OAK RIDGE NATIONAL LABORATORY
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UNION CARBIDE NUCLEAR COMPANY
Division of Union Carbide Corporation

Post Office Box X
Oak Ridge, Tennessee

ORNL
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FROM: C. D. Zerby, R. B. Curtis, and H. W. Bertini

Abstract

Several equations are derived which are useful in high-energy nuclear cascade calculations. Special attention is given to the motion of the nucleons in the nucleus which gives rise to the relativistic Doppler problem when a high-energy particle is incident on the nucleus. The effects of the exclusion principle are included in the derived equations.

Introduction

In nuclear cascade calculations the nucleus is often treated as a zero-temperature Fermi gas where the nucleons are free to move in any direction within the bounds of the nucleus. An incident particle passes through the sea of moving nucleons and interacts with them, giving rise to the Doppler problem. If the incident particle has high kinetic energy, then the problem becomes additionally complicated because relativistic mechanics must be used.

The purpose of this paper is to present the derivation of several relations which are useful in carrying out high-energy nuclear cascade calculations by Monte Carlo methods. For convenience, the units are selected so that $c = 1$.

Expression for the Cross Section

The cross section for a particle traveling through the sea of moving nucleons in the nucleus is most conveniently expressed in terms of the cross section in the frame of reference in which the struck particle is at rest. This is done because the experimental particle-particle cross sections are usually given for the case where the target is at rest. Thus, we must consider the effect on the cross section as one transforms from the rest frame of the struck particle, the primed frame, to the laboratory or unprimed frame of reference.

To simplify this discussion, we will first neglect the exclusion principle and consider it later by making appropriate changes in the equations that are derived.

The following definitions are useful in this discussion:

E_i = total energy of the incident particle,

$\vec{p}_i$ = momentum of the incident particle,

m = mass of the incident particle,

E = energy of a nucleon,

$\vec{p}$ = momentum of a nucleon,

M = mass of a nucleon,

ρ = density of incident particles,

$\vec{j} = \rho \frac{\vec{p}_i}{E_i}$ = incident current,

$\left(N(\vec{p})d\vec{p}\right)$ = nucleons per unit volume in $d\vec{p}$,

E_k = kinetic energy of the incident particle,

$\sigma^R(E_k')$ = microscopic cross section in the rest frame of the struck particle,

$\vec{v} = \frac{\vec{p}}{E}$ = velocity of the primed frame relative to the unprimed frame,

R = reactions per unit volume per unit time.

With the use of the above definitions, the expression for the reaction rate per unit volume in the primed system can be written as

$$R' = |\vec{j}'| \, \sigma^R(E_k') \left(N(\vec{p})d\vec{p}\right)' , \tag{1}$$

and it will be noted that it is also the expression for the reaction rate per unit volume in the unprimed frame since $\Delta x \Delta y \Delta z \Delta t$ is invariant to Lorentz transformations. The quantity $\left(N(\vec{p})d\vec{p} \right)'$ can be transformed to the unprimed frame by noting that $\Delta x' \Delta y' \Delta z' = \Delta x \Delta y \Delta z (1 - v^2)^{-1/2}$ so that the transformation yields $\left(N(\vec{p})d\vec{p} \right)' = \left(N(\vec{p})d\vec{p} \right)(1 - v^2)^{1/2}$. The remaining quantity to be transformed is the incident current which transforms like the spatial part of a four-vector. Thus, we have

$$\vec{j}' = \vec{j} + \frac{(\vec{j}\cdot\vec{v})\vec{v}}{v^2}\left(\frac{1}{\sqrt{1 - v^2}} - 1 \right) - \frac{\vec{v}\,\rho}{\sqrt{1 - v^2}} .$$

The macroscopic cross section, $d\sigma$, for the incident particle is obtained by dividing the reaction rate per unit volume by the incident current and using the relation $(1 - v^2)^{1/2} = M/E$ to get

$$d\sigma = \frac{|\vec{j}'|}{|\vec{j}|} \sigma^R(E_k') \frac{M}{E} N(\vec{p})\, d\vec{p} , \qquad (2)$$

where

$$\frac{|\vec{j}'|}{|\vec{j}|} = \frac{\left| \vec{p}_i + \frac{(\vec{p}_i\cdot\vec{p})\vec{p}}{p^2}\left(\frac{E}{M} - 1 \right) - \frac{\vec{p}E_i}{M} \right|}{|\vec{p}_i|} .$$

The kinetic energy, E_k', which is needed to determine the cross section, is obtained from the transformation

$$E_k' = E_i' - m = \frac{E_i - \vec{p}_i\cdot\vec{p}}{\sqrt{1 - v^2}} - m$$

$$= \frac{E_i E - \vec{p}_i\cdot\vec{p}}{M} - m .$$

Introduction of the Exclusion Principle

In Eq. 2 the cross section $\sigma^R(E'_k)$ is the total microscopic cross section, and if more than one type interaction is possible, then

$$\sigma^R(E'_k) = \sum_j \sigma_j^R(E'_k) \quad .$$

In the case where one of the interactions, the ℓ^{th} one say, results in a nucleon being ejected, then the energy of the nucleon must be above the Fermi energy, E_f, to have an acceptable interaction. This means that the cross section $\sigma_\ell^R(E'_k)$ is smaller than the measured cross section, $\sigma_\ell^{RM}(E'_k)$, where the exclusion principle is not operating.

Letting ϵ_n be the final energy of the nucleon resulting from the ℓ^{th} process, we can then write

$$\sigma_\ell^R(E'_k) = \int_{E_f}^{\infty} \left(\frac{d\sigma_\ell^R(E'_k)}{d\epsilon_n} \right) d\epsilon_n \tag{3}$$

and

$$\sigma_\ell^{RM}(E'_k) = \int_{0}^{\infty} \left(\frac{d\sigma_\ell^R(E'_k)}{d\epsilon_n} \right) d\epsilon_n \quad . \tag{4}$$

However, the integral given in Eq. 3 is usually difficult to find; therefore, we will introduce the cross section into Eq. 2 in its integral form so that

$$d\sigma = \frac{|\vec{J}'|}{|\vec{J}|} \frac{M}{E} \cdot \left\{ \sum_{j=1}^{k} \int_{E_f}^{\infty} \left(\frac{d\sigma_j^R(E'_k)}{d\epsilon_n} \right) d\epsilon_n + \sum_{j=k+1}^{m} \sigma_j^R(E'_k) \right\} N(\vec{p})d\vec{p}. \tag{5}$$

It was assumed in Eq. 5 that the first k of the m possible interactions had a nucleon present in the final state and the remainder are not restricted by the exclusion principle. For $j > k$, we have

$$\sigma_j^R(E_k') \equiv \sigma_j^{RM}(E_k') \quad .$$

The total macroscopic cross section is, of course,

$$\sigma = \sum_{j=1}^{m} \sigma_j \equiv \int \frac{|\vec{j}'|}{|\vec{j}\,|} \; \frac{M}{E} \sum_{j=1}^{m} \sigma_j^R(E_k')N(\vec{p}) \; d\vec{p} \quad . \tag{6}$$

The Monte Carlo Procedure

In this section the methods of randomly selecting the distance between interactions, the momentum of the struck nucleon, and the type of interaction are described. These methods are based in part on some Monte Carlo selection techniques described by Butcher and Messel.[1]

In order to simplify the discussion, Eq. 5 will be written in the standard form for application of a rejection technique:

$$d\sigma = \sum_{j=1}^{k} C_j Q_j(\vec{p})f(\vec{p})d\vec{p} \int_0^{\infty} V_j(\epsilon_n)g_j(\epsilon_n)d\epsilon_n + \sum_{j=k+1}^{m} C_j Q_j(\vec{p})f(\vec{p})d\vec{p} \quad , \tag{7}$$

where

$$f(\vec{p}) = \frac{N(\vec{p})d\vec{p}}{N_o}$$

$$N_o = \int N(\vec{p})d\vec{p} \quad ,$$

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[1] J. C. Butcher and H. Messel, Nuclear Physics 20, 45 (1960).

$$V_j(\epsilon_n) = 1 \text{ for } E_f \leq \epsilon_n$$

$$= 0 \text{ for } 0 \leq \epsilon_n < E_f$$

$$g(\epsilon_n) = \frac{\dfrac{d\sigma_j^R(E_k')}{d\epsilon_n}}{\sigma_j^{RM}(E_k')} \qquad \text{for } \epsilon_n \geq 0,$$

$$Q_j(\vec{p}) = \frac{\dfrac{|\vec{j}'|}{|\vec{j}|} \dfrac{M}{E} \sigma_j^{RM}(E_k')N_o}{C_j} \quad ,$$

and C_j is a constant such that

$$C_j \geq \text{maximum of } \frac{|\vec{j}'|}{|\vec{j}|} \frac{M}{E} \sigma_j^{RM}(E_k')N_o \quad .$$

The selection procedure for the type interaction and the momentum of the struck particle are based on Eq. 7 and are given below.* In these procedures, as a matter of notation, we use

$$S = \sum_{j=1}^{m} C_j \quad .$$

1. The type interaction is randomly selected from the m possibilities by selecting a random number R_1 and forming the sequence

$$y_1 = R_1 - \frac{C_1}{S}$$

*If one contribution to the cross section $\sigma_p(\vec{p}_i)$ is independent of the momentum of the nucleons in the nucleus it can be included in the set of interactions k+1 through m without loss of generality by letting $C_p = \sigma_p(\vec{p}_i)$ and $Q_p(\vec{p}) = 1$.

$$y_2 = y_1 - \frac{C_2}{S}$$

$$y_3 = y_2 - \frac{C_3}{S}$$

which is terminated when $y_\ell \leq 0$. The type interaction is then taken as ℓ.

2. The momentum $\vec{P}$ of the struck particle is randomly selected from the probability density function $f(\vec{p})$.
3. A random number R_2 is compared with $Q_\ell(\vec{P})$ and if $R_2 > Q_\ell(\vec{P})$, then ℓ and $\vec{P}$ are rejected and the process is started over. If $R_2 \leq Q_\ell(\vec{P})$ and $\ell > k$, then the interaction of type ℓ is allowed to occur. If $R_2 \leq Q_\ell(\vec{P})$ and $\ell \leq k$, then the process is continued in step 4.
4. The energy of the nucleon, E_n, is randomly selected from the probability density function $g_\ell(\epsilon_n)$ (see the following section) and compared with E_f. If $E_n < E_f$, then ℓ, $\vec{P}$, and E_n are rejected and the procedure is started over. If $E_n \geq E_f$, the complete process is accepted.

For $j > k$ in the above procedure, the probability of selecting j is C_j/S and the probability of selecting $\vec{P}$ and having it accepted contingent on having selected j is

$$\frac{\sigma_j}{C_j} = \int Q_j(\vec{p}) f(\vec{p}) d\vec{p} \quad ,$$

where σ_j is defined in Eq. 6. Thus, the probability of obtaining both j and $\vec{P}$ and having them both accepted is σ_j/S.

For $j \leq k$ the probability of selecting j is again C_j/S and, given $\vec{P}$, the contingent probability of accepting E_n is

$$\frac{\sigma_j^R(E_k')}{\sigma_j^{RM}(E_k')} = \int_0^\infty V_j(\epsilon_n) g_j(\epsilon_n) d\epsilon_n \quad .$$

Thus, given j, the probability of selecting $\vec{P}$ and E_n and having them both accepted is

$$\frac{\sigma_j}{C_j} = \int Q_j(\vec{p}) \frac{\sigma^R(E_k')}{\sigma^{RM}(E_k')} f(\vec{p}) d\vec{p} \quad .$$

Hence the probability of selecting j, $\vec{P}$ and E_n and having them all accepted is σ_j/S.

From the above discussion it is clear that the probability of getting an accepted interaction on the first attempt is σ/S where $\sigma = \sum_{j=1}^{m} \sigma_j$.

The manner of selecting the distance between collisions can now be described. Assuming a particle has just had a collision, a distance X is randomly selected from the exponential distribution $h(x) = Se^{-Sx}$, where the true cross section σ has been replaced by S. Then steps 1 through 4 are followed repeatedly until an accepted set of variables is obtained, as described above, with the provision that if a rejection occurs another distance X is selected from h(x) after each rejection. The sum of all X's is then the distance y traveled to the point of the next interaction.

To demonstrate that the above procedure is correct for selecting the distance, we note that if there was an acceptance on the n^{th} attempt, then y would be distributed as

$$\frac{S^n \, y^{n-1} \, e^{-Sy}}{(n-1)!}$$

But the probability of acceptance on the n^{th} attempt is

$$\frac{\sigma}{S}\left(1 - \frac{\sigma}{S}\right)^{n-1} \quad .$$

Thus the distance traveled will be distributed as

$$\sum_{n=1}^{\infty} \frac{S^n\, y^{n-1}\, e^{-Sy}}{(n-1)!} \; \frac{\sigma}{S}\left(1 - \frac{\sigma}{S}\right)^{n-1} = \sigma e^{-\sigma y} \quad .$$

Kinematics of a Collision

In the previous section one of the selection techniques required that the energy of the nucleon be randomly sampled from the distribution $g_j(\epsilon_n)$. The purpose of this section is to describe this operation more fully in the special case where a particle collides with a nucleon and yields a nucleon and another particle in the final state. The description will be generalized to the case where a particle of mass m_1, with momentum $\vec{p}_1$ and energy E_1, collides with a second particle described by the set $(m_2,\vec{p}_2,E_2)$ and yields two particles described by the sets $(m_3,\vec{p}_3,E_3)$ and $(m_4,\vec{p}_4,E_4)$.

We assume that the angular distribution function for ejection of particle 3 is given in the center-of-mass (C.M.) frame as a function of the C.M. energy M. Thus, the procedure will be to transform to the C.M. and obtain the expression for the C.M. energy and assume the angles of ejection are randomly selected from the distribution $\sigma_{m_1 m_2 \to m_3 m_4}(M)$. The description will be complete when expressions have been derived for $\vec{p}_3, E_3, \vec{p}_4$, and E_4.

If we take the C.M. frame as the primed frame and the laboratory (L) frame as the unprimed frame, the transformation to the C.M. frame is given by

$$\vec{p}_i' = \vec{p}_i + \frac{(\vec{p}_i \cdot \vec{v})\vec{v}}{v^2}\left(\frac{1}{\sqrt{1 - v^2}} - 1\right) - \frac{\vec{v}E_i}{\sqrt{1 - v^2}}, \quad i = 1,2,3,4 \tag{8}$$

and

$$E_i' = \frac{E_i - \vec{p}_i \cdot \vec{v}}{\sqrt{1 - v^2}}, \quad i = 1,2,3,4, \tag{9}$$

where $\vec{v}$ is the velocity of the C.M. relative to the L frame.

By letting $\vec{P} = \vec{p}_1 + \vec{p}_2$, $E = E_1 + E_2$, and $M = E_1' + E_2'$ and noting that $\vec{p}_1' + \vec{p}_2' = 0$, we can then use Eqs. 8 and 9 to get

$$\vec{P} + \frac{(\vec{P} \cdot \vec{v})\vec{v}}{v^2}\left(\frac{1}{\sqrt{1 - v^2}} - 1\right) - \frac{\vec{v}E}{\sqrt{1 - v^2}} = 0 \tag{10}$$

and

$$M = \frac{E - \vec{P} \cdot \vec{v}}{\sqrt{1 - v^2}} \tag{11}$$

Vector multiplication of Eq. 10 with $\vec{v}$ yields an expression which reduces to

$$\vec{P} \cdot \vec{v} = v^2 E \quad . \tag{12}$$

Substitution of Eq. 12 into Eq. 10 finally gives the expression for the relative velocity as

$$\vec{v} = \frac{\vec{P}}{E} = \frac{\vec{p}_1 + \vec{p}_2}{E} \quad . \tag{13}$$

From Eq. 11, again with the use of Eq. 12, we get

$$\frac{M}{E} = (1 - v^2)^{1/2} \equiv \gamma \quad . \tag{14}$$

And now we obtain the expression of the C.M. energy by substituting Eq. 13 into Eq. 14 to get

$$M = (E^2 - P^2)^{1/2} = \left[(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \right]^{1/2}$$

$$= \left[2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2) + m_1^2 + m_2^2 \right]^{1/2} \quad . \tag{15}$$

In the C. M. frame the conservation equations require

$$\vec{p}\,'_1 + \vec{p}\,'_2 = \vec{p}\,'_3 + \vec{p}\,'_4 = 0$$

and

$$E'_1 + E'_2 = E'_3 + E'_4 = M.$$

Thus

$$E'^2_2 = p'^2_2 + m^2_2 = M^2 + p'^2_1 + m^2_1 - 2E_1 M \quad ,$$

which, when it is noted that $p'^2_1 = p'^2_2$, reduces to

$$E'_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \quad , \tag{16}$$

so that

$$E'_2 = M - E'_1 \tag{17}$$

In a similar manner we get

$$E'_3 = \frac{M^2 + m_3^2 - m_4^2}{2M} \tag{18}$$

and

$$E'_4 = M - E'_3 \tag{19}$$

It is advantageous to define a coordinate system to facilitate the transformation back to the L frame after the angles of ejection of particle 3 are selected. The coordinate system is defined by

$$\vec{p}'_1 = p'_1\hat{z}$$

and

$$\vec{v} = \alpha\hat{x} + \beta\hat{z}$$

Thus, we have $\vec{p}'_1 \times \vec{v} = \alpha p'_1\hat{y}$, $\vec{p}'_1 \cdot \vec{v} = p'_1\beta$ and $v^2 = \alpha^2 + \beta^2$, and the coordinate unit vectors are easily shown to be

$$\hat{x} = \frac{\vec{v}}{\alpha} - \frac{\beta\,\vec{p}'_1}{\alpha\,p'_1} \tag{20}$$

$$\hat{y} = \frac{\vec{p}'_1 \times \vec{v}}{\alpha p'_1} \tag{21}$$

$$\hat{z} = \frac{\vec{p}'_1}{p'_1} \tag{22}$$

The various terms in Eqs. 20, 21, and 22 must be expressed in terms of known quantities to be useful in the subsequent development. The evaluation of $\vec{p}_1' \times \vec{v}$ is relatively simple if we take the cross product of $\vec{v}$ with $\vec{p}_1'$ as given in Eq. 8 which shows that $\vec{p}_1' \times \vec{v} = \vec{p}_1 \times \vec{v}$. The final expression for this quantity is obtained with the use of Eq. 13 and is given by

$$\vec{p}_1' \times \vec{v} = \vec{p}_1 \times \vec{v} = \frac{\vec{p}_1 \times \vec{p}_2}{E} \tag{23}$$

An acceptable form of $\vec{p}_1'$ can be derived by rearranging Eq. 9 to obtain $\vec{p}_1 \cdot \vec{v} = E_1 - \gamma E_1'$ and substituting it into Eq. 8 along with Eq. 13 to get

$$\vec{p}_1' = \vec{p}_1 + \frac{1}{\gamma(1+\gamma)E}(E_1 - \gamma E_1')(\vec{p}_1 + \vec{p}_2) - \frac{1}{\gamma E}(\vec{p}_1 + \vec{p}_2)E_1 ,$$

which reduces to

$$\vec{p}_1' = \left(\frac{E_2 + E_2'}{E + M}\right)\vec{p}_1 + \left(\frac{E_1 + E_1'}{E + M}\right)\vec{p}_2 \tag{24}$$

when $\gamma = M/E$ is used.

The expression for $\beta = \vec{p}_1' \cdot \vec{v}/p_1'$ is best derived by starting with the inverse transformation

$$E_1 = \frac{E_1' + \vec{p}_1' \cdot \vec{v}}{\gamma}$$

which can be rearranged to give $\vec{p}_1' \cdot \vec{v} = \gamma E_1 - E_1'$ so that

$$\beta = \frac{\vec{p}_1' \cdot \vec{v}}{p_1'} = \frac{1}{p_1'}(\gamma E_1 - E_1') \quad . \tag{25}$$

For completeness, the remaining quantities that need to be expressed in terms of known quantities are listed below

$$\alpha = (v^2 - \beta^2)^{1/2} \tag{26}$$

$$p_i' = (E_i'^2 - m_i^2)^{1/2} , \qquad i = 1,2,3,4 \quad . \tag{27}$$

Now we assume that θ and ϕ are the polar and azimuthal angles of ejection of particle 3 which have been randomly selected from the distribution $\sigma_{m_1 m_2 \to m_3 m_4}(M)$ and devote the remainder of this section to obtaining an expression for $\vec{p}_3, E_3, \vec{p}_4$, and E_4.

After selecting the angles of particle 3 in the C.M. frame, the expression for $\vec{p}_3'$ becomes

$$\vec{p}_3' = p_3'(\sin\theta \cos\phi \ \hat{x} + \sin\theta \sin\phi \ \hat{y} + \cos\theta \ \hat{z})$$

$$= p_3' \left[(\cos\theta - \frac{\beta}{\alpha} \sin\theta \cos\phi) \frac{\vec{p}_1'}{p_1'} + \frac{1}{\alpha} \sin\theta \cos\phi \ \vec{v} + \frac{1}{\alpha p_1'} \sin\theta \sin\phi \ (\vec{p}_1' \times \vec{v}) \right]. \tag{28}$$

By taking the vector product of $\vec{v}$ with $\vec{p}_3'$, as given in Eq. 28, and using Eqs. 25 and 26, we get

$$\vec{p}_3' \cdot \vec{v} = p_3' \left[(\cos\theta - \frac{\beta}{\alpha} \sin\theta \cos\phi) \frac{\vec{p}_1' \cdot \vec{v}}{p_1'} + \frac{v^2}{\alpha} \sin\theta \cos\phi \right]$$

$$= p_3' \left[\beta \cos\theta + \frac{(v^2 - \beta^2)}{\alpha} \sin\theta \cos\phi \right]$$

$$= p_3' \left[\beta \cos\theta + \alpha \sin\theta \cos\phi \right] , \tag{29}$$

which can be substituted into the inverse transformation

$$E_3 = \frac{E_3' + \vec{p}_3' \cdot \vec{v}}{\gamma} ,$$

to get

$$E_3 = \frac{E'_3 + p'_3(\beta \cos\theta + \alpha \sin\theta \cos\phi)}{\gamma} \tag{30}$$

Finally we have

$$E_4 = E - E_3 \quad . \tag{31}$$

Equations 14, 28, and 29 can also be substituted into the inverse transformation

$$\vec{p}_3 = \vec{p}'_3 + \frac{(\vec{p}'_3 \cdot \vec{v})\vec{v}}{v^2} \left(\frac{1}{\sqrt{1 - v^2}} - 1 \right) + \frac{E'_3\vec{v}}{\sqrt{1 - v^2}}$$

to get

$$\vec{p}_3 = p'_3 \left\{ \left(\cos\theta - \frac{\beta}{\alpha} \sin\theta \cos\phi\right) \frac{\vec{p}'_1}{p'_1} + \frac{1}{\alpha p'_1} \sin\theta \sin\phi \, (\vec{p}'_1 \times \vec{v}) + \left[\frac{1}{\alpha} \sin\theta \cos\phi + \frac{1}{\gamma(1 + \gamma)} (\beta \cos\theta + \alpha \sin\theta \cos\phi) \right] \vec{v} \right\} + \frac{E'_3\vec{v}}{\gamma} , \tag{32}$$

in which all terms have been defined above in terms of known quantities. The remaining expression for $\vec{p}_4$ is simply given by

$$\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3 \quad . \tag{33}$$

If we assume particle 4 is a nucleon, then the angles of ejection are selected as indicated above and E_4 is calculated by Eq. 31. The selected angles are rejected only if $E_4 \leq E_f$.

Distribution

1. E. P. Blizard
2-11. H. W. Bertini
12-21. R. B. Curtis
22-31. C. D. Zerby
32. W. E. Kinney
33. R. R. Coveyou
34. J. G. Sullivan
35. R. G. Alsmiller
36. F. S. Alsmiller
37. L. Dresner
38. H. S. Moran
39. F. C. Maienschein
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