PLASMA RESEARCH

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CONF-38-7

Longitudinal and Transversal Plasma Wave Instabilities in Two Counterstreaming Plasmas without External Fields*

by

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4th Annual Meeting, Division of Plasma Physics
American Physical Society
Atlantic City, New Jersey
November 28-December 1, 1962

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Technical Report No. A-16

March, 1963

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Abstract

Some aspects of the theory of longitudinal and transversal waves in a collisionless nonrelativistic plasma are treated in this paper. A dispersion relation for multicomponent plasmas is derived from the linearized Boltzmann-Vlasov equation using the full set of Maxwell's equations without an external field. The velocity distributions of the plasma streams are assumed to be Maxwellian. For the particular case of two counterstreaming plasmas it is shown that there exists transversal instabilities for all counterstreaming velocities whereas the well known two stream instabilities only exist for velocities greater than a critical velocity. Exact solutions for the onset of the instabilities can be given. This kind of instability may occur for any nonisotropic velocity distribution in a collisionless plasma.

* Supported in part by the National Aeronautics and Space Administration.

Introduction

Longitudinal wave instabilities in counterstreaming collisionless plasmas have been discussed by several authors.* They reported that these instabilities only occur above certain minimum counterstreaming velocities. One of the purposes of this paper is to find whether other instabilities of transversal character exist when the counterstreaming velocity is smaller. Taking the full set of Maxwell's equations into account this can indeed be shown. In Section I to III we derive the dispersion relation valid for an arbitrary multicomponent plasma. In section IV it is shown that there exists a complete equivalence between our method of Laplace transform technique and the eigenfunction expansion method. Section V brings a few remarks about the general problem of a plasma with boundaries. Section VI shows the application for a resting plasma whereas Section VII gives a detailed picture of all two stream instabilities.

I. Fundamental Equations

The set of Boltzmann-Vlasov (B.V.) equations for a multicomponent collisionless plasma is given by the expression

$$\frac{\partial F_s}{\partial t} + \underline{v} \text{ grad } F_s + \frac{e_s}{m_s} (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \text{grad}_v F_s = 0 \qquad (1)$$

$$(s = 1,2,\ldots,S)$$

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* J. R. Pierce, Possible Fluctuations in Electron Streams due to Ions." Journal of Appl. Phys. 19, 23, (1948).

D. Bohm and E. P. Gross, "Theory of Plasma Oscillations," Phys. Rev. 75, 1851 and 1864 (1949).

O. Buneman , Dissipation of Currents in Ionized Media, Phys. Rev. 115 503 - 517 (1959).

F_s is the distribution function for the particles of the component s of the plasma. The current $\underline{j}$ and the charge density ρ are given by the following expressions

$$\underline{j} = \sum_{s=1}^{S} e_s \int \underline{v}\, F_s(\underline{v})\, d^{(3)}v \qquad (2)$$

$$\rho = \sum_{s=1}^{S} e_s \int F_s(\underline{v})\, d^{(3)}v \qquad (3)$$

We assume the plasma to be neutral and without currents or fields just before we introduce some small perturbation at the time zero. This enables us to linearize the equations (1) by subtracting the analogous equations for the unperturbed quasi-stable distribution F_{os} with corresponding vanishing fields $\underline{E}_o$ and $\underline{B}_o$. Here F_{os} is to be assumed constant in space and time. For the small perturbation $f_s = F_s - F_{os}$ which in this case will be different from zero only for a positive time we get the linearized B.V. equations

$$\frac{\partial F_s}{\partial t} + \underline{v} \cdot \text{grad } f_s + \frac{e_s}{m_s} \left(\underline{E} + \frac{1}{c}\, \underline{v} \times \underline{B}\right) \cdot \text{grad}_v\, F_{os} = 0 \qquad (1')$$

$$(s = 1, 2, \ldots S)$$

The fields $\underline{E}$ and $\underline{B}$ in these equations are produced by the small perturbations only. Additionally we have to satisfy Maxwell's equations and the equation of continuity:

$$\text{div}\ \underline{E} = 4\pi\rho \tag{4}$$

$$\text{div}\ \underline{B} = 0 \tag{5}$$

$$\text{curl}\ \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \tag{6}$$

$$\text{curl}\ \underline{B} = \frac{4\pi \underline{j}}{c} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t} \tag{7}$$

$$\frac{\partial\rho}{\partial t} + \text{div}\ \underline{j} = 0 \tag{8}$$

It is our goal to solve the equations (1') together with (4) to (8). This solution will give us a complete and self-consistent picture of all possible plasma waves for the assumptions initially made. We now assume a time dependence proportional to

$$\exp\,(pt - \underline{\Gamma} \cdot \underline{r}) \tag{9}$$

where we have introduced the complex frequency $p = i\omega$ and the complex wave vector $\underline{\Gamma} = i\underline{k}$ for convenience sake.

As we deal with an initial value problem, we can use the Laplace transformation for our equations which we already indicated by using the letter p for the complex frequency. For the space domain we use the Fourier transformation regarding an infinite space without any boundary values.

II. Conductivity Tensor

Our next step is the elimination of $\underline{B}$ in the B.V. equations by means of Faraday's law (6) which now becomes

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$$p\underline{B} = c\underline{\Gamma} \times \underline{E} + \underline{B}(0^+) \, . \tag{10}$$

This gives us

$$\frac{1}{c}\,\underline{v} \times \underline{B} = \frac{1}{p}\,\underline{\Gamma}\,(\underline{v} \cdot \underline{E}) - \frac{1}{p}\,\underline{E}(\underline{v} \cdot \underline{\Gamma}) + \frac{1}{cp}\,\underline{v} \times \underline{B}(0^+) \tag{11}$$

Introducing this into the B.V. equations we get after division by $p - \underline{\Gamma} \cdot \underline{v}$

$$f_s(\underline{v}) = -\frac{n_s e_s}{p m_s}\,\underline{E} \cdot \left(\mathrm{grad}_v\, F_{os} + \underline{v}\,\frac{(\underline{\Gamma} \cdot \mathrm{grad}_v\, F_{os})}{p - \underline{\Gamma} \cdot \underline{v}}\right)$$

$$+ \frac{f_s(0^+)}{p - \underline{\Gamma} \cdot \underline{v}} + \frac{n_s e_s}{p m_s}\,\underline{B}(0^+) \cdot \frac{(\underline{v} \times \mathrm{grad}_v\, F_{os})}{p - \underline{\Gamma} \cdot \underline{v}} \tag{13}$$

If there are n_s particles per unit volume we normalize F_{os} by introducing $f_{os} = n_s^{-1} F_{os}$. Integration of

$$\sum_{s=1}^{S} \underline{v}\, e_s f_s(\underline{v})$$

over all velocities gives us the connection between current $\underline{j}$ and electric field $\underline{E}$

$$\underline{j} = \underline{E} \sum_{s=1}^{S} \frac{\omega_s^2}{4\pi p} - \underline{E} \cdot \sum_{s=1}^{S} \frac{\omega_s^2}{4\pi p} \int_{(v)} \frac{\underline{v}\,\underline{v}\,(\underline{\Gamma} \cdot \mathrm{grad}_v\, f_{os})}{p - \underline{\Gamma} \cdot \underline{v}}\, d^3v$$

$$+ \sum_{s=1}^{S} e_s \int_{(v)} \frac{\underline{v}\, f(0^+)}{p - \Gamma \cdot \underline{v}}\, d^3v + \underline{B}(0^+) \cdot \sum_{s=1}^{S} \frac{\omega_s^2}{4\pi} \int_{(v)} \frac{(\underline{v} \times \mathrm{grad}_v f_{os})\underline{v}}{p - \underline{\Gamma} \cdot \underline{v}}\, d^3v \tag{14}$$

The notation of two vectors written side by side without a connecting dot stands for a dyadic product. In this equation we used the identity

$$\int \underline{v}\, \mathrm{grad}_v\, f_{os}\, d^3v = -\underline{\underline{I}} \qquad (15)$$

and the abbreviation

$$\omega_s^2 = \frac{4\pi n_s e_s^2}{m_s} \qquad (16)$$

For electrons ω_s is equal to the plasma frequency. $\underline{\underline{I}}$ is the unit tensor of rank 2.

Introducing the "conductivity" tensor $\underline{\underline{\sigma}}$

$$\underline{\underline{\sigma}} = \sum_{s=1}^{S} \frac{\omega_s^2}{4\pi p} \left(\underline{\underline{I}} - \int_{(v)} \frac{\underline{v}\,\underline{v}\,(\underline{\Gamma} \cdot \mathrm{grad}_v\, f_{os})}{p - \underline{\Gamma} \cdot \underline{v}}\, d^3v \qquad (17)$$

and the vector $\underline{j}^{\mathrm{initial}}$ containing the initial values of the problem

$$\underline{j}^{\mathrm{initial}} = \sum_{s=1}^{S} e_s \int_{(v)} \frac{\underline{v}\, f(0^+)}{p - \Gamma \cdot v}\, d^3v + $$

$$+ \underline{B}(0^+) \cdot \sum_{s=1}^{S} \frac{\omega_s^2}{4\pi p} \int \frac{(\underline{v} \times \mathrm{grad}_v f_{os})\underline{v}}{p - \Gamma \cdot \underline{v}}\, d^3v \qquad (18)$$

we arrive to the equation for $\underline{j}$

$$\underline{j} = \underline{\underline{\sigma}} \cdot \underline{E} + j^{\mathrm{initial}} \qquad (19)$$

connecting the current with the electric field.

III. Dispersion Relations

Eliminating the magnetic field $\underline{B}$ from Faraday's and Ampére's law, we have another formula between the electric field and the current:

$$(p^2 - \Gamma^2 c^2)\underline{E}_\perp + p^2 \underline{E}_{||} + 4\pi p \underline{j} = p\,\underline{E}(0^+) + c\underline{\Gamma} \times \underline{B}(0^+) \tag{20}$$

$\underline{E}_{||}$ denotes the electric field parallel, $\underline{E}_\perp$ the electric field perpendicular to the vector $\underline{\Gamma}$. Additionally we used the identity

$$c\underline{\Gamma} \times \underline{B}(0^+) = \underline{E}'(0^+) + 4\pi\,\underline{j}(0^+) \tag{21}$$

Inserting the current $\underline{j}$ from equation (19) we have reduced everything to an equation for the electric field alone:

$$\begin{aligned}(p^2 + \sum_{s=1}^{S} \omega_s^2)\,\underline{E} - \Gamma^2 c^2\,\underline{E}_\perp - \sum_{s=1}^{S} \omega_s^2 \int_{(v)} \frac{(\underline{\Gamma} \cdot \mathrm{grad}_v f_{os})}{p - \underline{\Gamma} \cdot \underline{v}}\, \underline{v}\,\underline{v}\, d^3v \cdot E \\ = -4\pi p \sum_{s=1}^{S} e_s \int_{(v)} \frac{\underline{v}\, f_s(0^+)}{p - \underline{\Gamma} \cdot \underline{v}}\, d^3v + p\,\underline{E}(0^+) + c\underline{\Gamma} \times \underline{B}(0^+) \\ - \sum_{s=1}^{S} \omega_s^2 \int_{(v)} \frac{\underline{v}(\underline{v} \times \mathrm{grad}_v\, f_{os})}{p - \underline{\Gamma} \cdot \underline{v}}\, d^3v \cdot \underline{B}(0^+)\end{aligned} \tag{22}$$

Formally we write this equation in the following more elegant form

$$\underline{\underline{D}} \cdot \underline{E} = \underline{C} \tag{23}$$

The vector $\underline{C}$ represents the right side of (22) giving all initial values necessary. Assuming $\underline{\Gamma}$ in the x-direction and defining $\nu = p/\Gamma$ we get the following representation for the components of the dispersion tensor $\underline{\underline{D}}$:

<u>General expressions for the dispersion tensor $\underline{\underline{D}}$</u> :

$$D_{11} = p^2 \left(1 + \sum_s \frac{\omega_s^{\,2}}{\Gamma^2} \int \frac{\frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} d^3v\right) \tag{24}$$

$$D_{22} = p^2 - \Gamma^2 c^2 + \sum_s \omega_s^{\,2} \left(1 + \int \frac{v_y^{\,2} \frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} d^3v\right) \tag{25}$$

$$D_{33} = p^2 - \Gamma^2 c^2 + \sum_s \omega_s^{\,2} \left(1 + \int \frac{v_z^{\,2} \frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} \right. \tag{26}$$

$$D_{12} = D_{21} = p \sum_{s=1}^{S} \frac{\omega_s^{\,2}}{\Gamma} \int \frac{v_y \frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} d^3v \tag{27}$$

$$D_{13} = D_{31} = p \sum_{s=1}^{S} \frac{\omega_s^{\,2}}{\Gamma} \int \frac{v_z \frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} d^3v \tag{28}$$

$$D_{23} = D_{32} = \sum_{s=1}^{S} \omega_s^{\,2} \int \frac{v_y \, v_z \frac{\partial}{\partial v_x} f_{os}}{v_x - \nu} \tag{29}$$

For derivation of (24) and (26) we added and subtracted terms partly to cancel the factor $v_x - \nu$ in the denominator. Then there are only terms left without a factor $v_x - \nu$ in the numerator. Of course we used the identity

$$\int v_x \frac{\partial}{\partial v_x} f_{os} \, d^3v = -1 \tag{30}$$

The formulas are still valid for any normalized distribution functions f_{os}. We specialize these distribution functions to be Maxwellian in all directions:

$$f_{os} = \frac{1}{(2\pi < \underline{v}_s^2 >)^{3/2}} e^{-\frac{(\underline{v} - \underline{\bar{v}}_s)^2}{2 < \underline{v}_s^2 >}} \qquad (31)$$

and use the properties of the Maxwellian distributions:

$$\int \underline{v}\, f_{os}\, d^3v = \bar{v}_s \;, \quad \int (\underline{v} - \underline{v}_s)^2 f_{os}\, d^3v = < v_s^2 > \qquad (32)$$

$$\int \underline{v}^2 f_{os}\, d^3v = < v_s^2 > + \underline{\bar{v}}_s^2 \qquad (33)$$

Furthermore we introduce the following function

$$Z(\mu) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{x - \mu}\, d\mu \text{ for } \operatorname{Im} \mu > 0 \;;\; \mu_s = \frac{\frac{p}{\Gamma} - \bar{v}_{sx}}{2 < v_s^2 >} \qquad (34)$$

and its analytical continuation into the lower μ - halfplane which corresponds to the left p - halfplane for the Laplace transformation (Γ = ik, p = ikr). Some of the properties of $Z(\mu)$ are given in the appendix. For the dispersion matrix we use this analytical continuation as we treat our initial value problem with the Laplace transformation. Finally we get the following formulas for the dispersion matrix elements:

Dispersion matrix $\underline{\underline{D}}$ for Maxwellian velocity distributions:

$$D_{11} = p^2 \left\{ 1 + \sum_s \frac{\omega_s^2}{k^2 < v_s^2 >} (1 + \mu_s Z(\mu_s)) \right\} \tag{35}$$

$$D_{22} = p^2 + k^2 c^2 + \sum_s \omega_s^2 \left[1 - (1 + \frac{\bar{v}_{ys}^2}{< v_x^2 >}) \cdot (1 + \mu_s Z(\mu_s)) \right] \tag{36}$$

$$D_{33} = p^2 + k^2 c^2 + \sum_s \omega_s^2 \left[1 - (1 + \frac{\bar{v}_{zs}^2}{< v_x^2 >}) \cdot (1 + \mu_s Z(\mu_s)) \right] \tag{37}$$

$$D_{12} = D_{21} = p \sum_s \frac{\omega_s^2}{k^2 < v_s^2 >} \cdot ik \, \bar{v}_{sy} (1 + \mu_s Z(\mu_s)) \tag{38}$$

$$D_{13} = D_{31} = p \sum_s \frac{\omega_s^2}{k^2 < v_s^2 >} ik\bar{v}_{sz} (1 + \mu_s Z(\mu_s)) \tag{39}$$

$$D_{23} = D_{32} = - \sum_s \frac{\omega_s^2}{< v_s^2 >} \bar{v}_{sy} \cdot \bar{v}_{sz} (1 + \mu_s Z(\mu_s)) \tag{40}$$

Before discussing the dispersion relation we consider the case of a plasma beam at zero temperature (for the function $Z(\mu)$ see Appendix):

$$< v_s^2 > = 0 \; ; \; f_{os} = \delta(\underline{v} - \underline{\bar{v}}_s) \tag{41}$$

$$\lim \frac{1 + \mu_s Z(\mu_s)}{< v_s^2 >} = - \frac{1}{(\bar{v}_{sx} - \frac{p}{\Gamma})^2} \text{ for } < v_s^2 > \rightarrow 0$$

This has to be inserted into equations (35) to (40) for cold plasma streams. A Maxwellian plasma can be composed from singular velocity streams according to the identity

$$\frac{1}{< v_s^2 >} (1 + \mu_s \; Z(\mu_s)) = \int \frac{\frac{\partial}{\partial v_x} f_{os}}{v_x - r} d^3v = - \int \frac{f_{os}}{(v_x - \frac{p}{\Gamma})^2} d^3v \qquad (42)$$

A similar way of approach by superposition of single plasma streams was done by J. Neufeld and P. H. Doyle* using the polarization concept of a plasma instead of the B.V. equations. Their dispersion relation can be shown to be equivalent to a special case of our paper.

The time dependent electric field for a fixed $\underline{k}$ can be calculated by means of the inverse Laplace transformation:

$$\underline{E}(\underline{k}) = \lim_{\varepsilon \to 0} \int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} \left\{\underline{\underline{D}}(p)\right\}^{-1} \cdot \underline{\underline{C}}(p) e^{pt} dp + \sum_{\mathrm{Re}\, p_\nu > 0} \mathrm{Res} \left[\left\{\underline{\underline{D}}(p)\right\}^{-1} \cdot C(p) e^{pt}\right] \qquad (43)$$

The residues at the poles in the right p-plane correspond to instabilities given by the zeros of the determinant $|\underline{\underline{D}}(p)|$. The integral itself gives all contributions corresponding to more or less damped waves mainly dependent on the zeros of the analytical continuation of $\left\{\underline{\underline{D}}(p)\right\}^{-1}$ to the left half plane. Usually it is not possible to close the path of integration at infinity around the left half plane. This was shown by Landau

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* J. Neufeld and P. H. Doyle, Phys. Rev. 121, 654 (1961).

for the case of purely electrostatic waves in a Maxwellian plasma.

IV. Comparison with van Kampen's and Zelazny's Methods

A short remark may be given concerning methods developed by other authors. Extending van Kampen's eigenfunction expansion method* to the initial value problem of plasma oscillations including longitudinal and transversal waves, R. Zelazny gave a solution for an electron plasma**. The ions are treated as positive charge background. His method can be generalized without difficulties to a multicomponent plasma. The main difference between his method and our method is the fact that Zelazny- as we will show - implicitly used a two-sided Laplace transformation***.

If we assume the existence of two positive numbers p_1 and p_2 for which the functions

$$\underline{E}^{+}(t) = \underline{E}(t) \text{ for } t > 0 , \underline{E}^{+}(t) = 0 \text{ for } t < 0 \tag{44}$$

$$\underline{E}^{-}(t) = \underline{E}(t) \text{ for } t < 0 , \underline{E}^{-}(t) = 0 \text{ for } t > 0 \tag{45}$$

give finite values for the integrals

$$\int_{0}^{\infty} |\underline{E}(t)| e^{-p_1 t} dt \quad \text{and} \quad \int_{\infty}^{0} |\underline{E}(t)| e^{-p_2 t} dt \tag{46}$$

we can write the following formulas valid for any finite t:

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* N. G. van Kampen, Physica 21, 949 (1955).

** R. Zelazny, The initial value problem for longitudinal and transversal oscillations, Ann. of Phys. 20, 261-278 (1962).

*** Private communication from R. E. Collin and J. Gustincic.

$$\underline{E}^+(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-i\infty+p_1}^{+i\infty+p_1} \left\{ \int_0^\infty \underline{E}(t')e^{-pt'}dt' \right\} e^{+pt}dp \quad \text{for} \quad t > 0$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} \underline{E}^+(p)e^{pt}\,dp + \sum_{Re(p_\nu)>0} Res(\underline{E}^+(p_\nu)e^{p_\nu t}) \tag{47}$$

$$\underline{E}^-(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-i\infty-p_2}^{+i\infty-p_2} \left\{ \int_{-\infty}^{0} \underline{E}(t')e^{-pt'}dt' \right\} e^{pt}dp \quad \text{for} \quad t < 0$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-i\infty-\varepsilon}^{+i\infty-\varepsilon} \underline{E}^-(p)e^{pt}\,dp - \sum_{Re(p_\nu)<0} Res(\underline{E}^-(p_\nu)e^{p_\nu t}) \tag{48}$$

For $\underline{E}(p)$ we have to insert the expression

$$\left\{\underline{\underline{D}}(p)\right\}^{-1} \cdot \underline{C}(p) .$$

The sum of these functions is given by

$$\underline{E}(t) = \underline{E}^+(t) + \underline{E}^-(t) = \frac{1}{\pi i} P \int_{-i\infty}^{+i\infty} \underline{E}(p)e^{pt}\,dp +$$

$$+ \sum_{Re(p_\nu)\neq 0} Res(\underline{E}(p_\nu)e^{p_\nu t}) \cdot \text{sign}\left[Re(p_\nu)\right] \tag{49}$$

The residues on the imaginary axis are cancelled and the principal value of the integral is left over. This is exactly the expression which would have resulted from the singular integral method in which the sum over the residues expresses the action on the natural modes of the system.

But as we concern ourselves only with an initial value problem the method of the Laplace transform as given by (47) leads to the same result in a much simpler way. The same remark applies in calculating the distribution function.

V. The General Problem with Boundary Conditions

The B.V. equation describes the rate of change of the distribution function along the trajectories of the particles in phase space to be equal to the collision rate of the distribution function. We write the B.V. equation in a collisionless plasma for small perturbations $f_s = F_s - F_{os}$ in the following form

$$\frac{D}{Dt} f_s(\underline{r},\underline{v},t) + \underline{v} \operatorname{grad}_r F_{os}(\underline{r},\underline{v}) + \frac{e_s}{m_s} (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \operatorname{grad}_v F_{os}(\underline{r},\underline{v}) = 0 \tag{50}$$

$$s = 1,2,\ldots,S$$

We assume that $F_{os}(\underline{r}, \underline{v})$ is not dependent on time. Integration along the trajectories of the particles yields

$$\begin{aligned} f_s(\underline{r},\underline{v},t) &= \int_0^\infty \underline{v}' \cdot \operatorname{grad}_{r'} F_{os}(\underline{r}',\underline{v}') d\tau \\ &+ \frac{e_s}{m_s} \int_0^\infty \left| \underline{E}(\underline{r}',t-\tau) + \frac{1}{c} \underline{v}' \times \underline{B}(\underline{r}',t-\tau) \right| \cdot \operatorname{grad}_{v'} F_{os}(\underline{r}',\underline{v}') d\tau \end{aligned} \tag{51}$$

where

$$\begin{aligned} \underline{r}' &= \underline{r} - \int_0^\tau \underline{v}(t-\tau')\, d\tau' \\ \underline{v}' &= \underline{v} - \int_0^\tau \dot{\underline{v}}(t-\tau')\, d\tau' \end{aligned} \tag{52}$$

are the particle coordinates on their trajectories in phase space. Having no external fields we linearize our equations by assuming $\dot{v}$ oscillating with a very small amplitude and approximating $\underline{v}'$ by $\underline{v}$. This corresponds to a first linearization. The resulting equations*

$$f_s(\underline{r},\underline{v},t) = \frac{e_s}{m_s}\int_0^\infty d\tau\left[\underline{E}(\underline{r}-\underline{v}\tau,t-\tau)+\frac{1}{c}\,\underline{v}\times\underline{B}(\underline{r}-\underline{v}\tau,t-\tau)\right].\mathrm{grad}_v F_{os}(\underline{r}-\underline{v}\tau,\underline{v})d\tau + \int_0^\infty \underline{v}\cdot\mathrm{grad}\,F_{os}(\underline{r}-\underline{v}\tau,\underline{v})d\tau \tag{53}$$

$$s = 1,2,\ldots,s$$

are identical with the set of linearized B.V. equations (1) only if F_{os} do not depend on space. This last assumption corresponds to a second linearization of our equations. In this special case we can write them as

$$f_s(\underline{r},\underline{v},t) = \frac{e_s}{m_s}\int_0^\infty d\tau\left[\underline{E}(\underline{r}-\underline{v}\tau,t-\tau)+\frac{1}{c}\,\underline{v}\times\underline{B}(\underline{r}-\underline{v},t-\tau)\right].\,\mathrm{grad}_v\,F_{os}(v) \tag{53'}$$

$$s = 1,2,\ldots,S$$

or simply in the original differential form of the B.V. equation

$$\frac{\partial}{\partial\tau}f_s(\underline{r}-\underline{v}\tau,\underline{v},t-\tau) = -\frac{\partial}{\partial t}f_s(\underline{r}-\underline{v}\tau,\underline{v},t-\tau)-\underline{v}.\mathrm{grad}\,f_s(\underline{r}-\underline{v}\tau,\underline{v},t-\tau)$$

$$= \frac{e_s}{m_s}\left[\underline{E}(\underline{r}-\underline{v}\tau,t-\tau)+\frac{1}{c}\underline{v}\times\underline{B}(\underline{r}-\underline{v}\tau,t-\tau)\right].\mathrm{grad}\,F_{os}(\underline{v})$$

$$s = 1,2,\ldots,S$$

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* Collisions can be taken into account by an additional factor $e^{-\nu\tau}$ (ν = collision frequency) in the integrals.

which is identical to the previous used set of equations (1') for the argument $r' = \underline{r} - v\tau$ and $t - \tau$.

The neglect of boundaries and space variations in F_{os} implies that the description given in this paper is valid only for the following two conditions:

1. The wavelength is much smaller than the characteristic length of the system.
2. The time interval considered is short enough that the disturbance has not yet reached the boundaries and been reflected and transmitted. (Having introduced the disturbance at time zero we must take the time t instead of ∞ as upper limit of the integrals over τ).

Equation (53) has a simple interpretation: Only those velocity components of the right side integrand contribute for which $\underline{v}\tau$ just equals the distance of the space points $\underline{r}$ and $\underline{r}'$. This means that we could change the time integration to a one-dimensional space integration in the direction of $-\underline{v}$ beginning at r. At this integration all quantities are to be taken with a retarded time

$$t' = t - \tau = t - \frac{|r - r'|}{v}$$

Going back to the more general equation (51) we calculate the current in the plasma

$$\underline{j}(\underline{r},t) = \sum_s e_s \int \underline{v}\, f_s(\underline{r},\underline{v},t)\, d^3v$$

$$= \sum_s \frac{\omega_s^2}{4\pi} \int_{(v)} \int_{\tau=0}^{\infty} \underline{v}\,\underline{v}' \cdot \mathrm{grad}_r\, F(r',v')\, d^3v\, d\tau \qquad (54)$$

$$+ \sum_s \frac{\omega_s^2}{4\pi} \int_{(v)} \int_{\tau=0}^{\infty} \underline{v}\left[\underline{E}(r',t-\tau) + \frac{1}{c}\, \underline{v}' x \underline{B}(r',t-\tau)\right] \cdot \mathrm{grad}_v\, F(r',v') d^3v d\tau$$

where $\underline{r}'$, v' are given by (52). This current is "generated" by asymmetries of the distribution function in the space variables and by the action of the fields on the plasma producing a polarization at different points $\underline{r}'$. Both effects are carried along the trajectories by the particles. This gives a retardation time equal to the time in which the particles travel from $\underline{r}'$ to $\underline{r}$. By eliminating $\underline{B}$ from Maxwell's equations (6) and (7) and introducing the current (54) we get a complicated integro-differential equation for the electric field $\underline{E}$ which together with (51) gives a complete description of any collisionless plasma. However solutions to these equations can only be found for very simple special situations.

VI. Solutions for a Neutral Plasma at Rest

In order to clarify the importance of the dispersion relation we consider a plasma consisting of two components: electrons with mass m_e and charge e and ions with mass m_i and charge Ze. For convenience sake we introduce the following dimensionless parameters into the formulas (35) to (40):

$$\alpha = Z \frac{m_e}{m_i} \; ; \; \beta = \frac{v_e}{v_i} = \left(\frac{m_i}{m_e}\frac{T_e}{T_i}\right)^{1/2}$$

$$\mu = \frac{\omega}{\sqrt{2}\, k v_e} \; ; \; v_e = \left(\frac{\varkappa T}{m_e}\right)^{1/2} \; ; \; \tilde{k} = \frac{k v_e}{\omega_e} = k\lambda_D \text{ where } \lambda_D = \text{Debye length} \qquad (55)$$

$\varkappa$ = Boltzmann constant

Now the dispersion relation can be separated into

$$|\underline{\underline{D}}| = D_{11} \cdot D_{22} \cdot D_{33} = 0 \qquad (56)$$

where

$$p^2 D_{11} = \mathcal{L} = 1 + \tilde{k}^{-2} \, |(1 + \mu\, Z(\mu) + \alpha\beta^2(1 + \beta\mu Z(\beta\mu))| \qquad (57)$$

represents the condition for longitudinal waves and

$$\frac{1}{k^2 v_e^2} D_{22} = V = \left(\frac{c}{v_e}\right)^2 - 2\mu^2 - \tilde{k}^{-2} |\mu\, Z(\mu) + \alpha \cdot \beta\mu\, Z\, \beta(\mu)| \qquad (58)$$

the condition for transversal waves. As can be seen immediately, these equations have only solutions for damped waves, i.e. for Re $p < 0$ or Im $\omega > 0$. The fields are assumed to be proportional to

$$\exp\,(i(\omega t - \underline{k} \cdot \underline{r})) \qquad (59)$$

The time behavior of our plasma will be determined by the zeros of (57) and (58) which are closest to the real axis of the μ-plane. Now we assume large values of μ. As will be verified by the result we will have small damping, i.e. a small positive imaginary part ε of μ. With this condition we may approximate an equation $G(\mu) = 0$ by a linear expansion at the point μ_r = Re μ:

$$G(\mu_r) + i\varepsilon \frac{dG(\mu_r)}{d\mu_r} \triangleq 0 \qquad (60)$$

Neglecting terms of higher order the real part of this equation gives the following condition for μ_r:

$$\text{Re } G(\mu_r) \triangleq 0 \qquad (61)$$

After having solved this equation we conclude from the imaginary part of the equation (60):

$$\varepsilon \triangleq - \frac{y_m \, G(\mu_r)}{\text{Re } G'(\mu_r)} \qquad (62)$$

On the real axis of the μ-plane we approximate the function $Z(\mu_r)$ by the expansion given in the Appendix:

$$Z(\mu_r) \triangleq i\,\pi^{1/2} e^{-\mu_r^2} - \frac{1}{\mu_r}\left[1 + \frac{1}{2\mu_r^2} + \frac{3}{4\mu_r^4} + \ldots\right]$$

Now we set $G(\mu)$ equal to the longitudinal or transversal wave dispersion relation and get the following results:

a. <u>Longitudinal waves</u>

$$\omega^2 = 2(kv_e)^2 \mu_r^2 \mathrel{\hat{=}} \omega_p^2 \left[1 + Z\frac{m_e}{m_i}\right] + 3(kv_e)^2 \left[1 + Z(\frac{m_e}{m_i})^2 \frac{T_i}{T_e}\right] \qquad (63)$$

$$\delta_\ell = 2\,kv_e \cdot \varepsilon \triangleq (\frac{\pi}{8})^{1/2}\omega(\frac{\omega}{kv_e})^3 e^{-(\frac{\omega}{kv_e})^2}\left[(1+Z\frac{m_e}{m_i})(1+\beta e^{-(\beta^2-1)\frac{1}{2}(\frac{\omega}{kv_e})^2}\right] \qquad (64)$$

Because $m_e \ll m_i$ and $\beta^2 - 1 \gg 1$, $\omega \gg kv_e$ we might neglect the square brackets and our formulas reduce to the well known form of the dispersion relation and the Landau damping. However, there is some doubt about Landau damping which can be valid only for a limited time interval as we neglected the nonlinear terms of the B.V. equations in a first approximation assuming that the particles maintain their velocity.

b. Transversal waves

We can solve the equations for the transversal waves assuming very large values of μ. Neglecting all second order terms we get

$$\omega^2 \triangleq k^2c^2 + \omega_e^2 \tag{65}$$

$$\delta_t \triangleq \frac{1}{8}\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_e^2}{kv_e} e^{-\frac{1}{2}\left(\frac{\omega}{kv_e}\right)^2} \tag{66}$$

The phase velocity of these waves will always be faster than the speed of light. The contribution to the damping is due to particles having velocities in the vicinity of the phase velocity. In this respect we do a serious mistake as we use a nonrelativistic Maxwell distribution for our calculations: No particles can exist with higher velocities than the speed of light. Having this in mind we better leave the damping undetermined.

VII. Solutions for Two Counterstreaming Neutral Plasmas

In Section VI we considered a stable plasma at rest. From our dispersion relation we know that no coupling exists between longitudinal and transversal waves in a plasma as long as we have a symmetric distribution function. Therefore we consider in this section a special unsymmetric distribution: two counterstreaming initially neutral plasmas with the same kind of ions. Because of the uncertainties connected with all "damping" effects in a collisionless theory we will restrict ourselves to a consideration of the unstable solutions given by the zeros of our dispersion relation in the right p-plane. We assume plasma 1 at rest and plasma 2 moving with the velocity $\underline{v}_o$. For abbreviation we introduce the following dimensionless parameters

$$\alpha = Z\frac{m_e}{m_i}\ ;\ \beta_1 = \left(\frac{m_i\ T_e^{(1)}}{m_e\ T_i^{(1)}}\right)^{1/2}\ ;\ \beta_2 = \left(\frac{m_i\ T_e^{(2)}}{m_e\ T_i^{(2)}}\right)^{1/2}\ ;\ \gamma = \frac{N_e^{(2)}}{N_e^{(1)}}\ ;$$

$$\delta = \frac{v_e^{(1)}}{v_e^{(2)}};\ \mu = \frac{\omega}{2\ kv_e^{(1)}}\ ;\ \hat{v}_o = \frac{v_o}{\sqrt{2}\ v_e^{(1)}}\ ;\ a = \hat{v}_o\cos\vartheta;\ \tilde{k} = \frac{kv_e^{(1)}}{\omega_e} \tag{67}$$

and the function

$$Z(\mu) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{+\infty}\frac{e^{-x^2}}{x-\mu}\,dx \text{ for } \mathrm{Im}\,\mu > 0 \tag{68}$$

and its analytic continuation for the lower half plane (Appendix).

T_i and T_e are the temperatures of the ions and the electrons. For ω_e and v_e we used the definition:

$$v_e^{(1)} = \left(\frac{\varkappa T_e^{(1)}}{m_e}\right)^{1/2}\ ;\ v_e^{(2)} = \left(\frac{\varkappa T_e^{(2)}}{m_e}\right) \qquad \varkappa = \text{Boltzmann constant}$$

$$\omega_e = \omega_e^{(1)} = \frac{4\pi N_e^{(1)}e^2}{m_e} \qquad \omega_e = \text{plasma frequency for plasma "1"} \tag{69}$$

Furthermore we assume the axes $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$, of our coordinate system in the following way:

$\underline{\ell}_1$ parallel to $\underline{k}$

$\underline{\ell}_2$ in the plane of k and $\underline{v}$

$\underline{\ell}_3$ perpendicular to both

Using formulas (35) to (40) we get the following dispersion relation:

$$||D_{ik}|| = \begin{vmatrix} p^2\mathcal{L} & \sqrt{2}\omega_e p\tilde{k}C & 0 \\ \sqrt{2}\omega_e p\tilde{k}C & 2\omega_e^2\tilde{k}^2 V_{11} & 0 \\ 0 & 0 & 2\omega_e^2\tilde{k}^2 V_{22} \end{vmatrix} = 4\omega_e^4 p^2\tilde{k}^4(\mathcal{L}V_{11}-C^2)V_{22} = 0 \tag{70}$$

The term C characterizes the coupling. The different terms in this equation are given by the following expressions:

$$\mathcal{L} = 1 + \tilde{k}^{-2}(\mathcal{L}_1 + \mathcal{L}_2) \tag{71}$$

$$\mathcal{L}_1 = (1 + \mu\, Z(\mu)) + \alpha_1\beta_1^{\,2}\,(1 + \beta\mu\, Z(\beta\mu)) \tag{72}$$

$$\mathcal{L}_2 = \gamma\delta^2 \left[(1 + \delta(\mu-a)Z(\delta(\mu-a)) + \alpha_2\beta_2^{\,2}(1 + \beta_2\,\delta(\mu-a)\,Z(\delta(\mu-a))\right] \tag{73}$$

$$C = \frac{i\hat{v}_o \sin\vartheta}{\tilde{k}^2}\cdot\mathcal{L}_2 \tag{74}$$

$$V_{11} = V_{22} - (\hat{v}_o \sin\vartheta)^2\,\mathcal{L}_2 \tag{75}$$

$$V_{22} = \left(\frac{c}{v_e}\right)^2 - 2\mu^2 - \tilde{k}^{-2}\,(\mu\, Z(\mu) + \alpha_1\beta_1\mu\, Z(\beta_1\mu) + \gamma(\mu-a)\, Z(\mu-a) + $$
$$+ \alpha_2\beta_2(\mu-a)\, Z(\beta_2(\mu-a))) \tag{76}$$

The root factors of the dispersion relation can be written as

$$D_1 = V_{22} = 0 \tag{77}$$

and

$$D_2 = \tilde{k}^2 V_{22} \mathcal{L} - (\hat{v}_o \sin\vartheta)^2 (\tilde{k}^2 + \mathcal{L}_1)\mathcal{L}_2 \qquad (78)$$

The first condition (77) characterizes transversal waves with oscillations of the particles perpendicular to $\underline{k}$ and $\underline{v}$. It represents uncoupled transversal modes. The second condition characterizes the coupled longitudinal and transversal modes with particles oscillating in the plasma between $\underline{k}$ and $\underline{v}$. Let us first consider the limiting case $\Theta = 0$, where $\underline{v}_o$ is parallel to $\underline{k}$. Then we only get diagonal terms in the dispersion relation, the coupling disappears and we have to solve

$$\mathcal{L} = 0 \text{ for longitudinal waves}$$

$$V_{11} = V_{22} = 0 \text{ for transversal waves}$$

a. Longitudinal waves for $\Theta = 0$

The condition for longitudinal waves is

$$\tilde{k}^2 \mathcal{L} = \tilde{k}^2 + \mathcal{L}_1 + \mathcal{L}_2 = 0 \qquad (79)$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by the equations (72) and (73). At first we consider the limiting case of zero damping for which μ becomes real. For μ real we use the formula

$$\text{Im}\, \mu\, Z(\mu) = \pi^{1/2}\, \mu\, e^{-\mu^2} \qquad (80)$$

The imaginary part of equation (79) gives the following conditions:

$$0 = \mu e^{-\mu^2} + \alpha_1\beta_1{}^3 e^{-(\beta_1\mu)^2} + \gamma\delta^2|\delta(\mu-a)e^{-(\delta(\mu-a))^2} + \alpha_2\beta_2{}^3\delta(\mu-a)e^{-(\beta_2\delta(\mu-a))^2} \qquad (81)$$

In order to simplify the analysis we assume two identical counterstreaming plasmas:

$$\beta_1{}^2 = \beta_2{}^2 = \beta^2 \text{ and } \gamma = \delta = 1 \qquad (82)$$

or explicitly

$$T_e{}^{(1)} = T_e{}^{(2)} \; ; \; T_i{}^{(1)} = T_i{}^{(2)} \; , \; N_e{}^{(1)} = N_e{}^{(2)} \qquad (83)$$

For these conditions equation (81) has the solution

$$\mu_o = \frac{a\delta}{1 + \delta} \qquad (84)$$

Inserting this value μ_o into the real part of equation (79), and using the identity

$$\text{Re}\,(-\mu\, Z(-\mu)) = \text{Re}(\mu\, Z(\mu)) \text{ for } \mu \text{ real} \qquad (85)$$

we get the value of $\tilde{k}^2$ as a function of μ_o:

$$\tilde{k}^2(\mu_o) = -2|(1 + \mu_o \text{ Re } Z(\mu_o)) + \alpha\beta^2(1 + \beta\, \mu_o \text{ Re } Z(\beta\mu_o))| \qquad (86)$$

Only positive values of $k^2(\mu_o)$ are solutions to our problem. The curve $\tilde{k}^2(\mu_o)$ separates the instability region from the stability region. All $\tilde{k}$ smaller than $\tilde{k}(\mu_o)$ correspond to unstable solutions. In Fig. 1 the function $1 + \mu\, Z(\mu)$ is plotted for real μ. This function is negative only for $\mu > 0.93$. Its minimum value is about -0.28 for $\mu = 1.5$.

At first we assume equal temperatures of ions and electrons. Then our β is always greater than 42 and $\alpha\beta^2 = Z$. For $\beta\mu_o = 1.5$ we will have

$$0.98 \leq \mathrm{Re}\,(1 + \mu_o\, Z(\mu_o)) \leq 1$$

This means in (86) we get an ion-ion wave instability if the charge Z of the ions is greater or equal to 4. With greater charge number the tendency towards an instability for ion-ion oscillations increases because of the increasing interaction between the ions. The optimum relative velocity of the counterstreaming plasmas for this kind of instabilities is of the order of the thermal velocity of the ions. The electron-electron instabilities occur at much higher velocities where the second term in (86) can be neglected. The electron-electron instabilities are practically independent of the ions. In most discussions on two stream instabilities the effect of the ions can be neglected. In Fig. 2 the electron-electron instability for the special case of identical plasmas is plotted. Some curves for constant growing wave ratios have been obtained additionally. The dotted line gives the $\tilde{k}$ values for the maximum growing wave ratios. For $\hat{v}_o < 1.84$ or $v_o < 2.63 v_e$ there exists for ions with $Z \leq 3$ other instability regions if and only if the temperature of the electrons T_e is higher than the temperature T_i of the ions. For Z = 1 ion-ion instabilities exist for $T_e > 3.5\, T_i$ at about $v_o \simeq 0.025 v_e$ and ion-electron instabilities exist for $T_e > 1.4\, T_i$ at the region where v_o is of the order of the thermal velocity of the ions. These special cases have been calculated by T. E. Stringer* for a variety of different parameters.

* U.K.A.E.A., Harwell Report, August 1961.

b. Transversal waves for $\vartheta = 0$

The transversal waves for vanishing coupling are determined by the dispersion relation (76)

$$V_{11}(\vartheta = 0) = V_{22} = 0 \qquad (86)$$

As can be seen immediately, there are no unstable solutions possible. The phase velocity would always be greater than the velocity of light. No realistic prediction can be made about the damping in our treatment because of the ad hoc assumption of having a Maxwellian tail if velocities faster than the speed of light. A detailed relativistic calculation has to be made with respect to these difficulties.

c. Coupling effects. $\vartheta \neq 0$

For the coupling effects we have to admit solutions with $\vartheta \neq 0$. The dispersion relation for waves with particle oscillations in the plane of $\underline{k}$ and $\underline{v}$ includes the coupling with an extra term:

$$D_1 = (\tilde{k}^2 + \mathcal{L}_1 + \mathcal{L}_2)\tilde{k}^2 V_{22} - (\tilde{v}_o \sin\vartheta)^2 (\tilde{k}^2 + \mathcal{L}_1)\mathcal{L}_2 = 0 \qquad (87)$$

As before, we restrict ourselves to waves with non-relativistic phase velocities which mostly correspond to longitudinal waves. For these waves V_{22} is of the order of c^2/v_o^2 which we assumed to be large. Therefore we expect the effects to be proportional to

$$\varepsilon = \left(\frac{v_o \sin\vartheta}{c}\right)^2 \ll 1 \qquad (88)$$

because we deal exclusively with non-relativistic effects. Plotting the Nyquist curve as a criterion for the stability of D_1 we get the diagram shown schematically in Fig. 3 for the special case of a longitudinal instability. The most interesting part of this diagram is the vicinity of the zero point of the D_1-plane. It shows in this example damping for transversal waves and growing for special longitudinal waves. Taking Greek letters for the imaginary parts and indices 1 and 2 as indices for the plasmas, we get

$$\mathcal{L}_1 = r_1 + i\sigma_2 \; ; \; \mathcal{L}_2 = r_2 + i\sigma_2 \qquad (89)$$

$$\tilde{k}^2 V_{22} = \tilde{k}^2\left(\left(\frac{c}{v_e}\right)^2 - 2\mu^2\right)(1 - \delta_1) \text{ with } \delta_1 = \frac{-(t_1 + t_2) - i(\tau_1 + \tau_2)}{\tilde{k}^2\left(\left(\frac{c}{v_e}\right)^2 - 2\mu^2\right)} \qquad (90)$$

where r_1, r_2, σ_1, σ_2, t_1, t_2, τ_1, τ_2 are real and imaginary parts of corresponding quantities defined by (72), (73) and (76). As we are restricted to moderate phase velocities, our δ_1 is small compared to one except the case where $\tilde{k}$ is very small. Dividing the dispersion relation (87) by V_{22} we get immediately the result

$$\tilde{k}^2 + (r_1 + r_2) + i(\sigma_1 + \sigma_2) = \varepsilon \frac{(\tilde{k}^2 + r_1 + i\sigma_1)(r_2 + i\sigma_2)}{\tilde{k}^2 (1 - \delta_1)} \qquad (91)$$

For the treatment of this equation we consider 2 different cases:

1. $\underline{\tilde{k}c \gg v_e}$

From (90) we conclude that δ_1 is very small and can be neglected. Taking the imaginary part of this equation and using the solution for $\varepsilon = 0$ we get the approximation by expanding into a Taylor series

$$s_1 + s_2 \hat{=} (\sigma_1 + \sigma_2)_{\varepsilon=0} + \Delta\mu_o(\sigma'_1 + \sigma'_2) = \varepsilon\, r_2(s_1 - s_2)$$

or

$$\Delta\mu_o = \varepsilon \frac{\sigma_1 - \sigma_2}{\sigma_1' + \sigma_1'} \circ r_2 \tag{92}$$

Taking the real part of equation (91) we obtain similarly by expansion

$$\Delta\tilde{k}_o^2 \hat{=} -\Delta\mu_o(r'_1 + r'_2)_{\varepsilon=0} + \frac{\varepsilon}{\tilde{k}^2} \cdot \left[(\tilde{k}_o^2 + r_1)r_2 - s_1 s_2\right]_{\varepsilon=0}$$

or (93)

$$\Delta\tilde{k}_o^2 = \varepsilon \cdot \frac{1}{\tilde{k}^2} \left[(\tilde{k}_o^2 + r_1)r_2 - s_1 s_2 - \frac{r_1' + r_2'}{s_1' + s_2'}(s_1 - s_2)r_2\right]$$

and the new solution is given by the values

$$\mu_o(\varepsilon) = \mu_o + \Delta\mu_o$$

$$\tilde{k}_o^2(\varepsilon) = \tilde{k}_o^2 + \Delta\tilde{k}_o^2$$

For two counterstreaming identical plasmas with equal ion and electron temperatures these corrections can be seen to be extremely small for the range of validity of our calculation. Uncoupled longitudinal wave instabilities occur only for $\lambda \geq 8.3\ \lambda_D$ as can be seen from Fig. 2. The condition $k^2c^2 > v_e^2$ is a restriction for small wavelengths which combined with the first condition can be written as

$$8.3\ \lambda_D \leq \lambda \ll 2\pi \frac{c}{v_e} \lambda_D \tag{94}$$

where λ_D is the Debye wavelength.

2. Long wavelength approximation $\tilde{k} \ll 1$

We already discussed solutions for longitudinal and transversal waves which correspond to well known solutions. New solutions can be found if δ_1 cannot be neglected, i.e. the term

$$\tilde{k}^2 \left(\frac{c^2}{v_e^2} - 2\mu^2\right)$$

is not dominating. This means that we have to consider solutions for which $\tilde{k}^2 \ll 1$ or as a condition for the wavelengths

$$\lambda \gg \frac{2\pi v_e}{\omega_e} = 2\pi\lambda_D \tag{95}$$

Using previous notations, the dispersion relation becomes

$$\tilde{k}^2\left(\left(\frac{c}{v_e}\right)^2 - 2\mu^2\right) = t_1 + t_2 + i(\tau_1 + \tau_2) + \left(\frac{v_e}{v_e}\right)^2 \frac{(\tilde{k}^2 + r_1 + i\sigma_1)(r_2 + i\sigma_2)}{\tilde{k}^2 + r_1 + r_2 + i(\sigma_1 + \sigma_2)} \tag{96}$$

a. Complete Solution for $\vartheta = 90^\circ$

At first we discuss the solution for $\vartheta = 90^\circ$. In the argument of the Z-function we have $v_o \cos\vartheta = 0$ and the expressions become much simpler. The imaginary part of this equation vanishes if we take $\mu = iy$ purely imaginary as we have for this argument:

$$F(y) = -iy\, Z(iy) = \pi^{1/2}\, y\, e^{y^2}\, (1 - \mathrm{erf}(y)) \tag{97}$$

We consider the previous discussed example of two counterstreaming identical hydrogen plasmas. For $|y|$ small compared to β^{-1} the solution of (96) becomes

$$y = \pi^{1/2} \left(\frac{c}{v_e}\right)^2 \frac{\tilde{k}^2_{max} - \tilde{k}^2}{2(1 + \alpha\beta)} = 0.87 \left(\frac{c}{v_e}\right)^2 (\tilde{k}^2_{max} - \tilde{k}^2) \qquad (98)$$

which shows that k^2 is nearly identical to

$$\tilde{k}^2_{max} = \frac{\pi^{1/2}}{2} \left(\frac{v_o}{c}\right)^2 (1 + \alpha\beta) = 0.904 \left(\frac{v_o}{c}\right)^2 \ll 1 \qquad (99)$$

as both v_o and v_e are very small compared to c. We now proceed to solve the equation (96) for arbitrary values of y. The functions appearing in (96) can be expressed by F(y) in the following way:

$$t_1 + t_2 + i(\tau_1 + \tau_2) = -2\,(F(y) + \alpha\, F(\beta y)) \qquad (100)$$

and

$$r_1 + i\sigma_1 = r_2 + i\sigma_2 = 2 - F(y) - F(\beta y) \qquad (101)$$

Since we consider the case $\tilde{k}^2 \ll 1$ we can neglect $\tilde{k}^2$ on the right side of (96) which leads to the simpler equation

$$\tilde{k}^2 \left(\frac{c}{v_e}\right)^2 \left(1 + 2y^2 \left(\frac{v_e}{c}\right)^2\right) = -2\,(F(y) + \alpha\, F(\beta y)) +$$

$$+ \hat{v}_o^{\,2}\,(2 - F(y) - F(\beta y)) \qquad (102)$$

Introducing the following abbreviations

$$\hat{k} = \frac{kc}{\omega_e} = \tilde{k}\frac{c}{v_e} \; ; \; \hat{v}_{o\,m}^{\,2}(y) = \frac{2(F(y) + \alpha\, F(\beta y))}{2 - F(y) - F(\beta y)} \tag{103}$$

and neglecting the term with y on the left side of (102) we write the equation in a modified form

$$\hat{k}^2 = (\hat{v}_o^{\,2} - \hat{v}_{o\,m}^{\,2}(y)) \,.\, (2 - F(y) - F(\beta y)) \tag{104}$$

Now the argument y is given by

$$y = \frac{\mathrm{Im}\ \omega}{\sqrt{2}\, k v_e} = \frac{\mathrm{Im}\ \omega}{\omega_e} \frac{c}{\sqrt{2}\, v_e\, \hat{k}} \tag{105}$$

from which we deduce the growing wave ratio

$$\mathrm{Im}\ \omega = \omega_e \frac{\sqrt{2}\, v_e}{c} \,.\, \hat{k}y \tag{106}$$

Figure 4 shows a few curves of constant growing wave ratio (ky = constant).

The maximum growing wave ratio as a function of the counterstreaming velocity is plotted in Figure 4a. Comparing this with the growing wave ratio for longitudinal instabilities in Figure 2a, we conclude that the longitudinal instabilities grow much faster. The probability of excitation of the transversal instabilities depends on the initial distribution of Fourier components in space. The wavelength as the characteristic length λ_t for the transverse modes usually is much longer than that for longitudinal instabilities. According to (106) we get for λ_t the expression

$$\lambda_t \hat{=} \sqrt{2}\, 2\pi \frac{c}{v_e} \frac{v_e}{\omega_e} = 8.88 \frac{c}{v_o} \cdot \lambda_D = 4.7 \cdot 10^6 N_e^{-1/2} \frac{v_e}{v_o} \text{ in cm} \tag{107}$$

For the longitudinal modes λ_ℓ becomes (Fig. 2 and 2b)

$$\lambda_\ell = \frac{2\pi}{\tilde{k}_{\delta max}} \cdot \frac{v_e}{\omega_e} = \frac{2\pi}{\tilde{k}_{\delta max}} \cdot \lambda_D \tag{108}$$

where $\tilde{k}_{\delta max}$ is approximately $2 \cdot \tilde{k}_{max}$. From previous calculations we use

$$\tilde{k}_{max} \hat{=} -2(1 + \mu_o Z(\mu_o)) \text{ where } \mu_o = \frac{v_o}{\sqrt{2}\, v_e} \text{ for } \hat{\gamma} = 0$$

$$\hat{=} \mu_o^{-2} \text{ if } \mu_o \gtrsim 2$$

This gives for values $v_o \gtrsim 6v_e$ approximately

$$\lambda_\ell \hat{=} 4\pi \mu_o \frac{v_e}{\omega_e} \hat{=} \frac{1}{2} \frac{v_o^2}{c v_e} \cdot \lambda_t \tag{109}$$

proving that the characteristic length for the unstable transversal modes is much greater than that of the unstable longitudinal modes. In experimental devices λ_t might be even greater than the whole system. Fluctuations in the plasmas occur more often for characteristic lengths comparable to λ_ℓ than for very long lengths. This means that in most experiments only longitudinal modes are excited. They grow fast enough so that we must take into account nonlinear terms of the Boltzmann equation before dealing with a considerable amount of the transverse instability effect. However, for smaller v_o where no growing longitudinal modes exist the transversal mode will still serve as a strong relaxation process towards an isotropic velocity distribution in the plasma.

b. Maximum k^2_{max} for $\vartheta \neq 90^o$

For $\vartheta \neq 90^o$ we calculate the minimum value of k^2 which occurs for the special case y = 0 corresponding to vanishing damping. The resulting curve k^2_{max} separates the region of instability from the region of stability. We constrict ourselves to the case of two counterstreaming identical plasmas with equal temperatures of ions and electrons as we did for the evaluation of the uncoupled longitudinal waves. Additionally we assume Z = 1, i.e. hydrogen plasmas. The equation (96) now takes the form

$$\hat{k}^2 = \tilde{k}^2 \left(\frac{c}{v_e}\right)^2 = \left(\frac{kc}{\omega_e}\right)^2 = g(\mu,a) + \eta \frac{h(\mu)\, h(\mu-a)}{h(\mu) + h(\mu-a) + \tilde{k}^2} \qquad (104)$$

where

$$\eta = \left(\frac{v_o}{v_e}\right)^2 \sin^2\vartheta$$

We introduced the following functions

$$g(\mu,a) = \mu\, Z(\mu) + (\mu-a)\, Z(\mu-a) + \alpha(\beta\mu\, Z(\beta\mu) + \beta(\mu-a)\, Z(\beta(\mu-a))) \qquad (105)$$

$$h(\mu) = 2 + \mu\, Z(\mu) + \beta\mu\, Z(\beta\mu) \qquad (106)$$

As in the previously discussed equations for uncoupled longitudinal instabilities ($\vartheta = 0$) the imaginary part of equation (104) vanishes for $\mu = a/2$. This corresponds to a phase velocity of the wave equal to the velocity of the center of mass of the two counterstreaming plasmas. Of course we would get zero phase velocity if we had chosen a center of mass coordinate system. This clearly shows that the oscillations have relaxational character in the same sense as the longitudinal electrostatic two stream instabilities. For the functions (105) and (106) we use the property of symmetry:

$$h(-\mu_r) = h^*(-\mu_r) \text{ and } G(-(\mu_r - \tfrac{a}{2})) = G^*(\mu_r - \tfrac{a}{2}) \text{ for real argument } \mu \quad (107)$$

Now we define new functions

$$f(a, \tilde{k}^2) = a^2 \frac{|h(\frac{a}{2})|^2}{\mathrm{Re}\ h(\frac{a}{2}) + \tilde{k}^2} \quad (108)$$

and

$$g_1(a) = g(\tfrac{a}{2}, a) = -2(\tfrac{a}{2}\ \mathrm{Re}\ Z(\tfrac{a}{2}) + \alpha \tfrac{\beta a}{2}\ \mathrm{Re}\ (\tfrac{\beta a}{2})) \geq 0 \quad (109)$$

Using these expressions we can write the solutions in the following form:

$$\hat{k}^2 = \tilde{k}^2 \cdot (\frac{c}{v_e})^2 = tg^2 \vartheta \cdot f(a, (\frac{v_e}{c})^2 \cdot \hat{k}^2) - g_1(a) \quad (110)$$

Now we assumed $v_e \ll c$. This means we can neglect $(v_e/c)^2\ \hat{k}^2$ as long as $\hat{k}^2$ is of the order of one. Our solution therefore approximately becomes

$$\hat{k}^2 = (\frac{k_{max} c}{\omega_e})^2 \doteq (tg^2 \vartheta - tg^2 \vartheta_{min}) \cdot f(a, 0) \quad (110')$$

where

$$tg^2 \vartheta_{min} = \frac{g_1(a)}{f(a, 0)} \qquad (111)$$

This solution is valid only for $tg^2 \vartheta_{min} \geq 0$ and $\hat{k}^2 \ll (\frac{v_e}{c})^2 \cdot \mathrm{Re}\, h(\frac{a}{2})$. The last conditions is valid everywhere except the very close vicinity of the point where $\mathrm{Re}\, h(\frac{a}{2})$ vanishes i.e. $tg \vartheta_{min} = 0$. But this last condition corresponds to the onset of the well known longitudinal instabilities. It is, as we see by comparison, indeed identical to the earlier derived condition for the onset of longitudinal instabilities at $k_{max} = 0$ for the uncoupled longitudinal oscillations.

For $tg^2 \vartheta_{min} < 0$ we have no solution for ϑ_{min} and instabilities occur for all angles . The solution changes at the point where $\mathrm{Re}\, h(\frac{a}{2}) \leq 0$ to the condition for the longitudinal waves for which

$$\hat{k}^2 + \mathrm{Re}\, h(\tfrac{a}{2}) = 0 \text{ where } a = \frac{v_o}{\sqrt{2}\, v_e} \cdot \cos \vartheta \qquad (112)$$

Especially at $\vartheta = 0$ we get the well known solutions for k_{max} for uncoupled electrostatic oscillations. The function $f(a)$ is plotted in Fig. 5. The next figure presents ϑ_{min} as a function of a. Complete solutions for k^2_{max} as a function of the angle ϑ between $\underline{v}_o$ and $\underline{k}$ with v_o/v_e as parameter are plotted in Fig. 7-9. Near $\vartheta = 90^o$ we always have the main change of k^2_{max} due to the ions whereas corresponding to larger phase velocities k_{max} for smaller angles mainly depends on contributions due to the electrons. The growing wave ratio for $\hat{k} = \hat{k}_{max}$ is comparatively small for the "transversal instability" but increases with increasing counterstreaming velocity. It can be estimated from the limiting cases $\vartheta = 90^\circ$ for which we calculated its values.

Concluding Remarks

After the derivation of the dispersion relation for any anisotropic distribution of particles we considered the particular case of two counterstreaming neutral hydrogen plasmas without external fields. It is shown that there exist some transversal instabilities additionally to the well known longitudinal instabilities. However the probability for exciting them by natural fluctuations in the plasma is small because of their long characteristic lengths. We introduced dimensionless parameters related to the thermal velocity of the electrons and the plasma frequency. The angle ϑ between the counterstreaming velocity $\underline{v}_o$ and the wave vector $\underline{k}$ is very important. Longitudinal instabilities occur for all angles for which $\hat{v}_o \cos\vartheta$ is greater than 1.86 where $\hat{v}_o = v_o \cdot (\sqrt{2}\, v_e)^{-1}$. There exist however always transversal instabilities. Fig. 4 shows the limiting case $\vartheta = 90^\circ$ for which the curves of constant growing wave ratio are plotted as a function of the normalized counterstreaming velocity. The instability region extends from $\vartheta = 90^\circ$ to ϑ_{min}. Fig. 6 shows this angle ϑ_{min} as a function of $\hat{v}_o \cos\vartheta$. The H^+-ions contribute only if $\hat{v}_o \cos\vartheta$ is of the order of the thermal velocity of the ions which for increasing $\hat{v}_o$ restricts their effect to the vicinity of $\vartheta = 90^\circ$ (Fig. 7-9). As soon as $\hat{v}_o$ is greater than 1.3, the instabilities exist for all angles. The longitudinal instability is predominant only if ϑ is smaller than ϑ_o = arc cos $(1.86/\hat{v}_o)$ as can be seen in the example given in Fig. 9.

The results can be generalized to any anisotropic velocity distribution in a plasma.* An instability of the same character will occur as soon as there exists an unsymmetric velocity distribution perpendicular to certain directions of the vector of propagation k. This and the effect of external fields will be shown in a following paper.

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* In a different way it has been shown by J. Neufeld and F. H. Doyle l.c. and by E. G. Harris: Transversal Instabilities associated with Anisotropic Velocity Distributions. J. Nuclear Energy C, 2, 138 (1961).

Acknowledgements

This work has been done in connection with the Case Plasma Research Program. The author is very grateful for the hospitality extended to him at the Case Institute of Technology and to the Fulbright Commission who gave financial assistance for his travel expenses from Germany. Special thanks are due to Dr. O. K. Mawardi for helpful discussions and support of this work. Furthermore, I am indebted to Dr. R. Zelazny and J. Gustincic for valuable discussions concerning the general problem of transversal instabilities.

Appendix: The Plasma Dispersion Function $Z(\mu)$

For Maxwellian velocity distributions of the particles one has to introduce the function:

$$Z(\mu) = \pi^{-1/2} \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{x-\mu} dx \qquad \text{for Im } \mu > 0$$

and the analytical continuation of this for Im $\mu \leq 0$. This function can be written alternatively as

$$Z(\mu) = 2i\, e^{-\mu^2} \int_{-\infty}^{+\infty} e^{-t^2} dt$$

This representation is valid for either sign of Im μ and shows the close relation to the error function. The following formulas give a few expansions used in this paper.

1. Real Argument $\mu = \mu_r$

$$Z(\mu_r) = \text{Re } Z(\mu_r) + i \, \text{Im } Z(\mu_r)$$

$$\text{Re } Z(\mu_r) = -2e^{-\mu_r^2} \int_{0}^{\mu_r} e^{t^2} dt$$

$$\text{Im } Z(\mu_r) = \pi^{1/2} e^{-\mu_r^2}$$

2. Imaginary Argument $\mu = i\mu_r$

$$Z(i\mu_r) = i\pi^{1/2} e^{\mu_r^2} (1 - \text{erf } \mu_r)$$

3. Power Series

$$Z(\mu) = i\pi^{1/2} e^{-\mu^2} - 2\mu \left[1 - \frac{2\mu^2}{3} + \frac{2^2\mu^4}{3.5} - \frac{2^3\mu^6}{3.5.7} + \dots\right]$$

4. Asymptotic Series

$$Z(\mu) \hat{=} i\pi^{1/2}\,\sigma e^{-\mu^2} - \mu^{-1}\left[1 + \frac{1}{2\mu^2} + \frac{3}{2^2\mu^4} + \frac{3.5}{2^3\mu^6} + \ldots\right]$$

where

$$\sigma = \begin{cases} 0 \text{ for } \mathrm{Im}\,\mu > 0 \\ 1 \text{ for } \mathrm{Im}\,\mu = 0 \\ 2 \text{ for } \mathrm{Im}\,\mu < 0 \end{cases}$$

5. General Properties

$$Z(\mu^*) = -Z(-\mu)^*$$

for $\mathrm{Im}\,\mu > 0$:

$$Z(\mu^*) = Z(\mu) + 2i\pi^{1/2}\,e^{-(\mu^*)^2}$$

$$Z'(\mu) = -2(1 + \mu\,Z(\mu)),$$

$$Z(0) = i\pi^{1/2}\,.$$

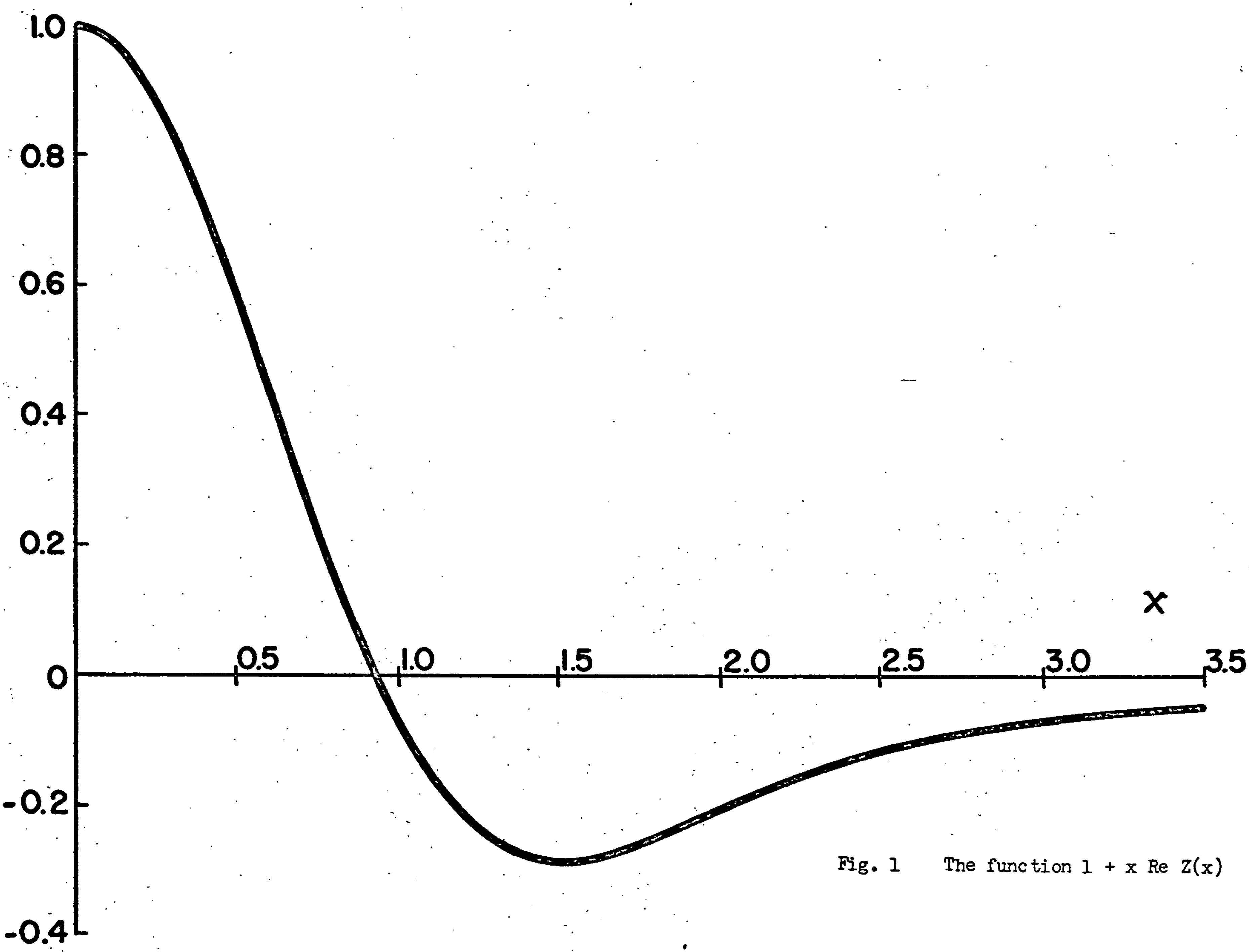

Fig. 1 The function 1 + x Re Z(x)

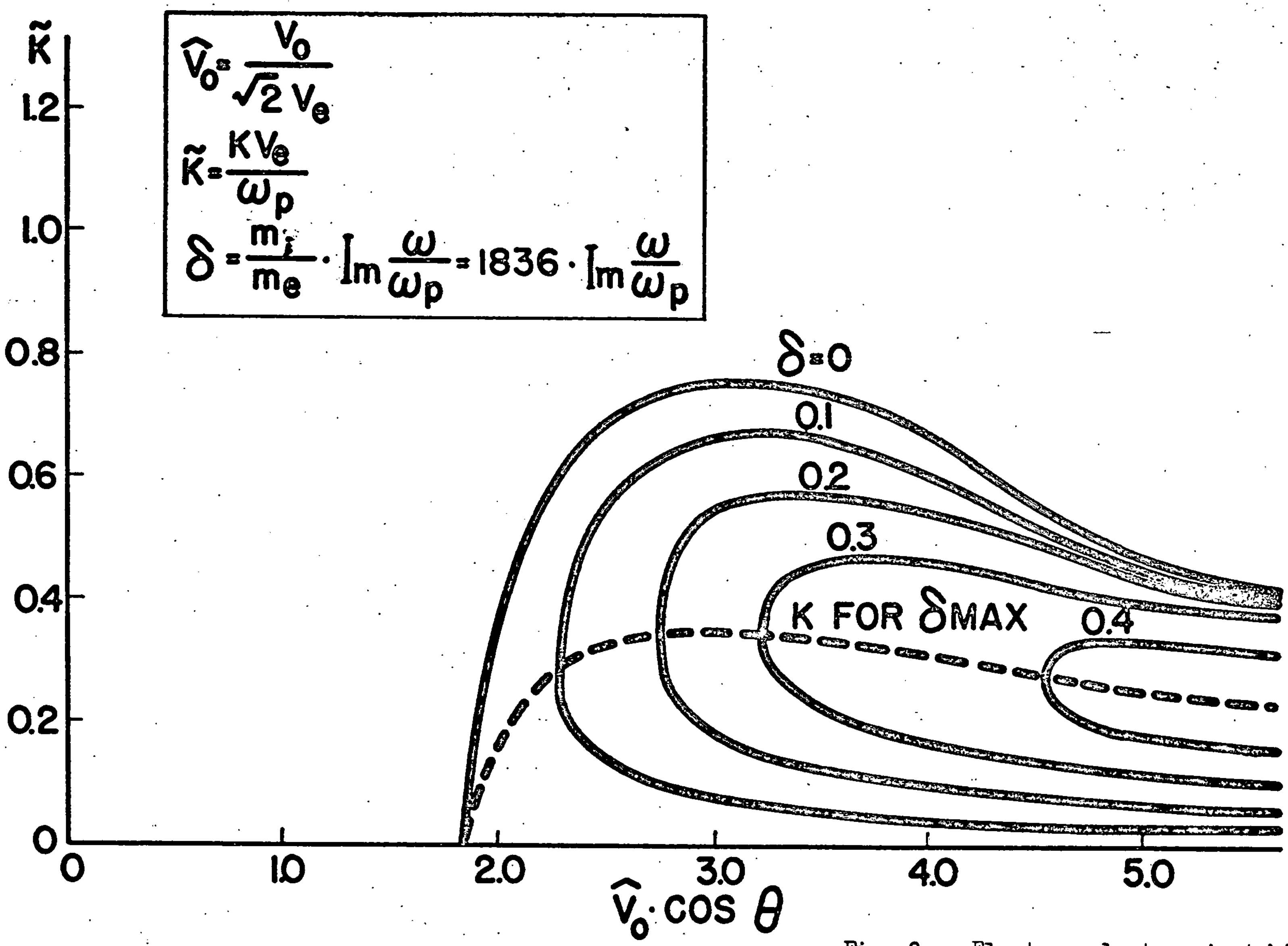

Fig. 2   Electron-electron instability for two identical counterstreaming plasmas.

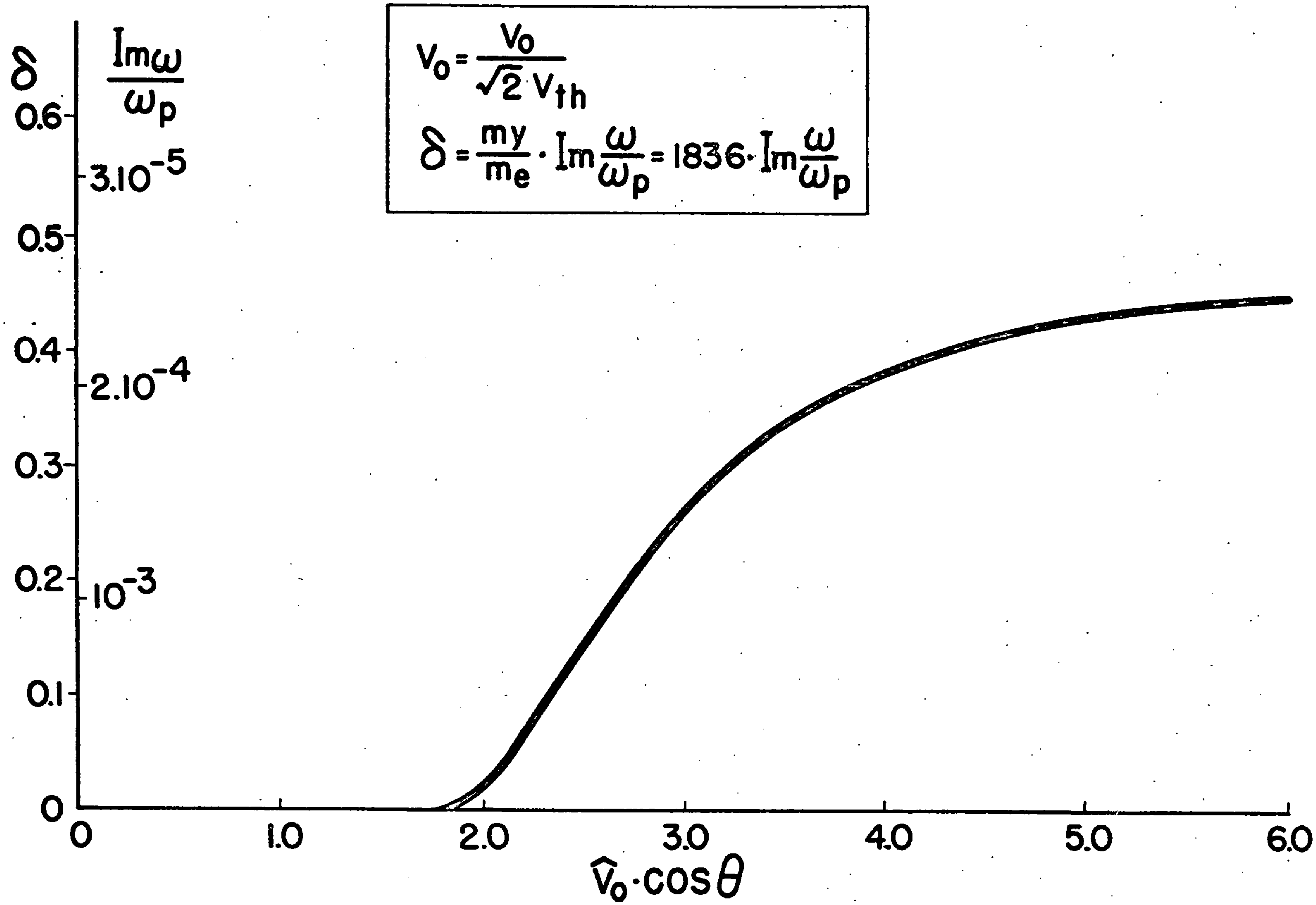

Fig. 2a Maximum growing wave ratio vs. counterstreaming velocity for longitudinal instabilities.

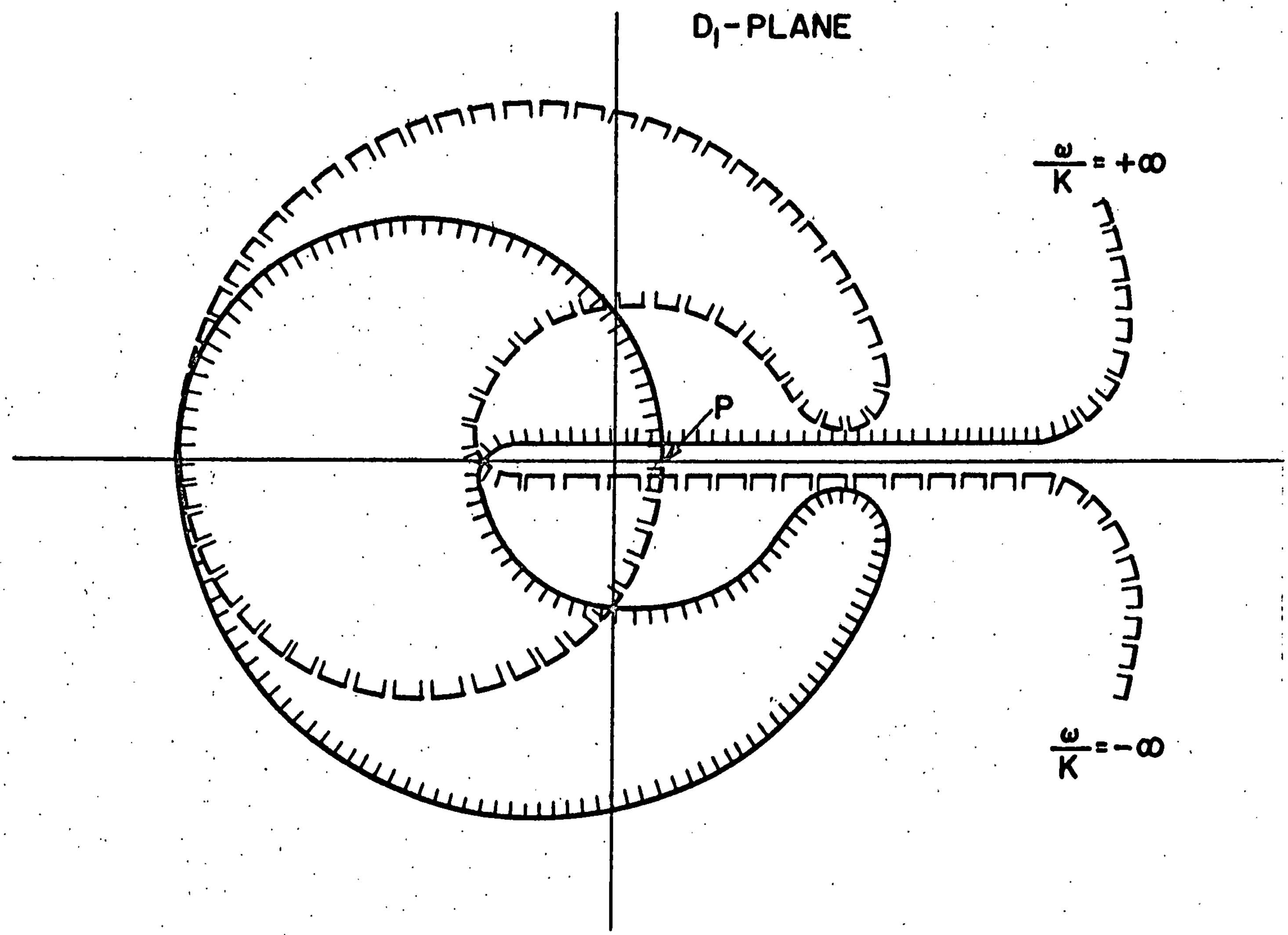

Fig. 3 Nyquist curve for a longitudinal instability (schematic)

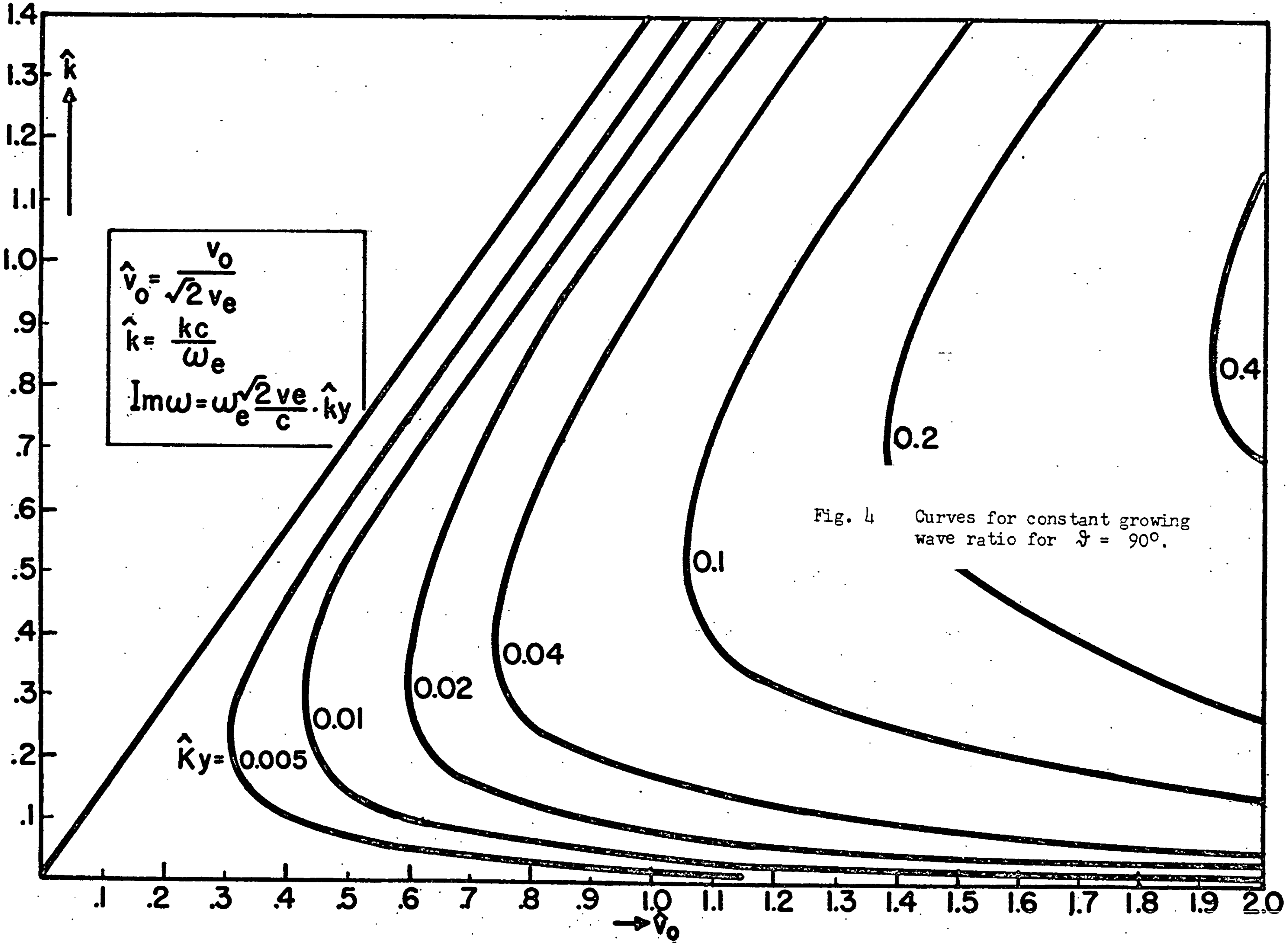

Fig. 4 Curves for constant growing wave ratio for $\vartheta = 90°$.

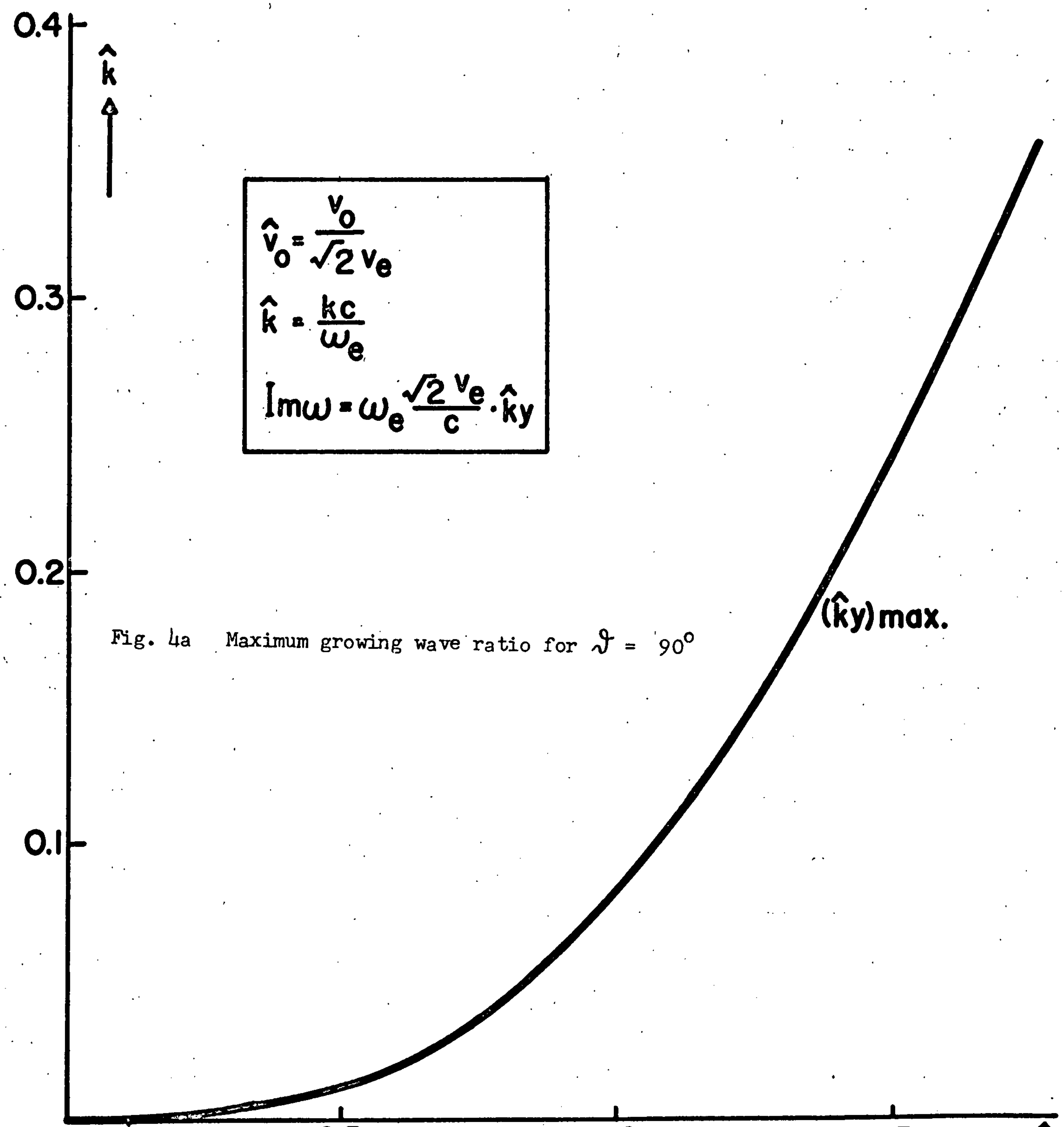

Fig. 4a Maximum growing wave ratio for $\vartheta = 90^{\circ}$

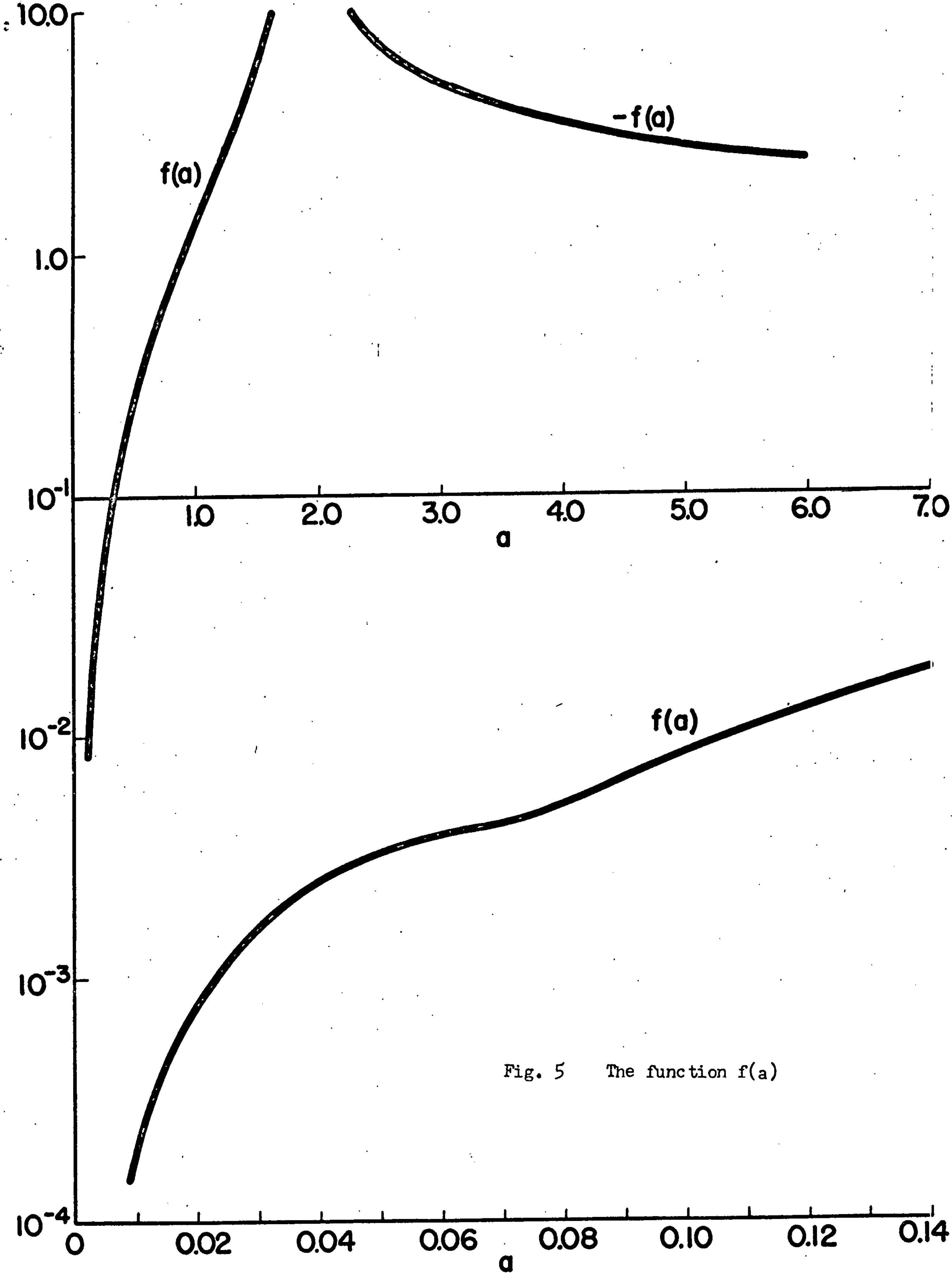

Fig. 5 The function f(a)

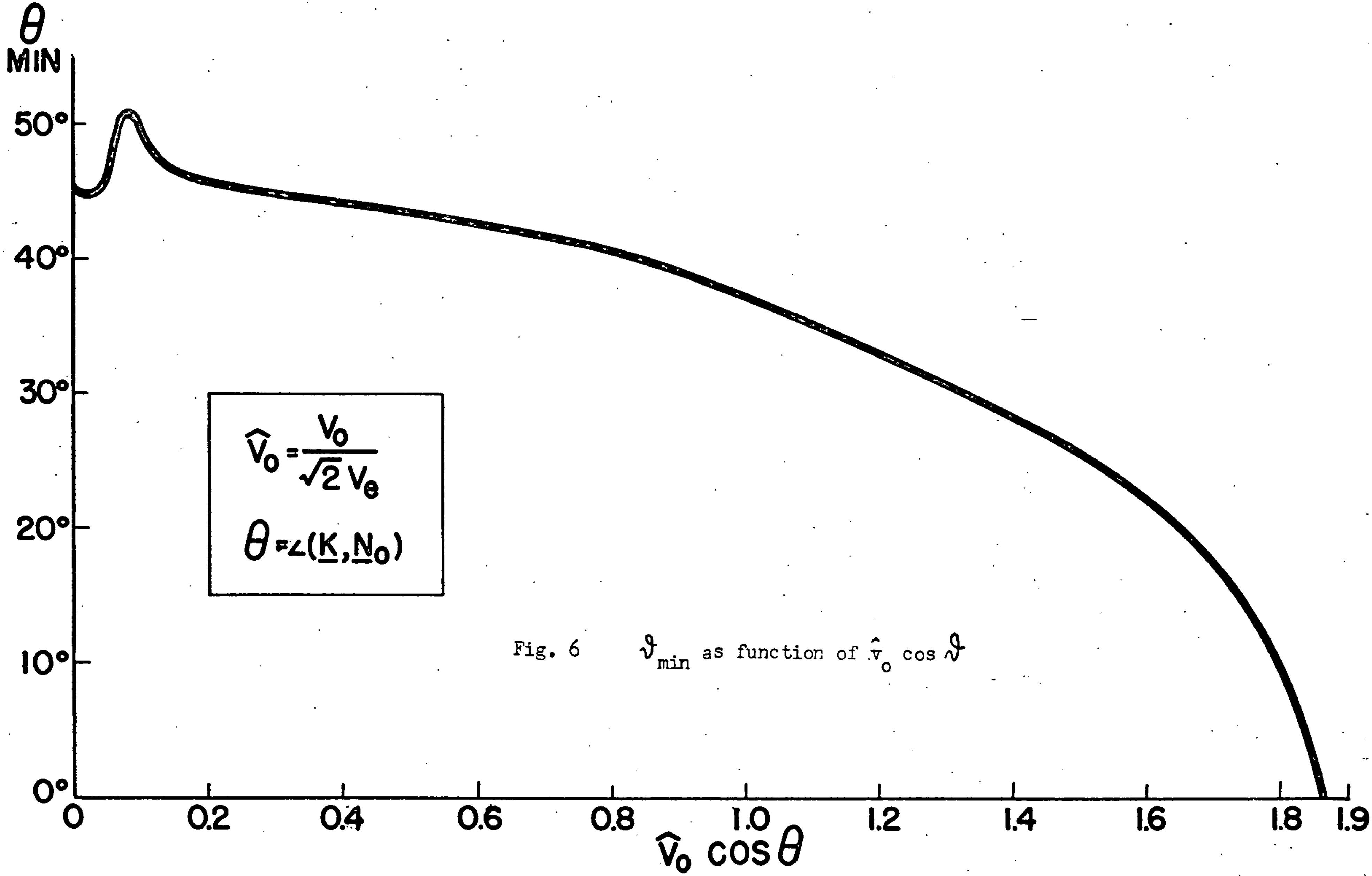

Fig. 6 ϑ_{min} as function of $\hat{v}_o \cos\vartheta$

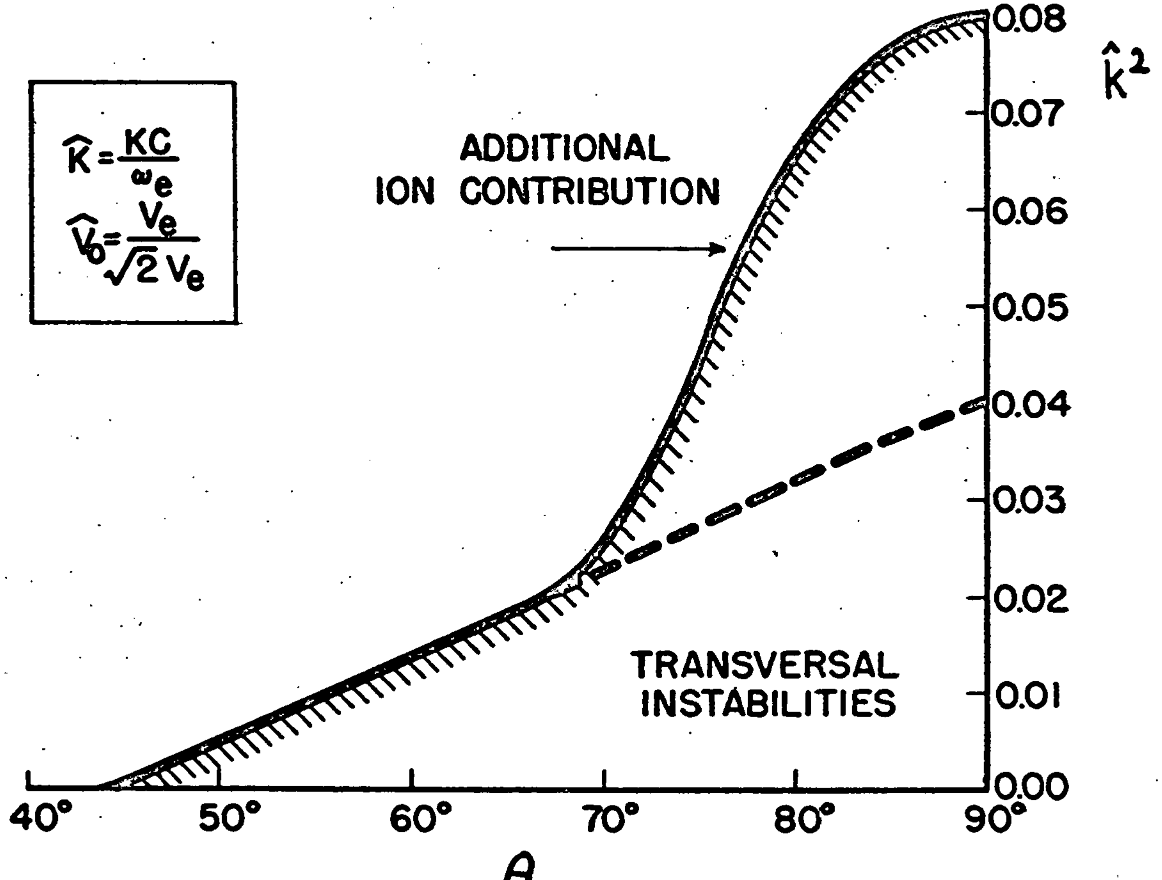

Fig. 7 Instability region for $\hat{v}_o = 0.2$

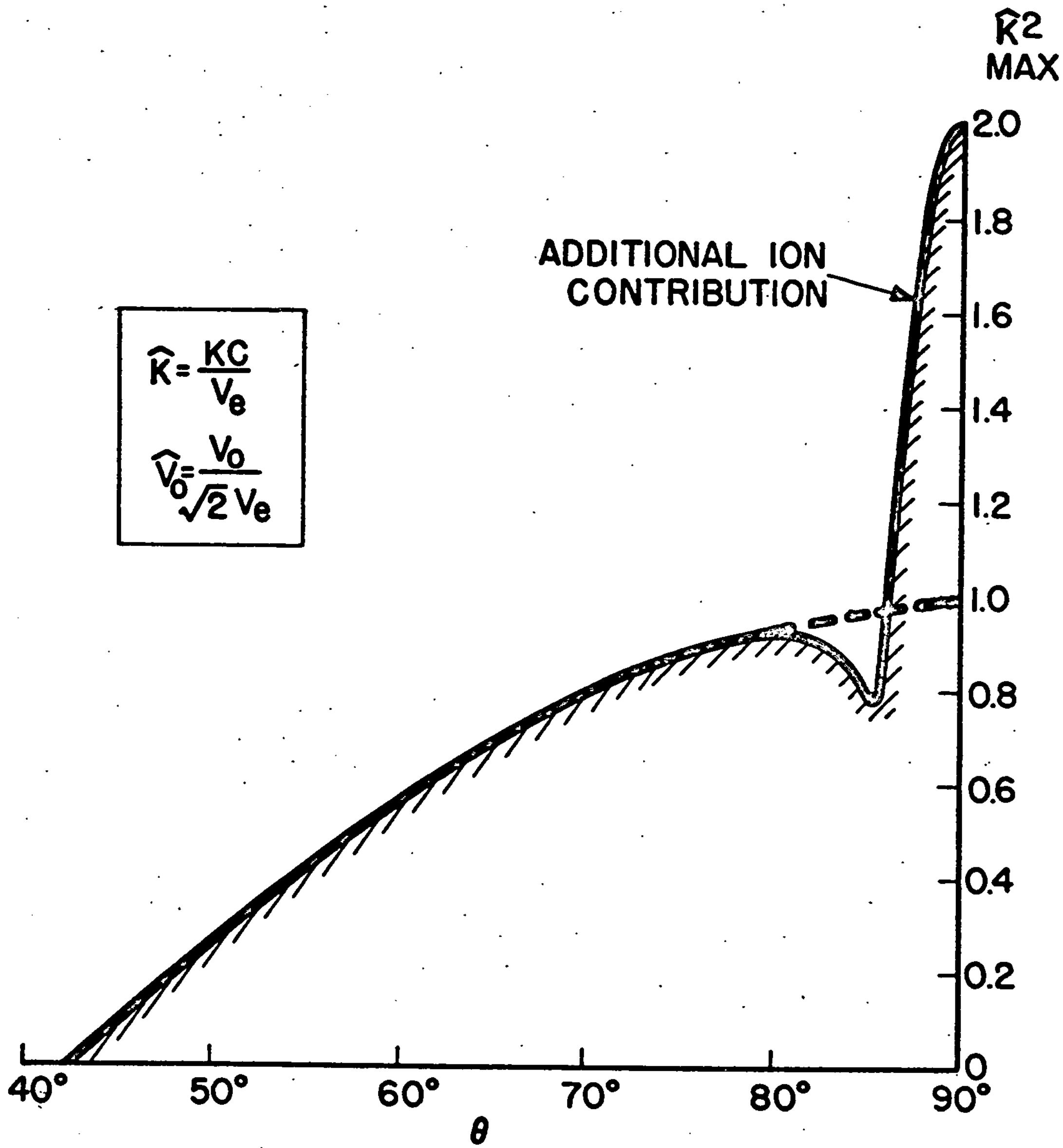

Fig. 8 Instability region for $\hat{v}_o = 1.0$

Fig. 9 Instability regions for $\hat{v}_o = 2.0$