SLAC/AP-106
November 1996

Transfer matrices of superimposed magnets and RF cavity*

Chun-xi Wang and Alex Chao
Stanford Linear Accelerator Center,
Stanford University, Stanford, CA 94309 USA

Superimposed magnets often occur in accelerators, such as in the interaction regions of colliders. This note presents the linear transfer matrices of various superimposed magnets. Since readers of this note are probably well informed, we simply list the results without derivation. The method used to calculate the linear transfer matrices is outlined in Ref. [1]. We list the Hamiltonian H and the corresponding matrix $e^{:-sH:}$ for several combinations of common magnets (only sector magnets without fringe field) used in accelerators. Magnetic fields and vector potentials (in Coulomb gauge) are also listed for reference. The usual $\{\hat{x}, \hat{y}, \hat{s}\}$ coordinate system is used.† The transfer matrices are for the commonly used canonical variables $\{x, P_x, y, P_y, z, \delta\}$. For definitions of the coordinate system and the dynamical variables, see Ref. [1, 2]. For those cases when the transfer matrices depend on the gauge selection, we also list the matrices for the coordinates and kinetic momenta $\{x, p_x, y, p_y\}$. The magnetic fields and Hamiltonians are correct only up to the order appropriate for linear optics. In this computer age, complicated analytical results may lose their usefulness and attractiveness in many applications. However, we hope our exact analytic results to linear order are still useful for particle dynamics studies in superimposed magnet systems.

1 Horizontal Dipole + Quadrupole (curved)

This is the simplest and well-known case, encountered for example in weak focusing synchrotron accelerators. It also occurs in combined function magnets. The Hamiltonian for

†In the Frenet-Serret coordinate system,

$$\nabla \times \vec{A} = [\partial_y A_s - (\frac{1}{1+x/\rho})\partial_s A_y]\hat{x} + (\frac{1}{1+x/\rho})[\partial_s A_x - \partial_x(1+\frac{x}{\rho})A_s]\hat{y} + (\partial_x A_y - \partial_y A_x)\hat{s}$$

$$\nabla \cdot \vec{A} = \frac{1}{1+x/\rho}\partial_x(1+\frac{x}{\rho})A_x + \partial_y A_y + \frac{1}{1+x/\rho}\partial_s A_s$$

and

$$H = -\frac{eA_s}{P_0}(1+\frac{x}{\rho}) - \left\{(1+\frac{x}{\rho})\sqrt{1-(\frac{P_x - eA_x/P_0}{1+\delta})^2 - (\frac{P_y - eA_y/P_0}{1+\delta})^2} - 1\right\}(1+\delta)$$

$$\simeq -\frac{eA_s}{P_0}(1+\frac{x}{\rho}) - \frac{x(1+\delta)}{\rho} + \frac{1}{2}(P_x - \frac{eA_x}{P_0})^2 + \frac{1}{2}(P_y - \frac{eA_y}{P_0})^2 + \cdots$$

where e, ρ are signed quantities and, P_0 is the nominal momentum.

* Work supported by the Department of Energy under contract DE-AC03-76SF00515.

such Dipole + Quadrupole system reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) - \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{1}{2}k(x^2 - y^2) \tag{1}$$

The magnetic field and vector potential used:

$$\vec{B} = \frac{P_0}{e}[ky\hat{x} + (\frac{1}{\rho} + kx)\hat{y}] + O(x^2), \qquad \vec{A} = -\frac{P_0}{2e}[1 + \frac{x}{\rho} + k(x^2 - y^2)]\hat{s}. \tag{2}$$

The transfer matrix reads:

$$\begin{pmatrix} \cos\sqrt{\tilde{k}}s & \frac{1}{\sqrt{\tilde{k}}}\sin\sqrt{\tilde{k}}s & 0 & 0 & 0 & \frac{2}{\rho\tilde{k}}\sin^2\frac{\sqrt{\tilde{k}}s}{2} \\ -\sqrt{\tilde{k}}\sin\sqrt{\tilde{k}}s & \cos\sqrt{\tilde{k}}s & 0 & 0 & 0 & \frac{1}{\rho\sqrt{\tilde{k}}}\sin\sqrt{\tilde{k}}s \\ 0 & 0 & \cosh\sqrt{k}s & \frac{1}{\sqrt{k}}\sinh\sqrt{k}s & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh\sqrt{k}s & \cosh\sqrt{k}s & 0 & 0 \\ -\frac{1}{\rho\sqrt{\tilde{k}}}\sin\sqrt{\tilde{k}}s & -\frac{2}{\rho\tilde{k}}\sin^2\frac{\sqrt{\tilde{k}}s}{2} & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where $\alpha = \frac{1}{\rho^2\tilde{k}}\left(\frac{1}{\sqrt{\tilde{k}}}\sin\sqrt{\tilde{k}}s - s\right)$ and $\tilde{k} \equiv k + \frac{1}{\rho^2}$.

2 Horizontal Dipole + Skew Quadrupole (curved)

The Hamiltonian reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) - \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} - kxy \tag{4}$$

The magnetic field and vector potential used:

$$\vec{B} = \frac{P_0}{e}[kx\hat{x} + (\frac{1}{\rho} - ky)\hat{y}] + O(xy), \qquad \vec{A} = -\frac{P_0}{2e}(1 + \frac{x}{\rho} - 2kxy)\hat{s}. \tag{5}$$

The transfer matrix R reads:

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & 0 & \frac{1}{k\rho}R_{13} \\ R_{21} & R_{11} & kR_{12} & R_{13} & 0 & \frac{1}{k\rho}R_{23} \\ R_{13} & R_{14} & R_{33} & R_{34} & 0 & \frac{1}{k\rho}(R_{33} - 1) \\ kR_{12} & R_{13} & R_{43} & R_{33} & 0 & \frac{1}{k\rho}R_{43} \\ -R_{26} & -R_{16} & -R_{46} & -R_{36} & 1 & \frac{1}{k\rho}R_{53} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

where

$$\left\{\begin{array}{lcl} R_{11} & = & \frac{1}{2a}[(a+1)\cos b_+ + (a-1)\cosh b_-] \\ R_{12} & = & \frac{\rho^2}{as}(b_+ \sin b_+ + b_- \sinh b_-) \\ R_{13} & = & \frac{k\rho^2}{a}(-\cos b_+ + \cosh b_-) \\ R_{14} & = & \frac{1}{2kas}[-(a-1)b_+ \sin b_+ + (a+1)b_- \sinh b_-] \\ R_{21} & = & \frac{1}{2as}[-(a+1)b_+ \sin b_+ + (a-1)b_- \sinh b_-] \\ R_{33} & = & \frac{1}{2a}[(a-1)\cos b_+ + (a+1)\cosh b_-] \\ R_{34} & = & \frac{1}{2|k|as}[(a-1)b_- \sin b_+ + (a+1)b_+ \sinh b_-] \\ R_{43} & = & \frac{|k|\rho^2}{as}(-b_- \sin b_+ + b_+ \sinh b_-) \end{array}\right. \tag{7}$$

and $a \equiv \sqrt{1+4k^2\rho^4}$, $b_\pm \equiv \sqrt{\frac{a\pm 1}{2}\frac{s}{\rho}}$.

3 Dipole + Solenoid (curved)

The Hamiltonian reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) - \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} - \frac{B_0}{2}(xP_y - yP_x) + \frac{B_0^2}{8}(x^2+y^2) \tag{8}$$

The magnetic field and vector potential used:[†]

$$\vec{B} = \frac{P_0}{e}\left(\frac{1}{\rho}\hat{y} + \frac{1}{1+x/\rho}B_0\hat{s}\right), \qquad \vec{A} = -\frac{P_0}{2e}\left[B_0\frac{y}{1+x/\rho}\hat{x} - B_0\rho\ln\left(1+\frac{x}{\rho}\right)\hat{y} + \left(1+\frac{x}{\rho}\right)\hat{s}\right]. \tag{9}$$

The transfer matrix R reads:

$$\begin{pmatrix} a_- + a_+\cos\tilde{B}s & \frac{1}{\tilde{B}}\sin\tilde{B}s & \frac{B_0}{2\tilde{B}}\sin\tilde{B}s & 2\frac{B_0}{\tilde{B}^2}\sin^2\frac{\tilde{B}s}{2} & 0 & R_{16} \\ -\frac{a_-s}{2\rho^2} - a_+^2\tilde{B}\sin\tilde{B}s & R_{11} & -a_+B_0\sin^2\frac{\tilde{B}s}{2} & -\frac{a_-s}{B_0\rho^2} + a_+\frac{B_0}{\tilde{B}}\sin\tilde{B}s & 0 & R_{26} \\ -R_{24} & -R_{14} & a_+ + a_-\cos\tilde{B}s & \frac{s}{\tilde{B}^2\rho^2} + \frac{2a_-}{\tilde{B}}\sin\tilde{B}s & 0 & R_{36} \\ -R_{23} & -R_{13} & -\frac{a_-}{2}\tilde{B}\sin\tilde{B}s & R_{33} & 0 & R_{46} \\ -R_{26} & -R_{16} & R_{46} & R_{36} & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

where

$$\left\{\begin{array}{lcl} R_{16} & = & \frac{2}{\tilde{B}^2\rho}\sin^2\frac{\tilde{B}s}{2} \\ R_{26} & = & \frac{1}{\tilde{B}\rho}(a_+\sin\tilde{B}s + a_-\tilde{B}s) \\ R_{36} & = & \frac{B_0}{\tilde{B}^3\rho}(\sin\tilde{B}s - \tilde{B}s) \\ R_{46} & = & -\frac{B_0}{\tilde{B}^2\rho}\sin^2\frac{\tilde{B}s}{2} \\ R_{56} & = & \frac{1}{\tilde{B}^3\rho^2}(\sin\tilde{B}s - \tilde{B}s) \end{array}\right. \tag{11}$$

and $\tilde{B} \equiv \sqrt{B_0^2 + \frac{1}{\rho^2}}$, $a_\pm \equiv \frac{1}{2}(1 \pm \frac{1}{\tilde{B}^2\rho^2})$.

[†]The magnetic field can not satisfy the Maxwell's equation $\nabla \times \vec{B} = 0$ without the factor $\frac{1}{1+x/\rho}$ in $\vec{B}$. It also guarantees the Coulomb's gauge $\nabla \cdot \vec{A} = 0$. However, this factor does not contribute to the linear Hamiltonian.

Since the vector potential components A_x and A_y are nonzero, the canonical momenta P_x and P_y are different from the kinetic momenta p_x and p_y. From Eq.(10) and $p_{x,y} = P_{x,y} - eA_{x,y}/P_0$, one can obtain the transfer matrix for $\{x, p_x, y, p_y, z, \delta\}$ via a similarity transformation:

$$\begin{pmatrix} (\frac{B_0}{\tilde{B}})^2 + \frac{1}{\tilde{B}^2\rho^2}\cos\tilde{B}s & \frac{1}{\tilde{B}}\sin\tilde{B}s & 0 & \frac{2B_0}{\tilde{B}^2}\sin^2\frac{\tilde{B}s}{2} & 0 & -R_{52} \\ -\frac{1}{\tilde{B}\rho^2}\sin\tilde{B}s & \cos\tilde{B}s & 0 & \frac{B_0}{\tilde{B}}\sin\tilde{B}s & 0 & \frac{1}{B_0\rho}R_{24} \\ \frac{B_0}{\tilde{B}^3\rho^2}(\tilde{B}s - \sin\tilde{B}s) & -\frac{2B_0}{\tilde{B}^2}\sin^2\frac{\tilde{B}s}{2} & 1 & \frac{s}{\tilde{B}^2\rho^2} + \frac{B_0^2}{\tilde{B}^3}\sin\tilde{B}s & 0 & R_{54} \\ \frac{2B_0}{\tilde{B}^2\rho^2}\sin^2\frac{\tilde{B}s}{2} & -\frac{B_0}{\tilde{B}}\sin\tilde{B}s & 0 & \frac{1}{\tilde{B}^2\rho^2} + (\frac{B_0}{\tilde{B}})^2\cos\tilde{B}s & 0 & -\rho R_{41} \\ -(\frac{B_0}{\tilde{B}})^2\frac{s}{\rho} - \frac{1}{\tilde{B}^3\rho^3}\sin\tilde{B}s & -\frac{2}{\tilde{B}^2\rho}\sin^2\frac{\tilde{B}s}{2} & 0 & -\frac{B_0}{\tilde{B}^3\rho}(\tilde{B}s - \sin\tilde{B}s) & 1 & \frac{1}{B_0\rho}R_{54} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

4 Solenoid + Quadrupole (co-axis)

The Hamiltonian reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) - \frac{B_0}{2}(xP_y - yP_x) + \frac{B_0^2}{8}(x^2 + y^2) + \frac{1}{2}k(x^2 - y^2) \tag{13}$$

The magnetic field and vector potential used:

$$\vec{B} = \frac{P_0}{e}(ky\hat{x} + kx\hat{y} + B_0\hat{s}), \qquad \vec{A} = -\frac{P_0}{2e}[B_0y\hat{x} - B_0x\hat{y} + k(x^2 - y^2)\hat{s}]. \tag{14}$$

The non-zero elements of the transfer matrix read:

$$\left\{\begin{array}{lclcl} R_{11} & = & R_{22} & = & \frac{1}{2a}(b_+\cos\phi_+ + b_-\cosh\phi_-) \\ R_{12} & = & & = & \frac{1}{2aB_0}\left(\frac{b_+ + 1}{a_+}\sin\phi_+ + \frac{b_- - 1}{a_-}\sinh\phi_-\right) \\ R_{13} & = & -R_{42} & = & \frac{1}{4a}\left(\frac{b_- + 1}{a_+}\sin\phi_+ + \frac{b_+ - 1}{a_-}\sinh\phi_-\right) \\ R_{14} & = & -R_{32} & = & -\frac{1}{aB_0}(\cos\phi_+ - \cosh\phi_-) \\ R_{21} & = & & = & -\frac{B_0}{2a}\left(\frac{a_+b_+}{b_- + 1}\sin\phi_+ + \frac{a_-b_-}{b_+ - 1}\sinh\phi_-\right) \\ R_{23} & = & -R_{41} & = & \frac{B_0}{4a}(\cos\phi_+ - \cosh\phi_-) \\ R_{24} & = & -R_{31} & = & \frac{1}{a}\left(\frac{a_+}{b_- + 1}\sin\phi_+ - \frac{a_-}{b_+ - 1}\sinh\phi_-\right) \\ R_{33} & = & R_{44} & = & \frac{1}{2a}(b_-\cos\phi_+ + b_+\cosh\phi_-) \\ R_{34} & = & & = & -\frac{1}{2aB_0}\left(\frac{b_- - 1}{a_-}\sin\phi_+ - \frac{b_+ + 1}{a_+}\sinh\phi_-\right) \\ R_{43} & = & & = & -\frac{B_0}{2a}\left(\frac{a_-b_-}{b_+ - 1}\sin\phi_+ - \frac{a_+b_+}{b_- + 1}\sinh\phi_-\right) \\ R_{55} & = & R_{66} & = & 1 \end{array}\right. \tag{15}$$

where $a \equiv \sqrt{1 + 4k^2/B_0^4}$, $a_+ \equiv \sqrt{\frac{a+1}{2}}$, $a_- \equiv \text{sgn}[k]\sqrt{\frac{a-1}{2}}$, $b_\pm \equiv a \pm 2k/B_0^2$, $\phi_\pm \equiv a_\pm B_0 s$.

In addition, the transfer matrix in terms of kinetic momenta reads:

$$\left\{\begin{array}{lllll} R_{11} & = & & = & \frac{1}{2a}\left[(b_+ - 1)\cos\phi_+ + (b_- + 1)\cosh\phi_-\right] \\ R_{12} & = & & = & \frac{B_0}{2ak}\left[a_-(b_+ + 1)\sin\phi_+ + a_+(b_- - 1)\sinh\phi_-\right] \\ R_{13} & = & R_{31} & = & \frac{1}{a}(-a_-\sin\phi_+ + a_+\sinh\phi_-) \\ R_{14} & = & -R_{32} & = & -\frac{1}{aB_0}(\cos\phi_+ - \cosh\phi_-) \\ R_{21} & = & & = & -\frac{B_0}{2a}\left[a_-(b_+ + 1)\sin\phi_+ + a_+(b_- - 1)\sinh\phi_-\right] \\ R_{22} & = & & = & \frac{1}{2a}\left[(b_+ + 1)\cos\phi_+ + (b_- - 1)\cosh\phi_-\right] \\ R_{23} & = & R_{41} & = & -\frac{k}{aB_0}(\cos\phi_+ - \cosh\phi_-) \\ R_{24} & = & -R_{42} & = & \frac{1}{a}(a_+\sin\phi_+ + a_-\sinh\phi_-) \\ R_{33} & = & & = & \frac{1}{2a}\left[(b_- - 1)\cos\phi_+ + (b_+ + 1)\cosh\phi_-\right] \\ R_{34} & = & & = & \frac{B_0}{2ak}\left[-a_+(b_- - 1)\sin\phi_+ + a_-(b_+ + 1)\sinh\phi_-\right] \\ R_{43} & = & & = & \frac{B_0}{2a}\left[a_-(b_- + 1)\sin\phi_+ + a_+(b_+ - 1)\sinh\phi_-\right] \\ R_{44} & = & & = & \frac{1}{2a}\left[(b_- + 1)\cos\phi_+ + (b_+ - 1)\cosh\phi_-\right] \\ R_{55} & = & R_{66} & = & 1 \end{array}\right. \tag{16}$$

Due to the axial symmetry of the solenoid field, one can get the Hamiltonian and transfer matrix of the **Solenoid + Skew Quadrupole** system by a 45° rotation.

5 Cavity + Quadrupole

We consider the well-known periodic cavity structure with cylindrical symmetry, whose accelerating field E_z and other non-zero field components read:[3][†]

$$\left\{\begin{array}{l} E_z = \sum_{n=-\infty}^{\infty} -iE_n J_0(k_{rn}r)e^{i(\omega t - k_{zn}z)} \;, \quad E_r = \sum_{n=-\infty}^{\infty} \frac{k_{zn}}{k_{rn}} E_n J_1(k_{rn}r)e^{i(\omega t - k_{zn}z)} \\ B_\theta = \sum_{n=-\infty}^{\infty} \frac{1}{c}\frac{k}{k_{rn}} E_n J_1(k_{rn}r)e^{i(\omega t - k_{zn}z)} \end{array}\right. \tag{17}$$

where E_n assumed to be constant, i.e. no fringe field, d is the period length, n is the space harmonic index, $k_{rn}^2 + k_{zn}^2 = k^2 = (\frac{\omega}{c})^2$, and $k_{zn} \equiv k_z + 2\pi n/d$. The vector potential of the cavity field can be chosen as:

$$A_z = \sum_{n=-\infty}^{\infty} \frac{E_n}{\omega} J_0(k_{rn}r)e^{i(\omega t - k_{zn}z)} \;, \quad A_r = \sum_{n=-\infty}^{\infty} \frac{i}{\omega}\frac{k_{zn}}{k_{rn}} E_n J_1(k_{rn}r)e^{i(\omega t - k_{zn}z)} \tag{18}$$

Since we are interested in linear dynamics here, we can expand the Bessel functions as $J_0(x) = 1 - x^2/4 + \cdots$ and $J_1(x) = x/2 + \cdots$, which yields

$$\vec{A}_{\text{cav}} = -\frac{P_0}{2e}\left\{K_r x\hat{x} + K_r y\hat{y} + \left[-\frac{2e}{P_0}A_z(r=0) + K_c(x^2 + y^2)\right]\hat{s}\right\} \tag{19}$$

[†]For cavities in storage rings, usually only one mode is of concern and commonly used field expression is much simpler, nonetheless, it is covered by the general form.

where $K_c \equiv \frac{2e}{P_0}\Re\left[\sum_n \frac{k_{rn}^2}{4}\frac{E_n}{\omega}e^{i(\omega t - k_{zn}z)}\right]_0$ and $K_r \equiv -\frac{2e}{P_0}\Re\left[\sum_n \frac{k_{zn}}{2}\frac{iE_n}{\omega}e^{i(\omega t - k_{zn}z)}\right]_0$; the subscript 0 means keeping the 0-th order term in $\{z, \delta\}$. We assume K_c and K_r are constant.†

It is not difficult to see that the longitudinal and transverse dynamics are decoupled at linear order. The superposition of quadrupole or solenoid with the cavity will not affect the linear dynamics longitudinally, thus we will compute only the 4×4 transfer matrix of the transverse motion. For this purpose, the Hamiltonian of the cavity+quadrupole system reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) + \frac{K_r}{2}(xP_x + yP_y) + \frac{4K_c + K_r^2}{8}(x^2 + y^2) + \frac{k}{2}(x^2 - y^2) \tag{20}$$

where k is the quadrupole strength.
The vector potential used:

$$\vec{A} = -\frac{P_0}{2e}\left\{K_r x\hat{x} + K_r y\hat{y} + \left[k\left(x^2 - y^2\right) + K_c\left(x^2 + y^2\right)\right]\hat{s}\right\} \tag{21}$$

The transfer matrix reads:

$$\begin{pmatrix} \cos\phi_+ + \frac{K_r}{2k_+}\sin\phi_+ & \frac{1}{k_+}\sin\phi_+ & 0 & 0 \\ -\frac{K_+}{k_+}\sin\phi_+ & \cos\phi_+ - \frac{K_r}{2k_+}\sin\phi_+ & 0 & 0 \\ 0 & 0 & \cosh\phi_- + \frac{K_r}{2k_-}\sinh\phi_- & \frac{1}{k_-}\sinh\phi_- \\ 0 & 0 & \frac{K_-}{k_-}\sinh\phi_- & \cosh\phi_- - \frac{K_r}{2k_-}\sinh\phi_- \end{pmatrix} \tag{22}$$

where $\phi_\pm \equiv k_\pm s$, $k_\pm \equiv \sqrt{k \pm K_c}$ and $K_\pm \equiv k \pm (K_c + K_r^2/4)$.
The transfer matrix for the kinetic momenta $\{x, p_x, y, p_y\}$ simply reads:

$$\begin{pmatrix} \cos k_+s & \frac{1}{k_+}\sin k_+s & 0 & 0 \\ -k_+\sin k_+s & \cos k_+s & 0 & 0 \\ 0 & 0 & \cosh k_-s & \frac{1}{k_-}\sinh k_-s \\ 0 & 0 & k_-\sinh k_-s & \cosh k_-s \end{pmatrix} \tag{23}$$

which is just a quadrupole with horizontal and vertical focusing strength modified by K_c as it should.

Although this transfer matrix is simplectic, damping effect is apparent if we remember that the momenta are normalized by a constant P_0. Usually it is the total momentum of a particle. But for this and next case, due to the existence of accelerating field, the total momentum of a particle is not conserved anymore. The increase of the total final momentum yields the dumping effect. However, care is needed when the energy gain is significant since $\delta \equiv \frac{P-P_0}{P_0}$ may not be a small quantity.

†In order for the $K_{c,r}$ to be constant, the longitudinal motion must be well synchronized such that the phase will not change (this may not be possible for different space harmonics). Otherwise, the Hamiltonian for the transverse dynamics will be time dependent and our treatment does not apply.

6 Cavity + Solenoid

Similar to the last section, the Hamiltonian reads:

$$H = \frac{1}{2}(P_x^2 + P_y^2) + \frac{K_r}{2}(xP_x + yP_y) - \frac{B_0}{2}(xP_y - yP_x) + \frac{1}{2}K(x^2 + y^2) \tag{24}$$

where B_0 is the solenoid strength and $K \equiv K_c + \frac{1}{4}(B_0^2 + K_r^2)$.
The vector potential used:

$$\vec{A} = -\frac{P_0}{2e}\left[(K_r x + B_0 y)\hat{x} + (K_r y - B_0 x)\hat{y} + K_c(x^2 + y^2)\hat{s}\right] \tag{25}$$

The transfer matrix reads:

$$\begin{pmatrix} \xi_+ \cos\phi & \xi\cos\phi & \xi_+ \sin\phi & \xi \sin\phi \\ -K\xi\cos\phi & \xi_- \cos\phi & -K\xi\sin\phi & \xi_- \sin\phi \\ -\xi_+ \sin\phi & -\xi\sin\phi & \xi_+ \cos\phi & \xi\cos\phi \\ K\xi\sin\phi & -\xi_- \sin\phi & -K\xi\cos\phi & \xi_- \cos\phi \end{pmatrix} \tag{26}$$

where $\xi_\pm = \cos a\phi \pm \frac{1}{2}K_r\xi$, $\xi = \frac{2}{aB_0}\sin a\phi$, and $\phi \equiv \frac{1}{2}B_0 s$, $a \equiv \sqrt{1 + 4K_c/B_0^2}$.
The transfer matrix for the kinetic momenta $\{x, p_x, y, p_y\}$ reads:

$$\begin{pmatrix} \cos\phi\cos a\phi + \frac{1}{a}\sin\phi\sin a\phi & \frac{2}{aB_0}\cos\phi\sin a\phi & -R_{31} & -R_{32} \\ -\frac{2K_c}{aB_0}\cos\phi\sin a\phi & cos\phi\cos a\phi - \frac{1}{a}\sin\phi\sin a\phi & -R_{41} & -R_{42} \\ -\sin\phi\cos a\phi + \frac{1}{a}\cos\phi\sin a\phi & -\frac{2}{aB_0}\sin\phi\sin a\phi & R_{11} & R_{12} \\ \frac{2K_c}{aB_0}\sin\phi\sin a\phi & -\sin\phi\cos a\phi - \frac{1}{a}\cos\phi\sin a\phi & R_{21} & R_{22} \end{pmatrix} \tag{27}$$

Note that both Eq.(27) and Eq.(23) are independent of the parameter K_r, i.e. the transverse electric field does not play a role in the transfer matrices. This is due to the cancellation of electric and magnetic forces.

7 Remarks

From these matrices, we see that, except for the dipole+quadrupole and cavity+quadrupole cases, all other cases are sources of linear coupling of transverse degree of freedoms. A quick look on the stability properties of these matrices will be interesting. The dipole+quadrupole system is stable provided that $\tilde{k} > 0$ and $k < 0$. This is the well known stability condition $0 < n < 1$ for a weak focusing synchrotron[4], where $k = -n/\rho^2$. The cavity+quadrupole case is stable if $|k| < K_c$. The cavity+solenoid system is stable provided that $B_0^2 > -4K_c$. The other systems are unstable in general. Note that when combined with dipole, solenoid and quadrupole axes are assumed to curve along the bending orbit. Dipole with straight solenoid or quadrupole are also practical cases but are not discussed here. Also note that the vertical dipole + quadrupole case can be obtained directly from the horizontal dipole + quadrupole case. So is the vertical dipole + skew quadrupole case.

References

[1] C. X. Wang and A. Chao, Notes on Lie algebraic analysis of achromats, (SLAC internal note AP-100)

[2] R. Ruth, Single particle dynamics in circular accelerators, SLAC-PUB-4103

[3] See for example, R. H. Helm and R. Miller, in Linear Accelerators, ed. P. M. Lapostolle and A.L.Septier, 1970

[4] See for example, H. Wiedemann, Particle Accelerator Physics, p.58, Springer-Verlag, 1993