that a large number of distinct unstable periodic orbits on a chaotic attractor can be determined [GLV,LK]. We then examine these unstable periodic orbits and select the one which gives the best performance. Again using an experimentally determined sequence, we obtain the stability properties of the chosen periodic orbit (cf. [GLV] and [LK] for discussion of how this can be done and for descriptions of its implementation in concrete experimental cases). For the purposes of simplicity, let us assume in what follows that this orbit is a fixed point of the surface of section map (i.e., period one; the case of higher period is a straightforward extension). Let λ_s and λ_u be the experimentally determined stable and unstable eigenvalues of the surface of section map at the chosen fixed point of the map ($|\lambda_u| > 1 > |\lambda_s|$). Let $\mathbf{e}_s$ and $\mathbf{e}_u$ be the experimentally determined unit vectors in the stable and unstable directions. Let $\boldsymbol{\xi} = \boldsymbol{\xi}_F \equiv 0$ be the desired fixed point. We then change p slightly from $p = 0$ to some other value $p = \bar{p}$. The fixed point coordinates in the experimental surface of section will shift from 0 to some nearby point $\boldsymbol{\xi}_F(\bar{p})$ and we determine this new position. For small $\bar{p}$ we approximate $\mathbf{g} \equiv \partial \boldsymbol{\xi}_F(p)/\partial p|_{p=0} \cong \bar{p}^{-1}\boldsymbol{\xi}_F(\bar{p})$, which allows an experimental determination of the vector $\mathbf{g}$.

Thus, in the surface of section, near $\boldsymbol{\xi} = 0$, we can use a linear approximation for the map, $(\boldsymbol{\xi}_{n+1} - \boldsymbol{\xi}_F(p)) \cong \mathbf{M} \cdot (\boldsymbol{\xi}_n - \boldsymbol{\xi}_F(p))$, where M is a 2 by 2 matrix. Using $\boldsymbol{\xi}_F(p) \cong p\mathbf{g}$ we have

$$\boldsymbol{\xi}_{n+1} \cong p_n\mathbf{g} + [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot [\boldsymbol{\xi}_n - p_n\mathbf{g}]. \tag{1}$$

(In the linearization (1), we have considered p_n to be small and of the same order as $\boldsymbol{\xi}_n$.) We emphasize that $\mathbf{g}$, $\mathbf{e}_u$, $\mathbf{e}_s$, λ_u and λ_s are all experimentally accessible by the embedding technique just discussed. In (1) $\mathbf{f}_u$ and $\mathbf{f}_s$ are contravariant basis vectors defined by $\mathbf{f}_s \cdot \mathbf{e}_s = \mathbf{f}_u \cdot \mathbf{e}_u = 1$, $\mathbf{f}_s \cdot \mathbf{e}_u = \mathbf{f}_u \cdot \mathbf{e}_s = 0$. Note that we have written the location of the fixed point as $p_n\mathbf{g}$, because we imagine that we adjust p to a new value p_n after each piercing of the surface of section. That is, we observe $\boldsymbol{\xi}_n$ and then adjust p to the value p_n. Thus p_n depends on $\boldsymbol{\xi}_n$. Further, we only envision making this adjustment when the orbit falls near the desired fixed point for $p = 0$.

Assume that $\boldsymbol{\xi}_n$ falls near the desired fixed point at $\boldsymbol{\xi} = 0$ so that (1) applies. We then attempt to pick p_n so that $\boldsymbol{\xi}_{n+1}$ falls on the stable manifold of $\boldsymbol{\xi} = 0$. That is, we choose p_n so that $\mathbf{f}_u \cdot \boldsymbol{\xi}_{n+1} = 0$. If $\boldsymbol{\xi}_{n+1}$ falls on the stable manifold of $\boldsymbol{\xi} = 0$, we can then set the parameter perturbations to zero, and the orbit for subsequent time will approach the fixed point at the geometrical rate λ_s. Thus for sufficiently small $\boldsymbol{\xi}_n$ we can dot (1) with $\mathbf{f}_u$, to obtain

$$p_n = \lambda_u(\lambda_u - 1)^{-1}(\boldsymbol{\xi}_n \cdot \mathbf{f}_u)/(\mathbf{g} \cdot \mathbf{f}_u), \tag{2}$$

which we use when the magnitude of the right-hand side of (2) is less than p_*. When it is greater than p_*, we set $p_n = 0$. We assume in (2) that the generic condition $\mathbf{g} \cdot \mathbf{f}_u \neq 0$ is satisfied. Thus, the parameter perturbations are activated (i.e., $p_n \neq 0$) only if $\boldsymbol{\xi}_n$ falls in a narrow strip $|\xi_n^u| < \xi_*$, where $\xi_n^u = \mathbf{f}_u \cdot \boldsymbol{\xi}_n$, and from (2) $\xi_* = p_*|(1 - \lambda_u^{-1})\mathbf{g} \cdot \mathbf{f}_u|$. Thus, for small p_*, a typical initial condition will execute a chaotic orbit, unchanged from the uncontrolled case, until $\boldsymbol{\xi}_n$ falls in the strip. Even then, because of nonlinearity not included in (1), the control may not be able to bring the orbit to the fixed point. In this case the orbit will leave the strip and continue to wander chaotically as if there was no control. Since the orbit on the uncontrolled chaotic attractor is ergodic, at some time it will eventually satisfy $|\xi_n^u| < \xi_*$ and also be sufficiently close to the desired fixed point that attraction to $\boldsymbol{\xi} = 0$ is achieved. (In rare cases applying Eq. (2) when the trajectory enters the strip, but is still far from 0, may result in stabilizing the wrong periodic orbit which visits the strip.)

Thus, we create a stable orbit, but, for a typical initial condition, it is preceded in time by a chaotic transient in which the orbit is similar to orbits on the uncontrolled chaotic attractor. The length τ of such a chaotic transient depends sensitively on the initial condition, and, for randomly chosen initial conditions, has an exponential probability distribution [GOY1, GORY] $P(\tau) \sim \exp -(\tau / <\tau>)$ for large τ. The average length of the chaotic transient $<\tau>$ increases with decreasing p_* and follows a power law relation [GOY1, GORY] for small p_*, $<\tau> \sim p_*^{-\gamma}$.

We will now derive a formula for the exponent γ. Dotting the linearized map for $\boldsymbol{\xi}_{n+1}$, Eq. (1), with $\mathbf{f}_u$, we obtain $\xi_{n+1}^u \cong 0$. In obtaining this result from (1) we have substituted p_n appropriate for $|\xi_n^u| < \xi_*$. We note that the result $\xi_{n+1}^n \cong 0$ is a linearization, and typically has a lowest order nonlinear correction that is quadratic. In particular, $\xi_n^s = \mathbf{f}_s \cdot \boldsymbol{\xi}_n$ is not restricted by $|\xi_n^u| < \xi_*$, and thus may not be small when the condition $|\xi_n^u| < \xi_*$ is satisfied. Hence the correction quadratic in ξ_n^s is most significant. Including such a correction we have $\xi_{n+1}^u \cong \kappa(\xi_n^s)^2$, where κ is a constant. Thus, if $|\kappa|(\xi_n^s)^2 > \xi_*$, then $|\xi_{n+1}^u| > \xi_*$, and attraction to $\boldsymbol{\xi} = 0$ is not achieved, even though $|\xi_n^u| < \xi_*$. Attraction to $\boldsymbol{\xi} = 0$ is achieved when the orbit falls in the small parallelogram P_c given by $|\xi_n^u| < \xi_*$, $|\xi_n^s| < (\xi_*/|\kappa|)^{1/2}$. For very small ξ_*, an initial condition will bounce around on the set comprising the uncontrolled chaotic attractor for a long time before it falls in the parallelogram P_c. At any given iterate the probability of falling in P_c is $\mu(P_c)$, the measure of the uncontrolled attractor contained in P_c. Thus, $<\tau>^{-1} = \mu(P_c)$. The scaling of $\mu(P_c)$ with ξ_* is $\mu(P_c) \sim (\xi_*)^{d_u}[(\xi_*/|\kappa|)^{1/2}]^{d_s} \sim \xi_*^{d_u + \frac{1}{2}d_s}$, where d_u and d_s are the partial pointwise dimensions for the uncontrolled chaotic attractor at $\boldsymbol{\xi} = 0$ in the unstable direction and the stable direction, respectively. Thus $\mu(P_c) = \xi_*^\gamma$, where $\gamma = d_u + (d_s/2)$. Since we assume the attractor to be effectively smooth in the unstable direction, $d_u = 1$. The partial pointwise dimension in the stable direction is given in terms of the eigenvalues [GOY1, GORY] at $\boldsymbol{\xi} = 0$, $d_s = (\ell n|\lambda_u|)/(\ell n|\lambda_s|^{-1})$. Thus

$$\gamma = 1 + \frac{1}{2}(\ell n|\lambda_u|)/(\ell n|\lambda_s|^{-1}). \tag{3}$$

To study the effect of noise we add a term $\epsilon\boldsymbol{\delta}_n$ to the right-hand side of the linearized equations for $\boldsymbol{\xi}_{n+1}$, Eq. (1), where $\boldsymbol{\delta}_n$ is a random variable and ϵ is a small parameter specifying the intensity of the noise. The quantities $\boldsymbol{\delta}_n$ are taken to have zero mean ($<\delta_n>=0$), be independent ($<\delta_n\delta_m>=0$ for $m\neq n$), and have a probability density independent of n. Dotting (1) with noise included with $\mathbf{f}_u$ we obtain $\xi^u_{n+1}=\epsilon\delta^u_n$, where $\delta^u_n\equiv\mathbf{f}_u\cdot\boldsymbol{\delta}_n$. Thus if the noise is bounded, $|\delta^u_n|<\delta_{\max}$, then the stability of $\boldsymbol{\xi}=0$ will not be affected by the noise if the bound is small enough, $\epsilon\delta_{\max}<\xi_*$. If this condition is not satisfied, then the noise can kick an orbit which is initially in the parallelogram P_c into the region outside P_c. We are particularly interested in the case where such kick-outs are caused by low probability tails on the probability density and are thus rare. (If they are frequent then our procedure is ineffective.) In such a case the average time to be kicked out $<\tau'>$ will be long. Thus an orbit will typically alternate between epochs of chaotic motion of average duration $<\tau>$ in which it is far from $\boldsymbol{\xi}=0$, and epochs of average length $<\tau'>$ in which the orbit lies in the parallelogram P_c. For small enough noise the orbit spends most of its time in P_c, $<\tau'>\gg<\tau>$, and one might then regard the procedure as being effective.

We now consider a specific numerical example. Our purpose is to illustrate and test our analyses of the average time to achieve control and the effect of noise. To do this we shall utilize the Henon map, $x_{n+1}=A-x_n^2+By_n$, $y_{n+1}=x_n$, where we take $B=0.3$. We assume that the quantity A can be varied by a small amount about some value A_0. Accordingly we write A as $A=A_0+p$, where p is the control parameter. For the values of A_0 which we investigate, the attractor for the map is chaotic and contains an unstable period one (fixed point) orbit. The coordinates (x_F,y_F) for the fixed point which is in the attractor along with the associated parameters and vectors appearing in Eq. (1) may be explicitly calculated. The quantity $\boldsymbol{\xi}_n$ appearing in (1) is $\boldsymbol{\xi}_n=(x_n-x_F)\mathbf{x}_0+(y_n-y_F)\mathbf{y}_0$. To test our prediction for the dependence of $<\tau>$, the average time to approach $\boldsymbol{\xi}=0$, on the maximum allowed size of the parameter perturbation p_*, we proceed as follows. We iterate the map with $p=0$ using a large number of randomly chosen initial conditions until all these initial conditions are distributed over the attractor (500 iterates were typically used). We then turn on the parameter perturbations and determine for each orbit how many further iterates τ are necessary before the orbit falls within a circle of radius $\frac{1}{2}\xi_*$ centered at the fixed point. We then calculate the average of these times. We do this for many different values of p_* and plot the results as a function of p_*. This is shown on the log-log plot in Fig. 1 along with the theoretical straight line of slope given by the exponent (3). We see that the agreement is good although there are significant variations about the general power law trend. These are to be believed due to the fractal nature of the attractor and have also been seen in numerical calculations of the pointwise dimension for points on chaotic attractors (cf. [GOY2] and [GOY3]).

Next we must consider the issue of noise. We add terms $\epsilon\delta_{xn}$ and $\epsilon\delta_{yn}$ to the right-hand sides of the Henon map equations. The random quantities δ_{xn} and δ_{yn} are independent of each other, have mean value zero, mean squared value one ($< \delta_x^2 > = < \delta_y^2 > = 1$), and have a gaussian probability density. Figure 2 shows orbit plots, x_n versus n for 1500 iterates of the noisy map with parameter perturbations given by (2), for two different noise levels and p_* held fixed at $p_* = 0.2$. As predicted the orbit stays near the fixed point with occasional bursts into the region far from $\boldsymbol{\xi} = 0$, and these bursts are less frequent for smaller noise levels.

We propose extensions of this work in the following directions. We emphasize that our numerical experiments to date have only involved the Henon map. In practice, one can anticipate difficulties as experience is gained with more typical dynamical systems. In particular, what are the implications of imperfect identification of the periodic orbit and its stability properties? What are the implications of different types of noise? Are there unforeseen problems in going to higher dimensional and infinite dimensional dynamical systems? How can the transient times to achieve the desired periodic orbit be most effectively reduced by small controls? These and other problems will be studied, and we anticipate that this area will be a rich source of very interesting practical and theoretical research with major potential technological benefits for a broad range of applications.

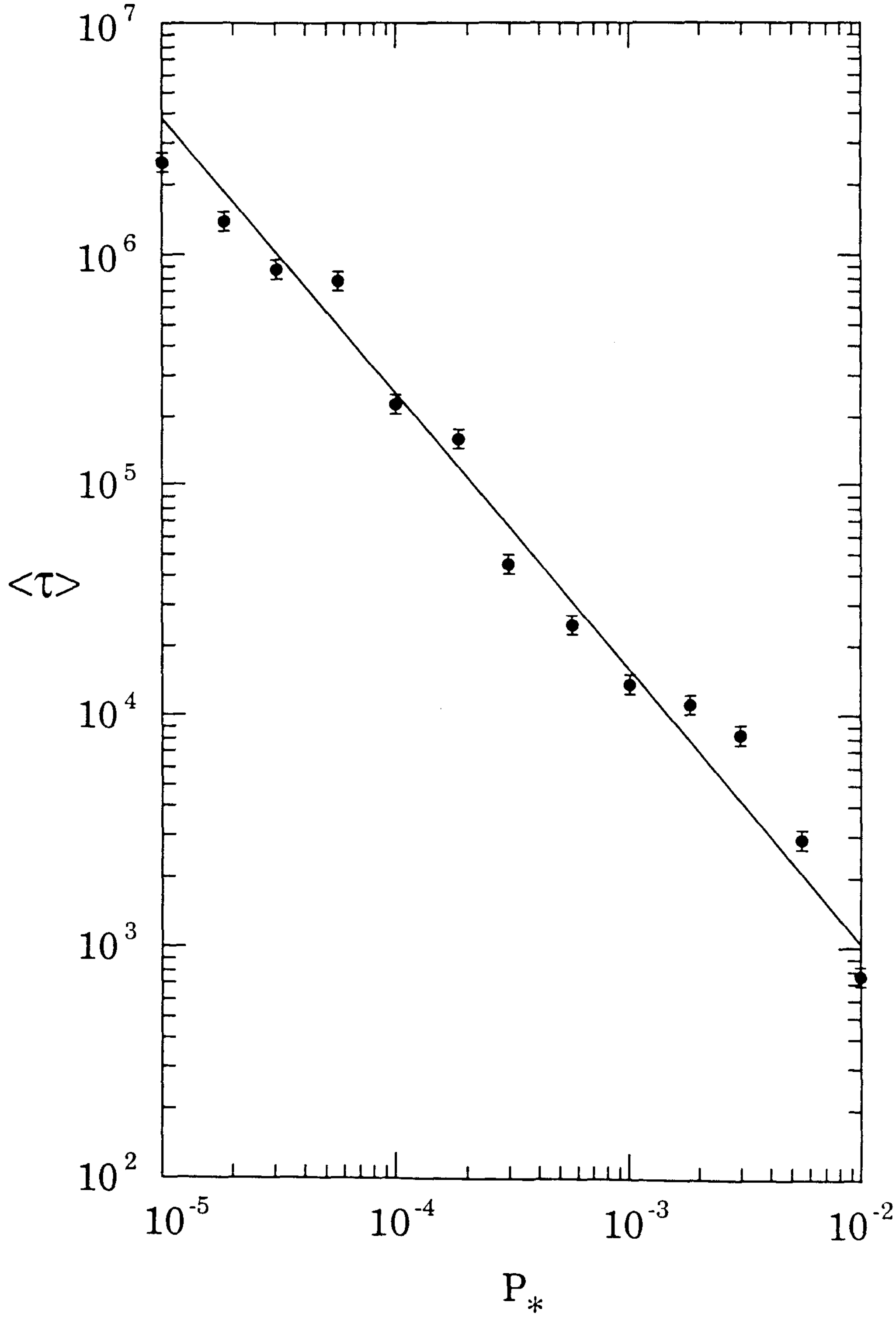

Fig. 1. $<\tau>$ versus p_*. Points were computed using 128 randomly selected initial conditions. $A_0 = 1.4$.

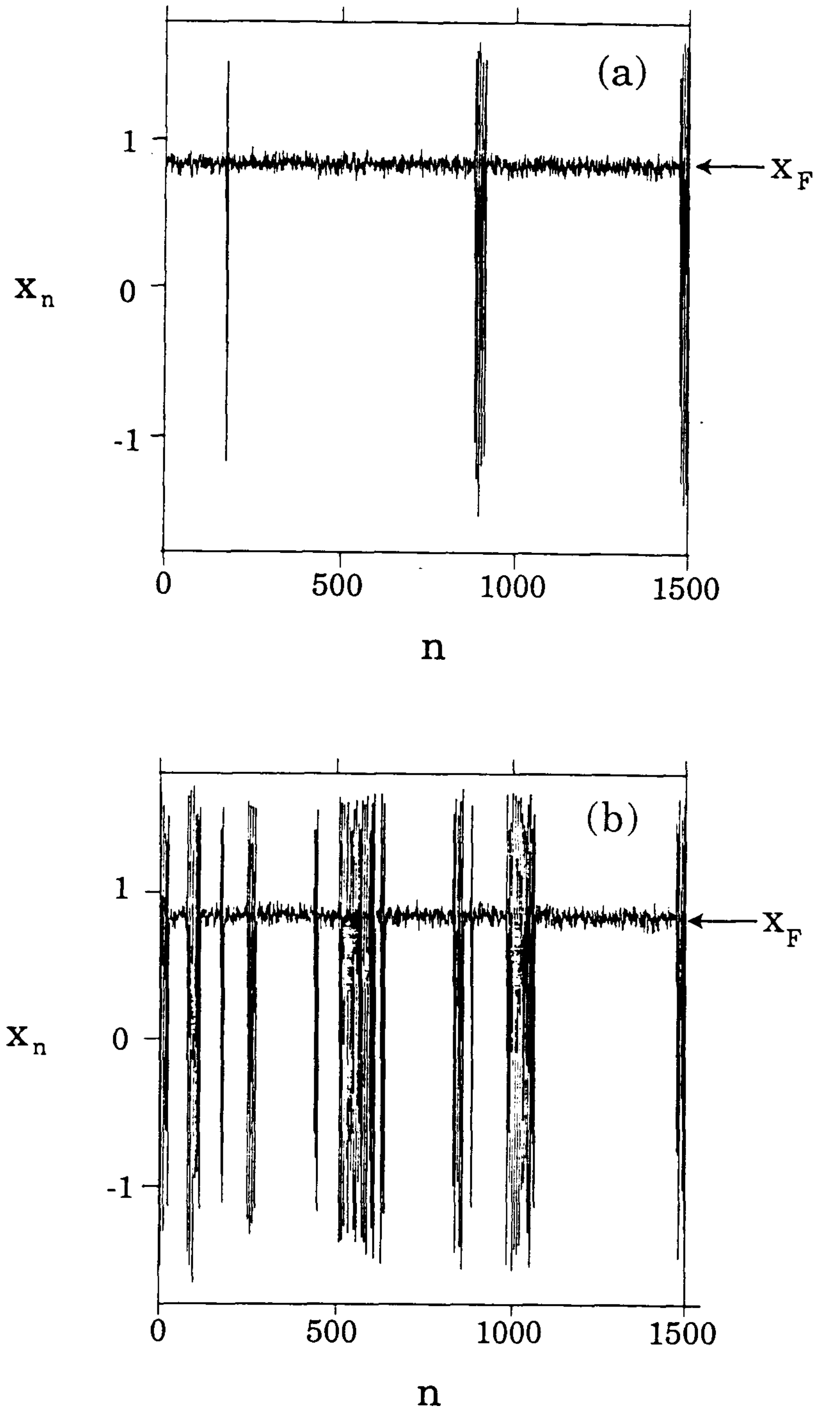

Fig. 2. x_n versus n for two cases with the same realization of the random vector δ. $p_* = 0.2$ and $A_0 = 1.29$ for both cases. (a) $\epsilon = 3.5 \times 10^{-2}$; (b) $\epsilon = 3.8 \times 10^{-2}$.