NONLINEAR TRANSVERSE WAVES IN PLASMAS

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ABSTRACT

The propagation of nonlinear stationary transverse waves in plasmas is investigated by first solving a relativistic Vlasov equation for the electrons under the influence of a Lorentz force due to a propagating 4-potential rigorously and without linearization. The solution, which reduces to a given equilibrium electron velocity distribution function, is then substituted into the Maxwell equations, and a set of wave equations is obtained. While nonlinearity couples the transverse and longitudinal modes except for one special

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case, propagation of plane-polarized transverse waves in both cold and hot (Maxwellian) plasmas is studied in the quasi-neutrality approximation. The conditions for the existence of periodic solutions for the nonlinear transverse wave equations indicate that propagation is possible only when the wave velocity exceeds the velocity of light (for plasmas free from external magnetic field). Expressions for waveform and frequency in terms of elliptical integrals are derived. Unlike the case of longitudinal waves, the nonlinear effect on transverse waves is manifested primarily in the reduction of frequency rather than distortion in waveform. Several typical examples of waveform and dispersion characteristics (frequency vs. phase velocity) are computed and plotted, ranging from cold to ultra-relativistically hot plasmas. The nonlinear effect is more pronounced at lower electron temperatures.

1. INTRODUCTION

The study of nonlinear wave propagation in plasmas has recently become an interesting subject of discussion among researchers. Published results based on rigorous kinetic solutions for transverse waves are, however, still scarce.[1] In this report we shall discuss the propagation of stationary waves in plasmas starting from a relativistic Vlasov equation for the electrons (ion motion is neglected) including the magnetic force term and without linearization. Solutions compatible with given equilibrium particle velocity distribution functions are obtained in the wave coordinate system, and dispersion relations for nonlinear transverse modes in both cold and hot plasmas are derived and plotted in terms of dimensionless variables. A number of interesting observations are made, especially when the nonlinear effect is different from that on the longitudinal waves.

2. SOLUTION OF THE RELATIVISTIC VLASOV EQUATION WITH GENERAL LORENTZ FORCES

We first find a general solution for the electron distribution function $f(u_\lambda, x_\lambda, t)$ under the influence of Lorentz forces due to propagating transverse and longitudinal waves in a plasma from the relativistic Vlasov equation[2]

$$\frac{\partial f}{\partial t} + \frac{cu_\lambda}{\gamma} \cdot \frac{\partial f}{\partial x_\lambda} + \frac{F_\lambda}{mc} \frac{\partial f}{\partial u_\lambda} + \frac{f}{mc} \frac{\partial F_\lambda}{\partial u_\lambda} = 0 \ . \qquad (2\text{-}1)$$

Here the time t and the coordinates x_λ form a 4-vector

$$x_i = (x_\lambda \ , \ ict) \quad , \qquad (2\text{-}2)$$

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[1] J. Enoch, Phys. Fluids 5, 467 (1962).

[2] P. C. Clemmow and A. J. Willson, Proc. Cambridge Phil. Soc. 53, 222 (1957).

u_λ is the spatial part of the 4-velocity

$$u_i = (\gamma v_\lambda/c \ , \ i\gamma) \quad , \tag{2-3}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = (1 + u^2)^{1/2} \quad , \tag{2-4}$$

and v_λ is the actual particle velocity. The summation convention over repeated indices is from 1 to 3 for Greek and from 1 to 4 for Latin indices. The Newtonian force F_λ per electron in our case is

$$F_\lambda = -\frac{eu_i}{\gamma}\left(\frac{\partial A_i}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_i}\right) \quad , \tag{2-5}$$

where $A_i = (A_\lambda, \ i\phi)$ is a 4-potential satisfying the Lorentz condition

$$\frac{\partial A_i}{\partial x_i} = 0 \quad . \tag{2-6}$$

The magnitude of charge and the rest mass of an electron are denoted by e and m, while c is the velocity of light in vacuum. For the Lorentz force

(2-5), $\dfrac{\partial F_\lambda}{\partial u_\lambda} = 0$ and (2-1) becomes

$$\frac{\partial f}{\partial t} + \frac{cu_\lambda}{\gamma}\frac{\partial f}{\partial x_\lambda} - \frac{e}{mc}\frac{u_i}{\gamma}\left(\frac{\partial A_i}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_i}\right)\frac{\partial f}{\partial u_\lambda} = 0 \quad . \tag{2-7}$$

If we assume plane wave solutions with a velocity of propagation βc in the x_1 direction, i.e.,

$$\left.\begin{aligned} f &= f(u_\lambda, \ \xi) \\ A_i &= (A_\lambda(\xi), \ i\phi(\xi)) \end{aligned}\right\} \tag{2-8}$$

with

$$\xi = x_1 - \beta c t \quad , \tag{2-9}$$

the Lorentz condition calls for

$$A_1 = \beta \phi \quad , \tag{2-10}$$

and (2-7) can be reduced to the following partial differential equation for f:

$$\left(\frac{u_2}{\gamma} \dot{A}_2 + \frac{u_3}{\gamma} \dot{A}_3 - \alpha^2 \dot{\phi} \right) \frac{\partial f}{\partial u_1}$$

$$+ \left(\beta - \frac{u_1}{\gamma}\right) \left\{ \dot{A}_2 \frac{\partial f}{\partial u_2} + \dot{A}_3 \frac{\partial f}{\partial u_3} + \frac{mc^2}{e} \frac{\partial f}{\partial \xi} \right\} = 0 \quad , \tag{2-11}$$

where

$$\alpha^2 = 1 - \beta^2 \quad , \tag{2-12}$$

and the dot indicates differentiation with respect to ξ. One can proceed in the usual manner by solving the auxiliary equations for (2-11) and finally obtain a general solution. It is, however, more interesting to derive such a solution by finding three constants of motion for the electrons in a reference frame S' moving with a velocity βc in the x_1 direction with respect to the laboratory frame S. Indeed, the physical principle used in the argument holds only for $\beta < 1$, the solution so obtained is mathematically correct even for $\beta > 1$. Since electrons move in conservative fields in S', the total energy or the fourth component of the conjugate momentum

$$p_i = m c u_i - \frac{e}{c} A_i \tag{2-13}$$

is a constant of motion when referred to S'. We have

$$p_4' = - \frac{i\beta}{\alpha} p_1 + \frac{1}{\alpha} p_4$$

$$= \frac{imc}{\alpha} \left\{ \gamma - \beta u_1 - \frac{e\alpha^2}{mc^2} \phi \right\} \quad ,$$

and the first constant can be taken as

$$\epsilon = \gamma - \beta u_1 - \psi \quad , \tag{2-14}$$

where

$$\psi = \frac{e\alpha^2}{mc^2} \phi \ . \tag{2-15}$$

Furthermore, since the Hamiltonian

$$H = \frac{1}{2m} (p_i + \frac{e}{c} A_i)^2 \tag{2-16}$$

for an electron does not involve the coordinates x_2 and x_3, the conjugate momenta p_2 and p_3 are constant and we take

$$\pi_\tau = u_\tau - \mu_\tau \ , \ \tau = 2, 3, \tag{2-17}$$

where

$$\mu_\tau = \frac{e}{mc^2} A_\tau \quad , \tag{2-18}$$

as the second and third constants. The general solution of (2-11) is therefore

$$f = f\ (\epsilon, \pi_2, \pi_3) \quad , \tag{2-19}$$

with ϵ, π_2 and π_3 given by (2-14) and (2-17). This can be verified to be the correct solution for both $\beta \lesseqgtr 1$. The condition that (2-19) reduces to the electron equilibrium distribution function for a plasma when the field potential A_i vanishes enables us to write f in definite functional form. Thus, for a cold plasma beam, the solution is

$$f = N\,\delta\left(\frac{\beta}{\alpha^2}\epsilon - \frac{1}{\alpha^2}\sqrt{\epsilon^2 - \alpha^2} - u_{10}\right) \delta(\pi_2)\,\delta(\pi_3) \quad , \tag{2-20}$$

where u_{10} is the reduced drift velocity in the absence of any fields and N is a normalizing constant.. For a plasma with Maxwellian electron velocity distribution,

$$f = \frac{a}{4\pi K_2(a)} \exp\left\{-\frac{a}{\alpha^2}\left[\epsilon - \beta\sqrt{\epsilon^2 - \alpha^2(1 + \pi_2^2 + \pi_3^2)}\right]\right\}, \quad (2\text{-}21)$$

where

$$a = mc^2 (\kappa_B T)^{-1} \quad , \qquad (2\text{-}22)$$

κ_B is the Boltzmann's constant, T is the electron temperature and $K_n(a)$ denotes the n^{th} order Bessel function for a purely imaginary argument.[3] In the limit of $A_i \to 0$, (2-20) and(2-21) reduce to the familiar unit-normalized distribution functions

$$f_{oo} = \delta(u_1 - u_{10})\,\delta(u_2)\,\delta(u_3) \qquad (2\text{-}23)$$

and

$$f_o = \frac{a}{4\pi K_2(a)} \cdot e^{-a\gamma} \qquad (2\text{-}24)$$

respectively.

3. WAVE EQUATIONS AND COUPLING DUE TO NONLINEARITY

The solution of the distribution function obtained in the last section will be used to evaluate the 4-current density, which together with the fields must satisfy the Maxwell equations. In terms of the antisymmetric electromagnetic field tensor

$$F_{ij} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix} , \qquad (3\text{-}1)$$

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[3] G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1952), 2nd ed.

Maxwell's equations can be written as

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{jk}}{\partial x_i} + \frac{\partial F_{ki}}{\partial x_j} = 0 \qquad (3\text{-}2)$$

and

$$\frac{\partial F_{ij}}{\partial x_j} = s_i \quad , \qquad (3\text{-}3)$$

where the 4-current density s_i are due to both the ions (assumed to be immobile) and the electrons:

$$s_i = s_i^{+} + s_i^{-} \quad , \qquad (3\text{-}4)$$

$$s_i^{+} = (0, 0, 0, i\, n_o\, e) \quad , \qquad (3\text{-}5)$$

$$s_i^{-} = -n_o e \iiint f\,(\epsilon, \pi_2, \pi_3)\, \frac{u_i}{\gamma}\, du_1 du_2 du_3 \quad . \qquad (3\text{-}6)$$

In (3-5) and (3-6), n_o is the number density of either species of particles under equilibrium condition. With the introduction of 4-potential and the use of Lorentz gauge, it is familiar that we only need to consider the wave equations

$$\frac{\partial^2}{\partial x_j \partial x_j} A_i = -4\pi s_i \quad , \qquad (3\text{-}7)$$

and the field tensor is given by

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \quad . \qquad (3\text{-}8)$$

Of the four equations of (3-7), the first and the fourth are not independent and it is sufficient to deal with the last three equations only. These are, due to our plane wave assumption and upon changing variables,

$$\ddot{\mu}_\tau = \frac{\omega_p^2}{\alpha^2 c^2} \int_{\epsilon_{min}}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\epsilon, \pi_2, \pi_3)\, \frac{(\pi_\tau + \mu_\tau)\, d\pi_2\, d\pi_3\, d\epsilon}{\sqrt{(\epsilon+\psi)^2 - \alpha^2[1 + (\pi_2+\mu_2)^2 + (\pi_3+\mu_3)^2]}}, \qquad (3\text{-}9)$$

$$\tau = 2, 3,$$

$$\ddot{\psi} = \frac{\omega_p^2}{c^2} \cdot \left\{ \frac{1}{\alpha^2} \int_{\epsilon_{min}}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\epsilon, \pi_2, \pi_3) \left(\frac{\epsilon + \psi}{\sqrt{(\epsilon+\psi)^2 - \alpha^2 [1 + (\pi_2 + \mu_2)^2 + (\pi_3 + \mu_3)^2]}} - \beta \right) d\pi_2 \, d\pi_3 \, d\epsilon - 1 \right\} , \tag{3-10}$$

where $\omega_p = (4\pi n_o e^2/m)^{1/2}$, and the lower limit of integration ϵ_{min} is such that there are no trapped electrons.[4]

Postponing detailed solution of these equations, we can observe certain general characteristics concerning nonlinear transverse waves in a plasma. If μ_2 and μ_3 are set equal to zero, (3-10) reduces to the same equation for longitudinal waves.[4] Since the integrand in Eq. (3-9) are odd in π_2 and π_3 when $\mu_2 = \mu_3 = 0$, the equations for the transverse waves are identically satisfied, and longitudinal waves can exist alone. The converse, however, is not true. Let us assume that $\mu_3 = 0$ for simplicity and $\mu_2 \neq 0$. The integral in (3-10) involves ξ and a variable space charge is always present wherever a transverse electric field exists. Equation (3-10) cannot be satisfied by a vanishing ψ and a longitudinal wave always accompanies the transverse waves in an exact nonlinear analysis.

In the case of a cold plasma, substitution of the distribution function (2-20) into (3-9, 10) yields

$$\ddot{\mu}_\tau = C \left\{ (\epsilon_o + \psi)^2 - \alpha^2 (1 + \mu_2^2 + \mu_3^2) \right\}^{-1/2} \mu_\tau , \quad \tau = 2, 3, \tag{3-11}$$

$$\ddot{\psi} = C \left(\frac{\epsilon_o + \psi}{\sqrt{(\epsilon_o + \psi)^2 - \alpha^2 (1 + \mu_2^2 + \mu_3^2)}} - \beta \right) - \frac{\omega_p^2}{c^2} , \tag{3-12}$$

where

$$C = \frac{N}{c^2 \alpha^2} \omega_p^2 \left| \alpha^2 \left(\beta - \frac{\epsilon_o}{\sqrt{\epsilon_o^2 - \alpha^2}} \right)^{-1} \right| , \tag{3-13}$$

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[4] H. S. C. Wang, "Nonlinear Stationary Waves in Relativistic Plasmas," to be published in the Phys. Fluids.

$$\epsilon_o = \sqrt{1 + u_{10}^2} - \beta u_{10} , \qquad (3\text{-}14)$$

and the choice of the normalizing constant N depends on further considerations. The discussion in the previous paragraph still applies except that one special case may be worth mentioning. If we let

$$\left.\begin{aligned} \mu_2 &= \mu \cos k\xi \\ \mu_3 &= \mu \sin k\xi \\ \psi &= 0 \end{aligned}\right\} , \qquad (3\text{-}15)$$

so that the transverse amplitude μ is a constant, and

$$N = \left(\beta - \frac{\epsilon_o}{\sqrt{\epsilon_o^2 - \alpha^2}}\right)\left(\beta - \frac{\epsilon_o}{\sqrt{\epsilon_o^2 - \alpha^2 (1 + \mu^2)}}\right)^{-1} , \qquad (3\text{-}16)$$

all the three equations given in (3-11, 12) can be satisfied for

$$k = \frac{\omega_p}{c}\left(\beta \sqrt{\epsilon_o^2 - \alpha^2 (1 + \mu^2)} - \epsilon_o\right)^{-1/2} , \qquad (3\text{-}17)$$

with the value of β limited to be greater than unity. The particular solution (3-15) represents a circularly polarized purely transverse wave with angular frequency

$$\frac{\omega}{\omega_p} = \beta (\beta^2 - 1)^{-1/2} (1 + u_1^2 + \mu^2)^{-1/4} , \qquad (3\text{-}18)$$

where

$$u_1 = \frac{1}{\alpha^2} \left\{\beta\epsilon_o - \sqrt{\epsilon_o^2 - \alpha^2 (1 + \mu^2)}\right\} \qquad (3\text{-}19)$$

is a new constant drift velocity increased due to the transverse wave amplitude μ. The trajectories of the electrons are helices with axis of symmetry in the direction of wave propagation. In the limit of $\mu \to 0$, $u_1 \to u_{10}$, $N \to 1$, and $\frac{\omega}{\omega_p} \to (1 - \beta^{-2})^{-1/2} (1 + u_{10}^2)^{-1/4}$; if in addition $u_{10} = 0$, $\frac{\omega}{\omega_p} \to (1 - \beta^{-2})^{-1/2}$, a result well known from linear theory. By choosing u_{10} suitably, i.e.,

$u_{10} = \frac{1}{\alpha^2}\left\{\beta\sqrt{1+\mu^2} - \sqrt{\beta^2+\mu^2}\right\}$, u_1 given by (3-19) can be made to vanish and the electrons move in circular orbits with a frequency $\frac{\omega}{\omega_p} = \beta\,(\beta^2-1)^{-1/2}\,(1+\mu^2)^{-1/4}$. This is the special case pointed out by Akhiezer and Polovin.[5]

4. NONLINEAR PLANE-POLARIZED TRANSVERSE WAVES IN A COLD PLASMA

In this section we wish to investigate the nonlinear effects on the plane-polarized transverse waves. As was mentioned before, the coupling between such waves and longitudinal waves would necessitate the solution of two second order nonlinear differential equations simultaneously. To the knowledge of these writers, a completely rigorous analytic method does not exist. We can, however, learn a great deal by first ignoring ψ in Eq. (3-11), say, for the μ_2 mode ($\mu_3 = 0$). This is roughly equivalent to the quasi-neutrality assumption used either explicitly or implicitly by several authors[1,6,7] in studying similar problems. It is also shown in the Appendix that the frequency of nonlinear transverse mode so obtained agrees with that derived from Cesari's theorem and method of successive approximation[8] for an autonomous system to the order of amplitude squared.

Since the amplitude of plane-polarized transverse waves vanishes somewhere in the plasma, we take $N = 1$, $\mu_3 = \psi = 0$ in (3-11), $u_{10} = 0$ in (3-14) for simplicity, and write

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[5] A. I. Akhiezer and R. V. Polovin, J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 915 (1956).

[6] A. D. Pataraya, Technical Phys. (U.S.S.R.) 7, 97 (1962).

[7] Ts. D. Loladze and N. L. Tsintsadze, Technical Phys. (U.S.S.R.) 6, 944 (1962).

[8] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations (Springer-Verlag, Berlin, 1959).

$$\ddot{\eta} = -k_o^2 \frac{\eta}{\sqrt{1 \mp \eta^2}} \quad , \quad \beta \lessgtr 1 \quad , \tag{4-1}$$

where

$$\eta = \frac{1}{\beta} \sqrt{\pm \alpha^2} \; \mu_2 \quad , \qquad \beta \lessgtr 1 \quad , \tag{4-2}$$

and

$$k_o^2 = - \frac{\omega_p^2}{\alpha^2 c^2} \quad . \tag{4-3}$$

The first integral of (4-1) from η_{min} to η is

$$\dot{\eta}^2 = Y(\eta) = \pm 2 k_o^2 \left(\sqrt{1 \mp \eta^2} - \sqrt{1 \mp \eta_{min}^2} \right) , \quad \beta \lessgtr 1 \quad . \tag{4-4}$$

The function Y in (4-4) vanishes at η_{min} and $\eta_{max} = -\eta_{min}$, and is positive in between only when the lower signs are used. From the discussion in a previous paper,[4] we conclude that periodic solutions or oscillations are possible only for $\beta > 1$. If we choose the origin of the ξ-axis such that $\eta(o) = 0$, the solution of (4-4) is

$$\xi = \frac{\sqrt{2(r+1)}}{k_o} \left\{ E(\kappa) - E(\chi,\kappa) - \frac{1}{r+1} \left[K(\kappa) - F(\chi,\kappa) \right] \right\} ,$$

$$0 < \xi < \xi(\eta_{max}) \quad , \tag{4-5}$$

where

$$r = \sqrt{1 + \eta_m^2} \quad , \quad \eta_m = \left| \eta_{min} \right| \quad , \tag{4-6}$$

$$\chi = \sin^{-1} \sqrt{\frac{r - \sqrt{1 + \eta^2}}{r - 1}} \quad , \tag{4-7}$$

E and F are respectively the normal elliptical integrals of the second and the first kind with the modulus

$$\kappa = \sqrt{\frac{r-1}{r+1}} \quad . \tag{4-8}$$

The curve of η vs. ξ is, due to the evenness of $Y(\eta)$, odd with respect to the origin, and of course symmetric with respect to the lines $\xi = \xi(\eta_{max})$. In the limit of infinitesimal amplitude, (4-5) reduces to a pure sinusoidal wave $\eta = \eta_m \sin(k_o \xi)$. In Fig. 1, we plot one cycle of the μ_2 wave vs. $\frac{\omega_p}{c}\xi$ as abscissa for $\beta = 1.5$ and $\eta_m \approx 0$, $\eta_m = 0.5, 1.0$, based on Eq. (4-5). The curve labeled $\eta_m \approx 0$ represents a sine wave. As the amplitude increases, the nonlinear effect is felt primarily through the increase in wavelength for a fixed velocity of propagation rather than distortion in waveform. The last conclusion is different from that for nonlinear longitudinal waves[9] and is largely due to the evenness of the function $Y(\eta)$. The wavelength λ and frequency ω are given by

$$\lambda = \frac{4}{k_o \kappa'} \left[2E(\kappa) - \kappa'^2 K(\kappa) \right] , \tag{4-9}$$

and

$$\frac{\omega}{\omega_p} = \frac{\pi}{2} \cdot \frac{\beta}{\sqrt{\beta^2 - 1}} \cdot \frac{\kappa'}{2E(\kappa) - \kappa'^2 K(\kappa)} , \tag{4-10}$$

where

$$\kappa' = \sqrt{1 - \kappa^2} \tag{4-11}$$

is the complementary modulus. In the limit of vanishing η_m, (4-10) reduces to the frequency of linear theory $\frac{\omega}{\omega_p} = \beta(\beta^2 - 1)^{-1/2}$. The dispersion relation (4-10) for nonlinear transverse waves is computed and plotted in Fig. 2 for values of $\eta_m \approx 0$, $\eta_m = 0.5$ and 1.0. Similar to the case of nonlinear longitudinal waves,[4] the frequency decreases as amplitude is increased over wide range of the wave velocity. For small values of η_m and hence κ, we expand the complete elliptical integrals in (4-10) into series of κ and obtain

$$\frac{\omega}{\omega_p} = \frac{\beta}{\sqrt{\beta^2 - 1}} \left(1 - \frac{3}{4}\kappa^2 + O(\kappa^4)\right) , \tag{4-12}$$

or in terms of η_m and then A_{2m} ,

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[9] A. Cavaliere, Nuovo cimento 23, 440 (1962).

$$\frac{\omega}{\omega_p} \approx \frac{\beta}{\sqrt{\beta^2 - 1}} \left\{ 1 + \frac{3\alpha^2}{16\beta^2} \left(\frac{e}{mc^2} \right)^2 A_{2m}^{\;2} \right\} . \tag{4-13}$$

The agreement between this and the result obtained (see Appendix) by using Cesari's successive approximation to the order of $A_{2m}^{\;2}$ serves as a justification for the quasi-neutrality assumption introduced earlier.

5. NONLINEAR TRANSVERSE WAVES IN HOT PLASMAS

Following the same reasoning in the previous sections, we may investigate the propagation characteristics of nonlinear transverse waves in plasmas of finite electron temperature. Let us consider the A_2 or μ_2 mode in the quasi-neutrality approximation by setting $\mu_3 = \psi = 0$ in Eq. (3- 9):

$$\ddot{\mu}_2 = \frac{\omega_p^{\;2}}{\alpha^2 c^2} \iiint \frac{f(\epsilon, \pi_2, \pi_3)\,(\pi_2 + \mu_2)}{\sqrt{\epsilon^2 - \alpha^2 [1 + (\pi_2 + \mu_2)^2 + \pi_3^{\;2}]}}\, d\pi_2\, d\pi_3\, d\epsilon . \tag{5-1}$$

The first integral of (5-1) is, assuming $\mu_2\,(o) = 0$,

$$\dot{\mu}_2^{\;2} = Y(\mu_2)$$

$$= \frac{\omega_p^{\;2}}{c^2} \cdot \left\{ a_o - \frac{2}{\alpha^4} \iiint f(\epsilon, \pi_2, \pi_3) \left(\sqrt{\epsilon^2 - \alpha^2 [1 + (\pi_2 + \mu_2)^2 + \pi_3^{\;2}]} - \sqrt{\epsilon^2 - \alpha^2 (1 + \pi_2^{\;2} + \pi_3^{\;2})} \right) d\pi_2\, d\pi_3\, d\epsilon \right\} , \tag{5-2}$$

where

$$a_o = \frac{c^2}{\omega_p^{\;2}} \, [\dot{\mu}_2 (o)]^2 \tag{5-3}$$

is proportional to the square of the maximum transverse electric or magnetic fields, and the possibility of electron trapping is excluded from consideration. It is easy to see that the second derivative of Y with respect to μ_2

$$Y''(\mu_2) = \frac{2\omega_p^{\;2}}{\alpha^2 c^2} \iiint f(\epsilon, \pi_2, \pi_3) \cdot \frac{\epsilon^2 - \alpha^2 (1 + \pi_3^{\;2})}{\left\{ \epsilon^2 - \alpha^2 [1 + (\pi_2 + \mu_2)^2 + \pi_3^{\;2}] \right\}^{3/2}} d\pi_2\, d\pi_3\, d\epsilon \tag{5-4}$$

is always negative for $\beta > 1$, but positive for $\beta < 1$. Wave propagation is therefore possible for phase velocities greater than c, as in the case of a cold plasma. We expand the function $Y(\mu_2)$ into a Taylor's series

$$Y(\mu_2) = \frac{\omega_p^2}{c^2} \sum_{n=0}^{\infty} a_n \mu_2^n \quad , \tag{5-5}$$

with

$$a_n = \frac{c^2}{\omega_p^2} Y^{(n)}(0) / n! \quad . \tag{5-6}$$

For plasmas with isotropic equilibrium electron velocity distribution, all odd order coefficients vanish, i.e., $a_n = 0$, $n = 1, 3, 5, \cdots$, and Y is even with respect to μ_2. We shall consider the nonlinear effects up to the μ_2^4 term. In the case of a plasma with Maxwellian electron distribution, we found

$$a_2 = \frac{1}{\alpha^2} \iiint \frac{f_o(u)}{\gamma} \left[1 + \frac{\alpha^2 u_2^2}{(\beta\gamma - u_1)^2} \right] du_1\, du_2\, du_3 \quad ,$$

$$a_4 = \frac{1}{4} \iiint \frac{f_o(u)}{\gamma} \left[\frac{1}{(\beta\gamma - u_1)^2} + \frac{6\alpha^2 u_2^2}{(\beta\gamma - u_1)^4} + \frac{5\alpha^4 u_2^4}{(\beta\gamma - u_1)^6} \right] du_1\, du_2\, du_3 \quad ,$$

or, upon substituting from (2-24) and integrating over the angular part,[4]

$$a_2 = \frac{\beta^2}{\alpha^2} \frac{K_1(a)}{K_2(a)} + \frac{a\beta}{2K_2(a)} \int_o^{\infty} e^{-a\cosh\tau} \sinh\tau \cosh\tau \ln\left(\frac{\beta\coth\tau + 1}{\beta\coth\tau - 1}\right) d\tau, \tag{5-7}$$

$$a_4 = \frac{a}{4\beta^2 K_2(a)} \int_o^{\infty} e^{-a\cosh\tau} \sinh^{-4}\tau \left(\coth^2\tau - \frac{1}{\beta^2}\right)^{-3} d\tau \quad . \tag{5-8}$$

For $a_o < \dfrac{a_2^2}{4a_4}$, the quartic approximation of Y has four real zeros

$$\left.\begin{aligned} Z_1 &= -Z_4 \\ Z_2 &= -Z_3 \end{aligned}\right\} = \left\{ \frac{1}{2a_4} \left[-a_2 \pm (a_2^2 - 4a_o a_4)^{1/2} \right] \right\}^{1/2} \quad . \tag{5-9}$$

Integration of (5-2) with Y expressed in terms of these zeros yields a quarter cycle

$$\xi = \frac{2c}{\omega_p a_4^{1/2} (Z_1 + Z_2)} \left\{ F(\chi_o, \kappa) - F(\chi, \kappa) \right\} , \qquad (5\text{-}10)$$

$$o < \xi < \xi (\mu_{2\,max}) ,$$

where

$$\chi = \sin^{-1} \sqrt{\frac{(Z_1 + Z_2)(Z_2 - \mu_2)}{2 Z_2 (Z_1 - \mu_2)}} , \quad \mu_2 < Z_2 ,$$

$$\chi_o = \sin^{-1} \sqrt{\frac{Z_1 + Z_2}{2Z_1}} ,$$

and

$$\kappa = \frac{2}{Z_1 + Z_2} \sqrt{Z_1 Z_2} .$$

In the limit of infinitesimal amplitude, i.e., $a_o \to o$, (5-10) approaches a pure sinusoidal wave

$$\mu_2 = \left(- \frac{a_o}{a_2}\right)^{1/2} \sin \left\{ \frac{\omega_p}{c} (-a_2)^{1/2} \xi \right\} .$$

Typical examples of nonlinear transverse waveform in hot plasmas have also been computed from (5-10); they are of the same general shape as those plotted in Fig. 1. The low distortion in the waveform here is primarily due to the lack of the cubic or, for that matter, the odd power terms in the series expansion (5-5). The wavelength and frequency are given by

$$\lambda = \frac{8c\, F(\chi_o, \kappa)}{\omega_p a_4^{1/2} (Z_1 + Z_2)} ,$$

and

$$\frac{\omega}{\omega_p} = \frac{\pi}{4} \cdot \frac{\beta a_4^{1/2} (Z_1 + Z_2)}{F(\chi_o, \kappa)} . \qquad (5\text{-}11)$$

In the limit of $a_o \to o$, (5-11) tends to the limiting value $\beta\,(-a_2)^{1/2}$ or

$$\frac{\omega}{\omega_p} \to -\frac{\beta^4}{\alpha^2} \cdot \frac{K_1(a)}{K_2(a)} - \frac{a\beta^3}{2K_2(a)} \int_o^\infty e^{-a\cosh\tau} \sinh\tau \cosh\tau$$

$$\ln\left(\frac{\beta\coth\tau + 1}{\beta\coth\tau - 1}\right) d\tau \quad .$$

It can be demonstrated that the last relation is equivalent to a dispersion equation derived by Imre[10] based on linearized analysis. In Fig. 3, 4, and 5, dispersion characteristics for nonlinear transverse waves in plasmas of different temperatures (a = 0.05, 1.0, and 10) are plotted based on computed results from Eq. (5-11) for values of a_o labeled on each curve. The solid curve in each figure represents the characteristic for a wave of infinitesimal amplitude. It is again seen that nonlinearity causes a general decrease of frequency of oscillation or an increase in the wavelength as in the case of a cold plasma. The nonlinear effect is, however, more pronounced for plasmas of lower electron temperature.

6. ACKNOWLEDGMENTS

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APPENDIX

NONLINEAR TRANSVERSE WAVE SOLUTION BY SUCCESSIVE APPROXIMATION

We show here that Cesari's[8] method of successive approximation for nonlinear autonomous systems applied to the case of transverse-longitudinal

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[10] K. Imre, Phys. Fluids 5, 459 (1962).

waves in a cold plasma yields the same frequency (to the order of wave amplitude squared) as that derived in Section 4. For $\mu_3 = 0$, μ_2 and ψ relatively small, $u_{10} = 0$, and $N = 1$, equations (3-11, 12) can be written as

$$\left.\begin{aligned} \ddot{\mu}_2 + k_o^2 \mu_2 &= \tilde{\epsilon}\left(-\frac{2}{\alpha^2}\mu_2\psi + \mu_2^3\right) \\ \ddot{\psi} + k_\psi^2 \psi &= \tilde{\epsilon}\,\mu_2^2 \end{aligned}\right\}, \qquad \text{(A-1)}$$

with $k_\psi^2 = \dfrac{\omega_p^2}{\beta^2 c^2}$ and $\tilde{\epsilon} = \dfrac{1}{2}\cdot\dfrac{\omega_p^2}{\beta^2 c^2}$, assumed to be a small quantity.

(A-1) is equivalent to the following canonical system:

$$\dot{y}_j = i\tau_j y_j + \tilde{\epsilon} f_j, \quad j = 1, \cdots, 4, \qquad \text{(A-2)}$$

where

$$\left.\begin{aligned} y_j &= i k_o \mu_2 + (-1)^{j-1}\dot{\mu}_2, \quad j = 1, 2, \\ y_j &= i k_\psi \psi + (-1)^{j-1}\dot{\psi}, \quad j = 3, 4, \end{aligned}\right\} \qquad \text{(A-3)}$$

$$\tau_1 = -\tau_2 = k_o, \qquad \tau_3 = -\tau_4 = k_\psi, \qquad \text{(A-4)}$$

and

$$\left.\begin{aligned} f_1 &= -f_2 = \frac{1}{2\alpha^2 k_o k_\psi}(y_1 + y_2)(y_3 + y_4) + \frac{i}{8k_o^3}(y_1 + y_2)^3 \\ f_3 &= -f_4 = -\frac{1}{4k_o^2}(y_1 + y_2)^2 \end{aligned}\right\}. \qquad \text{(A-5)}$$

Taking for the zeroth order approximation

$$\left.\begin{aligned} y_1^{(o)} &= \mathcal{A} e^{ik\xi}, \quad y_2^{(o)} = -\mathcal{A} e^{-ik\xi}, \\ y_3^{(o)} &= y_4^{(o)} = 0, \end{aligned}\right\} \qquad \text{(A-6)}$$

with real $\mathcal{A}$, we have, from the determining equation

$$i\,k + \frac{3}{8} \cdot \frac{i\,\tilde{\epsilon}}{k_o^{\,3}}\,\mathcal{A}^2 = i\,k_o \quad , \tag{A-7}$$

the frequency of wave solution to the first order of $\tilde{\epsilon}$

$$\frac{\omega}{\omega_p} = \frac{\beta}{\sqrt{\beta^2 - 1}} \left(1 - \frac{3}{16} \cdot \frac{c^2 \alpha^4}{\omega_p^{\,2} \beta^2}\,\mathcal{A}^2 \right) \quad , \tag{A-8}$$

which is equivalent to (4-13) in view of (A-3).

LIST OF FIGURE CAPTIONS

Fig. 1. Waveform for nonlinear transverse waves in a cold plasma, $\beta = 1.5$.

Fig. 2. Dispersion characteristics for nonlinear transverse waves in a cold plasma.

Fig. 3. Dispersion characteristics for nonlinear transverse waves in a hot plasma, a = 0.05.

Fig. 4. Dispersion characteristics for nonlinear transverse waves in a hot plasma, a = 1.0.

Fig. 5. Dispersion characteristics for nonlinear transverse waves in a hot plasma, a = 10.

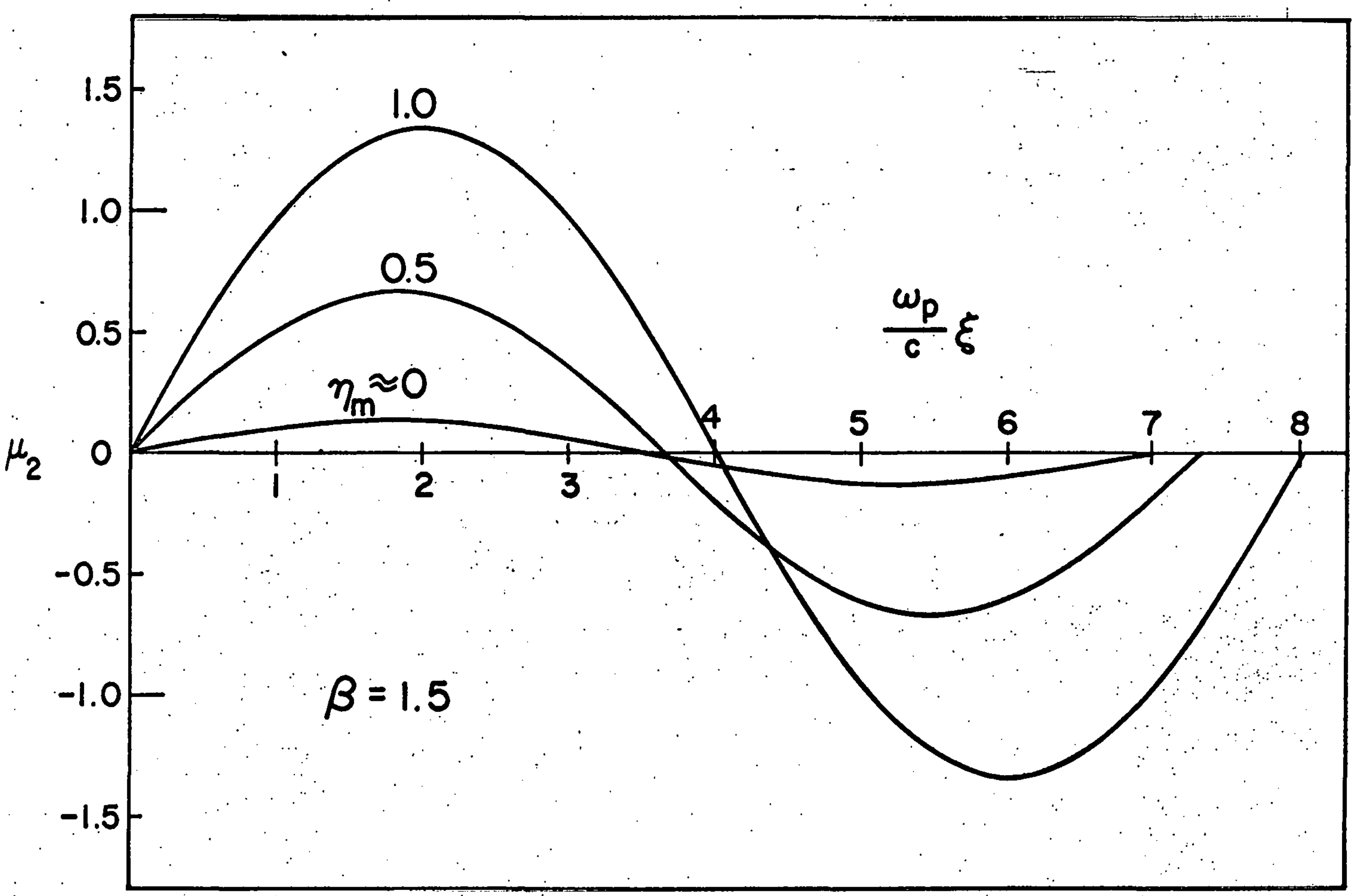

Fig. I. Waveform for Nonlinear Transverse Waves in a Cold Plasma, $\beta = 1.5$

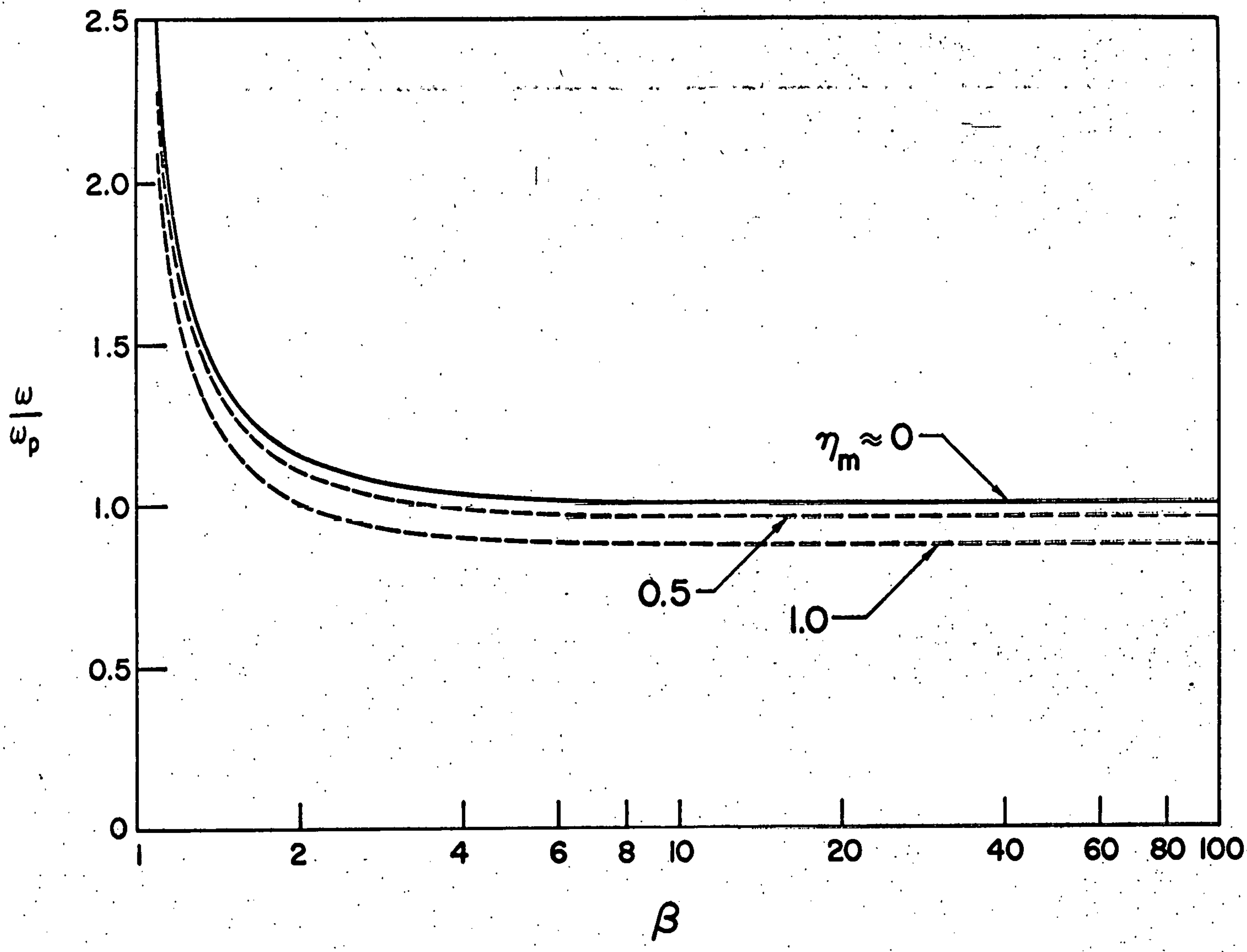

Fig. 2 Dispersion Characteristics for Nonlinear Transverse Waves in a Cold Plasma.

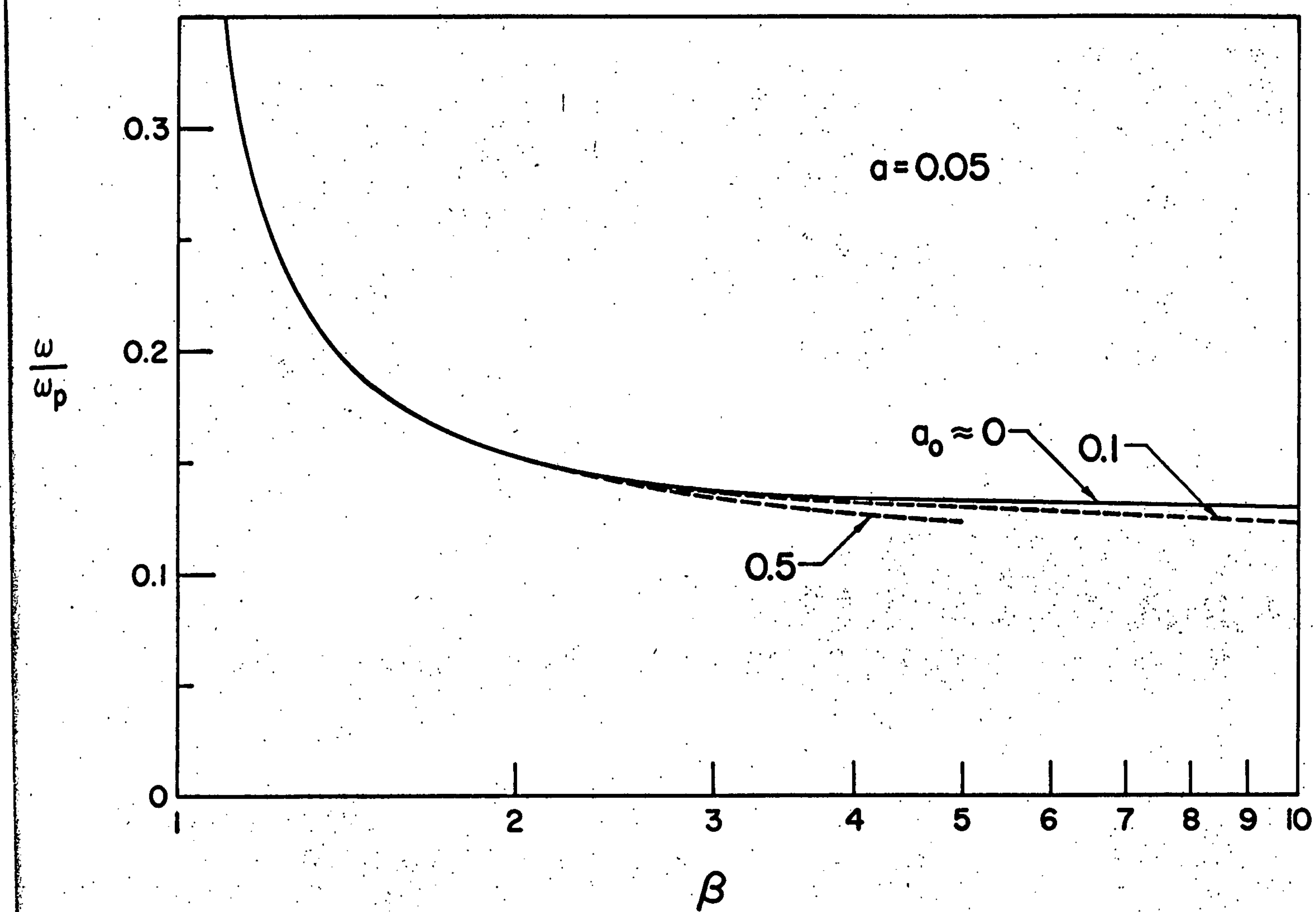

Fig. 3 Dispersion Characteristics for Nonlinear Transverse Waves in a Hot Plasma, a = 0.05

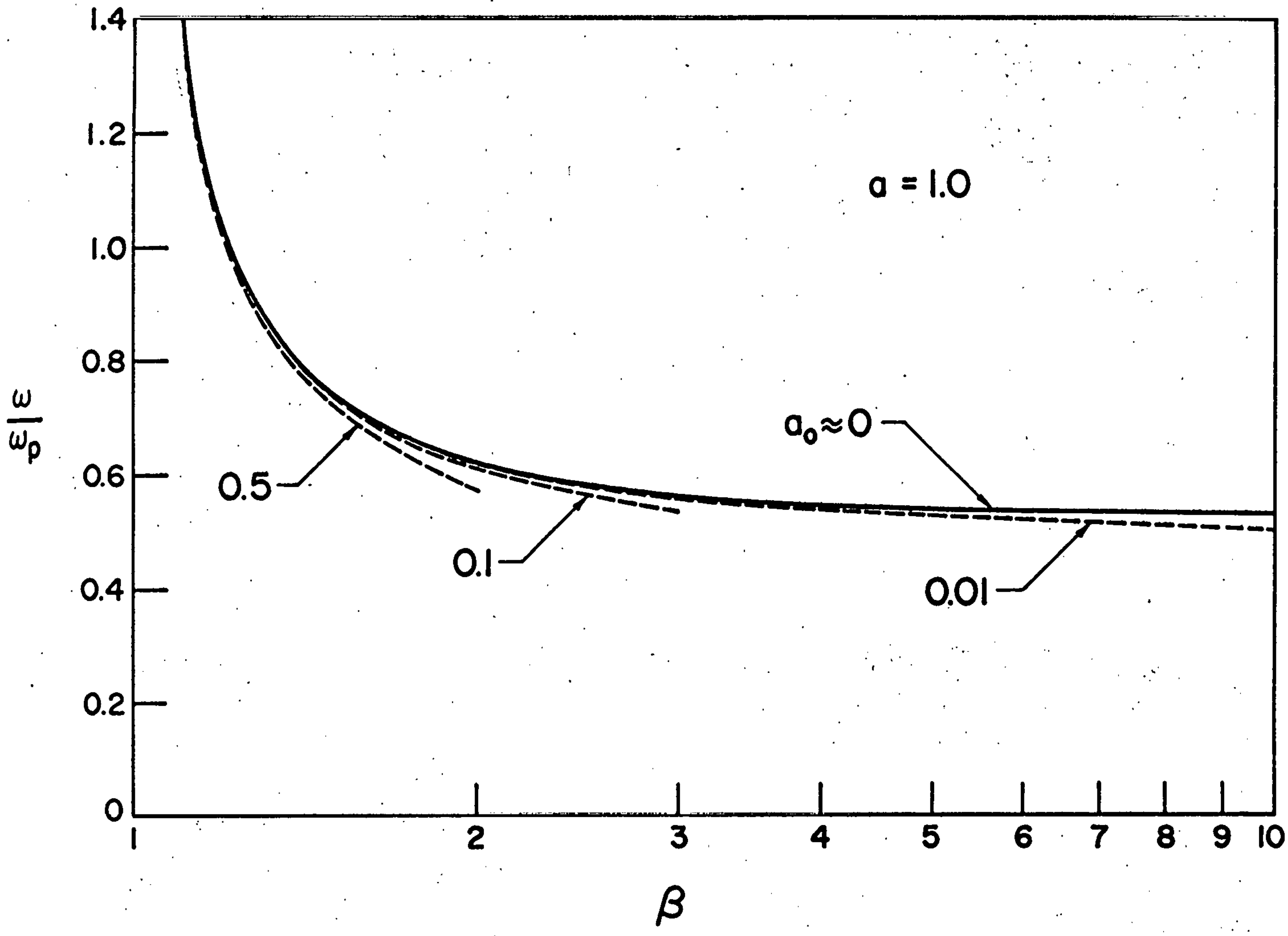

Fig. 4 Dispersion Characteristics for Nonlinear Transverse Waves in a Hot Plasma, $a = 1.0$

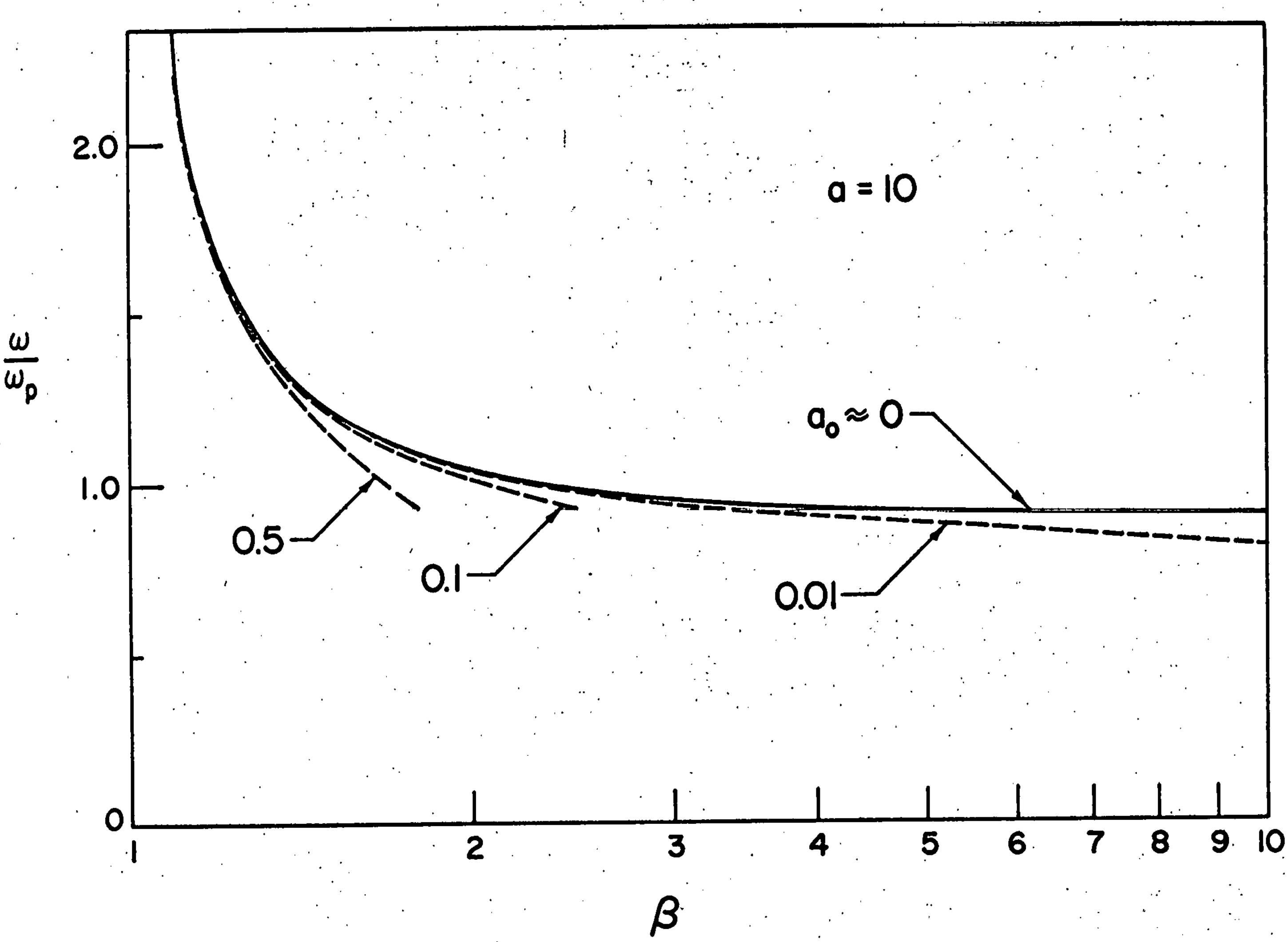

Fig. 5 Dispersion Characteristics for Nonlinear Transverse Waves in a Hot Plasma, a = 10