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APPLICATION OF THE MANDELSTAM REPRESENTATION TO PION-NUCLEON SCATTERING

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APPLICATION OF THE MANDELSTAM REPRESENTATION TO PION-NUCLEON SCATTERING[*,†]

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Historically, pion-nucleon scattering was the first problem on which the modern methods for making calculations via analyticity and unitarity achieved significant success. Chew and Low[1] were able to find from the cutoff model a simple effective-range formula which accounted for the shape of the (3,3) resonance in terms of a single parameter, the pion-nucleon coupling constant. Dispersion relations at fixed momentum transfer gave some understanding of the success of the cutoff model. In view of this relative simplicity which the pion-nucleon problem showed it is rather ironic to find that it is quite a complicated problem within the new framework of calculation developed by Chew and Mandelstam.[2] The main results of Chew and Low have been given a more systematic justification,[3,4] but I have no new information of comparable significance to present. This talk will therefore be limited to a presentation of methods and prospects for wringing some theoretical predictions out of the Mandelstam representation for pion-nucleon scattering.

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I shall describe two complementary methods of attack on the problem: first, the partial-wave dispersion relations; and second, a modification of the method of Chew, Goldberger, Low, and Nambu.[5] The former is the more basic approach, but also the more complicated. Let us try to get some feeling for the capabilities of this method.

In the standard notation,[5] the matrix element is proportional to

$$\bar{u}(p_2)\ \{\ \delta_{\beta\alpha}\,[-A^{(+)} + \tfrac{1}{2}\, i\, \gamma\cdot(q_1 + q_2)B^{(+)}]$$

$$+ \tfrac{1}{2}\,[\tau_\beta, \tau_\alpha][-A^{(-)} + \tfrac{1}{2}\, i\, \gamma\cdot(q_1 + q_2)B^{(-)}]\ u(p_1)\ , \tag{1}$$

where p_1 and p_2 are the four-momenta of the incident and outgoing nucleons, q_1 and q_2 are those of the pions, and β and α are the isotopic-spin indices of the pions. The invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$, which are functions of the variables

$$\begin{aligned} s &= (p_1 + q_1)^2 , \\ \bar{s} &= (p_2 - q_1)^2 , \\ t &= (q_2 - q_1)^2 , \end{aligned} \tag{2}$$

are assumed to satisfy the Mandelstam representation.[2]

Consider now the partial-wave amplitude $f_{\ell\pm}^{(\pm)}$; that is, the amplitude with parity $-(-1)^\ell$ and the total angular momentum $j = \ell \pm \frac{1}{2}$. In the physical region, unitarity tells us

$$f_{\ell\pm} = \frac{e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm}}{q} , \tag{3}$$

where q is the magnitude of the momentum in the barycentric system. The relation between $f_{\ell\pm}$, A, and B is

$$f_{\ell\pm}(W) = \frac{1}{8\pi W} \{(E + m) [A_{\ell} + (W - m)B_{\ell}] + (E - m)[-A_{\ell\pm1} + (W + m)B_{\ell\pm1}]\} , \tag{4}$$

where $W = \sqrt{s}$,

$$E = \sqrt{q^2 + m^2} = \frac{W^2 + m^2 - 1}{2 W} , \tag{5}$$

and

$$A_{\ell}(s) = \frac{1}{2} \int_{-1}^{1} d \cos \theta \, P_{\ell}(\cos \theta) \, A(s, \bar{s}, t) . \tag{6}$$

The pion mass has been set equal to unity. The singularities of $A_{\ell}(s)$ can be found in the standard manner[2] from the Mandelstam representation.

In order to exploit the simple unitarity condition, Eq. (3), we wish to write a dispersion relation for $f_{\ell\pm}(W)$, or perhaps for a simple multiple thereof. We shall not discuss the time-consuming question of just exactly what amplitude is most convenient.[3,4] We can avoid extra singularities from the kinematical factors in Eq. (4) if we work in the W plane. One can, of course, use such variables as s or q^2 if he is willing to work on a Riemann surface of two sheets. In the W plane the physical cut for $s \geqslant (m + 1)^2$ becomes two cuts, the physical cut $W \geqslant m + 1$ plus a cut along the negative real axis, $W \leqslant -m - 1$. This cut causes no trouble, because we can use the symmetry noticed by MacDowell,[6]

$$f_{\ell+}(-W) = -f_{(\ell+1)-}(W) , \tag{7}$$

to write an expression of the form of Eq. (3) on this cut. Notice that Eq. (7) relates the two states with the same j.

The rest of the cuts are somewhat frightening. The pole terms, which are identical to the Born approximation, give rise to a pole at $W = -m$ (in the amplitude f_{0+} only), plus a cut along the imaginary axis and short branch cuts in the regions

$$-(m^2 + 2)^{1/2} \leqslant W \leqslant -m + 1/m ,$$
$$m - 1/m \leqslant W \leqslant (m^2 + 2)^{1/2} . \tag{8}$$

The singularities in $\bar{s}$, arising from crossed pion-nucleon scattering, lie along the imaginary axis plus the region $-m + 1 \leqslant W \leqslant m - 1$. Finally, the singularities in t, corresponding to the process $\pi + \pi \rightarrow N + \bar{N}$, lie along the imaginary axis plus a circle of radius $W = (m^2 - 1)^{1/2}$.

The discontinuities across all these cuts have been evaluated in terms of absorptive parts of pion-nucleon scattering and the process $\pi + \pi \rightarrow N + \bar{N}$.[3,4] This process has been evaluated in terms of pion-pion scattering, and some information about it is available from the electromagnetic structure of the nucleon.[7,8,9] This evaluation involves pion-nucleon scattering itself as input information, so that on the most basic level the two processes must be determined simultaneously from a set of coupled integral equations.

In order to express the discontinuities in terms of physical absorptive parts an analytic continuation in momentum transfer is required. The only method yet devised is an expansion in Legendre polynomials, which immediately leads to difficulties. The expansion converges only in a limited region of the W plane near the physical thresholds. Furthermore, if higher partial waves than S waves are important in the absorptive parts, the integrals that occur in the N/D solution[2] fail to exist, forcing the introduction of additional parameters. The situation here is completely analogous to the singularity of the Chew-Mandelstam equations for pion-pion scattering.[10] It seems clear that the techniques available at present will not permit a convincing calculation to be made without the introduction of at least one parameter in addition to the coupling constant. The exact number of parameters required is an open question, to which I shall return later.

At this point those of you who are hearing about this morass for the first time may be wondering how a theory so simple as that of Chew and Low gave meaningful results. The answer suggested by the work of Frautschi and Walecka[4] is that the only significant long-range force (i.e., "near-by" singularity) in the (3,3) state comes from the short branch cut in Eq. (8). This short branch cut, which is approximated by a pole in the static model, controls the width of the resonance, but not its position. Although the result is the same as the cutoff model, it is not insignificant that it has at last been given justification in a more systematic theory.

I have mentioned above the rough inverse correlation which exists between distance of singularities from the physical threshold and range of an equivalent "force." We believe we have reliable information about the long-range forces, but not about the short-range ones. It has already become clear, however, that some of the most interesting aspects of the low-energy pion-nucleon problem, such as S-wave scattering lengths and the position of the (3,3) resonance, depend critically on these short-range forces. The important question is: How many parameters will we need to introduce to describe the short-range forces?

Crossing symmetry will provide powerful help in reducing the number of parameters. We shall treat the short-range forces phenomenologically, perhaps by representing them by poles or by subtraction constants. These phenomenological forces will be constrained, however, to obey certain crossing conditions, which we shall now discuss.

The amplitudes A and B have the following crossing properties:

$$A^{(\pm)}(s, \bar{s}, t) = \pm A^{(\pm)}(\bar{s}, s, t) ,$$
$$B^{(\pm)}(s, \bar{s}, t) = \mp B^{(\pm)}(\bar{s}, s, t) . \tag{9}$$

If we consider a "symmetry point" s_0 , such that $s = \bar{s} = s_0$, $t = t_0$, we have

$$A^{(-)}(s_0, s_0, t_0) = B^{(+)}(s_0, s_0, t_0) = 0 . \tag{10}$$

Conditions can also be written down for derivatives. A convenient choice of s_0 is such that $t_0 = -2k_0^2$, so that $\cos\theta_0 = 0$. Then $s_0 \approx m^2$, or $W_0 \approx m$; and we find, neglecting D waves and higher,

$$f_{0+}^{(-)} = -\frac{1}{4m^2}\left(f_{1-}^{(-)} - f_{1+}^{(-)} \right) , \tag{11}$$

$$f_{0+}^{(+)\prime} = \frac{1}{m}\left(3f_{1+}^{(+)} - f_{0+}^{(+)}\right) , \tag{12}$$

$$f_{1-}^{(+)} - f_{1+}^{(+)} = \frac{1}{4m^2}\, f_{0+}^{(+)} , \tag{13}$$

$$f_{1-}^{(-)\prime} - f_{1+}^{(-)\prime} = -\frac{1}{m}\left(f_{1-}^{(-)} - f_{1+}^{(-)}\right) , \tag{14}$$

$$2f_{1+}^{(-)} + f_{1-}^{(-)} = -\frac{1}{2m}\, f_{0+}^{(-)\prime} , \tag{15}$$

$$2f_{1+}^{(+)\prime} + f_{1-}^{(+)\prime} = -\frac{1}{m}\left(5f_{1+}^{(+)} + f_{1-}^{(+)}\right) , \tag{16}$$

where terms of relative order $1/m^2$ have been neglected. All quantities are evaluated at $W = W_0$, and the prime means d/dW. Higher-derivative conditions also exist, but these are less interesting because they should be dominated by the long-range forces. The four P wave conditions above agree in the static limit with the cutoff model.

The pole terms, plus the t spectrum, should satisfy crossing symmetry separately. Therefore, we can subtract them from the amplitude before applying the crossing conditions. We must, at any rate, subtract the pole terms in order to get rid of the short branch cut on which W_0 lies.

At this point we can make a very crude and rather pessimistic guess at the number of arbitrary parameters that will remain in the theory after the crossing conditions are used. Suppose we introduce as parameters the values of the P waves at W_0 (for the (3,3) state this parameter is essentially the position of the resonance), and the values and derivatives of the S waves. We expect that at low energies the higher partial waves will be less sensitive to the unknown long-range forces. That adds up to eight parameters, whereas we have six conditions. Unfortunately two of these conditions, Eqs. (14) and (16), involve the P waves only as $1/m$ corrections, and cannot reliably be used to determine them. By this method we guess that four parameters will be necessary in addition to the coupling constant and knowledge of the pion-pion phase shifts if our theory is to agree with experiment within the usual accuracy expected of dispersion-relation calculations. There is, of course, a considerable body of experimental information for the theory to fit. It is easy to overlook, in the shadow of the dominant (3,3) resonance, how little theoretical knowledge is presently available about pion-nucleon scattering.

Let us now turn to a simpler, more phenomenological approach, in which we shall derive correction terms to the CGLN equations. These authors begin with the fixed momentum-transfer dispersion relations, such as

$$B^{(\pm)}(s, \bar{s}, t) = \frac{1}{\pi} \int_{m^2}^{\infty} ds' \, \mathrm{Im}\, B^{(\pm)}(s', \Sigma - s' - t, t) \left[\frac{1}{s' - s} \mp \frac{1}{s' - \bar{s}} \right], \tag{17}$$

where $\Sigma \equiv 2m^2 + 2$, and where the pole terms are to be understood as a delta-function contribution to Im B. In CGLN a polynomial expansion of the continuum part of Im B was made, and only the (3,3) term in the expansion was kept. A puzzling aspect of the method was that the pion-pion interaction never appeared explicitly. The two most obvious places where it was hidden are in possible subtraction terms in Eq. (17) and in corrections to the polynomial approximation. We shall exhibit a method, based on a suggestion by Mandelstam,[2] for the approximate evaluation of these pion-pion corrections.

Suppose we make a subtraction in Eq. (17) to improve the convergence. Notice that the leading term as $s' \to \infty$ is a function of t only; i.e., it is related to the coefficient of $P_0(\cos\theta_3)$ in an expansion of $B(s, \bar{s}, t)$ in Legendre polynomials in the scattering angle of the process $\pi + \pi \to N + \bar{N}$. In detail, one finds, considering--for example--$B^{(-)}$,

$$B^{(-)}(s, \bar{s}, t) = \frac{12\pi}{\sqrt{2}} f_-^{\,1}(t) + \frac{1}{\pi} \int_{m^2}^{\infty} ds' \, \mathrm{Im}\, B^{(-)}(s', \Sigma - s' - t, t) \times$$

$$\left\{ \frac{1}{s' - s} + \frac{1}{s' - \bar{s}} - \int_{-1}^{1} d\cos\theta_3 [1 - P_2(\cos\theta_3)] \times \right.$$

$$\left. \left[\frac{1}{s' - s(\cos\theta_3)} \quad \frac{1}{s' - \bar{s}(\cos\theta_3)} \right] \right\}, \tag{18}$$

which can easily be seen to be more convergent. The notation $f_{\pm}^{J}(t)$, which is taken from Ref. 7, refers to the partial-wave amplitude for the process $\pi + \pi \rightarrow N + \bar{N}$ with given J and helicity. One could effect still better convergence by subtracting the $J = 3$ amplitude, but we shall assume that only $J = 0,1$ states are important. The amplitude $A^{(-)}$ has a similar form, involving both f_{+}^{1} and f_{-}^{1}. The $A^{(+)}$ amplitude involves f_{+}^{0}, which is related to the pion-pion S-wave phase shift. For $B^{(+)}$, which involves only $J = 2$ and higher, we assume that the unsubtracted CGLN representation is adequate.

We now make the usual Legendre polynomial expansion of Im B in Eq. (18), but with somewhat more confidence than usual because of the improved convergence of the integral. If we now assume, following CGLN, that the (3,3) resonance dominates the integral, we can calculate $B^{(-)}$, and, in a similar manner, the other three amplitudes. For $f_{\pm}^{1}(t)$ we can use the improved calculation of these quantities from the nucleon structure now being carried out by Ball and Wong.[9] We must also have some knowledge or make some assumption about the S-wave pion-pion phase shift.

The number of parameters (in addition to the coupling constant) in this approach is roughly four. The three amplitudes $f_{\pm}^{1}$, f_{+}^{0} cannot in general be calculated reliably without using the normalization procedure of Ball and Wong. They calculate the normalization constants in terms of pion-nucleon scattering, so we must regard these constants as parameters. One parameter is required for each P wave, and two for the S wave. In addition we have assumed the dominance of the (3,3) resonance.

A modification of this procedure can be obtained by manipulation of Eq. (18). Suppose we write a partial-wave dispersion relation for $f_{-}^{1}(t)$.

We must write a subtracted dispersion relation if the integral over the left cut is to be convergent when we keep P waves in the polynomial expansion on this cut. If it were necessary to keep ℓ waves, we would need ℓ subtractions. Now it can easily be seen that the integral over the left cut cancels the last term in Eq. (18) (the term integrated over $\cos\theta_3$) except for a constant, provided Im B is expanded in Legendre polynomials up to and including P waves.[11] One obtains

$$B^{(-)}(s, \bar{s}, t) = \text{CGLN} + C_B^{(-)} + \frac{t}{\pi} \int_4^\infty dt' \frac{b^{(-)}(t')}{t'(t' - t)} , \tag{19}$$

$$B^{(+)}(s, \bar{s}, t) = \text{CGLN} , \tag{20}$$

$$A^{(-)}(s, \bar{s}, t) = \text{CGLN} + C_A^{(-)}(s - \bar{s}) + \frac{(s - \bar{s})t}{\pi} \int_4^\infty dt' \frac{a^{(-)}(t')}{t'(t' - t)} , \tag{21}$$

$$A^{(+)}(s, \bar{s}, t) = \text{CGLN} + C_A^{(+)} + tC_A^{(+)\prime} + \frac{t^2}{\pi} \int_4^\infty dt' \frac{a^{(+)}(t')}{t'^2(t' - t)} , \tag{22}$$

where CGLN refers to the unsubtracted, fixed momentum-transfer dispersion relations with absorptive parts expanded in S and P waves, and where $b^{(-)}$ and $a^{(-)}$ are linear combinations of $f_\pm^{\,1}$ and $a^{(+)}$ is a multiple of $f_+^{\,0}$. These equations are of the one-dimensional form derived as an approximation to the Mandelstam representation by Cini and Fubini,[12] and applied to the pion-nucleon problem by Bowcock, Cottingham, and Lurie.[13] They differ slightly in that these authors do not have the term in $C_A^{(-)}$, which we seem to find a necessary consequence of the importance of P waves in pion-nucleon scattering, and in that we have explicitly included the pion-pion S wave.

An apparent difference is the subtracted form of the t integrals, but this is merely a redefinition of the constants $C_{A,B}$.

A further modification of these equations can be made by relating the t integrals in the (-) amplitudes to isotopic-vector nucleon form factors. It is clear that some such relation must exist, because written schematically the absorptive part of the two-pion contribution to the nucleon structure is of the form $\langle N\bar{N} \mid \pi\pi \rangle \langle \pi\pi \mid \gamma \rangle$, whereas the absorptive part of the two-pion contribution to pion-nucleon scattering is of the form $\langle N\bar{N} \mid \pi\pi \rangle \langle \pi\pi \mid \pi\pi \rangle$, with all amplitudes in the J = 1, I = 1 state. In the notation of Chew and Mandelstam,[2] $\langle \pi\pi \mid \pi\pi \rangle \propto N/D$ whereas $\langle \pi\pi \mid \gamma \rangle \propto 1/D$.[8] The only difference is the pion-pion numerator function. If we approximate this function by a series of poles (not an essential approximation),

$$N(t) = \Sigma \frac{a_i}{t + t_i} , \tag{23}$$

we find, by algebraic manipulations,

$$\frac{t}{\pi} \int_4^\infty dt' \frac{b^{(-)}(t')}{t'(t' - t)} = \sum_i \frac{24\pi a_i}{e} \left[\frac{G_T^V(-t_i) - G_T^V(0)}{t_i} + \frac{G_T^V(t) - G_T^V(-t_i)}{t + t_i} \right] , \tag{24}$$

$$\frac{t}{\pi} \int_4^\infty dt' \frac{a^{(-)}(t')}{t'(t' - t)} = \sum_i \frac{12\pi a_i}{e} \left[\frac{G_2^V(-t_i) - G_2^V(0)}{t_i} + \frac{G_2^V(t) - G_2^V(-t_i)}{t + t_i} \right] , \tag{25}$$

where $G_T^V(t)$ is 2m times the isovector total magnetic moment form factor,

$$G_T^V(t) = G_1^V(t) + 2m\, G_2^V(t) . \tag{26}$$

The notation for the form factors is that of Ref. 8. The numbers a_i can also be inferred from the nucleon structure.[8,9] With the one-pole formula used in Ref. 8, t_i is large and

$$\frac{a_i}{t + t_i} \approx \frac{a_i}{t_i} = \frac{\Gamma}{\nu_r} \approx \frac{1}{4} . \tag{27}$$

We are now in a position to make a fit to all the low-energy pion-nucleon scattering data, plus the high angular momenta at higher energies, in terms of the six parameters f^2, $C_A^{(\pm)}$, $C_B^{(-)}$, $C_A^{(+)\prime}$ and the position of the (3,3) resonance, plus some assumption about the pion-pion S wave. Although one can make the comparison with experiment directly in Eq. (18), which is clearer from the point of view of subtractions and number of parameters necessary, equations such as Eqs. (19)-(22) of the one-dimensional Cini-Fubini form which explicitly exhibit the pion-pion term should be quite useful.

It is clear that the approach we have just finished discussing, the modified CGLN approach, is much farther from being a dynamical theory than is the partial-wave method. In the former we must use our knowledge, which at present comes only from experiment, of which phase shifts are large enough that we should put them into the absorptive part in Eq. (18). It is the task of the partial-wave method to supply this knowledge, and to calculate the large phase shifts. Then Eq. (18), or Eqs. (19)-(22), are convenient formulas for calculating the small phase shifts.

FOOTNOTES

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9. James S. Ball and D. Y. Wong (University of California, La Jolla), private communication.
10. G. F. Chew and S. Mandelstam, "Theory of the Low-Energy Pion-Pion Interaction, Part II," Lawrence Radiation Laboratory Report UCRL-9126, March 24, 1960.
11. A point of uncertainty in this derivation which must be examined further is how the derivation must be modified if subtractions are necessary in the fixed momentum-transfer dispersion relations. It is probably necessary to introduce additional constants in this case in the following equations, whereas Eq. (18) will remain valid.
12. M. Cini and S. Fubini, Annals of Physics 3, 352 (1960).
13. Bowcock, Cottingham, and Lurie, Nuovo cimento 16, 918 (1960), and CERN Preprint (1960).