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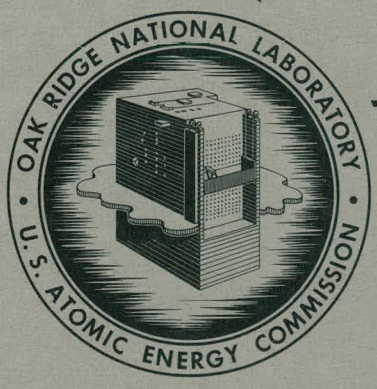
NEW SOLUTIONS OF THE BOLTZMANN EQUATION FOR MONOENERGETIC NEUTRON TRANSPORT IN SPHERICAL GEOMETRY

Walter Kofink



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Introduction

Solving the Boltzmann equation for problems of neutron transport in media with sources or boundaries by approximations, one faces the appearance of the transient solutions in addition to the asymptotic solutions. E. P. Wigner[8] suggested that in plane geometry for infinitely high order of approximation, i.e., for the exact treatment, these functions form the set of solutions which belongs to the spectrum of the Boltzmann operator. The asymptotic solutions belong to the discrete part and the transients to the continuous part of the spectrum. K. M. Case, "Annals of Physics", 9, 1-23 (1960), uttered this idea independently and proved that these solutions form a complete set.

The analogous problem for spherical geometry is still not solved. Similarly to the two different kinds of solutions of the equation $\Delta f = \kappa^2 f$, for which one has regular and singular solutions at the center of the sphere, one also has to expect both kinds of solutions for the Boltzmann equation. In part I of this report the singular solutions are derived. The Boltzmann equation is solved in section B by two steps. First, a partial differential equation with the desired density on the right hand side will be solved. In general, the partial solution, found by this way, will not yield the desired density and one has to add a suitable solution of the homogeneous differential equation to obtain the desired density. This addition leads to a Sonine integral equation. Second, this integral equation has to be solved; it gives the right additional solution of the homogeneous partial differential equation to fit the desired density. In section C the uniqueness of the total solution is shown in the sense that a different choice of the original partial solution does not influence the total solution. There is a further interesting property of these solutions: they do not involve a requirement to satisfy a characteristic equation. This fact implies that all those

terms in the total solution which contain the constant of multiplication c as a factor do not contribute to the density. Hence in section D it is shown that the densities arising from the originally chosen partial solution and from its Sonine transform cancel one another. The total density is given by a third term in the total solution which originates from the desired density by the Sonine procedure. Of course it is also independent from the choice of the partial solution of the inhomogeneous partial differential equation. The same fact can be observed for the solutions in plane geometry which belong to the continuous spectrum by comparison of equations (86) and (87). This fact permits solutions which avoid the satisfaction of a characteristic equation.

In section E a partial solution containing an arbitrary parameter is given and in section F three different specific partial solutions are considered. The partial solution treated in section F1 has the attribute that its density is easily calculable and that the Sonine transformation can be performed analytically. Hence it is used to write down the total singular solution of the Boltzmann equation in equation (35). The term of the solution, which is singular at the center of the sphere, is

$$\frac{\cos d_0}{d_0} = \frac{\cos(\rho \sin\vartheta)}{\rho \sin\vartheta} .$$

Of course it is invariant against a rotation of the coordinate system around the center of the sphere because $d_0 = \rho \sin\vartheta$ is the invariant distance of a neutron ray from the center of the sphere.

The partial solution (under F1), however, contains two terms which are solutions of the homogeneous differential equation. They contain a log ρ-term and the Sonine transform contains them with the opposite sign. Therefore they are removed from the total solution and we are sure that the total solution does not contain a singular term proportional to log ρ.

A second partial solution (under F2) does not contain a log ρ-term from the outset, but it contains still some presumably superfluous terms which satisfy the homogeneous partial differential equation. To calculate the density belonging to it is not easy and this solution is not persued further.

Finally, a third partial solution (under F3) is chosen, dropping all superfluous parts which satisfy the homogeneous differential equation. The density belonging to it is the sum of the equations (49) and (53), which are given in integral form. The total solution, shown in equation (57), would appear rather lengthy if written down explicitly. Of course it has to be identical with the solution under F1 in the form of equation (35) or (40) according to the uniqueness theorem.

In part II the solutions of the Boltzmann equation in spherical geometry, which are regular at the center of the sphere, are considered. In section A the regular solution which satisfies the characteristic equation is given in an integral form by equation (63). This case distinguishes itself as the only one in this report for which the Sonine transform must not be applied; the solution (63) yields already the desired density (66) after the application of the characteristic equation (5). One can, however, write this solution in the form of equation (67) in which the first term gives already the whole density and satisfies the homogeneous differential equation (8), whereas the second term gives the density zero and satisfies the inhomogeneous differential equation (59) with the right hand side (62). This suggests to construct regular solutions as the difference of the two singular solutions for κ and $-\kappa$. For instance, one may use equation (35) as the total solution for $-\kappa$, reverse the sign of κ in it and take the difference of both. Then the application of Bessel's integral (36) to this difference leads quickly to the regular solution (71). It has a form identical

with (67) with the only exception that $\frac{c}{4} \log \frac{1+\kappa}{1-\kappa}$ is replaced by $\frac{\kappa}{2}$. This has the meaning that the solution (71) yields the desired density $\frac{\sin \kappa\rho}{\rho}$ without the requirement of fulfilling a characteristic equation. Furthermore, the integral form (72) of this solution shows no hint which excludes its validity over the whole complex κ-plane. To show the invariance of all regular solutions against a rotation of the coordinate system around the center of the sphere, it is proved in section C that they satisfy also the equation $\Delta f = \kappa^2 f$.

Finally, the regular solutions in spherical geometry are constructed in section D by superposition of solutions in plane geometry which belong to the same κ. Of course, this can be done only with solutions which belong to the discrete and continuous spectrum of the Boltzmann operator in plane geometry and, for instance, not for complex κ-values. The superposition of plane solutions belonging to a κ-value of the discrete spectrum yields immediately to that regular solution in spherical geometry which has to fulfill the characteristic equation. The solutions belonging to the continuum in plane geometry are given in the symbolic form (83) or (85) of a series of a Cauchy principle value and a Dirac δ-function. The superposition of those solutions to a solution in spherical geometry removes the symbolic form and one obtains for the regular solution an ordinary function - see equation (100) - which is identical with (71) obtained in section B as the difference of two singular spherical solutions for κ and $-\kappa$.

No method is given to obtain the singular spherical solutions by superposition of plane solutions. There is also no suggestion how one may find the spherical solutions for those κ-values which are different from the κ's of the spectrum in plane geometry, as a linear combination of solutions with κ's belonging to it. Hence a statement about the spectrum of the Boltzmann operator in spherical geometry and its complete set of eigenfunctions is still missing.

In Part III the new solutions will be compared with their well-known representations by a series of spherical harmonics. A simple proof is given for the equality of the two regular solutions. In the case of the singular solutions, however, one has to cross out all terms with negative powers of κ as factors in the divergent series of spherical harmonics. Then one obtains a convergent series which is equal to the new singular solution.

I. Solutions of the Boltzmann Equation for Monoenergetic Neutron Transport in Spherical Geometry which are Singular at the Center of the Sphere.

A. Preliminary remarks. The Boltzmann equation in spherical geometry has the form

$$\mu \frac{\partial f}{\partial \rho} + \frac{1 - \mu^2}{\rho} \frac{\partial f}{\partial \mu} + f(\rho,\mu) = \frac{c}{2} \int_{-1}^{+1} f(\rho,\mu')\, d\mu', \qquad (1)$$

if scattering, absorption and multiplication are assumed to be isotropic. The constant c of multiplication

$$c = \frac{1}{\Sigma}(\Sigma+\nu\Sigma_f-\Sigma_a) = \frac{1}{\Sigma}(\Sigma_s+\nu\Sigma_f) = 1 - \frac{1}{\Sigma}(\Sigma_a-\nu\Sigma_f),$$

in which $\Sigma = \Sigma_s + \Sigma_a = \Sigma_s + \Sigma_c + \Sigma_f$ is the total macroscopic cross section, Σ_s is the macroscopic cross section for pure scattering, Σ_c for capture, Σ_f for fission, $\Sigma_a = \Sigma_c + \Sigma_f$ for absorption by capture without and with fission together. ν is the average number of neutrons produced in one fission process. $\rho = \Sigma r$ is the dimensionless measure of the distance from the center of the sphere on a radiusvector $\vec{r}$ and $\mu = \cos\vartheta$ is the cosine of the angle ϑ between the direction $\vec{\Omega}$ of a neutron and the radiusvector $\vec{r}$.

Solutions $f(\rho,\mu)$ of the Boltzmann equation will be called "regular" if they are finite and "singular" if they are infinite at the center of the sphere.

Examples of solutions of both kinds are well known, for instance, in the form of a spherical harmonics series[1] for the discrete spectrum of the Boltzmann operator. Such solutions are

$$f(\rho,\mu) = \sum_{\ell=0}^{\infty} \Psi_\ell(\rho)\, P_\ell(\mu), \qquad (2)$$

---

(1) B. Davison and J. B. Sykes, "Neutron Transport Theory", p. 146, Oxford Press, 1957; Alvin M. Weinberg and E. P. Wigner, "The Physical Theory of Neutron Chain Reactors", p. 273, University of Chicago Press, 1958.

with

$$\psi_\ell(\rho) = \frac{\kappa}{z}(2\ell+1)\, G_\ell(-\tfrac{1}{\kappa})\, f_\ell(-\kappa\rho)$$

and

$$G_o(-\tfrac{1}{\kappa}) = 1, \quad G_\ell(-\tfrac{1}{\kappa}) = P_\ell(\tfrac{1}{\kappa}) - \frac{c}{\kappa} W_{\ell-1}(\tfrac{1}{\kappa}) \quad \text{for } \ell = 1,2,\ldots$$

For a regular solution one chooses for f_ℓ the functions

$$f_\ell^{I}(-\kappa\rho) = \sqrt{\frac{\pi}{2}}\, \frac{I_{\ell+\frac{1}{2}}(-\kappa\rho)}{\sqrt{-\kappa\rho}} \tag{3}$$

and for a singular solution the functions

$$f_\ell^{II}(-\kappa\rho) = \sqrt{\frac{2}{\pi}}\, \frac{K_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}}\,. \tag{4}$$

c and κ are related by a characteristic equation

$$\frac{c}{2\kappa} \log \frac{1+\kappa}{1-\kappa} = 1 \tag{5}$$

with a pair of eigenvalues $\pm\kappa$ for every $c > 0$, which represent the discrete spectrum of the Boltzmann operator. The densities (or also the fluxes of velocity $v = 1$) which belong to these solutions are proportional to

$$P(\rho) = \int_{-1}^{+1} f(\rho,\mu)\, d\mu = 2\psi_0(\rho) \equiv \kappa f_o(-\kappa\rho),$$

i.e.,

$$P^{I}(\rho) = \kappa\sqrt{\frac{\pi}{2}}\, \frac{I_{\frac{1}{2}}(-\kappa\rho)}{\sqrt{-\kappa\rho}} \equiv \frac{\sinh \kappa\rho}{\rho} \quad \text{for the "regular" solution}$$

and

$$P^{II}(\rho) = \kappa\sqrt{\frac{2}{\pi}}\, \frac{K_{\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} \equiv \frac{e^{-\kappa\rho}}{\rho} \quad \text{for the "singular" solution.}$$

A representation as an integral[2] is known at least for the regular solution of the discrete spectrum; a new one will be given here (equation 63).

B. The Sonine transformation. The problem will be attacked now by a different method to find other solutions. To obtain singular solutions, one prescribes the density $P(\rho) = \frac{e^{-\kappa\rho}}{\rho}$ and solves the partial differential equation

$$\mu\frac{\partial f}{\partial \rho} + \frac{1-\mu^2}{\rho}\frac{\partial f}{\partial \mu} + f(\rho,\mu) = \frac{c}{2}\frac{e^{-\kappa\rho}}{\rho} . \tag{6}$$

One finds a partial solution $f_p(\rho,\mu)$ of this equation and has to investigate whether it is compatible with the condition:

$$P_p(\rho) \equiv \int_{-1}^{+1} f_p(\rho,\mu')\, d\mu' \quad \text{should be} \quad \frac{e^{-\kappa\rho}}{\rho} .$$

In general, however, this will not be the case and one has to add the suitable solution

$$f_H(\rho,\mu) = e^{-\mu\rho}\,\phi(\rho\sqrt{1-\mu^2}) \tag{7}$$

of the homogeneous differential equation

$$\mu\frac{\partial f_H}{\partial \rho} + \frac{1-\mu^2}{\rho}\frac{\partial f_H}{\partial \mu} + f_H(\rho,\mu) = 0 \tag{8}$$

to the original partial solution $f_p(\rho,\mu)$ of the inhomogeneous differential equation to satisfy the equation of compatibility by the sum of both:

$$\int_{-1}^{+1}\left[f_p(\rho,\mu) + e^{-\mu\rho}\,\phi(\rho\sqrt{1-\mu^2})\right] d\mu = \frac{e^{-\kappa\rho}}{\rho} .$$

Hence one has to find the function ϕ in the integral equation

$$\int_{-1}^{+1} e^{-\mu\rho}\,\phi(\rho\sqrt{1-\mu^2})\, d\mu = \frac{e^{-\kappa\rho}}{\rho} - P_p(\rho) . \tag{9}$$

(2) H. Stittgen, Beitraege zur Loesung von Neutronentransport - Problemen in kugelfoermigen Medien, Diplomarbeit, Technische Hochschule Karlsruhe 1958 (unpublished).

To have a common denominator of both terms on the right hand side of this equation and to remember always that the partial solution f_p contains $\frac{c}{2}$ as a factor, one must introduce another abbreviation instead of $P_p(\rho)$, namely,

$$\rho P_p(\rho) \equiv \rho \int_{-1}^{+1} f_p(\rho,\mu')\, d\mu' = \frac{c}{2} D_p(\rho) \tag{10}$$

into the second term on the right hand side of equation (9). Furthermore, the function $\phi(\rho\sqrt{1-\mu^2})$ is symmetric in respect to a change of the sign of μ; this fact permits writing the equation in the form

$$2\int_0^1 \cosh \mu\rho\, \phi(\rho\sqrt{1-\mu^2})\, d\mu = \frac{1}{\rho}\left[e^{-\kappa\rho} - \frac{c}{2} D_p(\rho)\right]. \tag{11}$$

$D_p(\rho)$ is proportional to ρ times the density of the chosen partial solution $f_p(\rho,\mu)$, a known function of ρ.

To solve the integral equation one puts $\rho^2 = s$ and chooses a new variable of integration $t = \rho^2(1-\mu^2)$. Then one has

$$\rho^2\mu^2 = \rho^2 - t = s - t, \quad \rho\mu = \sqrt{s-t}, \quad d\mu = -\frac{1}{2}\frac{dt}{\sqrt{s(s-t)}}$$

and the limits of integration become

$$t = \rho^2 = s \text{ for } \mu = 0 \text{ and } t = 0 \text{ for } \mu = 1.$$

The integral equation (11) takes the Sonine[3] form

$$\int_0^s \frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\, \phi(\sqrt{t})\, dt = e^{-\kappa\sqrt{s}} - \frac{c}{2} D_p(\sqrt{s}). \tag{12}$$

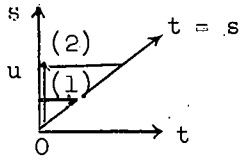
(3) N. Sonine, "Acta Mathematica", 4, 171 (1884).

Following Sonine, one multiplies this equation on both sides with the factor

$$\frac{1}{\pi}\int_0^u \frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\,ds$$

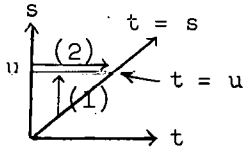
and obtains

$$\frac{1}{\pi}\int_0^u ds\,\frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\left[e^{-\kappa\sqrt{s}} - \frac{c}{2}D_p(\sqrt{s})\right] = \int_0^u ds\,\frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}}\int_0^s dt\,\frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\,\phi(\sqrt{t}). \tag{13}$$



An exchange of the order of integration on the right hand side of this equation yields

$$= \int_0^u dt\,\phi(\sqrt{t})\int_t^u \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}}\,\frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\,ds = \int_0^u dt\,\phi(\sqrt{t}),$$



because

$$\int_t^u \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}}\,\frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\,ds = 1. \tag{14}$$

To release the reader from a study of Sonine's work, the proof of the last equation will be given here. One introduces a new variable of integration x by putting

$$s = t + (u-t)\,x, \qquad x = (s-t)/(u-t)$$

$$ds = (u-t)\,dx$$

$$\sqrt{u-s} = \sqrt{u-t}\,\sqrt{1-x}, \qquad \sqrt{s-t} = \sqrt{u-t}\,\sqrt{x}$$

with the new limits of integration $x = 0$ for $s = t$ and $x = 1$ for $s = u$. This transforms the integral on the left hand side of (14) into

$$\frac{1}{\pi}\int_0^1 \frac{\cos(y\sqrt{1-x})\,\cosh(y\sqrt{x})}{\sqrt{x(1-x)}}\,dx$$

with the abbreviation $y = \sqrt{u-t}$. A series development of cos and cosh and term-by-term integration yields

$$\frac{1}{\pi}\sum_{\ell,m=0}\frac{(-1)^\ell\, y^{2(\ell+m)}}{(2\ell)!\,(2m)!}\int_0^1 (1-x)^{\ell-\frac{1}{2}}\, x^{m-\frac{1}{2}}\,dx$$

$$= \sum_{\ell,m=0}(-1)^\ell\binom{\ell+m}{\ell}\frac{(y/2)^{2(\ell+m)}}{\left[(\ell+m)!\right]^2} = 1 + \sum_{s=1}\frac{(y/2)^{2s}}{(s!)^2}\sum_{\ell=0}^{s}(-1)^\ell\binom{s}{\ell} = 1$$

q.e.d. The following formulas were used:

$$\int_0^1 (1-x)^{\ell-\frac{1}{2}}\, x^{m-\frac{1}{2}}\,dx = \frac{\Gamma(\ell+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(\ell+m+1)} = \frac{\pi\,(2\ell-1)!\,(2m-1)!}{2^{2(\ell-m-1)}\,(\ell-1)!\,(m-1)!\,(\ell+m)!}$$

and

$$\sum_{\ell=0}^{s}(-1)^\ell\binom{s}{\ell} = (1-1)^s = 0.$$

The solution of the integral equation (12) has now the form

$$\int_0^u \phi(\sqrt{t})\,dt = \frac{1}{\pi}\int_0^u ds\,\frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\left\{e^{-\kappa\sqrt{s}} - \frac{c}{2}\,D_p(\sqrt{s})\right\} \tag{15}$$

and the <u>function ϕ</u> itself will be given by differentiation in respect to u

$$\phi(\sqrt{u}) = \frac{1}{\pi}\,\frac{d}{du}\int_0^u ds\,\frac{\cos(\sqrt{u-s})}{\sqrt{u-s}}\left[e^{-\kappa\sqrt{s}} - \frac{c}{2}\,D_p(\sqrt{s})\right] \tag{16}$$

$$\text{with}\quad \frac{c}{2}\,D_p(\sqrt{s}) = \sqrt{s}\int_{-1}^{+1} f_p(\sqrt{s},\mu')\,d\mu'$$

$$\text{and}\quad u = \rho^2(1-\mu^2).$$

There are other forms of ϕ which are sometimes more convenient. After a partial integration on the right hand side of equation (16), one can perform the differentiation d/du:

$$\phi(\sqrt{u}) = \frac{1}{\pi}\frac{d}{du}\left\{\left[-2\sin(\sqrt{u-s})\left(e^{-\kappa\sqrt{s}} - \frac{c}{2}D_p(\sqrt{s})\right)\right]_{s=0}^{s=u} + 2\int_0^u \sin(\sqrt{u-s})\frac{d}{ds}\left(e^{-\kappa\sqrt{s}} - \frac{c}{2}D_p(\sqrt{s})\right)ds\right\}$$

$$= \frac{1}{\pi}\frac{d}{du}\left\{2\sin\sqrt{u}\left(1 - \lim_{s\to 0}\frac{c}{2}D_p(\sqrt{s})\right) - 2\left[\lim_{s\to u}\sin(\sqrt{u-s})\right]\left(e^{-\kappa\sqrt{u}} - \frac{c}{2}D_p(\sqrt{u})\right)\right.$$

$$\left. + \frac{2}{\pi}\frac{d}{du}\int_0^u \sin(\sqrt{u-s})\frac{d}{ds}\left(e^{-\kappa\sqrt{s}} - \frac{c}{2}D_p(\sqrt{s})\right)ds.\right.$$

The second term is zero, because $e^{-\kappa\sqrt{u}} - \frac{c}{2}D_p(\sqrt{u})$ is <u>finite</u>. The differentiation in respect to u yields now

$$\phi(\sqrt{u}) = \frac{1}{\pi}\frac{\cos\sqrt{u}}{\sqrt{u}}\left[1 - \lim_{s\to 0}\frac{c}{2}D_p(\sqrt{s})\right] + \frac{1}{\pi}\int_0^u \frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\frac{d}{ds}\left(e^{-\kappa\sqrt{s}} - \frac{c}{2}D_p(\sqrt{s})\right)ds + \frac{2}{\pi}\left[\lim_{s\to u}\sin(\sqrt{u-s})\right]\frac{d}{du}\left(e^{-\kappa\sqrt{u}} - \frac{c}{2}D_p(\sqrt{u})\right).$$

The last term of this equation vanishes because $\frac{d}{du}\left(e^{-\kappa\sqrt{u}} - \frac{c}{2}D_p(\sqrt{u})\right)$ is finite, and one obtains a second form of ϕ

$$\phi(\sqrt{u}) = \frac{1}{\pi}\,\frac{\cos\sqrt{u}}{\sqrt{u}}\left[1 - \frac{c}{2}\,D_p(0)\right] + \frac{1}{\pi}\int_0^u \frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\,\frac{d}{ds}\left[e^{-\kappa\sqrt{s}} - \frac{c}{2}\,D_p(\sqrt{s})\right] ds$$

with $\frac{c}{2}\,D_p(0) = \lim_{\rho\to 0}\left(\rho\int_{-1}^{+1} f_p(\rho,\mu')\,d\mu'\right)$.

In this kind of writing the first term shows a singularity at the center of the sphere, whereas the second term is regular there for the different specific partial solutions f_p, which will be considered later. Furthermore, $D_p(0)$ will be zero for these f_p's also.

Inserting $u = \rho^2(1-\mu^2) = \rho^2 \sin^2\vartheta$ we have

$$\phi(\rho\sqrt{1-\mu^2}) = \frac{1}{\pi}\,\frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}}\left[1 - \frac{c}{2}\,D_p(0)\right] + \frac{1}{\pi}\int_0^{\rho^2(1-\mu^2)} \frac{\cos\left(\sqrt{\rho^2(1-\mu^2)-s}\right)}{\sqrt{\rho^2(1-\mu^2)-s}}\,\frac{d}{ds}\left[e^{-\kappa\sqrt{s}} - \frac{c}{2}\,D_p(\sqrt{s})\right] ds. \tag{18}$$

One may perform the differentiation in the second term and may introduce another variable of integration v by $s = \rho^2(1-\mu^2)\,v^2$ to transform the upper limit of the integral into $v = 1$. Furthermore, one may introduce the distance $d_0 = \rho\sqrt{1-\mu^2} = \rho\sin\vartheta$ of the "neutron-ray" from the center of the sphere as an abbreviation. Then one obtains

$$\phi(\rho\sqrt{1-\mu^2}) \equiv \phi(d_0) = \frac{1}{\pi}\,\frac{\cos d_0}{d_0}\left[1 - \frac{c}{2}\,D_p(0)\right] - \frac{1}{\pi}\int_0^1 dv\,\frac{\cos(d_0\sqrt{1-v^2})}{\sqrt{1-v^2}}\left[\kappa e^{-\kappa d_0 v} + \frac{c}{2}\,D_p'(d_0 v)\right] \tag{19}$$

where

$$\frac{c}{2}\,D'(x) \equiv \frac{c}{2}\,\frac{d\,D(x)}{dx} = \frac{d}{dx}\left\{x\int_{-1}^{+1} f_p(x,\mu')\,d\mu'\right\} \cdot$$

$$= \int_{-1}^{+1} f_p(x,\mu')\,d\mu' + x\int_{-1}^{+1} \frac{\partial}{\partial x}\,f_p(x,\mu')\,d\mu'.$$

$f_p(x,\mu')$ is the originally chosen partial solution $f_p(\rho,\mu)$, in which one has replaced ρ by x and μ by μ' (a variable of integration here). Hence the solution $f^{(s)}(\rho,\mu)$ of the Boltzmann equation for monoenergetic neutron transport with isotropic scattering and absorption or multiplication in an infinite medium in spherical geometry, which is singular at the origin, is the sum of a partial solution $f_p(\rho,\mu)$ of the inhomogeneous partial differential equation (6) and of the solution $f_H(\rho,\mu)$ of the homogeneous equation (8) containing the "Sonine transform" $\phi\left(\rho\sqrt{1-\mu^2}\right)$ given by equation (16) or (18) or (19), which corresponds to the chosen partial solution f_p:

$$f^{(s)}(\rho,\mu) = f_p(\rho,\mu) + e^{-\mu\rho}\,\phi(\rho\sqrt{1-\mu^2}). \tag{20}$$

The choice of the original partial solution $f_p(\rho,\mu)$ is restricted by the requirement that 1)

$$\frac{c}{2} D_p(0) = \lim_{\rho\to 0}\left[\rho\int_{-1}^{+1} f_p(\rho,\mu)\,d\mu\right]$$

is finite, and 2) $\frac{c}{2} D_p'(d_0 v)$ diverges at $v = 1$ at most as $\frac{1}{(1-v)^\alpha}$ with $\alpha < \frac{1}{2}$ and has no pole in the remaining interval $0 \le v < 1$ to guarantee the convergence of the integral in equation (19).

It is sometimes convenient to split the term $e^{-\mu\rho}\,\phi(\rho\sqrt{1-\mu^2})$ into its two parts: the first, which is produced by the Sonine transformation from the originally chosen partial solution $f_p(\rho,\mu)$

$$S_0(f_p) = -\frac{1}{\pi} e^{-\mu\rho}\left\{\frac{c}{2} D_p(0)\,\frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}} + \int_0^{\rho^2(1-\mu^2)} ds\,\frac{\cos\sqrt{\rho^2(1-\mu^2)-s}}{\sqrt{\rho^2(1-\mu^2)-s}}\,\frac{d}{ds}\left(\frac{c}{2} D_p(\sqrt{s})\right)\right.$$

$$\left. = -\frac{1}{\pi} e^{-\mu\rho}\left\{\frac{c}{2} D_p(0)\,\frac{\cos d_0}{d_0} + \int_0^1 dv\,\frac{\cos(d_0\sqrt{1-v^2})}{\sqrt{1-v^2}}\,\frac{c}{2} D_p(d_0 v)\right\}\right. \tag{21}$$

and the second, which is produced by the Sonine transformation of the required density $\frac{1}{\rho} e^{-\kappa\rho}$.

$$e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) = \frac{1}{\pi} e^{-\mu\rho} \left\{ \frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}} + \int_0^{\rho^2(1-\mu^2)} ds \, \frac{\cos\sqrt{\rho^2(1-\mu^2) - s}}{\sqrt{\rho^2(1-\mu^2) - s}} \, \frac{d}{ds} e^{-\kappa\sqrt{s}} \right\}$$

$$= \frac{1}{\pi} e^{-\mu\rho} \left\{ \frac{\cos d_0}{d_0} - \kappa \int_0^1 dv \, \frac{\cos(d_0\sqrt{1-v^2})}{\sqrt{1-v^2}} \, e^{-\kappa d_0 v} \right\}. \tag{22}$$

With this notation the singular solution may be written

$$f^{(s)}(\rho,\mu) = f_p(\rho,\mu) + S_0(f_p) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}). \tag{23}$$

C. The effect of the Sonine-transformation upon solutions of the homogeneous partial differential equation (8). Uniqueness of the solution. If one has taken fortunately that solution $F^{(s)}(\rho,\mu)$ of the inhomogeneous partial differential equation (6) which gives the required density, no need of an application of the Sonine transformation would arise at all. Every other partial solution $f_p(\rho,\mu)$ of the inhomogeneous equation will differ from $F^{(s)}(\rho,\mu)$ by a solution

$$f_H(\rho,\mu) = e^{-\mu\rho} \psi(\rho\sqrt{1-\mu^2}) \tag{7a}$$

of the homogeneous equation (8). It will be shown, theorem I, that the Sonine transform of such a function is just the opposite of itself

$$S_0(f_H) = -f_H. \tag{24}$$

Proof. The Sonine transformation applied to f_H in the form of equation (16) contains the double integral

$$\phi_H(\sqrt{u}) = \frac{1}{\pi} \frac{d}{du} \int_0^u ds \, \frac{\cos\sqrt{u-s}}{\sqrt{u-s}} \left\{ -\sqrt{s} \int_{-1}^{+1} e^{-\mu'\sqrt{s}} \psi\left(\sqrt{s(1-\mu'^2)}\right) d\mu' \right\}$$

in which the expression in curly brackets replaces the corresponding expression in equation (16) and ϕ_H indicates that ϕ_H is the transform of f_H alone. The Sonine transform of f_H will be

$$S_0(f_H) = e^{-\mu\rho}\,\phi_H(\sqrt{u}) \tag{25}$$

where $u = \rho^2(1-\mu^2)$. To evaluate $\phi_H(\sqrt{u})$ one changes first the interval of integration over μ' in $0 \le \mu' \le 1$; this yields for f_H of equation (7a)

$$\phi_H(\sqrt{u}) = -\frac{2}{\pi}\frac{d}{du}\int_0^u ds\,\frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\sqrt{s}\int_0^1 \cosh(\mu'\sqrt{s})\,\psi(\sqrt{s(1-\mu'^2)})\,d\mu'.$$

One introduces a new variable of integration $t = s(1-\mu'^2)$, $\mu'\sqrt{s} = s - t$, $d\mu' = -dt/(2\sqrt{s(s-t)})$ with the new limits $t = s$ for $\mu' = 0$ and $t = 0$ for $\mu' = 1$. Then the integral becomes

$$\phi_H(\sqrt{u}) = -\frac{1}{\pi}\frac{d}{du}\int_0^u ds\,\frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\,\frac{\sqrt{s}}{\sqrt{s}}\int_0^s \frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\,\psi(\sqrt{t})\,dt$$

$$= -\frac{d}{du}\int_0^u dt\,\psi(\sqrt{t})\int_t^u \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}}\,\frac{\cosh\sqrt{s-t}}{\sqrt{s-t}}\,ds$$

$$= -\psi(\sqrt{u}) = -\psi(\rho\sqrt{1-\mu^2}),$$

remembering that the last integral over s is 1 by equation (14). After insertion of this result into equation (25) one obtains the Sonine transform of f_H

$$S_0(f_H) = -e^{-\mu\rho}\,\psi(\rho\sqrt{1-\mu^2}) = -f_H(\rho,\mu),$$

which is in fact the opposite of $f_H(\rho,\mu)$ itself.

This theorem provides the uniqueness of the solution of the Boltzmann equation, if a definite density is given. Two different partial solutions, $f_{p_1}(\rho,\mu)$ and $f_{p_2}(\rho,\mu)$ of the inhomogeneous differential equation (6), differ one from

another just by a solution $f_H(\rho,\mu)$ of the homogeneous equation. Hence, if one writes down that part of the total solution (23) of the Boltzmann equation which depends on the choice of the original partial solution f_p, namely,

$$f_p(\rho,\mu) + S_0(f_p),$$

one recognizes that it is independent of this choice: If $f_{p_2}(\rho,\mu) = f_{p_1}(\rho,\mu) + f_H(\rho,\mu)$ is a second partial solution, the expression

$$\begin{aligned} f_{p_2}(\rho,\mu) + S_0(f_{p_2}) &= f_{p_1}(\rho,\mu) + f_H(\rho,\mu) + S_0(f_{p_1}) + S_0(f_H) \\ &= f_{p_1}(\rho,\mu) + S_0(f_{p_1}) \end{aligned}$$

will be unchanged since $S_0(f_H) = - f_H$. Hence the solution $f^{(s)}(\rho,\mu)$ is unique.

If $F^{(s)}(\rho,\mu)$, which yields fortunately the right density, contains already a term f_{H_0} which satisfies the homogeneous differential equation (8), in addition to another term, $F_0^{(s)}(\rho,\mu)$, which satisfies the inhomogeneous equation (6)

$$F^{(s)}(\rho,\mu) = F_0^{(s)}(\rho,\mu) + f_{H_0} \tag{26}$$

one may apply the Sonine procedure to find f_{H_0}. One considers to this aim $F^{(s)}(\rho,\mu)$ as a partial solution f_p and applies the Sonine procedure to obtain the total solution

$$\begin{aligned} f^{(s)}(\rho,\mu) &= F^{(s)}(\rho,\mu) + S_0(F^{(s)}) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) \\ &= F_0^{(s)}(\rho,\mu) + f_{H_0} + S_0(F_0^{(s)}) + S_0(f_{H_0}) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) \end{aligned}$$

which is by equation (24)

$$= F_0^{(s)}(\rho,\mu) + S_0(F_0^{(s)}) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}).$$

Because $F^{(s)}(\rho,\mu)$ of equation (26) is in this case already the right solution, we have $f^{(s)}(\rho,\mu) = F^{(s)}(\rho,\mu)$, and by comparison with equation (26)

$$f_{H_0} = S_0(F_0^{(s)}) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}).$$

If f_{H_0} would be zero fortunately, one obtains the relation

$$\rho^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) = -S_0(F_0^{(s)}).$$

D. Statement about the density. The method of obtaining this solution does not contain any requirement to satisfy a characteristic equation. This fact must have the consequence that the density of this part of the solution $f^{(s)}(\rho,\mu)$, which contains c as a factor, vanishes, whereas the remaining part gives the whole density. The partial solution $f_p(\rho,\mu)$ of the inhomogeneous equation (6) is proportional to c and the quantities $\frac{c}{2} D_p(0)$ and $\frac{c}{2} D_p(\sqrt{s})$, depending on f_p by the Sonine transformation, also contain c as a factor. Hence one has Theorem II: the remaining part $R(\rho\sqrt{1-\mu^2})$ of $\phi(\rho\sqrt{1-\mu^2})$ in the form of equation (22)

$$R(\rho\sqrt{1-\mu^2}) = \frac{1}{\pi} \frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}} + \frac{1}{\pi}\int_0^{\rho^2(1-\mu^2)} ds \frac{\cos\sqrt{\rho^2(1-\mu^2)-s}}{\sqrt{\rho^2(1-\mu^2)-s}} \left(\frac{d}{ds} e^{-\kappa\sqrt{s}}\right)$$

should give the whole density $P^{(s)}(\rho)$ of the total singular solution $f^{(s)}(\rho,\mu)$, i.e.,

$$\int_{-1}^{+1} e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})\, d\mu$$

$$= 2\int_0^1 d\mu \cosh\mu\rho \left\{ \frac{\cos(\rho\sqrt{1-\mu^2})}{\pi\rho\sqrt{1-\mu^2}} + \int_0^{\rho^2(1-\mu^2)} ds \frac{\cos\sqrt{\rho^2(1-\mu^2)-s}}{\pi\sqrt{\rho^2(1-\mu^2)-s}} \left(\frac{d}{ds} e^{-\kappa\sqrt{s}}\right) \right\}$$

should be equal to

$$P^{(s)}(\rho) = \int_{-1}^{+1} f^{(s)}(\rho,\mu)\, d\mu = \frac{e^{-\kappa\rho}}{\rho}.$$

Proof. (a) The first term of the integral yields the singularity of the density at the center

$$2\int_0^1 d\mu \cosh \mu\rho \frac{\cos(\rho\sqrt{1-\mu^2})}{\pi\rho\sqrt{1-\mu^2}} = 2 \cdot \frac{1}{2\rho} \int_t^{\rho^2+t} \frac{\cosh\sqrt{s-t}}{\sqrt{s-t}} \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}} ds$$

$$= \frac{1}{\rho}\int_t^u \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}} \frac{\cosh\sqrt{s-t}}{\sqrt{s-t}} ds = \frac{1}{\rho}$$

after transformation of the variable of integration μ into a new variable s by $\mu = \frac{1}{\rho}\sqrt{s-t}$, $d\mu = \frac{1}{2\rho}\frac{ds}{\sqrt{s-t}}$, with the introduction of the notation $u = s + \rho^2(1-\mu^2) = \rho^2 + t$ and use of equation (14).

(b) The second term of the integral gives the nonsingular part of the density. One introduces $u = \rho^2(1-\mu^2)$ as a new variable of integration instead of μ, then one has $\mu = \frac{1}{\rho}\sqrt{\rho^2-u}$, $d\mu = -\frac{1}{2\rho}\frac{du}{\sqrt{\rho^2-u}}$ and the limits of integration over μ will be

$$u = \rho^2 \text{ for } \mu = 0 \text{ and } u = 0 \text{ for } \mu = 1.$$

Hence it follows that

$$2\int_0^1 d\mu \cosh \mu\rho \int_0^{\rho^2(1-\mu^2)} ds \frac{\cos\sqrt{\rho^2(1-\mu)-s}}{\pi\sqrt{\rho^2(1-\mu^2)-s}} \left(\frac{d}{ds} e^{-\kappa\sqrt{s}}\right)$$

$$= \frac{1}{\rho}\int_0^{\rho^2} du \frac{\cosh\sqrt{\rho^2-u}}{\sqrt{\rho^2-u}} \int_0^u ds \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}} \left(\frac{d}{ds} e^{-\kappa\sqrt{s}}\right)$$

$$= \frac{1}{\rho}\int_0^{\rho^2} ds \frac{d}{ds}(e^{-\kappa\sqrt{s}}) \int_s^{\rho^2} du \frac{\cosh\sqrt{\rho^2-u}}{\sqrt{\rho^2-u}} \frac{\cos\sqrt{u-s}}{\pi\sqrt{u-s}}$$

with the same exchange of the order of integration which was applied to the double integral (13). The last integral over u from s to ρ^2 is equal to 1 according to equation (14). Therefore the integration over ds yields

$$= \frac{1}{\rho}\left[e^{-\kappa\sqrt{s}}\right]_{s=0}^{s=\rho^2} = \frac{1}{\rho}\left(e^{-\kappa\rho} - 1\right).$$

This is indeed non-singular at the origin. If one adds now the two contributions (a) and (b) one obtains the total contribution of the $R(\rho\sqrt{1-\mu^2})$-part of $\phi(\rho\sqrt{1-\mu^2})$ to the density

$$\int_{-1}^{+1} e^{-\mu\rho}\, R(\rho\sqrt{1-\mu^2})\, d\mu = \frac{e^{-\kappa\rho}}{\rho}. \qquad (27)$$

This is in fact the whole density $P^{(s)}(\rho)$ belonging to the singular solution $f^{(s)}(\rho,\mu)$. If one remembers the representation (23) of $f^{(s)}(\rho,\mu)$, one recognizes that the density belonging to

$$f_p(\rho,\mu) + S_0(f_p)$$

must vanish. The following Theorem III will be proved: <u>the density which belongs to a partial solution $f_p(\rho,\mu)$ of the inhomogeneous differential equation is the opposite of the density which belongs to its Sonine transform</u>

$$\int_{-1}^{+1} f_p(\rho,\mu)\, d\mu = -\int_{-1}^{+1} S_0(f_p)\, d\mu.$$

<u>Proof</u>. Consider the second integral using the form (17) of the Sonine transform

$$-\int_{-1}^{+1} S_0(f_p)\, d\mu = \frac{c}{2}\,\frac{1}{\pi}\int_{-1}^{+1} d\mu\; e^{-\mu\rho}\,\frac{d}{du}\int_0^u ds\,\frac{\cos\sqrt{u-s}}{\sqrt{u-s}}\, D_p(\sqrt{s})$$

and remember that $u = \rho^2(1-\mu^2)$, $\rho\mu = \sqrt{\rho^2 - u}$, $d\mu = -du/(2\rho\sqrt{\rho^2-u})$ and that $u = \rho^2$ for $\mu = 0$, $u = 0$ for $\mu = 1$. Then the integral becomes

$$= \frac{c}{2\pi\rho}\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\, \frac{d}{du}\int_0^u \frac{\cos\sqrt{u - s}}{\sqrt{u - s}}\, D_p(\sqrt{s})\, ds$$

and by partial integration in the interior of the second integral

$$= -\frac{c}{\pi\rho}\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\, \frac{d}{du}\left\{\left[\sin\sqrt{u - s}\, D_p(\sqrt{s})\right]_{s=0}^{s=u} - \int_0^u ds\, \sin\sqrt{u - s}\left(\frac{d}{ds} D_p(\sqrt{s})\right)\right\}$$

$$= \frac{c}{\pi\rho}\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\left\{\frac{\cos\sqrt{u}}{2\sqrt{u}} D_p(0) + \frac{d}{du}\int_0^u ds\, \sin\sqrt{u - s}\, \frac{d}{ds} D_p(\sqrt{s})\right\}$$

$$= \frac{c}{2\rho} D_p(0)\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\, \frac{\cos\sqrt{u}}{\pi\sqrt{u}}$$

$$+ \frac{c}{\pi\rho}\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\left\{\lim_{s\to u}\left[\sin\sqrt{u - s}\, \frac{d}{ds} D_p(\sqrt{s})\right] + \int_0^u \frac{\cos\sqrt{u - s}}{2\sqrt{u - s}}\left(\frac{d}{ds} D_p(\sqrt{s})\right) ds\right\}.$$

The integral in the first term is 1 according to equation (14); the limit of the square brackets expression in the second term is supposed to be zero along the interval $0 \le u \le \rho^2$. Then the integral becomes

$$= \frac{c}{2\rho} D_p(0) + \frac{c}{2\rho}\int_0^{\rho^2} du\, \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\int_0^u \frac{\cos\sqrt{u - s}}{\pi\sqrt{u - s}}\left[\frac{d}{ds} D_p(\sqrt{s})\right] ds$$

$$= \frac{c}{2\rho} D_p(0) + \frac{c}{2\rho}\int_0^{\rho^2} ds\left[\frac{d}{ds} D_p(\sqrt{s})\right]\int_s^{\rho^2} \frac{\cosh\sqrt{\rho^2 - u}}{\sqrt{\rho^2 - u}}\, \frac{\cos\sqrt{u - s}}{\pi\sqrt{u - s}}\, du.$$

The last integral over u is 1 again, and the whole expression becomes

$$= \frac{c}{2\rho} D_p(0) + \frac{c}{2\rho}\left[D_p(\sqrt{s})\right]_{s=0}^{s=\rho^2} = \frac{c}{2\rho} D_p(\rho)$$

which is according to the definition (10) of $D_p(\rho)$

$$= \int_{-1}^{+1} f_p(\rho,\mu)\, d\mu \qquad \text{q.e.d.}$$

Theorem III is a counterpart to Theorem II and provides an independent check of the statement about the density.

E. General solution of the partial differential equation (6). The general solution of the inhomogeneous partial differential equation

$$\mu \frac{\partial f}{\partial \rho} + \frac{1 - \mu^2}{\rho} \frac{\partial f}{\partial \mu} + f(\rho,\mu) = \frac{c}{2} \frac{e^{-\kappa\rho}}{\rho} \tag{6}$$

will be derived by the well-known method of Cauchy: the equivalent system of differential equations is

$$\frac{d\rho}{\mu} = \frac{d\mu}{\frac{1 - \mu^2}{\rho}} = \frac{df}{-f + \frac{c}{2} \frac{e^{-\kappa\rho}}{\rho}} :$$

From the first equation one obtains $\rho\sqrt{1 - \mu^2} = C_1$ (a constant). This equation may be used to eliminate ρ from the second equation, which goes over into a linear differential equation of first order

$$\frac{df}{d\mu} + \frac{C_1}{(1-\mu^2)^{3/2}} f = \frac{c}{2} \frac{e^{-\kappa C_1/\sqrt{1 - \mu^2}}}{1 - \mu^2} .$$

Its solution is

$$f = \exp\left(- \frac{C_1\mu}{\sqrt{1 - \mu^2}}\right)\left\{\frac{c}{2}\int_{\mu_1}^{\mu} \exp\left(\frac{(\nu-\kappa)\, C_1}{\sqrt{1 - \nu^2}}\right) \cdot \frac{d\nu}{1 - \nu^2} + C_2 \exp\left(\frac{C_1\mu_0}{\sqrt{1 \quad \mu_0^2}}\right)\right\}$$

C_2, μ_0, μ_1 are constants.

Replacing C_1 now by $\rho\sqrt{1-\mu^2}$ one obtains for the constant C_2

$$C_2 = \exp\left(\frac{-\mu\,\rho\sqrt{1-\mu^2}}{\sqrt{1-\mu_0^2}}\right)\left\{e^{\mu\rho}\,f - \frac{c}{2}\int_{\mu_1}^{\mu}\exp\left(\frac{(\nu-\kappa)\,\rho\sqrt{1-\mu^2}}{\sqrt{1-\nu^2}}\right)\cdot\frac{d\nu}{1-\nu^2}\right\}.$$

The general solution of the partial differential equation is given by

$$C_2 = W(C_1)$$

with an arbitrary function W. This yields

$$f(\rho,\mu) = e^{-\mu\rho}\left\{W(\rho\sqrt{1-\mu^2})\exp\left(\frac{\mu_0\rho\sqrt{1-\mu^2}}{\sqrt{1-\mu_0^2}}\right) + \frac{c}{2}\int_{\mu_1}^{\mu}\exp\left(\frac{(\nu-\kappa)\,\rho\sqrt{1-\mu^2}}{\sqrt{1-\nu^2}}\right)\cdot\frac{d\nu}{1-\nu^2}\right\}.$$

The first term in the curly bracket with inclusion of the exp-factor is still an arbitrary function of $\rho\sqrt{1-\mu^2}$; so we denote the whole term by $\phi(\rho\sqrt{1-\mu^2})$. Hence the general solution of equation (6) is

$$f(\rho,\mu) = e^{-\mu\rho}\left\{\phi(\rho\sqrt{1-\mu^2}) + \frac{c}{2}\int_{\mu_1}^{\mu}\exp\left(\frac{(\nu-\kappa)\,\rho\sqrt{1-\mu^2}}{\sqrt{1-\nu^2}}\right)\cdot\frac{d\nu}{1-\nu^2}\right\}.$$

One may transform the partial solution, which occurs in it, namely:

$$f_p(\rho,\mu) = \frac{c}{2}\,e^{-\mu\rho}\int_{\mu_1}^{\mu}\exp\left(\frac{(\nu-\kappa)\,\rho\sqrt{1-\mu^2}}{\sqrt{1-\nu^2}}\right)\frac{d\nu}{1-\nu^2}$$

into a more convenient form, putting

$$\nu = \frac{(1+\mu)\,s^2-(1-\mu)}{(1+\mu)\,s^2+(1-\mu)} \quad \text{or} \quad s = \sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\nu}{1-\nu}},$$

$$\frac{d\nu}{1-\nu^2} = \frac{ds}{s} \quad \text{and} \quad \frac{\rho(\nu-\kappa)\sqrt{1-\mu^2}}{\sqrt{1-\nu^2}} = \frac{\rho}{2}\left[(1-\kappa)(1+\mu)\,s - (1+\kappa)(1-\mu)\,\frac{1}{s}\right].$$

With this transformation of the variable of integration the partial solution takes the form

$$f_p(\rho,\mu) \equiv f_{(\mu_1)}(\rho,\mu) = \frac{c}{2}\, e^{-\mu\rho} \int_{\sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\mu_1}{1-\mu_1}}}^{1} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s - (1+\kappa)(1-\mu)\,\frac{1}{s}\right]} \frac{ds}{s} . \quad (28)$$

Different partial solutions can be chosen by giving μ_1 different values. μ_1 is an arbitrary constant; one can prove, however, that μ_1 could also be an arbitrary function of $\rho\sqrt{1 - \mu^2}$.

F. Specific partial solutions.

1. Let us consider first a partial solution, for which the density can be calculated easily. One obtains it putting $\mu_1 = -1$ in equation (28). It will turn out that this partial solution contains a term log ρ, which actually will not occur in the total solution. In fact, this term appears in the combination $\rho\sqrt{1 - \mu^2}$; hence it is reproduced by the Sonine procedure with the opposite sign and cancels out of the total solution $f^{(s)}(\rho,\mu)$. We have no logarithmic singularity at the center of the sphere. But the analysis is simpler for this partial solution than for any other choice and the efficiency of the Theorem I can be shown easily also.

(a) Taking $\mu_1 = -1$ one obtains the partial solution of equation (6)

$$f_{(-1)}(\rho,\mu) = \frac{c}{2}\, e^{-\mu\rho} \int_0^1 e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s - (1+\kappa)(1-\mu)\,\frac{1}{s}\right]} \frac{ds}{s} . \quad (29)$$

It is well known that an integral of this type is related to a series of Bessel-functions. The development of the integrand in such a series would be invalid at the lower limit $s = 0$ of the integral. Therefore one splits the path of integration in two parts. The first part from $s = 0$ to $s = \beta$ with

$$\beta = \sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\kappa}{1-\kappa}} \tag{30}$$

can be integrated as it stands:

$$\frac{c}{2}\, e^{-\mu\rho} \int_0^{\beta} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s \,-\, (1+\kappa)(1-\mu)\, \frac{1}{s}\right]} \frac{ds}{s}$$

$$= \frac{c}{2}\, e^{-\mu\rho} \int_0^{1} e^{\frac{h}{2}\left(w - \frac{1}{w}\right)} \frac{dw}{w} \qquad \text{with } s = \beta w$$

$$= \frac{c}{2}\, e^{-\mu\rho} \int_0^{\infty} e^{-h \sinh t}\, dt \qquad \text{with } w = e^{-t}$$

$$= \frac{c}{2}\, e^{-\mu\rho}\, S_{00}(h).$$

$h = \rho\sqrt{(1-\kappa^2)(1-\mu^2)}$ is an abbreviation, $S_{00}(h)$ is the notation for a Lommelfunction defined in "Higher Transcendental Functions".(4)

In the second part of the integral one develops the integrand in a series of Besselfunctions(5), and integrates term by term:

$$\frac{c}{2}\, e^{-\mu\rho} \int_{\beta}^{1} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s \,-\, (1+\kappa)(1-\mu)\, \frac{1}{s}\right]} \frac{ds}{s}$$

$$= \frac{c}{2}\, e^{-\mu\rho} \int_1^{\frac{1}{\beta}} e^{\frac{h}{2}\left(w - \frac{1}{w}\right)} \frac{dw}{w}$$

$$= \frac{c}{2}\, e^{-\mu\rho} \int_1^{\frac{1}{\beta}} \frac{dw}{w} \left\{ J_0(h) + \sum_{n=1}^{\infty} J_n(h) \left[w^n + \frac{(-1)^n}{w^n} \right] \right\}$$

(4) A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, "Higher Transcendental Functions", II, p.40 formula (25), and p. 84 formula (50), New York (1955).

(5) Ibid, p. 7 formula (25).

$$= \frac{c}{2}\, e^{-\mu\rho} \left\{ J_0(h) \log w + \sum_{n=1}^{\infty} \frac{1}{n} J_n(h) \left[w^n - \frac{(-1)^n}{w^n} \right] \right\}_{w=1}^{w=\frac{1}{\beta}}$$

$$= \frac{c}{2}\, e^{-\mu\rho} \left\{ - J_0(h) \log \beta + \sum_{n=1}^{\infty} \frac{1}{n} J_n(h) \left[\frac{1}{\beta^n} - (-\beta)^n \right] - 2 \sum_{n=0}^{\infty} \frac{J_{2n+1}(h)}{2n+1} \right\} .$$

According to page 64, formula (7) of "Higher Transcendental Functions", II,(4) the last series represents another Lommelfunction

$$S_{00}(h) = 2 \sum_{n=0}^{\infty} \frac{J_{2n+1}(h)}{2n+1}$$

and the difference of both Lommelfunctions

$$S_{00}(h) - s_{00}(h) = - \frac{\pi}{2} Y_0(h)$$

is just $(-\frac{\pi}{2})$ times the Besselfunction of second kind $Y_0(h)$ [called $N_0(h)$ by Jahnke-Emde(6)]. By this remark one gets rid of the Lommelfunctions in the representation of the partial solution which will be

$$f_{(-1)}(\rho,\mu) = \frac{c}{2}\, e^{-\mu\rho} \left\{ - \frac{\pi}{2} Y_0(h) - J_0(h) \log \beta + \sum_{n=1}^{\infty} \frac{1}{n} J_n(h) \left[\frac{1}{\beta^n} - (-\beta)^n \right] \right. \qquad (31)$$

with $h = \rho\sqrt{(1-\kappa^2)(1-\mu^2)}$ and $\beta = \sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\kappa}{1-\kappa}}$.

(b) We calculate now the density $P_{(-1)}(\rho)$ which belongs to this partial solution

$$P_{(-1)}(\rho) \equiv \int_{-1}^{+1} f_{(-1)}(\rho,\mu)\, d\mu = \frac{c}{2} \int_{-1}^{+1} d\mu\, e^{-\mu\rho} \int_0^1 e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s - (1+\kappa)(1-\mu)\, \frac{1}{s}\right]} \frac{ds}{s} .$$

(6) Jahnke-Emde, "Tafeln hoeherer Funktionen", 5th edition, p. 131 (1952).

After an exchange of the order of integration one performs first the integration over μ; this yields

$$P_{(-1)}(\rho) = \frac{c}{\rho} e^{-\kappa\rho} \int_0^1 ds \frac{(e^{-(1-\kappa)\rho(1-s)} - 1) - (e^{-(1+\kappa)\rho\frac{1-s}{s}} - 1)}{(1-s)\left[1 + \kappa - (1-\kappa)s\right]} .$$

On replaces in the first part of the integrand the variable of integration s by $s = \frac{(1-\kappa)\rho - x}{(1-\kappa)\rho}$, in the second part by $s = \frac{(1+\kappa)\rho}{(1+\kappa)\rho + x}$. Then one obtains

$$P_{(-1)}(\rho) = -\frac{c}{\rho} e^{-\kappa\rho} \int_{(1-\kappa)\rho}^{\infty} \frac{e^{-x} - 1}{2\kappa + \frac{x}{\rho}} \frac{dx}{x}$$

$$= \frac{c}{2\kappa\rho}\left[e^{\kappa\rho}\int_{(1+\kappa)\rho}^{\infty} dx \frac{e^{-x}}{x} - e^{-\kappa\rho}\int_{(1-\kappa)\rho}^{\infty} dx \frac{e^{-x}}{x} + e^{-\kappa\rho}\int_{(1-\kappa)\rho}^{(1+\kappa)\rho} \frac{dx}{x}\right]$$

$$= \frac{c}{2\kappa\rho}\left\{e^{\kappa\rho} E_1(\left[1+\kappa\right]\rho) - e^{-\kappa\rho} E_1(\left[1-\kappa\right]\rho) + e^{-\kappa\rho}\log\frac{1+\kappa}{1-\kappa}\right\} \quad \kappa \leq 1 \qquad (32)$$

where $E_1(x)$ is defined by (see reference (4), page 143)

$$E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u} du = -\left[\gamma + \log x\right] + e^{-x}\sum_{m=1}^{\infty} \frac{h_m x^m}{m!} . \qquad (33)$$

The expression $P_{(-1)}(\rho)$ shows that the chosen partial solution $f_{(-1)}(\rho,\mu)$ does not give the desired density $\frac{e^{-\kappa\rho}}{\rho}$ on account of the added two terms which contain the "exponential-integral". If both these terms would be absent, the third term would give the desired result under the assumption of the characteristic equation for the discrete spectrum

$$\frac{c}{2\kappa}\log\frac{1+\kappa}{1-\kappa} = 1. \qquad (5)$$

(c) We shall apply now the Sonine transformation. We show first that $D_{(-1)}(0) = 0$, using the series (33) for $E_1(x)$:

$$\frac{c}{2} D_{(-1)}(\rho) \equiv \rho P_{(-1)}(\rho) = \frac{c}{2\kappa}\left\{\left[-\gamma - \log((1+\kappa)\rho)\right] e^{\kappa\rho} + e^{-\rho} \sum_{m=1}^{\infty} \frac{h_m (1+\kappa)^m \rho^m}{m!}\right.$$

$$+\left[\gamma + \log([1-\kappa]\rho)\right] e^{-\kappa\rho} - e^{-\rho} \cdot \sum_{m=1}^{\infty} \frac{h_m (1-\kappa)^m \rho^m}{m!}$$

$$\left. + e^{-\kappa\rho} \log \frac{1+\kappa}{1-\kappa} \right\}$$

$$= \frac{c}{2\kappa}\left\{-2\left[\gamma + \log\rho + \log(1+\kappa)\right] \sinh\kappa\rho + e^{-\rho} \sum_{m=1}^{\infty} \frac{h_m\left[(1+\kappa)^m - (1-\kappa)^m\right] \rho^m}{m!}\right\}.$$

This expression vanished for $\rho \to 0$: $D_{(-1)}(0) = 0$.

Furthermore, one needs the derivative of $D_{(-1)}(\rho)$ in the Sonine transform (19)

$$\frac{c}{2} D'_{(-1)}(\rho) = \frac{c}{2}\left\{e^{\kappa\rho} E_1([1+\kappa]\rho) + e^{-\kappa\rho} E_1([1-\kappa]\rho) - e^{-\kappa\rho} \log \frac{1+\kappa}{1-\kappa}\right\}$$

$$= \frac{c}{2}\left\{-2\left[\gamma + \log(1+\kappa)\,\rho\right] \cosh\kappa\rho + e^{-\rho} \sum_{m=1}^{\infty} \frac{h_m\left[(1+\kappa)^m + (1-\kappa)^m\right] \rho^m}{m!}\right\}.$$

$\frac{c}{2} D'_{(-1)}(d_0 v)$ is regular in the interval $0 < v \leq 1$ and the series development shows that it diverges like $-c \log v$ for $v \to 0$. The Sonine transformation allows, however, a much stronger divergence like $\frac{1}{v^\alpha}$ with $\alpha < 1$. This ensures the convergence of the integral and one obtains from equation (19)

$$S_0(f_{(-1)}) = -\frac{c}{2\pi} e^{-\mu\rho} \int_0^1 dv \frac{\cos(d_0\sqrt{1-v^2})}{\sqrt{1-v^2}}\left\{e^{\kappa d_0 v} E_1([1+\kappa]\, d_0 v)\right.$$

$$\left. + e^{-\kappa d_0 v}\left[E_1([1-\kappa]\, d_0 v) - \log\frac{1+\kappa}{1-\kappa}\right]\right\} \qquad (34)$$

with $d_0 = \rho\sqrt{1-\mu^2} = \rho \sin\vartheta$.

Hence the total solution $f^{(s)}(\rho,\mu)$ of the Boltzmann equation (1), which is singular at $\rho = 0$ and has the density $\int_{-1}^{+1} f^{(s)}(\rho,\mu)\, d\mu = \frac{e^{-\kappa\rho}}{\rho}$, may be represented by

$$f^{(s)}(\rho,\mu) = f_{(-1)}(\rho,\mu) + S_0(f_{(-1)}) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})$$

$$= \frac{c}{2} e^{-\mu\rho}\Bigg\{ -\frac{\pi}{2} Y_0(h) - J_0(h) \log\beta + \sum_{n=1} \frac{1}{n} J_n(h)\left[\frac{1}{\beta^n} - (-\beta)^n\right]$$

$$- \frac{1}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}}\Bigg\{ e^{\kappa d_0 s} E_1([1+\kappa]\, d_0 s)$$

$$+ e^{-\kappa d_0 s}\left[E_1([1-\kappa]\, d_0 s) - \log\frac{1+\kappa}{1-\kappa}\right]\Bigg\}\Bigg\}$$

$$+ \frac{1}{\pi} e^{-\mu\rho}\left\{ \frac{\cos d_0}{d_0} - \kappa\int_0^1 \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}}\, e^{-\kappa d_0 s}\, ds \right\} \qquad (35)$$

with $d_0 = \rho\sqrt{1-\mu^2}$ (distance of the neutron ray from the origin)

$$h = \rho\sqrt{(1-\mu^2)(1-\kappa^2)} = d_0\sqrt{1-\kappa^2}.$$

$$\beta = \sqrt{\frac{1-\mu}{1+\mu}\cdot\frac{1+\kappa}{1-\kappa}}$$

$$|\kappa| < 1.$$

This is the desired solution; one sees, however, that in the originally chosen partial solution $f_{(-1)}(\rho,\mu)$ the terms

$$-\frac{\pi}{2} Y_0(h) - J_0(h) \log\sqrt{\frac{1+\kappa}{1-\kappa}}$$

are functions of the variables $\rho\sqrt{1-\mu^2}$ alone and satisfy the homogeneous partial differential equation (8). They are of the type f_H. Hence the Sonine transform

will contain these terms with the negative sign. They are superfluous in principle, but by removing them one obtains a rather lengthy expression into the Sonine transform. We shall be contented here with the removal of the log ρ-term which is easily recognized in the Sonine transform. Using the series development of $Y_0(h)$

$$-\frac{\pi}{2} Y_0(h) = -\left(\gamma + \log\frac{h}{2}\right) J_0(h) + \sum_{m=1}^{\infty} \frac{(-1)^m h_m}{(m!)^2} \left(\frac{h}{2}\right)^{2m}$$

with $h_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \quad m \geq 1$

and $\gamma = \lim_{m\to\infty} (h_m - \log m) = 0.577216 \ldots$

one finds

$$-\frac{\pi}{2} Y_0(h) - J_0(h) \log\sqrt{\frac{1+\kappa}{1-\kappa}} = -\left[\gamma + \log d_0 + \log(1+\kappa) - \log 2\right] J_0(h) + \sum_{m=1}^{\infty} \frac{(-1)^m h_m}{(m!)^2} \left(\frac{h}{2}\right)^{2m}.$$

With Bessel's integral[7]

$$J_0(h) = \frac{2}{\pi}\int_0^1 ds \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} \cosh \kappa d_0 s \tag{36}$$

this expression may be written

$$-- Y_0(h) - J_0(h) \log\sqrt{\frac{1+\kappa}{1-\kappa}}$$

$$= \frac{2}{\pi}\int_0^1 ds \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} \left[-\gamma - \log d_0 - \log(1+\kappa) + \log 2\right] \cosh \kappa d_0 s$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m h_m}{(m!)^2} \left(\frac{h}{2}\right)^{2m} \tag{37}$$

(7) G. N. Watson, "A Treatise on the Theory of Bessel Functions", p. 21 equation (1), Cambridge University Press (1948).

Furthermore, using the series development (33) of $E_1(x)$ to express the E_1-functions in the Sonine transform, one finds

$$- \frac{1}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} \left\{ e^{\kappa d_0 s}\, E_1(\left[1+\kappa\right] d_0 s) + e^{-\kappa d_0 s}\left[E_1(\left[1-\kappa\right] d_0 s) - \log\frac{1+\kappa}{1-\kappa}\right]\right\}$$

$$= \frac{2}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} \left\{ \left[\gamma + \log d_0 + \log(1+\kappa) + \log s\right] \cosh \kappa d_0 s \right.$$

$$\left. - \frac{1}{2}\, e^{-d_0 s} \sum_{m=1}^{\infty} \frac{h_m\left[(1+\kappa)^m + (1-\kappa)^m\right] d_0^m\, s^m}{m!} \right\}. \qquad (38)$$

The log ρ-part is contained in log d_0 in both expressions but with the opposite sign. The log ρ-terms are omitted at all from the total solution, if one writes the corresponding part of the solution

$$f_I^{(s)}(\rho,\mu) = \frac{c}{2}\, e^{-\mu\rho} \left(-\frac{\pi}{2} Y_0(h) - J_0(h) \log\sqrt{\frac{1+\kappa}{1-\kappa}} \right.$$

$$- \frac{1}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} \left\{ e^{\kappa d_0 s}\, E_1(\left[1+\kappa\right] d_0 s) \right.$$

$$\left.\left. + e^{-\kappa d_0 s}\left[E_1(\left[1-\kappa\right] d_0 s) - \log\frac{1+\kappa}{1-\kappa}\right]\right\}\right)$$

$$= \frac{c}{2}\, e^{-\mu\rho} \left(\sum_{m=1}^{\infty} \frac{(-1)^m\, h_m}{(m!)^2} \left(\frac{h}{2}\right)^{2m} \right.$$

$$+ \frac{2}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} (\log 2s) \cosh \kappa d_0 s$$

(39)

$$\left. - \frac{1}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}}\, e^{-d_0 s} \sum_{m=1}^{\infty} \frac{h_m\left[(1+\kappa)^m + (1-\kappa)^m\right] d_0^m\, s^m}{m!} \right).$$

The removal of superfluous terms occuring in $f_I^{(s)}(\rho,\mu)$ is <u>not</u> complete because the first term of the three terms above will be contained in the other two terms with the opposite sign. But $f_I^{(s)}(\rho,\mu)$ contains now only positive powers of ρ. The total solution for $\kappa < 1$ may be represented

$$f^{(s)}(\rho,\mu) = f_I^{(s)}(\rho,\mu) + f_{II}^{(s)}(\rho,\mu) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) \tag{40}$$

with $f_I^{(s)}(\rho,\mu)$ in the form (39) and with

$$f_{II}^{(s)}(\rho,\mu) = \frac{c}{2} e^{-\mu\rho}\left\{J_0(h) \log\sqrt{\frac{1+\mu}{1-\mu}} + \sum_{n=1}^{\infty} \frac{J_n(h)}{n}\left[\beta^{-n} - (-\beta)^n\right]\right\} \tag{41}$$

and finally

$$e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) = \frac{1}{\pi} e^{-\mu\rho}\left\{\frac{\cos d_0}{d_0} - \kappa \int_0^1 \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} e^{-\kappa d_0 s}\, ds\right\}. \tag{42}$$

The notations d_0, h, β have the meaning:

$d_0 = \rho\sqrt{1-\mu^2}$, $h = d_0\sqrt{1-\kappa^2}$, $\beta = \sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\kappa}{1-\kappa}}$. The singularity of $f^{(s)}(\rho,\mu)$ at the <u>center</u> of the sphere $\rho = 0$ is given by

$$\frac{1}{\pi}\,\frac{\cos d_0}{d_0} = \frac{1}{\pi}\,\frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}}.$$

No other singular term does occur with importance at $\rho = 0$.

2. There exists a way to find another partial solution which does not have a log ρ-term. One obtains it by insertion of κ for μ_1 into the lower limit of the integral (28). It is for $|\kappa| < 1$

$$f_{(\kappa)}(\rho,\mu) = \frac{c}{2} e^{-\mu\rho} \int_\beta^1 e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)\, s - (1+\kappa)(1-\kappa)\,\frac{1}{s}\right]} \frac{ds}{s} \tag{29a}$$

$$= \frac{c}{2} e^{-\mu\rho}\left\{- J_0(h) \log\beta + \sum_{n=1}^{\infty} \frac{J_n(h)}{n}\left[\frac{1}{\beta^n} - (-\beta)^n\right] - s_{00}(h)\right\}.$$

This solution does not contain a log ρ-term from the outset because the Lommel-function $s_{00}(z)$, which is proportional [for the indices (0,0)] to the Weber function $E_0(z)$ - see reference (4) page 42 formula (83), page 40 formula (70), and page 36 formula (37):

$$s_{00}(z) = -\frac{\pi}{2}E_0(z) = \frac{\pi}{2}\left[Y_0(z)\int_0^z J_0(z)\,dz - J_0(z)\int_0^z Y_0(z)\,dz\right]$$

$$= \sum_{n=0} \frac{(-1)^n\, z^{2n+1}}{\left[1\cdot 3\cdot 5\cdot \ldots\ (2n+1)\right]^2}$$

is a power series around z = 0. The log ρ-term in the partial solution $f_{(-1)}(\rho,\mu)$ under 1(a) was caused just by the lower limit s = 0 of the integral (29). The <u>disadvantage</u> of the partial solution $f_{(\kappa)}(\rho,\mu)$ on the other hand is that the density is not more easily calculable. Furthermore, it also contains superfluous terms which depend on $\rho\sqrt{1-\mu^2}$ only, namely,

$$-J_0(h)\,\log\sqrt{\frac{1+\kappa}{1-\kappa}} - s_{00}(h).$$

These would be reproduced with the opposite sign by the Sonine transformation. Of course, if this partial solution would give the desired density fortunately, one would not have to apply the Sonine transformation at all. It remains a <u>problem to be solved</u> which selection of the lower limit μ_1 yields the desired density without the application of the Sonine transformation.

3. Applying the Sonine transformation, it seems reasonable to relate the partial solution not more to the integral (28) and a special choice of μ_1, but to take just the function

$$f_{II}^{(s)}(\rho,\mu) = \frac{c}{2}\,e^{-\mu\rho}\left\{J_0(h)\,\log\sqrt{\frac{1+\mu}{1-\mu}} + \sum_{n=1}^{\infty}\frac{J_n(h)}{n}\left[\frac{1}{\beta^n} - (-\beta)^n\right]\right\} \qquad (41)$$

for the original partial solution. The other terms of $f^{(s)}(\rho,\mu)$ in equation (40) are solutions of the homogeneous partial differential equation because they have the form $f_H(\rho,\mu)$ of equation (7). Hence $f_{II}^{(s)}(\rho,\mu)$ is indeed a partial solution of the inhomogeneous differential equation (6). An advantage of this solution is that its region of validity can be extended to the whole complex κ-plane.

As a check, which is independent of all calculations done before, one may verify that $f_{II}^{(s)}(\rho,\mu)$ is a solution of equation (6). For this check remember

$$h = \rho\sqrt{(1-\mu^2)(1-\kappa^2)}, \qquad \beta \equiv \sqrt{\frac{1-\mu}{1+\mu}\,\frac{1+\kappa}{1-\kappa}},$$

$$\frac{\partial h}{\partial \rho} = \frac{h}{\rho}, \quad \frac{\partial h}{\partial \mu} = -\frac{\mu h}{1-\mu^2}, \quad \frac{\partial \beta}{\partial \mu} = -\frac{\beta}{1-\mu^2}, \quad \frac{\partial \log\sqrt{\frac{1+\mu}{1-\mu}}}{\partial \mu} = \frac{1}{1-\mu^2}.$$

Then the derivatives involved in the equation will be $\left(' = \frac{\partial}{\partial h}\right)$

$$\frac{\partial f_{II}^{(s)}}{\partial \rho} = -\mu f_{II}^{(s)} + \frac{c}{2}\, e^{-\mu\rho}\,\frac{h}{\rho}\left[\left(\log\sqrt{\frac{1+\mu}{1-\mu}}\right)J_0'(h) + \sum_{n=1}^{\infty}\frac{J_n'(h)}{n}\left(\frac{1}{\beta^n} - (-\beta)^n\right)\right]$$

$$\frac{\partial f_{II}^{(s)}}{\partial \mu} = -\rho f_{II}^{(s)} + \frac{c}{2}\, e^{-\mu\rho}\left\{-\frac{\mu h}{1-\mu^2}\left[\left(\log\sqrt{\frac{1+\mu}{1-\mu}}\right)J_0'(h) + \sum_{n=1}^{\infty}\frac{J_n'(h)}{n}\left(\frac{1}{\beta^n} - (-\beta)^n\right)\right]\right.$$
$$\left. + \frac{J_0(h)}{1-\mu^2} + \sum_{n=1}^{\infty}\frac{J_n(h)}{1-\mu^2}\left(\frac{1}{\beta^n} + (-\beta)^n\right)\right\}.$$

Insertion of these expressions into the left hand side of equation (6) yields

$$\mu\frac{\partial f_{II}^{(s)}}{\partial \rho} + \frac{1-\mu^2}{\rho}\,\frac{\partial f_{II}^{(s)}}{\partial \mu} + f_{II}^{(s)} = \overbrace{\left(-\mu^2 - 1 + \mu^2 + 1\right)}^{0} f_{II}^{(s)}$$
$$+ \frac{c}{2}\, e^{-\mu\rho}\underbrace{\left\{\frac{\mu h}{\rho} + \frac{1-\mu^2}{\rho}\left(-\frac{\mu h}{1-\mu^2}\right)\right\}}_{0}\left[\cdots\right]$$
$$+ \frac{c}{2}\,\frac{e^{-\mu\rho}}{\rho}\left[J_0(h) + \sum_{n=1}^{\infty}\left(\frac{1}{\beta^n} + (-\beta)^n\right)J_n(h)\right].$$

The last term should be equal to $\frac{c}{2}\ \frac{e^{-\kappa\rho}}{\rho}$. Hence it remains to show that

$$J_0(h) + \sum_{n=1}^{\infty} \left(\frac{1}{\beta^n} + (-\beta)^n \right) J_n(h) = e^{(\mu-\kappa)\rho} \quad . \tag{43}$$

Verification: Put $x = 1-\kappa\mu$, $y = \mu-\kappa$ and consequently $\beta = \sqrt{\frac{x-y}{x+y}}$ and $h = \rho\sqrt{x^2-y^2}$. Then the left hand side of the equation will take the form

$$J_o\left(\rho\sqrt{x^2-y^2}\right) + \sum_{n=1}^{\infty} \left[\left(\frac{x+y}{x-y}\right)^{n/2} + (-1)^n \left(\frac{x-y}{x+y}\right)^{n/2} \right] J_n\left(\rho\sqrt{x^2-y^2}\right)$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell!)^2} \left(\frac{\rho}{2}\right)^{2\ell} (x^2-y^2)^\ell + \sum_{\substack{n=1 \\ \ell=0}}^{\infty} \left[\left(\frac{x+y}{x-y}\right)^{n/2} + (-1)^n \left(\frac{x-y}{x+y}\right)^{n/2} \right] \frac{(-1)^\ell \left(\frac{\rho}{2}\right)^{n+2\ell}}{\ell!(\ell+n)!} (x^2-y^2)^{\frac{n}{2}+\ell} .$$

One can include the first series into the second double series, obtaining

$$= \sum_{n,\,\ell=0}^{\infty} \frac{(-1)^\ell \left(1 - \frac{1}{2}\delta_{n0}\right)}{\ell!(\ell+n)!} \left[(y+x)^n + (y-x)^n\right] (x^2-y^2)^\ell \left(\frac{\rho}{2}\right)^{n+2\ell}$$

$$= \sum_{n,\,\ell=0}^{\infty} \frac{\left(\frac{\rho y}{2}\right)^{n+2\ell}}{(n+2\ell)!} \binom{n+2\ell}{\ell} \left(1-\tfrac{1}{2}\delta_{n0}\right) \left[\left(1+\frac{x}{y}\right)^{n+\ell} \left(1-\frac{x}{y}\right)^\ell + \left(1-\frac{x}{y}\right)^{n+\ell} \left(1+\frac{x}{y}\right)^\ell \right]$$

A change of the indices of summation to $S = n + 2\ell$, $n = S - 2\ell$ yields

$$= \sum_{S=0}^{\infty} \frac{(\rho y)^S}{S!\ 2^S} \sum_{\ell=0}^{\left[\frac{S}{2}\right]}{}' \binom{S}{\ell} \left(1 - \tfrac{1}{2}\delta_{S,2\ell}\right) \left[\left(1+\frac{x}{y}\right)^{S-\ell} \left(1-\frac{x}{y}\right)^\ell + \left(1-\frac{x}{y}\right)^{S-\ell} \left(1+\frac{x}{y}\right)^\ell \right]$$

with $\left[\frac{S}{2}\right] = \begin{cases} \frac{S}{2} \text{ for S even} \\ \frac{S-1}{2} \text{ for S odd} \end{cases}$

The factor $1 - \frac{1}{2}\delta_{S,2\ell}$ ensures that the term $\ell = \frac{S}{2}$ occurs only for even S and only half as often as the other process; so the sum over S is just a binominal series

$$= \sum_{S=0}^{\infty} \frac{(\rho y)^S}{S!} \cdot \frac{1}{2^S} \sum_{\ell=0}^{S} \binom{S}{\ell} \left(1 + \frac{x}{y}\right)^{S-\ell} \left(1 - \frac{x}{y}\right)^{\ell} = \sum_{S=0}^{\infty} \frac{(\rho y)^S}{S!} \frac{1}{2^S} \left(1 + \frac{x}{y} + 1 - \frac{x}{y}\right)^S$$

$$= \sum_{S=0}^{\infty} \frac{(\rho y)^S}{S!} = e^{\rho y} = e^{\rho(\mu-\kappa)} \cdot \quad \text{q. e. d.}$$

To recognize that this partial solution is generally valid in the whole complex κ-plane one may transform the series, contained in it, into an integral by application of the formula[8]

$$J_n(Z) = \frac{Z^n}{2^{n-1}(n-1)!} \int_0^1 J_0(Zt)\, t(1 - t^2)^{n-1}\, dt. \tag{44}$$

It yields

$$\sum_{n=1}^{\infty} \frac{J_n(h)}{n} \left[\frac{1}{\beta^n} - (-\beta)^n\right]$$

$$= \sum_{n=1}^{\infty} \frac{h^n}{2^{n-1} n!} \left[\frac{1}{\beta^n} - (-\beta)^n\right] \int_0^1 J_0(ht)\, t(1 - t^2)^{n-1}\, dt$$

$$= 2 \int_0^1 \frac{t\,dt\, J_0(ht)}{1 - t^2} \sum_{n=1}^{\infty} \frac{1}{n!} \left\{\left[\frac{h}{2\beta}(1 - t^2)\right]^n - \left[-\frac{h\beta}{2}(1 - t^2)\right]^n\right\}$$

$$= 2 \int_0^1 \frac{t\,dt\, J_0(ht)}{1 - t^2} \left[e^{\frac{h}{2\beta}(1-t^2)} - e^{-\frac{h\beta}{2}(1-t^2)}\right]$$

$$= \int_0^1 \frac{dv}{v} J_0(h\sqrt{1-v}) \left[e^{\frac{h}{2\beta} v} - e^{-\frac{h\beta}{2} v}\right]$$

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(8) G. Petiau, "La Théorie des Fonctions de Bessel," Paris 1955, p. 21, formula (115).

$$= \int_0^1 \frac{dv}{v} J_0(h\sqrt{1-v}) \left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v} \right]$$

remembering $\frac{h}{\beta} = \rho(1-\kappa)(1+\mu)$ and $h\beta = \rho(1+\kappa)(1-\mu)$. The - sign between the exponentials ensures the convergence of the integral at $v = 0$; this integral is convergent in the whole complex κ-plane. Inserting it into equation (41), the partial solution $f_{II}^{(s)}(\rho,\mu)$ becomes

$$f_{II}^{(s)}(\rho,\mu) = \frac{c}{2} e^{-\mu\rho} \left\{ J_0(h) \log\sqrt{\frac{1+\mu}{1-\mu}} + \int_0^1 \frac{dv}{v} J_0(h\sqrt{1-v}) \left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v} \right] \right\} . \quad (45)$$

The density which belongs to this partial solution

$$P_{II}^{(s)}(\rho) = \int_{-1}^{+1} f_{II}^{(s)}(\rho,\mu) d\mu = P_a(\rho) + P_b(\rho) \quad (46)$$

consists of the following two terms:

$$(a) \quad P_a(\rho) = \frac{c}{2} \int_{-1}^{+1} e^{-\mu\rho} J_0(h) \log\sqrt{\frac{1+\mu}{1-\mu}} d\mu \quad (47)$$

$$= -2 \cdot \frac{c}{2} \int_0^1 \left(\sinh \mu\rho\right) J_0\left(\rho\sqrt{(1-\mu^2)(1-\kappa^2)}\right) \log\sqrt{\frac{1+\mu}{1-\mu}} \, d\mu .$$

The last expression shows that $P_a(\rho)$ depends on κ^2 and is negative for positive ρ. One may represent the log-term by

$$\log\sqrt{\frac{1+\mu}{1-\mu}} = \frac{1}{2} \int_0^\infty \frac{dv}{v} e^{-v} \left(e^{\mu v} - e^{-\mu v} \right)$$

and one obtains from the first equation

$$P_a(\rho) = -\frac{c}{4}\int_0^\infty \frac{dv}{v} e^{-v} \int_{-1}^{+1} d\mu \left[e^{-\mu(\rho+v)} - e^{-\mu(\rho-v)}\right] J_0\left(\rho\sqrt{(1-\kappa^2)(1-\mu^2)}\right)$$

The application of an aid formula to the second integration

$$\int_{-1}^{+1} e^{-\mu t} J_0(\rho\sqrt{(1-\kappa^2)(1-\mu^2)}\ d\mu = 2\,\frac{\sinh\sqrt{t^2-(1-\kappa^2)\rho^2}}{\sqrt{t^2-(1-\kappa^2)\rho^2}} \tag{48}$$

with $t = \rho + v$ respectively $t = \rho - v$, yields

$$P_a(\rho) = -\frac{c}{2}\int_0^\infty \frac{dv}{v}\, e^{-v}\left\{\frac{\sinh\sqrt{v^2+2\rho v+\kappa^2\rho^2}}{\sqrt{v^2+2\rho v+\kappa^2\rho^2}} - \frac{\sinh\sqrt{v^2-2\rho v+\kappa^2\rho^2}}{\sqrt{v^2-2\rho v+\kappa^2\rho^2}}\right\}. \tag{49}$$

The integrand vanishes at the upper limit $v = \infty$ proportional to $\frac{1}{v^2}$ and the expression in the curly brackets contains a factor v, which counterbalances the corresponding factor v in the denominator of the integrand. The integral exists for all values of κ and ρ.

Proof of the aid formula (48):

$$I = \int_{-1}^{+1} e^{-\mu t} J_0(\rho\sqrt{(1-\kappa^2)(1-\mu^2)}\ d\mu$$

$$= 2\int_0^1 \cosh \mu t\, J_0(\rho\sqrt{(1-\kappa^2)(1-\mu^2)}\ d\mu$$

$$= 2\sum_{n,\ell=0}^{\infty} \frac{t^{2n}}{(2n)!}\,\frac{(-1)^\ell}{(\ell!)^2}\left|\frac{\rho}{2}\right|^{2\ell}(1-\kappa^2)^\ell \int_0^1 \mu^{2n}(1-\mu^2)^\ell d\mu$$

With $\int_0^1 \mu^{2n}(1-\mu^2)^{\ell} d\mu = \frac{1}{2} \frac{\Gamma(n+\frac{1}{2})\ \ell!}{\Gamma(n+\ell+\frac{3}{2})}$

$$= \frac{2^{2\ell+1}(2n-1)!(n+\ell)!\ \ell!}{(n-1)!(2n+2\ell+1)\ !}$$

the integral I will become

$$I = 2 \sum_{n,\ell=0}^{\infty} \frac{(-1)^{\ell}(1-\kappa^2)^{\ell}(n+\ell)!\ t^{2n}\rho^{2\ell}}{n!\ \ell!(2n+2\ell+1)!}$$

and a change of the indices of summation to $S = n + \ell$ and ℓ yields

$$I = 2 \sum_{S=0}^{\infty} \frac{t^{2S}}{(2S+1)!} \sum_{\ell=0}^{S} \binom{S}{\ell} (-1)^{\ell}(1-\kappa^2)^{\ell} \left(\frac{\rho}{t}\right)^{2\ell}$$

$$= 2 \sum_{S=0}^{\infty} \frac{t^{2S}}{(2S+1)!} \left[1 - (1-\kappa^2)\left(\frac{\rho}{t}\right)^2\right]^S = 2 \sum_{S=0}^{\infty} \frac{\left[\sqrt{t^2 - (1-\kappa^2)\rho^2}\ \right]^{2S}}{(2S+1)!}$$

$$= 2 \frac{\sinh\sqrt{t^2 - (1-\kappa^2)\rho^2}}{\sqrt{t^2 - (1-\kappa^2)\rho^2}} \text{ q.e.d.}$$

For $t = \rho$ we have simply

$$\int_{-1}^{+1} e^{-\mu\rho} J_0\left(\rho\sqrt{(1-\kappa^2)(1-\mu^2)}\right) d\mu = 2 \frac{\sinh \kappa\rho}{\kappa\rho} . \qquad (48a)$$

The first part $P_a(\rho)$ of the density may be represented by some power series of ρ

$$P_a(\rho) = -2c\rho \sum_{\ell,m,n=0}^{\infty} \frac{(-1)^m(1-\kappa^2)^m(2\ell+2n+1)!(\ell+m+n+1)!}{(2\ell+1)!m!(\ell+n)!(2\ell+2m+2n+3)!} \frac{\rho^{2(\ell+m)}}{2n+1} \tag{50}$$

or

$$= -4c\rho \sum_{n=0}^{\infty}{}' \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \sum_{m=0}^{n-2j-2} \frac{n!(2n-2j-2m-4)!\,2^{2j}}{(2n)!(n-2j-m-2)!(2j+1)!m!} \kappa^{2m} \rho^{2(j+m)},$$

which show that $P_a(\rho)$ is proportional ρ for small ρ and vanishes at $\rho = 0$. Therefore, the corresponding expression $\frac{c}{2} D_a(\rho) = \rho P_a(\rho)$ is proportional ρ^2 for small ρ. This stronger kind of going to zero at $\rho = 0$ is a benefit of taking a partial solution, in which the variable S of integration - compare equations (29a, 29) - does not meet the essential singularity of the integrand at $S = 0$.

(b) The second part of the density is

$$P_b(\rho) = \frac{c}{2} \int_{-1}^{+1} d\mu\, e^{-\mu\rho} \int_0^1 \frac{dv}{v} J_0\left(h\sqrt{1-v}\right)\left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v}\right] \tag{51}$$

If one changes the order of integration, one may apply the aid formula (48) inserting $\rho\sqrt{1-v}$ for ρ and $\rho(1 - \frac{1-\kappa}{2} v)$ respectively $\rho(1 - \frac{1+\kappa}{2} v)$ for t. This yields

$$P_b(\rho) = \frac{c}{\rho} \int_0^1 \frac{dv}{v} \left\{ e^{\frac{\rho}{2}(1-\kappa)v} \frac{\sinh\left(\rho\sqrt{\left(1 - \left[\frac{1-\kappa}{2}\right]v\right)^2 - (1-\kappa^2)(1-v)}\right)}{\sqrt{\left(1 - \left[\frac{1-\kappa}{2}\right]v\right)^2 - (1-\kappa^2)(1-v)}} \right.$$

$$- e^{-\frac{\rho}{2}(1+\kappa)v} \frac{\sinh\left(\rho\sqrt{(1-\frac{1+\kappa}{2}v)^2 - (1-\kappa^2)(1-v)}\right)}{\sqrt{(1-\frac{1+\kappa}{2}v)^2 - (1-\kappa^2)(1-v)}}$$

$$= \frac{c}{\rho} e^{-\kappa\rho} \int_0^1 \frac{dv}{v} \left\{ \frac{e^{\rho\left[2\kappa+(1-\kappa)v\right]} - 1}{2\kappa+(1-\kappa)v} - \frac{e^{\rho\left[2\kappa-(1+\kappa)v\right]} - 1}{2\kappa-(1+\kappa)v} \right\}$$

$$= \frac{c}{2\kappa\rho} \left\langle e^{+\kappa\rho} \left\{ \int_0^{(1-\kappa)\rho} du \frac{(\cosh u)-1}{u} - \int_0^{(1+\kappa)\rho} du \frac{(\cosh u)-1}{u} + \int_0^{(1-\kappa)\rho} du \frac{\sinh u}{u} \right.\right.$$

$$\left. + \int_0^{(1+\kappa)\rho} du \frac{\sinh u}{u} \right\}$$

$$- e^{-\kappa\rho} \left\{ \int_{2\kappa\rho}^{(1+\kappa)\rho} du \frac{(\cosh u)-1}{u} - \int_{2\kappa\rho}^{-(1-\kappa)\rho} du \frac{(\cosh u)-1}{u} \right.$$

$$\left.\left. + \int_{2\kappa\rho}^{(1+\kappa)\rho} du \frac{\sinh u}{u} - \int_{2\kappa\rho}^{-(1-\kappa)\rho} du \frac{\sinh u}{u} \right\} \right\rangle$$

$$= \frac{c}{2\kappa\rho} \left\langle e^{\kappa\rho} \left\{ \mathrm{Shi}(1+\kappa)\rho + \mathrm{Shi}(1-\kappa)\rho - \left[\widetilde{\mathrm{Chi}}(1+\kappa)\rho - \widetilde{\mathrm{Chi}}(1-\kappa)\rho\right] \right\} \right.$$

$$\left. - e^{-\kappa\rho} \left\{ \mathrm{Shi}(1+\kappa)\rho + \mathrm{Shi}(1-\kappa)\rho + \left[\widetilde{\mathrm{Chi}}(1+\kappa)\rho - \widetilde{\mathrm{Chi}}(1-\kappa)\rho\right] \right\} \right\rangle$$

$$= \frac{c}{\kappa\rho} \left\{ \sinh\kappa\rho \left[\mathrm{Shi}(1+\kappa)\rho + \mathrm{Shi}(1-\kappa)\rho\right] - \cosh\kappa\rho \left[\widetilde{\mathrm{Chi}}(1+\kappa)\rho - \widetilde{\mathrm{Chi}}(1-\kappa)\rho\right] \right\}, \qquad (52)$$

in which $\mathrm{Shi}\ x = \int_0^x \frac{\sinh t}{t}\,dt \equiv \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n+1)!}$

and $\widetilde{\widetilde{\mathrm{Chi}}}\ x = \int_0^x \frac{(\cosh t)-1}{t}\,dt \equiv \sum_{n=0}^{\infty} \frac{x^{2n}}{2n\cdot(2n)!}$.

Using once more the integral representations for Shi x and $\widetilde{\widetilde{\mathrm{Chi}}}$ x one obtains

$$\mathrm{Shi}(1+\kappa)\rho + \mathrm{Shi}(1-\kappa)\rho = 2\int_0^{\rho} (\sinh u)\cosh(\kappa u)\,\frac{du}{u} \quad \text{and}$$

$$\widetilde{\widetilde{\mathrm{Chi}}}(1+\kappa)\rho - \widetilde{\widetilde{\mathrm{Chi}}}(1-\kappa)\rho = 2\int_0^{\rho} (\sinh u)\sinh(\kappa u)\,\frac{du}{u}.$$

If one inserts these expressions into the last version (52) of $P_b(\rho)$ one gets a short integral representation

$$P_b(\rho) = \frac{2c}{\kappa\rho}\int_0^{\rho} \frac{du}{u}(\sinh u)\sinh(\kappa[\rho-u]). \tag{53}$$

This integral is proportional ρ for small ρ and $\frac{c}{2} D_b(\rho) = \rho\, P_b(\rho)$ is proportional ρ^2 similar to $\frac{c}{2} D_a(\rho)$. Hence certainly $\frac{c}{2} D_b(0) = 0$. One obtains altogether

$$\frac{c}{2} D_{II}^{(s)}(0) \equiv \lim_{\rho\to 0}\left[\rho\, P_{II}^{(s)}(\rho)\right] = \frac{c}{2}\left(D_a(0) + D_b(0)\right) = 0. \tag{54}$$

and

$$\frac{c}{2} D_{II}^{(s)}(\rho) \equiv \rho\, P_{II}^{(s)}(\rho) =$$

$$= \frac{c}{2}\left\{ -\rho\int_0^{\infty} \frac{du}{u} e^{-u}\left\{ \frac{\sinh\sqrt{u^2+2\rho u+\kappa^2\rho^2}}{\sqrt{u^2+2\rho u+\kappa^2\rho^2}} - \frac{\sinh\sqrt{u^2-2\rho u+\kappa^2\rho^2}}{\sqrt{u^2-2\rho u+\kappa^2\rho^2}} \right\} + \right.$$

$$+ \frac{4}{\kappa} \int_0^{\rho} \frac{du}{u} \sinh u \cdot \sinh(\kappa[\rho - u]) \Bigg\} . \qquad (55)$$

Hence the Sonine transform (21) is

$$S_0(f_{II}^{(s)}) = - \frac{c}{2} e^{-\mu\rho} \frac{1}{\pi} \int_0^1 \frac{\cos(d_0\sqrt{1-v^2})}{\sqrt{1-v^2}} D_{II}^{(s)\prime}(d_0 v)dv \qquad (56)$$

with $d_0 = \rho\sqrt{1-\mu^2}$. One has to derive $D_{II}^{(s)}(\rho)$ in respect to ρ and to insert $d_0 v$ instead of ρ in the derivative before inserting it into the integrand of the last integral.

The total solution, singular at the centre $\rho = 0$, is now

$$f^{(s)}(\rho,\mu) = f_{II}^{(s)}(\rho,\mu) + S_0\left(f_{II}^{(s)}\right) + e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})$$

with $f_{II}^{(s)}(\rho,\mu)$ from equation (41) in the form of a series of Bessel functions or from equation (45) in an integral representation. Then one has to add the Sonine transform $S_0(f_{II}^{(s)})$ from equation (56), which is of course a function of $d_0 = \rho\sqrt{1-\mu^2}$. The last terms $e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})$, which is caused by the desired density, remains untouched by the choice of the original partial solution; according to equation (22) it carries the important singularity at the centre of the sphere $e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})$

$$= \frac{1}{\pi} e^{-\mu\rho} \left\{ \frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}} - \kappa \int_0^1 \frac{\cos(\rho\sqrt{1-\mu^2}(1-s^2))}{\sqrt{1-s^2}} e^{-\kappa\rho\sqrt{1-\mu^2}s} \, ds \right\} .$$

This amount is given irrespective to the value of the constant c of multiplication, whereas the other two terms $f_{II}^{(s)}(\rho,\mu) + S_0(f_{II}^{(s)})$ contain c as a factor. Furthermore, the term $e^{-\mu\rho} R(\rho\sqrt{1-\mu^2})$ gives the whole density, whereas the densities belonging to $f_{II}^{(s)}$ and $S_0(f_{II}^{(s)})$ cancel one another in consequence of

the theorem III.

No relation between κ and c is imposed upon the solution, i.e. no characteristic equation is necessary for these solutions. They are valid for any value in the complex κ-plane.

II. The Solutions which are Regular at the Centre of the Sphere.

A. Regular solutions belonging to the discrete spectrum.

$$f(\rho,\mu) = \frac{c}{2} e^{-\mu\rho} \int_0^{a(\mu)} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]} \frac{dS}{S} \tag{58}$$

satisfies the differential equation

$$\mu \frac{\partial f}{\partial \rho} + \frac{1-\mu^2}{\rho} \frac{\partial f}{\partial \mu} + f(\rho,\mu) = \frac{c}{2} \frac{e^{-\mu\rho}}{\rho} \left(1 + \rho \frac{da(\mu)}{d\mu}\right) e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)a(\mu) - (1+\kappa)(1-\mu)\frac{1}{a(\mu)}\right]} \tag{59}$$

1. If one puts a = 1, then one has $\frac{da}{d\mu} = 0$ and the exponent in the last factor on the right hand side becomes

$$\frac{\rho}{2}\left[(1-\kappa)(1+\mu) - (1+\kappa)(1-\mu)\right] = \rho(\mu-\kappa)$$

Therefore $$f(\rho,\mu;-\kappa) = \frac{c}{2} e^{-\mu\rho} \int_0^1 e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]} \frac{dS}{S} \tag{60}$$

is a partial solution of the differential equation above with the right-hand side $\frac{c}{2} \frac{e^{-\kappa\rho}}{\rho}$. This is our previous result in equation (29).

2. If one puts $a = \frac{1+\kappa}{1-\kappa}$, then one has $\frac{da(\mu)}{d\mu} = 0$ again, but the exponent in the last factor on the right-hand side becomes

$$\frac{\rho}{2}\left[(1+\kappa)(1+\mu) - (1-\kappa)(1-\mu)\right] = \rho(\mu+\kappa).$$

Therefore $f(\rho,\mu;+\kappa) = \frac{c}{2} e^{-\mu\rho} \int_0^{\frac{1+\kappa}{1-\kappa}} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]} \frac{dS}{S}$ (61)

is a partial solution of the differential equation above with the right-hand side $\frac{c}{2} \frac{e^{\kappa\rho}}{\rho}$.

3. Half the difference of the second and the first integral satisfies therefore the differential equation above with the right-hand side

$$\frac{c}{2} \frac{\sinh \kappa\rho}{\rho} . \tag{62}$$

It is the solution $f^{(r)}(\rho,\mu)$ which is regular at the centre $\rho = 0$, in the form of the following integral, which is valid for $|k| < 1$:

$$f^{(r)}(\rho,\mu) = \tfrac{1}{2}\left[f(\rho,\mu;\kappa) - f(\rho,\mu;-\kappa)\right]$$

$$= \frac{c}{4} e^{-\mu\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} \frac{dS}{S} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]} \tag{63}$$

Whereas the two partial solutions $f(\rho,\mu;\kappa)$ and $f(\rho,\mu;-\kappa)$ do not satisfy their compatibility equations, because their densities

$$P(\rho;\kappa) = \int_{-1}^{+1} f(\rho,\mu;\kappa)d\mu = \frac{c}{2\kappa\rho}\left\{e^{\kappa\rho} E_1\left(\left[1+\kappa\right]\rho\right) - e^{-\kappa\rho} E_1\left(\left[1-\kappa\right]\rho\right) + e^{+\kappa\rho} \log \frac{1+\kappa}{1-\kappa}\right\} \tag{64}$$

and

$$P(\rho;-\kappa) = \int_{-1}^{+1} f(\rho,\mu;-\kappa)d\mu = \frac{c}{2\kappa\rho}\left\{e^{\kappa\rho} E_1\left(\left[1+\kappa\right]\rho\right) - e^{-\kappa\rho} E_1\left(\left[1-\kappa\right]\rho\right) + e^{-\kappa\rho} \log \frac{1+\kappa}{1-\kappa}\right\} \tag{65}$$

are not equal to $\frac{e^{+\kappa\rho}}{\rho}$ respectively to $\frac{e^{-\kappa\rho}}{\rho}$, their difference $f^{(r)}(\rho,\mu)$ does it. The terms containing an E_1-function drop out of its density $P^{(r)}(\rho) = \frac{1}{2}\left[P(\rho;\kappa) - P(\rho;-\kappa)\right]$, which becomes

$$p^{(r)}(\rho) = \int_{-1}^{+1} f^{(r)}(\rho,\mu)d\mu = \left(\frac{c}{2\kappa}\log\frac{1+\kappa}{1-\kappa}\right)\ \frac{\sinh\kappa\rho}{\rho}\ . \tag{66}$$

The factor in brackets on the right-hand side of this equation must be equal to 1, to give the desired result. This is just the characteristic equation

$$\frac{c}{2\kappa}\log\frac{1+\kappa}{1-\kappa} = 1. \tag{5}$$

Its two roots $\kappa = \pm\kappa_0$ form the discrete spectrum of the Boltzmann-operator.

Here the Sonine-transformation must not be applied, because $f^{(r)}(\rho,\mu)$ is fortunately that partial solution which also fulfills the equation of compatibility, if the validity of the characteristic equation is supposed.

It will be shown later, however, that there exist regular solutions also, for every κ of the whole complex κ-plane.

To proceed in this direction, the regular solution $f^{(r)}(\rho,\mu)$ above will be written in a form, in which the first term gives the whole density $p^{(r)}(\rho) = \frac{\sinh\kappa\rho}{\rho}$, whereas the second term gives just the density 0. If one puts again $S = t\sqrt{\frac{1-\mu}{1+\mu}\frac{1+\kappa}{1-\kappa}}$ the equation (63) for $f^{(r)}(\rho,\mu)$ will become

$$f^{(r)}(\rho,\mu) = \frac{c}{4}\,e^{-\mu\rho}\int_{\sqrt{\frac{1+\mu}{1-\mu}\frac{1-\kappa}{1+\kappa}}}^{\sqrt{\frac{1+\mu}{1-\mu}\frac{1+\kappa}{1-\kappa}}} \frac{dt}{t}\,e^{\frac{h}{2}\left(t-\frac{1}{t}\right)}.$$

with $h = \rho\sqrt{(1-\kappa^2)(1-\mu^2)}$.

The expansion of the integrand in a series of Bessel functions and term by term integration yields

$$f^{(r)}(\rho,\mu) = \frac{c}{4} e^{-\mu\rho} \left\{ J_0(h) \log \frac{1+\kappa}{1-\kappa} + \sum_{n=1}^{\infty} \frac{J_n(h)}{n} \left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}} \right] \left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} + (-1)^n \left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}} \right] \right\} . \tag{67}$$

Using the aid formula (48a) one recognizes that the first-term of the curly bracket gives already the <u>whole</u> density

$$\frac{c}{4} \left(\log \frac{1+\kappa}{1-\kappa} \right) \int_{-1}^{+1} e^{-\mu\rho} J_0(h) d\mu = \frac{c}{2} \left(\log \frac{1+\kappa}{1-\kappa} \right) \frac{\sinh \kappa\rho}{\kappa\rho}$$

$$= \frac{\sinh \kappa\rho}{\rho} ,$$

if the characteristic equation (5) is supposed.

It is easy to show, that the second term of $f^{(r)}(\rho,\mu)$, namely the series from n=1 through n = ∞, has zero-density. This will be proved in the following quite independently. Using formula (44) one may write

$$\sum_{n=1}^{\infty} \frac{J_n(h)}{n} \left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}} \right] \left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} + (-1)^n \left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}} \right]$$

$$= 2 \int_0^1 \frac{J_0(ht)\, t\, dt}{1-t^2} \left\{ e^{\frac{\rho}{2}(1+\kappa)(1+\mu)(1-t^2)} - e^{\frac{\rho}{2}(1-\kappa)(1+\mu)(1-t^2)} \right.$$

$$+ e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)(1-t^2)} - e^{-\frac{\rho}{2}(1-\kappa)(1-\mu)(1-t^2)} \Bigg\}$$

$$= \int_0^1 \frac{dv}{v} J_0(\hbar\sqrt{1-v}) \Bigg\{ \left[e^{\frac{\rho}{2}(1+\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1-\kappa)(1-\mu)v} \right]$$

$$- \left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v} \right] \Bigg\} \qquad (68)$$

after a transformation of the variable of integration to $v = 1 - t^2$ and after arranging the terms in such a way that the second square bracket in tho integrand can be obtained from the first by exchanging κ with $-\kappa$.

We know, however, already the density, which belongs to the second square bracket by comparison with $P_b(\rho)$ in equation (51) and (53), it is just

$$-\tfrac{1}{2} P_b(\rho) = -\frac{c}{\kappa\rho} \int_0^\rho \frac{du}{u} \left(\sinh u\right) \sinh(\kappa\left[\rho-\mu\right]) .$$

This expression remains unchanged, if one replaces κ by $-\kappa$; it gives the contribution of the first square bracket, and the difference of both is zero. Hence the density 0 belongs to the series in $f^{(r)}(\rho,\mu)$.

B. Regular solutions, belonging to every κ in the complex κ-plane. We start from the singular solution $f_{II}^{(s)}(\rho,\mu)$ in the form (41) as a sum and in the form (45) as an integral. Let us denote it now by

$$f_{II}^{(s)}(\rho,\mu;-\kappa)$$

to show that it is a solution of the inhomogeneous partial differential equation (6) with $\frac{c}{2} \frac{e^{-\kappa\rho}}{\rho}$ on the right-hand side. Reversing the sign of κ in this

solution one obtains a solution $f_{II}^{(s)}(\rho,\mu;\kappa)$ of the partial differential equation with the same left-hand side as (6), but with $\frac{c}{2}\ \frac{e^{\kappa\rho}}{\rho}$ on the right-hand side. Finally

$$f_{II}^{(r)}(\rho,\mu) = \frac{1}{2}\left[f_{II}^{(s)}(\rho,\mu;\kappa) - f_{II}^{(s)}(\rho,\mu;\kappa)\right] \tag{69}$$

will satisfy this differential equation with the right-hand side

$$\frac{c}{2}\ \frac{\sinh \kappa\rho}{\rho}$$

as a partial solution which is regular at the center of the sphere. It is in series and in integral representation

$$f_{II}^{(r)}(\rho,\mu) = \frac{c}{4}\, e^{-\mu\rho} \sum_{n=1}^{\infty} \frac{J_n(h)}{n}\left[\left(\frac{1+\mu}{1-\mu}\,\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - (-1)^n\left(\frac{1-\mu}{1+\mu}\,\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}}\right.$$

$$\left. - \left(\frac{1+\mu}{1-\mu}\,\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}} + (-1)^n\left(\frac{1-\mu}{1+\mu}\,\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}}\right]$$

$$= \frac{c}{4}\, e^{-\mu\rho} \int_0^1 \frac{dv}{v}\, J_0(h\sqrt{1-v})\left\{\left[e^{\frac{\rho}{2}(1+\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1-\kappa)(1-\mu)v}\right]\right.$$

$$\left. - \left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v}\right]\right\}.$$

The last expression is identical with the integral in equation (68) to which the density 0 belongs. Hence the partial solution $f_{II}^{(r)}(\rho,\mu)$ does <u>not</u> contribute to the whole desired density

$$\frac{\sinh \kappa\rho}{\rho}$$

and its Sonine transform vanishes

$$S_0\left(f_{II}^{(r)}\right) = 0\ .$$

Only the term $e^{-\mu\rho}\, R(\rho\sqrt{1-\mu^2})$ of $f^{(s)}(\rho,\mu)$ gives a contribution, namely

$$\tfrac{1}{2}\, e^{-\mu\rho}\left\{ R(\rho\sqrt{1-\mu^2};\kappa) - R(\rho\sqrt{1-\mu^2};-\kappa)\right\}$$

$$= e^{-\mu\rho}\cdot\tfrac{1}{2}\left\{ \frac{1}{\pi}\,\frac{\cos d_o}{d_o} + \frac{\kappa}{\pi}\int_0^1 \frac{\cos\left(d_o\sqrt{1-S^2}\right)}{\sqrt{1-S^2}}\, e^{\kappa d_o S}\, dS \right.$$

$$\left. -\left[\frac{1}{\pi}\,\frac{\cos d_o}{d_o} - \frac{\kappa}{\pi}\int_0^1 \frac{\cos(d_o\sqrt{1-S^2})}{\sqrt{1-S^2}}\, e^{-\kappa d_o S}\, dS\right]\right\}$$

$$= \frac{\kappa}{2}\, e^{-\mu\rho}\, J_0(h) \qquad (70)$$

by application of formula (36). The term $\dfrac{\cos d_o}{d_o}$, which is singular at $\rho = 0$, drops out of the solution. The density which corresponds to this part of the solution is

$$\tfrac{1}{2}\int_{-1}^{+1} e^{-\mu\rho}\left[R(\rho\sqrt{1-\mu^2};\kappa) - R(\rho\sqrt{1-\mu^2};-\kappa)\right] d\mu = \frac{\kappa}{2}\int_{-1}^{+1} e^{-\mu\rho}\, J_0(h)\, d\mu = \frac{\sinh \kappa\rho}{\rho}$$

according to equation (48a). Hence the <u>total solution</u>, which is regular at the center of the sphere and which is valid for arbitrary κ's, becomes

$$f^{(r)}(\rho,\mu;\kappa) = \tfrac{1}{2}\, e^{-\mu\rho}\left\{ R(\rho\sqrt{1-\mu^2};\kappa) - R(\rho\sqrt{1-\mu^2};-\kappa)\right\} + f_{II}^{(r)}(\rho,\mu)$$

$$= e^{-\mu\rho}\left\{ \frac{\kappa}{2}\, J_0(h) + \frac{c}{4}\sum_{n=1}^{\infty}\frac{J_n(h)}{n}\left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} + (-1)^n\left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}}\right]\right\} \qquad (71)$$

in the form containing a series, or

$$= e^{-\mu\rho}\left\{\frac{\kappa}{2} J_0(h) + \frac{c}{4}\int_0^1 \frac{dv}{v} J_0(h\sqrt{1-v})\cdot\left\{\left[e^{\frac{\rho}{2}(1+\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1-\kappa)(1-\mu)v}\right]\right.\right.$$
$$\left.\left. - \left[e^{\frac{\rho}{2}(1-\kappa)(1+\mu)v} - e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)v}\right]\right\}\right\} \qquad (72)$$

in the form containing an integral.

κ is not restricted by a characteristic equation, it can be every κ of the complex κ-plane. For that special κ, however, which satisfies the characteristic equation (5), the solution $f^{(r)}(\rho,\mu;\kappa)$ is identical with the solution $f^{(r)}(\rho,\mu)$, which is represented in equation (67). One has to replace $\frac{\kappa}{2}$ in the first term of $f^{(r)}(\rho,\mu;\kappa)$ just by the value which the characteristic equation

$$\frac{\kappa}{2} = \frac{c}{4}\log\frac{1+\kappa}{1-\kappa}$$

gives.

The discrete spectrum is therefore completely embedded in a continuous manifold of solutions with neighboring κ-parameters in spherical geometry. The density belonging to a solution is given by the first term in (67) or (71), whereas the density of the second term is zero.

C. Proof that the regular solutions satisfy the equation $\Delta f = \kappa^2 f$.

1. The operator Δ expressed by our coordinates ρ and μ. Remember the definition of $\rho = \sqrt{x^2+y^2+z^2}$ and $\mu = \cos\vartheta$, where ϑ is the angle between the direction of the neutron $\vec{v}$ (with $|\vec{v}| = 1$) and the radius vector $\vec{\rho}$. Therefore we have $\rho\mu = \vec{\rho}\cdot\vec{v} = xv_x + yv_y + zv_z$

$$\frac{\partial(\rho\mu)}{\partial x} = v_x, \quad \frac{\partial\mu}{\partial x} = \frac{v_x}{\rho} - \frac{\mu x}{\rho^2},$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial f}{\partial\mu}\frac{\partial\mu}{\partial x} = \frac{x}{\rho}\frac{\partial f}{\partial\rho} + \left(\frac{v_x}{\rho} - \frac{\mu x}{\rho^2}\right)\frac{\partial f}{\partial\mu}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} + \frac{x}{\rho}\frac{\partial^2 f}{\partial x \partial \rho} + \left(-2\frac{v_x x}{\rho^3} + 3\frac{\mu x^2}{\rho^4} - \frac{\mu}{\rho^2}\right)\frac{\partial f}{\partial \mu} + \left(\frac{v_x}{\rho} - \frac{\mu x}{\rho^2}\right)\frac{\partial^2 f}{\partial x \partial \mu}$$

$$= \frac{\rho^2 - x^2}{\rho^3}\cdot\frac{\partial f}{\partial \rho} + \frac{x^2}{\rho^2}\frac{\partial^2 f}{\partial \rho^2} + 2\left(\frac{v_x x}{\rho^2} - \frac{\mu x^2}{\rho^3}\right)\frac{\partial^2 f}{\partial \mu \partial \rho} - \left(\frac{2v_x x}{\rho^3} - \frac{\mu(\rho^2 - 3x^2)}{\rho^4}\right)\frac{\partial f}{\partial \mu}$$

$$+ \left(\frac{v_x^2}{\rho^2} - \frac{2\mu v_x x}{\rho^3} + \frac{\mu^2 x^2}{\rho^4}\right)\frac{\partial^2 f}{\partial \mu^2}$$

and finally

$$\Delta f = \frac{1}{\rho^2}\left\{\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right]\right\} \tag{73}$$

2. It is easy to show that the solution $f^{(r)}(\rho,\mu)$ of equation (63), which belongs to the discrete spectrum, satisfies the equation $\Delta f = \kappa^2 f$. Denoting the exponent in its integrand by E for abbreviation

$$E = \exp\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]$$

we have

$$f^{(r)}(\rho,\mu) = \frac{c}{4}\, e^{-\mu\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} E\,\frac{dS}{S}$$

and by applying differentiation under the integral sign

$$\Delta f^{(r)} = f^{(r)} + \frac{c}{4}\, e^{-\mu\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} E\,\frac{dS}{S}\left\{\frac{1}{\rho}\left[(1-\kappa)S - (1+\kappa)\frac{1}{S}\right] - \left[(1-\kappa)(1+\mu)S + (1+\kappa)(1-\mu)\frac{1}{S}\right]\right.$$

$$\left. + \frac{(1-\kappa)^2}{2}(1+\mu)S^2 + \frac{(1+\kappa)^2}{2}(1-\mu)\frac{1}{S^2}\right\}$$

Partial integration of the $\frac{1}{\rho}$ - term yields

$$\Delta f^{(r)} = f^{(r)} + \frac{c}{4} e^{-\mu\rho} \left\{ \frac{1}{\rho} \left[(1-\kappa)S + (1+\kappa)\frac{1}{S} \right] E \right\}_{S=1}^{S=\frac{1+\kappa}{1-\kappa}}$$

$$+ \frac{c}{4} e^{-\mu\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} \left\{ \kappa^2 - 1 - \left[(1-\kappa)(1+\mu)S + (1+\kappa)(1-\mu)\frac{1}{S} \right] \right\} E \, \frac{dS}{S}$$

$$= \kappa^2 f^{(r)} + c \, \frac{\sinh \kappa\rho}{\rho} - \frac{c}{4} e^{-\mu\rho} \cdot \frac{2}{\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} \left(\frac{d}{dS} E \right) ds$$

$$= \kappa^2 f^{(r)} + c \, \frac{\sinh \kappa\rho}{\rho} - c \, \frac{\sinh \kappa\rho}{\rho} = \kappa^2 \, f^{(r)}(\rho,\mu) \quad \text{q.e. d.}$$

3. To show that also the regular solutions $f^{(r)}(\rho,\mu;-\kappa)$ of equation (71) satisfy the equation $\Delta f = \kappa^2 f$, it is now <u>sufficient to show that its first term</u>

$$e^{-\mu\rho} \, J_0(h)$$

satisfies this equation. Then one recognizes that also the second term in the series development (67) of $f^{(r)}(\rho,\mu)$ satisfies this equation. But this second term is identical with the corresponding in the series development (71) of $f^{(r)}(\rho,\mu;-\kappa)$. Hence it is true that also the regular solutions $f^{(r)}(\rho,\mu;-\kappa)$ with continuous κ satisfy the equation $\Delta f = \kappa^2 f$.

One has therefore to verify finally

$$\Delta f = \kappa^2 f \quad \text{for} \quad f = e^{-\mu\rho} J_0(h) \quad \text{with} \quad h = \rho\sqrt{(1-\mu^2)(1-\kappa^2)}$$

One finds

$$\Delta f = f + (1-\kappa^2)\, e^{-\mu\rho} \left(\frac{d^2 J_0(h)}{dh^2} + \frac{1}{h} \frac{d\, J_0(h)}{dh} \right)$$

$$= f - (1-\kappa^2)\, e^{-\mu\rho} J_0(h) = f - (1-\kappa^2) f = \kappa^2 f \qquad \text{q.e.d.}$$

D. Representation of the regular solutions in spherical geometry by superposition of solutions in plane geometry.

1. The solutions belonging to the discrete spectrum. The Boltzmann equation in plane geometry may be written

$$\eta \frac{\partial \psi(\zeta,\eta)}{\partial \zeta} + \psi(\zeta,\eta) = \frac{c}{2} \int_{-1}^{+1} \psi(\zeta,\eta')d\eta' \tag{74}$$

where ζ is the distance in space on an axis perpendicular to the planes from a chosen origin, and $\eta = \cos\Theta$ is the cosine of the angle between the $\vec{\zeta}$-axis and the direction $\vec{v}$ of the velocity of a neutron. A solution belonging to the discrete spectrum is the angular distribution of neutrons

$$\psi(\zeta,\eta) = \frac{1}{4\pi} \cdot \frac{c}{2} \quad \frac{\kappa}{1-\kappa\eta}\, e^{-\kappa\zeta} \,. \tag{75}$$

To construct a solution in spherical geometry one has to fix the centre of the sphere and to measure the distance in space by the distance ζ of the plane from the centre of the sphere. If one wants the angular distribution in spherical geometry at a chosen point A with the distance ρ from the centre, one draws the radius vector $\vec{\rho}$ through A. The angle between this radius vector and the ζ-direction may be denoted by ϑ'; then the distance of the plane

through A from the centre of the sphere is

$$\zeta = \rho \cos \vartheta' . \tag{76}$$

It is a measure for the "phase," with which a plane solution contributes to the spherical. To get a solution in spherical geometry with a fixed direction of the neutron-velocity $\vec{v}$, one has to superpose plane solutions, which belong to this fixed direction of velocity, whereas the normal $\vec{\zeta}$ of the plane runs all over the directions of space. The parameter $\eta = \cos \Theta$ in the plane solution is the scalar product of the unit vectors in $\vec{\zeta}$- and $\vec{v}$-directions. If one supposes a Cartesian coordinate system with the z-axis in $\vec{\rho}$-direction and the x, z-plane identical with the $\vec{v}, \vec{\rho}$-plane, the direction of the $\vec{\zeta}$-axis may be described by the angle ϑ' between $\vec{\zeta}$ and $\vec{z}$ and by the angle φ' between the x, z-plane and $\vec{\zeta}, \vec{z}$-plane. Then the components of a unit vector $\vec{e}$ along $\vec{\zeta}$ in these Cartesian coordinate systems will be

$$e_x = \sin \vartheta' \cos \varphi', \quad e_y = \sin \vartheta' \sin \varphi', \quad e_z = \cos \vartheta',$$

and the components of a unit vector $\vec{v}$ in the fixed direction of the velocity will be

$$v_x = \sin \vartheta = \sqrt{1-\mu^2} \qquad v_y = 0 \qquad v_z = \cos \vartheta = \mu .$$

Hence the scalar product of both is

$$\eta = \cos \Theta = \vec{e} \cdot \vec{v} = \mu \cos \vartheta' + \sqrt{1-\mu^2} \sin \vartheta' \cos \varphi' . \tag{77}$$

Therefore one gives the plane solutions, which shall contribute at a definite point A with the distance ρ from the centre of the sphere to a solution in spherical geometry with a fixed direction of neutron-velocity, the form

$$\psi(\zeta,\eta) = \psi\left(\rho\cos\vartheta',\ \mu\cos\vartheta' + \sqrt{1-\mu^2}\sin\vartheta'\cos\varphi'\right). \tag{78}$$

$$= \frac{1}{4\pi}\,\frac{c\kappa}{2}\,\frac{e^{-\kappa\rho\cos\vartheta'}}{1-\kappa\left[\mu\cos\vartheta' + \sqrt{1-\mu^2}\sin\vartheta'\cos\varphi'\right]} \tag{79}$$

By superposition of such solutions for all directions, i.e. by integration over φ' from 0 to 2π and over ϑ' from 0 to π one obtains a solution $S(\rho,\mu)$ in spherical geometry for $|\kappa| < 1$

$$S(\rho,\mu) = \int_0^{\pi} \sin\vartheta'\,d\vartheta' \int_0^{2\pi} d\varphi'\ \psi(\zeta,\eta)$$

$$= \frac{1}{4\pi}\ \frac{c\kappa}{2}\ \int_0^{\pi} \sin\vartheta'\,d\vartheta' \int_0^{2\pi} d\varphi'\ \frac{e^{-\kappa\rho\cos\vartheta'}}{1-\kappa(\mu\cos\vartheta' + \sqrt{1-\mu^2}\sin\vartheta'\cos\varphi')}$$

$$= \frac{\kappa c}{4}\int_0^{\pi} \frac{\sin\vartheta'\,d\vartheta'\,e^{-\kappa\rho\cos\vartheta'}}{\left[(1-\kappa\,\mu\cos\vartheta')^2 - \kappa^2(1-\mu^2)\sin^2\vartheta'\right]^{1/2}}$$

$$= \frac{\kappa c}{4}\int_{-1}^{+1} \frac{dv\,e^{-\kappa\rho v}}{\sqrt{1-\kappa^2(1-\mu^2) - 2\kappa\,\mu v + \kappa^2 v^2}}$$

The transformation of the variable of integration

$$v = \frac{1}{\kappa}\left\{\mu - \tfrac{1}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]\right\} \tag{80}$$

into a new variable S transforms the upper limit $v_1 = 1$ into $S_1 = 1$ and the lower limit $v_2 = -1$ into $S_2 = \frac{1+\kappa}{1-\kappa}$. The square root in the denominator of the integrand becomes

$$\sqrt{1-\kappa^2(1-\mu^2) - 2\kappa\,\mu v + \kappa^2 v^2} = \tfrac{1}{2}\left[(1-\kappa)(1+\mu)S + (1+\kappa)(1-\mu)\frac{1}{S}\right]$$

and the differential $dv = -\frac{1}{2\kappa}\left[(1-\kappa)(1+\mu) + (1+\kappa)(1-\mu)\frac{1}{S^2}\right] dS$.
This leads to the remarkably simple expression for

$$\frac{dv}{\sqrt{1-\kappa^2(1-\mu^2) - 2\kappa\mu v + \kappa^2 v^2}} = -\frac{1}{\kappa}\frac{dS}{S}$$

and the solution takes the form

$$S(\rho,\mu) = \frac{c}{4} e^{-\mu\rho} \int_1^{\frac{1+\kappa}{1-\kappa}} \frac{dS}{S} e^{\frac{\rho}{2}\left[(1-\kappa)(1+\mu)S - (1+\kappa)(1-\mu)\frac{1}{S}\right]} \tag{81}$$

which is identical with the regular solution (63) belonging to the discrete spectrum of the Boltzmann operator in spherical geometry.

$$S(\rho,\mu) = f^{(r)}(\rho,\mu). \tag{82}$$

2. Superposition of the solutions belonging to the continuous spectrum in plane geometry. E. P. Wigner[9] showed in his lecture on Mathematical Problems of Nuclear Reactor Theory that the Boltzmann operator for monoenergetic neutron transport in plane geometry has a continuous spectrum. In approximations, for instance by the Gauss quadrature or by the spherical harmonics method, this continuous spectrum makes itself conspicuous by those eigenvalues of the approximate characteristic equation, which belong to the transient solutions.

The continuous spectrum of the Boltzmann operator extends from $|\kappa| = 1$ until $|\kappa| = \infty$ on both sides of the real axis in the complex κ-plane. The eigen-

[9] E. P. Wigner at the Meeting on Mathematical Aspects of Reactor Theory in New York, April 23-24, 1959. It is published in a Colloquium Publication of the American Mathematical Society under the title, "Nuclear Reactor Theory", Garrett Birkhoff and E. P. Wigner, editors, p. 89 (1961).

function belonging to a specific κ-value of this spectrum may be written in a symbolic form

$$\psi(\zeta,\eta;\kappa) = \lim_{\varepsilon\to 0}\left\{\frac{c_1\kappa}{\kappa\eta-1+i\varepsilon\kappa} + \frac{c_2\kappa}{\kappa\eta-1-i\varepsilon\kappa}\right\} e^{-\kappa\zeta} \tag{83}$$

with the coefficients

$$c_1 = -\frac{\pi c}{2} + i\left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right] \tag{84a}$$

$$c_2 = -\frac{\pi c}{2} - i\left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right]. \tag{84b}$$

ε is a small real (positive) quantity; ζ is the space coordinate and $\eta = \cos\Theta$ is the cosine of the angle Θ of direction of the neutrons against the ζ-axis in the supposed plane geometry as in the preceding section 1.

Inserting the coefficients c_1 and c_2 the symbolic eigen-function takes the form

$$\psi(\zeta,\eta;\kappa) = \lim_{\varepsilon\to 0}\left\{\pi c\kappa\ \frac{1-\kappa\eta}{(1-\kappa\eta)^2+\varepsilon^2\kappa^2} + \left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right]\ \frac{2\varepsilon\kappa^2}{(1-\kappa\eta)^2+\varepsilon^2\kappa^2}\right\} e^{-\kappa\zeta} \tag{85}$$

After multiplication with an arbitrary weight function $g\left(\frac{1}{\kappa}\right)$, which ensures the convergence of the integral, the contribution of the continuous spectrum to a total solution of the Boltzmann equation may be represented by the following integral over the continuous spectrum

$$\phi(\zeta,\eta) = \pi c\kappa \int_{-1}^{+1} \frac{g(\frac{1}{\kappa})}{1-\kappa\eta}\, e^{-\kappa\zeta}\, d(\tfrac{1}{\kappa}) + 2\pi\left[\frac{1}{\eta} - \frac{c}{2}\log\frac{1+\eta}{1-\eta}\right] g(\eta)\, e^{-\zeta/\eta} \qquad (86)$$

and the corresponding contribution to the density (or flux at velocity v = 1) by

$$\phi(\zeta) = \frac{1}{2}\int_{-1}^{+1} \phi(\zeta,\eta)\, d\eta = \pi \int_{-1}^{+1} \frac{1}{\eta}\, g(\eta)\, e^{-\zeta/\eta}\, d\eta\ . \qquad (87)$$

By comparison of the last two equations one recognizes that only <u>that</u> term of the angular distribution $\phi(\zeta,\eta)$ contributes to the density $\phi(\zeta)$, which does <u>not</u> contain the constant of multiplication c as a factor. I owe Dr. E. Inönü the observation of this fact. The same fact was noticed already at the solutions in spherical geometry for all κ-values which do <u>not</u> satisfy the characteristic equation.

In spherical geometry, however, it is not necessary to write the solution for a specific κ in a symbolic form, as equation (83) is in plane geometry, and one has not to integrate over at least a part of the continuous spectrum to obtain ordinary functions for every single κ. In spherical geometry the solutions are already ordinary functions for every single κ. Furthermore, there are certainly two different kinds of solutions for every $|\kappa|$, one, which behaves <u>regular</u> at the origin, and another which is <u>singular</u> at the origin of the sphere. It will be shown in the following that a superposition of solutions belonging to the continuous spectrum of the Boltzmann operator in plane geometry for a specific κ-value ($|\kappa| \geq 1$), similar to the superposition in the last section, yields the corresponding <u>regular</u> solution in spherical geometry. It is interesting to observe how the integration over all space directions already leads to the elimination of the ε for every single specific κ.

We superpose solutions of the kind (83) in the way described to obtain a solution in spherical geometry

$$S(\rho,\mu;\kappa) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \sin\vartheta' d\vartheta' d\varphi' \, e^{-\kappa\rho\cos\vartheta'} \lim_{\varepsilon\to 0}\left\{\frac{c_1\kappa}{\kappa\mu\cos\vartheta'+\kappa\sqrt{1-\mu^2}\sin\vartheta'\cos\varphi'-1+i\varepsilon\kappa}\right.$$

$$\left. + \frac{c_1^*\kappa}{\kappa\mu\cos\vartheta'+\kappa\sqrt{1-\mu^2}\sin\vartheta'\cos\varphi'-1-i\varepsilon\kappa}\right\}. \qquad (88)$$

Noticing that $c_2 = c_1^*$ for real κ and

putting $a = -1 + \kappa\mu\cos\vartheta'$, $b = \kappa\sqrt{1-\mu^2}\sin\vartheta'$ as an abbreviation, one has

$$S(\rho,\mu;\kappa) = \frac{\kappa}{4\pi}\lim_{\varepsilon\to 0}\int_0^{\pi}\sin\vartheta' d\vartheta' e^{-\kappa\rho\cos\vartheta'}\int_{-\pi}^{\pi}\left\{\frac{c_1}{a+i\varepsilon\kappa+b\cos\varphi'}\right.$$

$$\left. + \frac{c_1^*}{a-i\varepsilon\kappa+b\cos\varphi'}\right\} d\varphi'.$$

It is transformed by $u = \operatorname{tg}\frac{\varphi'}{2}$, $d\varphi' = \frac{2du}{1+u^2}$, $\cos\varphi' = \frac{1-u^2}{1+u^2}$ into the form

$$= \frac{\kappa}{2\pi}\lim_{\varepsilon\to 0}\int_0^{\pi}\sin\vartheta' d\vartheta' e^{-\kappa\rho\cos\vartheta'}\int_{-\infty}^{+\infty} du\left\{\frac{c_1}{a+b+i\varepsilon\kappa+(a-b+i\varepsilon\kappa)u^2}\right.$$

$$\left. + \frac{c_1^*}{a+b-i\varepsilon\kappa+(a-b-i\varepsilon\kappa)u^2}\right\}. \qquad (89)$$

The integrand vanishes for large u sufficiently strong that the path of integration can be closed by a half circle in the infinite of the upper half complex u-plane. The denominator of the first term of the integrand has a pair of roots u_0 and $-u_0$, in which

$$u_0 = \sqrt{\frac{b+a+i\varepsilon\kappa}{b-a-i\varepsilon\kappa}} = \sqrt{\frac{b^2-a^2-\varepsilon^2\kappa^2+2i\varepsilon\kappa b}{(b-a)^2+\varepsilon^2\kappa^2}} \tag{90}$$

The second term of the integrand is the complex conjugate of the first and the roots of its denominator are u_0^* and $-u_0^*$. All 4 roots lie in the 4 corners of a rectangle symmetric against the real and complex axis of the u-plane. The second expression for u_0 in (90) shows, that the imaginary part of u_0^2 is positive, because b is positive for $0 < \vartheta' < \pi$, if one chooses furthermore a positive κ from the continuous spectrum. Then the imaginary part of u_0 itself is positive also and the two roots u_0 and $-u_0^*$ lie in the upper half of the complex u-plane, whereas the other two roots $-u_0$ and u_0^* lie in the lower half plane. Only the poles at u_0 and $-u_0^*$ contribute to the integral with their residues by application of Cauchy's theorem to the upper half plane, which is enclosed by the path C of integration:

$$S(\rho,\mu;\kappa) = \frac{\kappa}{4\pi} \lim_{\varepsilon\to 0} \int_0^{\pi} \sin\vartheta' d\vartheta' e^{-\kappa\rho\cos\vartheta'} \int_C du \left\{ \frac{c_1}{(a-b+i\varepsilon\kappa)u_0} \left(\frac{1}{u-u_0} - \frac{1}{u+u_0} \right) \right.$$

$$\left. + \frac{c_1^*}{(a-b-i\varepsilon\kappa)u_0^*} \left(\frac{1}{u-u_0^*} - \frac{1}{u+u_0^*} \right) \right\} \tag{91}$$

$$= \frac{\kappa}{2} \lim_{\varepsilon \to 0} \int_0^{\pi} \sin\vartheta' d\vartheta' e^{-\kappa\rho\cos\vartheta'} \left\{ \frac{ic_1}{(a-b+i\varepsilon\kappa)u_0} + \frac{-ic_1^*}{(a-b-i\varepsilon\kappa)u_0^*} \right\}$$

$$= \frac{\kappa}{2} \lim_{\varepsilon \to 0} \int_0^{\pi} \sin\vartheta' d\vartheta' e^{-\kappa\rho\cos\vartheta'} \left\{ \frac{c_1}{\sqrt{a^2-b^2+2i\varepsilon\kappa a-\varepsilon^2\kappa^2}} + \frac{c_1^*}{\sqrt{a^2-b^2-2i\varepsilon\kappa a-\varepsilon^2\kappa^2}} \right\} \qquad (92)$$

One may still apply the same transformation of the variable of integration ϑ' into a new variable S

$$\cos\vartheta' = \frac{1}{\kappa}\left\{\mu + \frac{1}{2}\left[(\kappa-1)(1+\mu)S + (\kappa+1)(1-\mu)\frac{1}{S}\right]\right. \qquad (80)$$

as in the preceding section to perform the second integration. One has to remember, however, that $\kappa \geq 1$ holds this time: the path of integration in the S-plane starts for $\vartheta' = 0$ at $S = 1$ and ends for $\vartheta' = \pi$ at $S = -\frac{\kappa+1}{\kappa-1}$ on the other side of the origin. When ϑ' increases from 0 to π, $\cos\vartheta'$ decreases monotonously from 1 to -1. To maintain this property also on the path of integration in the complex S-plane, we have to proceed along the real axis from $S = 1$ to $S_1 = \sqrt{\frac{\kappa+1}{\kappa-1}\frac{1-\mu}{1+\mu}}$, then along a half circle with the radius S_1 around the origin of the S-plane until $S_2 = -S_1$ and finally from $-S_1$ until the end-point of the path at $S = -\frac{\kappa+1}{\kappa-1}$ on the real axis. At the points $S = \pm S_1$ the expression on the right hand side of equation (80) has an extremum, because

$$\frac{\partial\cos v'}{\partial S} = \frac{1}{2\kappa}\left[(\kappa-1)(1+\mu) - (\kappa+1)(1-\mu)\frac{1}{S^2}\right] \qquad (93)$$

in zero at $S = \pm S_1$ with

$$S_1 = \sqrt{\frac{\kappa+1}{\kappa-1}\,\frac{1-\mu}{1+\mu}}$$

and the quantity $a^2 - b^2$, which occurs in the radicands of the square roots in the denominators of the integrand

$$a^2-b^2 = 1-\kappa^2(1-\mu^2) - 2\kappa\mu\cos\vartheta' + \kappa^2\cos^2\vartheta'$$

$$= \frac{1}{4}\left\{(\kappa-1)(1+\mu)S - (\kappa+1)(1-\mu)\frac{1}{S}\right\}^2 \qquad (94)$$

is zero just at the same two points $S = \pm S_1$. We denote the corresponding ϑ'-values with ϑ'_1 for $S = S_1$ and ϑ'_2 for $S = S_2 = -S_1$. Equation (81) yields

$$\cos\vartheta'_1 = \frac{1}{\kappa}\left[\mu + \sqrt{(\kappa^2-1)(1-\mu^2)}\right] \quad \text{for } S = S_1$$

and

$$\cos\vartheta'_2 = \frac{1}{\kappa}\left[\mu - \sqrt{(\kappa^2-1)(1-\mu^2)}\right] \quad \text{for } S = S_2 = -S_1 \ .$$

In the interval $\vartheta'_1 > \vartheta' > \vartheta'_2$ the quantity a^2-b^2 is negative; this follows from the first expression of equation (94) for instance by inserting the mean value $\cos\vartheta'_m = \frac{\mu}{\kappa}$ of both values at the boundary of the interval; one obtains for this angle $(a^2-b^2)_{\vartheta=\vartheta_m} = -(\kappa^2-1)(1-\mu^2)$. The second expression of equation (94) obtains indeed negative values for complex values of S. Hence we assume finally

$$S = \begin{cases} S_1 \cdot t & \text{a) for the real S-intervals } (1, S_1) \text{ and } (-S_1, -\frac{\kappa+1}{\kappa-1}) \\ S_1 \cdot e^{i\chi} & \text{b) for the half-circle } 0 \le \chi \le \pi \end{cases}$$

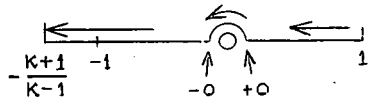
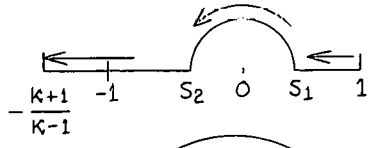
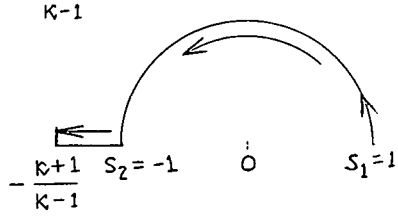
This supposition yields on the corresponding parts of the path of integration

$$\cos\vartheta' = \begin{cases} \frac{1}{\kappa}\left\{\mu + \frac{1}{2}\sqrt{(\kappa^2-1)(1-\mu^2)}\ \left(t + \frac{1}{t}\right)\right\} & \text{(a)} \\ \frac{1}{\kappa}\left\{\mu + \sqrt{(\kappa^2-1)(1-\mu^2)}\ \cos\chi\right\} & \text{(b)} \end{cases}$$

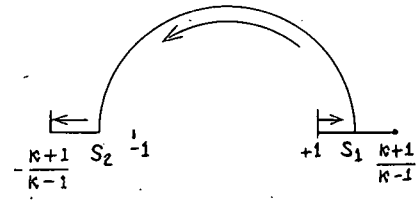
$$a^2-b^2 = \begin{cases} \frac{1}{4}\ (\kappa^2-1)(1-\mu^2)\ \left(t - \frac{1}{t}\right)^2 & \text{(a)} \\ -\ (\kappa^2-1)(1-\mu^2)\ \sin^2\chi & \text{(b)} \end{cases}$$

$$\frac{\sin\vartheta'\,d\vartheta'}{\sqrt{a^2-b^2}} = \begin{cases} -\ \frac{1}{\kappa}\ \frac{dt}{t} & \text{(a)} \\ \frac{-i}{\kappa}\ d\chi & \text{(b)} \end{cases}$$

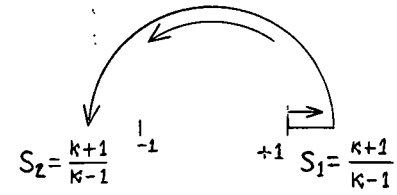
The path of integration in the complex S-plane looks different for 5 regions of μ-values. The following table shows it:

region of μ	corner at positive S	corner at negative S	
1a) $\mu = 1$	$S_1 = +0$	$S_2 = -0$	
1b) $\frac{1}{\kappa} < \mu < 1$	$0 < S_1 < 1$	$-\frac{\kappa+1}{\kappa-1} < -1 < S_2 < 0$	
2) $\mu = \frac{1}{\kappa}$	$S_1 = 1$	$S_2 = -1$	

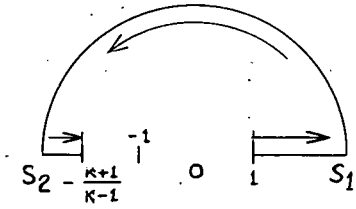
3) $-\frac{1}{\kappa} < \mu < \frac{1}{\kappa}$ $\quad 1 < S_1 < \frac{\kappa+1}{\kappa-1}$ $\quad -\frac{\kappa+1}{\kappa-1} < S_2 < -1$



4) $\mu = -\frac{1}{\kappa}$ $\quad S_1 = \frac{\kappa+1}{\kappa-1}$ $\quad S_2 = -\frac{\kappa+1}{\kappa-1}$



5a) $-1 < \mu < -\frac{1}{\kappa}$ $\quad \frac{\kappa+1}{\kappa-1} < S_1 < \infty$ $\quad -\infty < S_2 < -\frac{\kappa+1}{\kappa-1}$



5b) $\mu = -1$ $\quad S_1 = \infty$ $\quad S_2 = -\infty$

To obtain the whole integral, which is real, <u>one has to add the integral over the conjugate complex path</u>.

Then the integral takes the form

$$S(\rho,\mu;\kappa) = \frac{-1}{2}\, e^{-\mu\rho}\left\{\left(\int_{\frac{1}{S_1}}^{1} + \int_{-1}^{-\frac{\kappa+1}{\kappa-1}\frac{1}{S_1}}\right)\frac{dt}{t}\, e^{-\frac{\rho}{2}\sqrt{(\kappa^2-1)(1-\mu^2)}\,\left(t+\frac{1}{t}\right)}\,(c_1+c_1^*)\right.$$

$$\left. + \int_0^{\pi} d\chi\, e^{-\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\,\cos\chi}\; i(c_1-c_1^*)\right\}. \qquad (95)$$

With $c_1 + c_1^* = -\pi c$ and $i(c_1 - c_1^*) = -2\left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right]$ it becomes

$$S(\rho,\mu;\kappa) = e^{-\mu\rho}\left\{\frac{\pi c}{2}\left(\int_{-\sqrt{\frac{\kappa-1}{\kappa+1}\frac{1+\mu}{1-\mu}}}^{1} + \int_{-1}^{-\sqrt{\frac{\kappa+1}{\kappa-1}\frac{1+\mu}{1-\mu}}}\right) e^{-\frac{\rho}{2}\sqrt{(\kappa^2-1)(1-\mu^2)}\,(t+\frac{1}{t})}\,\frac{dt}{t}\right.$$

$$\left. + \left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right]\cdot\int_0^{\pi} e^{-\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\cos\chi}\,d\chi\right\}. \qquad (96)$$

The <u>first integrand</u> may be represented by a series of modified Bessel-functions using the formula

$$e^{-\frac{z}{2}(t+\frac{1}{t})} = I_0(z) + \sum_{n=1}^{\infty}(t^n + \frac{1}{t^n})(-1)^n I_n(z) \qquad (97)$$

and its integrals give $S(\rho,\mu;\kappa)$ the following contributions

$$e^{-\mu\rho}\,\frac{\pi c}{2}\left\{I_0\left(\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\right)\log\frac{\kappa+1}{\kappa-1}\right.$$

$$\left. + \sum_{n=1}^{\infty}{}' \frac{I_n\left(\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\right)}{n}\left[\left(\frac{\kappa+1}{\kappa-1}\right)^{n/2} - \left(\frac{\kappa-1}{\kappa+1}\right)^{n/2}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{n/2} + (-1)^n\left(\frac{1-\mu}{1+\mu}\right)^{n/2}\right]\right\}. \qquad (98)$$

The <u>second integral</u> yields

$$e^{-\mu\rho}\left[\kappa - \frac{c}{2}\log\frac{\kappa+1}{\kappa-1}\right]\pi I_0\left(\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\right) \qquad (99)$$

and its second part cancels just the term with the factor $\log \frac{\kappa+1}{\kappa-1}$ of the first integral. The total integral is now

$$S(\rho,\mu;\kappa) = 2\pi e^{-\mu\rho}\left\{\frac{\kappa}{2} I_0(\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\right.$$

$$\left. + \frac{c}{4} \sum_{n=1}^{\infty} \frac{I_n\left(\rho\sqrt{(\kappa^2-1)(1-\mu^2)}\right)}{n}\left[\left(\frac{\kappa+1}{\kappa-1}\right)^{n/2} - \left(\frac{\kappa-1}{\kappa+1}\right)^{n/2}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{n/2} + (-1)^n\left(\frac{1-\mu}{1+\mu}\right)^{n/2}\right]\right\} \quad (100)$$

By comparison with equation (71) one recognizes that

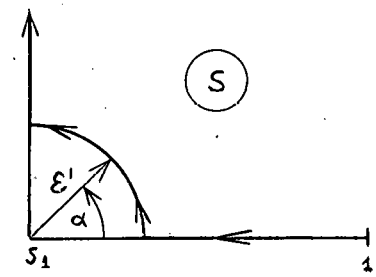
$$S(\rho,\mu;\kappa) = 2\pi \; f^{(r)}(\rho,\mu;\kappa) \quad (101)$$

is 2π times the total solution $f^{(r)}(\rho,\mu;\kappa)$ for the regular case of the Boltzmann equation in spherical geometry.

To complete this integration, one has to justify that the integration through the "corners," in which the half circle meets the real axis in the S-plane and a^2-b^2 changes its sign, does not give a contribution. It will be sufficient to show this in one corner, which may lie at $S_1 < 1$. We encircle it by a quarter of a circle with the radius ε' and its center at S_1. Hence S may be represented along this quarter of a circle by

$$S = \sqrt{\frac{\kappa+1}{\kappa-1}\frac{1-\mu}{1+\mu}}\,(1 + \varepsilon' e^{i\alpha})$$

and $$\frac{1}{S} = \sqrt{\frac{\kappa-1}{\kappa+1}\frac{1+1}{1-\mu}}\,(1 - \varepsilon' e^{i\alpha} + \varepsilon'^2 e^{2i\alpha} + \ldots).$$



The quantities, involved in the integral, are by this supposition

$$\cos\vartheta' = \frac{1}{\kappa}\left\{\mu + \frac{1}{2}\sqrt{(\kappa^2-1)(1-\mu^2)}\,(2 + \varepsilon'^2 e^{2i\alpha})\right\}$$

$$\sin \vartheta' \, d\vartheta' = -\frac{i}{\kappa} \sqrt{(\kappa^2-1)(1-\mu^2)} \, \varepsilon'^2 e^{2i\alpha} \, d\alpha$$

$$a^2-b^2 = (\kappa^2-1)(1-\mu^2) \, \varepsilon'^2 e^{2i\alpha}$$

and the first part of the integral (92) along the quarter of the circle around the corner at $\vartheta' = \vartheta'_1$, will be

$$\lim_{\varepsilon \to 0} \int_{\left(\substack{\text{corner} \\ \text{at } \vartheta'_1}\right)} \frac{e^{-\kappa\rho \cos \vartheta'} \sin \vartheta' \, d\vartheta'}{\sqrt{a^2-b^2+2i\varepsilon\kappa \, a-\varepsilon^2\kappa^2}}$$

$$= \lim_{\varepsilon \to 0} e^{-\kappa\rho \cos \vartheta'_1} \int_0^{\pi/2} \frac{\varepsilon'^2 \, e^{2i\alpha} \, d\alpha}{\sqrt{\varepsilon'^2 e^{2i\alpha} - 2i\varepsilon \frac{\kappa}{\sqrt{\kappa^2-1}} \frac{1}{\sqrt{\kappa^2-1}} - \frac{\mu}{\sqrt{1-\mu^2}}}}$$

ε, ε' are arbitrary small constants; if we choose $(\varepsilon')^2 = C \cdot \varepsilon$ (C = const.), we join their limits for $\varepsilon \to 0$. The integral will tend to zero, when $\varepsilon' = \sqrt{C\varepsilon}$ tends to zero. The cause of this behavior is that $\cos \vartheta'$ is stationary in respect to S at all corners (see equation (93)), in consequence of which the differential $\sin \vartheta' \, d\vartheta'$ at the corners is quadratic in ε'. Hence one obtains no contribution to the integral from the neighborhood of the corners.

III. Comparison of the New Solutions with their Representations by a Series of Spherical Harmonics

A. Proof of the equality of two solutions of the Boltzmann equation which yield the same density. The difference $\delta(\rho,\mu)$ of two solutions of the Boltzmann equation (1) with the same density satisfies the homogeneous partial differential equation (8). Therefore it has the form

$$\delta(\rho,\mu) = e^{-\mu\rho} F(\rho\sqrt{1-\mu^2})$$

with some function F of $\rho\sqrt{1-\mu^2}$, about which one knows that it yields the density zero:

$$\int_{-1}^{+1} \delta(\rho,\mu)\, d\mu = \int_{-1}^{+1} e^{-\mu\rho} F(\rho\sqrt{1-\mu^2})\, d\mu = 2\int_{0}^{1} (\cosh \mu\rho)\, F(\rho\sqrt{1-\mu^2})\, d\mu = 0.$$

This is a Sonine integral equation again. If we replace zero on the right hand side of this equation by a constant C, its solution would be

$$F(\rho\sqrt{1-\mu^2}) = C \cdot \frac{1}{\pi} \frac{d}{du} \int_0^u \frac{\cos\sqrt{u-s}}{\sqrt{u-s}} \sqrt{s}\, ds \qquad \text{with } u = \rho^2(1-\mu^2)$$

$$= C\,\frac{1}{\pi} \int_0^1 \frac{\cos(\rho\sqrt{(1-\mu^2)(1-t^2)})}{\sqrt{1-t^2}}\, dt = C \cdot \frac{1}{2} J_0(\rho\sqrt{1-\mu^2}).$$

One recognizes by this calculation that the expression accompanying C does not diverge. Hence the difference $\delta(\rho,\mu)$ of both solutions vanishes together with $C = 0$. This means that two solutions of (1) with equal densities are equal. Of course the theorem is not applicable to the comparison of two solutions, which differ by a diverging part. Such a case appears in section D of this chapter.

B. Application of the theorem of equality to two regular solutions of the Boltzmann equation with the same density. According to the equations (2) and (3) the development of the regular solution in a series of spherical harmonics is

$$f^{(r)}_{(S.H.)}(\rho,\mu) = \frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, G_\ell(-\tfrac{1}{\kappa}) \frac{I_{\ell+\frac{1}{2}}(-\kappa\rho)}{\sqrt{-\kappa\rho}} P_\ell(\mu) \tag{102a}$$

$$\equiv \frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(-\tfrac{1}{\kappa}) \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} P_\ell(\mu)$$

$$+ \frac{c}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=1}^{\infty} (2\ell+1)\, W_{\ell-1}(-\tfrac{1}{\kappa}) \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} P_\ell(\mu). \tag{102b}$$

Its density integral becomes

$$\int_{-1}^{+1} f^{(r)}_{(S.H.)}(\rho,\mu)\, d\mu = 2 \cdot \frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \frac{I_{\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} = \frac{\sinh \kappa\rho}{\rho}. \tag{103}$$

On the other hand, our corresponding regular solution is $f^{(r)}(\rho,\mu;\kappa)$ in the form of the equations (71) or (72). It yields by the use of the integral (48) the same density (103). Therefore our regular solution is equal to the spherical harmonics series (102a). This equality gives two relations, a first between the parts <u>without the factor c</u> and a second between the parts <u>with the factor c</u>:

$$1)\quad \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(-\tfrac{1}{\kappa})\, P_\ell(\mu) \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} = e^{-\mu\rho} J_0(h) \tag{104}$$

$$2)\quad \sqrt{2\pi} \sum_{\ell=1}^{\infty} (2\ell+1)\, W_{\ell-1}(-\tfrac{1}{\kappa})\, P_\ell(\mu) \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}}$$

$$= e^{-\mu\rho} \sum_{n=1}^{\infty} \frac{J_n(h)}{n} \left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} + (-1)^n \left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}}\right] \tag{105a}$$

$$= e^{-\mu\rho} \int_0^1 \frac{dv}{v} J_0(h\sqrt{1-v}) \left\{ e^{\frac{\rho}{2}(1+\kappa)(1+\mu)\, v} - e^{-\frac{\rho}{2}(1-\kappa)(1-\mu)\, v} \right.$$

$$\left. - e^{\frac{\rho}{2}(1-\kappa)(1+\mu)\, v} + e^{-\frac{\rho}{2}(1+\kappa)(1-\mu)\, v} \right\} \tag{105b}$$

with $h = \rho\sqrt{(1-\mu^2)(1-\kappa^2)}$. We obtained two equations because c and κ are independent

variables in both representations of the regular solution. A connection between c and κ, i.e., a characteristic equation $G_{L+1}(-\frac{1}{\kappa}) = 0$, would appear spontaneously only by truncation of the series (102a) after the L-th term. The extrapolation to $L \to \infty$ in this characteristic equation can be made or not. If one performs the extrapolation, then the discrete and the continuous spectrum follow at least in the case of plane geometry. Everybody expects that the spectrum of the Boltzmann-operator depends on the qualities of the material and does not depend on the geometry as long as one considers only solutions in the infinite space without boundaries and sources. The truncation of the spherical harmonics solution after the L-th term means in fact the addition of a source, namely of an error-source term[10], to the original Boltzmann equation (1). To this hidden addition one owes the characteristic equation and by extrapolation to $L \to \infty$ one obtains the hint at the spectrum. But for the untruncated series (102a) and the solution (71) or (72) the error source does not exist or has lost its importance - in the view from the extrapolation of finite L - because of the convergence of the series (102a) for a problem without boundaries. Hence they are solutions for every κ, for which they are convergent and differentiable in respect to κ and μ.

C. Direct verification of equation (104). Of course a verification of equations (104) and (105) as a check of the calculations would be of some value. I restrict myself to the verification of the simpler equation (104) because I found only in this case a suitable aid formula. In absence of a complete aid formula for a treatment of (105), however, one could use the developments of G. Bauer for a verification of (105) by steps from ρ^n to ρ^{n+1} similar to that which will be used in section E.

Multiplying both sides of (104) with one of the spherical harmonics and using their orthonormality relations, one obtains

$$\frac{1}{2}\int_{-1}^{+1} e^{-\mu\rho}\, J_0\left(\rho\sqrt{(1-\mu^2)(1-\kappa^2)}\right)\, P_\ell(\mu)\, d\mu = \sqrt{\frac{\pi}{2}}\, P_\ell(-\tfrac{1}{\kappa})\, \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} \,. \qquad (106)$$

[10] W. Kofink, Oak Ridge National Laboratory report 2358, p. 3 (1957) and Nuovo Cimento Supplement 2 to Vol. 9, p. 499 (1958).

The proof of (106) is equivalent to the proof of (104). Equation (106) can be derived from the following extension of the aid-formula (48) by putting $t = \rho$:

$$\frac{1}{2}\int_{-1}^{+1} e^{-\mu t}\, J_0\left(\rho\sqrt{(1-\mu^2)(1-\kappa^2)}\right) P_\ell(\mu)\, d\mu$$

$$= P_\ell\left(\frac{-t}{\sqrt{t^2-(1-\kappa^2)\,\rho^2}}\right)\sqrt{\frac{\pi}{2}}\,\frac{I_{\ell+\frac{1}{2}}\left(\sqrt{t^2-(1-\kappa^2)\,\rho^2}\right)}{\left[t^2-(1-\kappa^2)\,\rho^2\right]^{\frac{1}{4}}}\,. \qquad (107)$$

The equation (107) will be proved by complete induction. For $\ell = 0$ it is identical with equation (48). For $\ell = 1$ it is the derivative $-\partial/\partial t$ of equation (48). Supposing furthermore the validity of the equation (107) until ℓ, one has finally to show its validity for $\ell + 1$. The left-hand side is in this case

$$\frac{1}{2}\int_{-1}^{+1} e^{-\mu t}\, J_0(h)\, P_{\ell+1}(\mu)\, d\mu$$

$$= \frac{1}{2}\int_{-1}^{+1} e^{-\mu t}\, J_0(h)\left\{\frac{2\ell+1}{\ell+1}\,\mu\, P_\ell(\mu) - \frac{\ell}{\ell+1}\, P_{\ell-1}(\mu)\right\} d\mu$$

$$= -\frac{2\ell+1}{\ell+1}\,\frac{\partial}{\partial t}\,\frac{1}{2}\int_{-1}^{+1} e^{-\mu t}\, J_0(h)\, P_\ell(\mu)\, d\mu - \frac{\ell}{\ell+1}\,\frac{1}{2}\int_{-1}^{+1} e^{-\mu t}\, J_0(h)\, P_{\ell-1}(\mu)\, d\mu$$

$$= -\frac{2\ell+1}{\ell+1}\,\frac{\partial}{\partial t}\left\{P_\ell\left(-\frac{t}{x}\right)\sqrt{\frac{\pi}{2}}\,\frac{I_{\ell+\frac{1}{2}}(x)}{\sqrt{x}}\right\} - \frac{\ell}{\ell+1}\,P_{\ell-1}\left(-\frac{t}{x}\right)\sqrt{\frac{\pi}{2}}\,\frac{I_{\ell-\frac{1}{2}}(x)}{\sqrt{x}}$$

with $x = \sqrt{t^2-(1-\kappa^2)\,\rho^2}$ as abbreviation. This expression should be equal to the right-hand side of equation (107) for $\ell + 1$. Hence it remains to show that

$$(\ell+1)\, P_{\ell+1}\left(-\frac{t}{x}\right) I_{\ell+\frac{3}{2}}(x)$$

$$= -(2\ell+1)\sqrt{x}\,\frac{\partial}{\partial t}\left\{P_\ell\left(-\frac{t}{x}\right)\frac{I_{\ell+\frac{1}{2}}(x)}{\sqrt{x}}\right\} - \ell P_{\ell-1}\left(-\frac{t}{x}\right) I_{\ell-\frac{1}{2}}(x).$$

The right-hand side of this equation is

$$= -(2\ell+1)\left\{P'_\ell\left(\frac{-t}{x}\right)\cdot\frac{t^2-x^2}{x^3}\,I_{\ell+\frac{1}{2}}(x)+P_\ell\left(\frac{-t}{x}\right)\cdot\frac{t}{x}\,I'_{\ell+\frac{1}{2}}(x)\right.$$

$$\left.-P_\ell\left(\frac{-t}{x}\right)\frac{t}{2x^2}\,I_{\ell+\frac{1}{2}}(x)\right\}-\ell P_{\ell-1}\left(\frac{-t}{x}\right)I_{\ell-\frac{1}{2}}(x)$$

$$= \left\{\left(1-\frac{t^2}{x^2}\right)P'_\ell\left(\frac{-t}{x}\right)-\ell\,\frac{t}{x}\,P_\ell\left(\frac{-t}{x}\right)-\ell P_{\ell-1}\left(\frac{-t}{x}\right)\right\}I_{\ell-\frac{1}{2}}(x)$$

$$+\left\{\left(\frac{t^2}{x^2}-1\right)P'_\ell\left(\frac{-t}{x}\right)-(\ell+1)\,\frac{t}{x}\,P_\ell\left(\frac{-t}{x}\right)\right\}I_{\ell+\frac{3}{2}}(x).$$

The first curly bracket is zero, whereas the second curly bracket gives the desired result. It is

$$= (\ell+1)\,P_{\ell+1}\left(\frac{-t}{x}\right)I_{\ell+\frac{3}{2}}(x). \qquad \text{q.e.d.}$$

D. Comparison of two singular solutions. The following theorem will be suggested and partially verified: if one omits from the singular solution in its representation by a series of spherical harmonics all terms which contain negative powers of κ as factors, one obtains the new singular solution (35) or (40).

Some evidence exists for the validity of this theorem. The density-integral for the new solution is

$$P^{(s)}(\rho) = \int_{-1}^{+1} f^{(s)}(\rho,\mu)\,d\mu = \int_{-1}^{+1} e^{-\mu\rho}\,R(\rho\sqrt{1-\mu^2})\,d\mu = \frac{e^{-\kappa\rho}}{\rho} \qquad (108)$$

as it was shown in equation (27). If one develops the right-hand side of the Boltzmann equation in this case in a power series of κ

$$\mu\frac{\partial f}{\partial\rho}+\frac{1-\mu^2}{\rho}\frac{\partial f}{\partial\mu}+f(\rho,\mu) = \frac{c}{2}\cdot\frac{e^{-\kappa\rho}}{\rho}$$

$$= \frac{c}{2}\left\{\frac{1}{\rho}-\kappa+\frac{1}{2!}\kappa^2\rho-+\ldots+\frac{(-1)^n}{n!}\kappa^n\rho^{n-1}+\ldots\right\} \qquad (109)$$

and if one supposes a solution as a power series of κ

$$f(\rho,\mu) = \sum_{n=-\infty}^{n=+\infty} \kappa^n \varphi_n(\rho,\mu) \tag{110}$$

one recognizes that there is no need in f for terms with negative powers of κ because the right-hand side of (109) contains only the non-negative powers of κ. The singular solution of (1) in spherical harmonics

$$f^{(s)}_{(S.H.)}(\rho,\mu) = \frac{\kappa}{\sqrt{2\pi}} \sum_{\ell=0}^{\infty} (2\ell+1)\, G_\ell(-\tfrac{1}{\kappa})\, \frac{K_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_\ell(\mu) \tag{111}$$

with the modified Bessel functions of the second kind

$$K_{\ell+\frac{1}{2}}(\kappa\rho) = \frac{\pi}{2}(-1)^\ell \left[I_{-(\ell+\frac{1}{2})}(\kappa\rho) - I_{\ell+\frac{1}{2}}(\kappa\rho)\right] \tag{112}$$

contains, however, in the first term of (112) negative powers of κ. This may be seen in the power series

$$\frac{\kappa}{\sqrt{2}} \cdot \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}} \qquad \frac{1}{\rho} \sum_{m=0}^{\infty} \frac{(\frac{\kappa\rho}{2})^{2m-\ell}}{m!\ \Gamma(m-\ell+\frac{1}{2})} \,. \tag{113}$$

The singular solution (111) in spherical harmonics yields the same density-integral as the solution (35), namely,

$$\int_{-1}^{+1} f^{(s)}_{(S.H.)}(\rho,\mu)\, d\mu = \kappa\sqrt{\frac{2}{\pi}}\, \frac{K_{\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} = \frac{e^{-\kappa\rho}}{\rho} \,. \tag{114}$$

According to the theorem in section A of this chapter both solutions (111) and (35) should be equal if the series (111) would be convergent. Consequently its divergence originates in the useless terms which contain κ in negative powers. This is the matchless conclusion which is compatible with the facts that the new solution (35) is convergent, contains only non-negative powers of κ and yields the same density. Therefore one has to cross out all terms in the singular solution (111) which have negative powers of κ as factors. The remainder of the series is supposed to converge and to be equal

to the new solution (35) and (40).

Using (112) one may write the total solution in spherical harmonics (111) as the sum of a singular solution with the density-integral $(\cosh \kappa\rho)/\rho$ and of a regular solution with the density-integral $(-\sinh \kappa\rho)/\rho$:

$$f^{(s)}_{(S.H.)}(\rho,\mu) = \frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \left\{ \sum_{\ell=0}^{\infty} (-1)^{\ell}(2\ell+1)\, G_{\ell}(-\tfrac{1}{\kappa}) \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}} P_{\ell}(\mu) \right.$$

$$\left. - \sum_{\ell=0}^{\infty} (-1)^{\ell}(2\ell+1)\, G_{\ell}(-\tfrac{1}{\kappa}) \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} P_{\ell}(\mu) \right\} . \qquad (115)$$

Because of

$$(-1)^{\ell} \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}} = \frac{I_{\ell+\frac{1}{2}}(-\kappa\rho)}{\sqrt{-\kappa\rho}}$$

the second series in (115) is just the regular solution (102a) with the opposite sign. It contains only <u>positive odd</u> powers of κ and, using equations (104) and (105a), one easily finds its contributions to the new singular solution (40). To recognize them in the equation (40), one writes one part of (40), namely (42), in the following form:

$$e^{-\mu\rho} R(\rho\sqrt{1-\mu^2}) = e^{-\mu\rho} \left[\frac{\cos d_0}{\pi d_0} - \frac{\kappa}{\pi} \int_0^1 ds \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}} (\cosh \kappa d_0 s - \sinh \kappa d_0 s) \right]$$

$$= e^{-\mu\rho} \left\{ \frac{\cos d_0}{\pi d_0} - \frac{\kappa}{2} J_0(h) + \kappa^2 \sqrt{\frac{d_0}{2\pi}} \sum_{\ell=0}^{\infty} \frac{\ell!\,(2\kappa^2 d_0)^{\ell}}{(2\ell+1)!} J_{\ell+\frac{1}{2}}(d_0) \right\} \qquad (116)$$

in which $h = d_0\sqrt{1-\kappa^2} = \rho\sqrt{(1-\mu^2)(1-\kappa^2)}$. The second term in the curly brackets of (116) contains the odd powers of κ and is the contribution of (104)

$$-\frac{\kappa}{2} e^{-\mu\rho} J_0(h)$$

to equation (42). The second contribution is the more lengthy expression (105a); it gives that part of the equation (41) which is odd in κ. Hence the identification of all terms which contain the positive odd powers of κ as factors in the series of

spherical harmonics (111), with the corresponding terms in the new solution (40) is simple and complete:

$$\left\{\text{Part } \underline{\text{odd in } \kappa} \text{ of } f^{(s)}_{(S.H.)}(\rho,\mu)\right\}$$

$$\equiv -\frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (-1)^{\ell} (2\ell+1)\, G_{\ell}(-\tfrac{1}{\kappa})\, \frac{I_{\ell+\frac{1}{2}}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_{\ell}(\mu) \tag{117}$$

$$= -e^{-\mu\rho}\left\{\frac{\kappa}{2} J_0(h) + \frac{c}{4} \sum_{n=1}^{\infty} \frac{J_n(h)}{n}\left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} - \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} + (-1)^n \left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}}\right]\right\}$$

$= -f^{(r)}(\rho,\mu;\kappa)$ according to equation (71). Negative odd powers of κ do not occur in (111); thus there is of course nothing to cross out.

Furthermore one has to compare the parts of the singular solutions (111) and (40) which are even in κ. The series of spherical harmonics has the

$$\left\{\text{Part } \underline{\text{even in } \kappa} \text{ of } f^{(s)}_{(S.H.)}(\rho,\mu)\right\}$$

$$\equiv \frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (-1)^{\ell} (2\ell+1)\, G_{\ell}(-\tfrac{1}{\kappa})\, \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_{\ell}(\mu). \tag{118}$$

It is the first series in the curly bracket of (115). A glance at

$$G_0(-\tfrac{1}{\kappa}) = 1$$

$$(-1)^{\ell} G_{\ell}(-\tfrac{1}{\kappa}) = (-1)^{\ell} P_{\ell}(\tfrac{1}{\kappa}) + \frac{c}{\kappa}(-1)^{\ell-1} W_{\ell-1}(\tfrac{1}{\kappa})$$

$$= P_{\ell}(-\tfrac{1}{\kappa}) + \frac{c}{\kappa} W_{\ell-1}(-\tfrac{1}{\kappa}) \qquad \text{for } \ell \geq 1,$$

shows that (118) contains two parts, one without a factor c and a second with a factor c:

$$\left\{\text{Part even in } \kappa \text{ of } f^{(s)}_{(S.H.)}(\rho,\mu)\right\}$$

$$\equiv \sqrt{\frac{\pi}{2}} \left\{ \frac{\kappa}{2} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(-\tfrac{1}{\kappa})\, \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_\ell(\mu) \right.$$

$$\left. + \frac{c}{2} \sum_{\ell=1}^{\infty} (2\ell+1)\, W_{\ell-1}(-\tfrac{1}{\kappa})\, \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_\ell(\mu) \right\} . \qquad (119)$$

Here the superfluous terms with negative powers of κ arise from the introduction of the power series (113) for the modified Bessel functions into (119). Of course the rearrangement of the double series in a power series is only a formal way. It leads nevertheless to something reasonable, namely to the new singular solution (40) after dropping all terms with negative powers of κ. The largest negative power of κ in the ℓ-th term of (119) is $\kappa^{-2\ell}$. One has to omit ℓ expressions in the ℓ-th term, namely those with $\kappa^{-2\ell}$, $\kappa^{-2\ell+2}$, $\kappa^{-2\ell+4}$, ... κ^{-4}, κ^{-2} as factors. It is sufficient to keep the terms with the non-negative powers κ^0, κ^2, κ^4, ... in (119).

E. Verification of statement of section D about the even parts in κ of the singular solutions in the two different representations. We consider first the part of (119) which is independent of c:

$$\frac{\kappa}{2}\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(-\tfrac{1}{\kappa})\, \frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}}\, P_\ell(\mu)$$

$$= \frac{\sqrt{\pi}}{2\rho} \left\{ \sum_{s=1}^{\infty} (\tfrac{\kappa\rho}{2})^{-s} \sum_{r=0}^{\infty} \frac{(4r+2s+1)\, P_{2r+s}(-\tfrac{1}{\kappa})\, P_{2r+s}(\mu)}{r!\ \Gamma(-s-r+\frac{1}{2})} \right. \qquad (120)$$

$$\left. + \sum_{s=0}^{\infty} (\tfrac{\kappa\rho}{2})^{s} \sum_{r=[\frac{s}{2}]+1}^{\infty} (4r-2s+1)\, \frac{P_{2r-s}(-\tfrac{1}{\kappa})\, P_{2r-s}(\mu)}{r!\ \Gamma(s-r+\frac{1}{2})} \right\} .$$

$[\frac{s}{2}]$ is the largest integer $< \frac{s}{2}$; e.g., for s = 0 it is $[\frac{s}{2}] = -1$, for s = 1 or 2 it is 0, for s = 3 or 4 it is 1 and so on. The first series of (120) contains only negative powers of κ; hence we omit it. It does not also contribute to the density. We investigate only the second series of (120) and write down its first four terms

$$\rho^{-1}\frac{\sqrt{\pi}}{2}\sum_{r=0}^{\infty}(4r+1)\frac{P_{2r}(-\frac{1}{\kappa})\,P_{2r}(\mu)}{r!\,\Gamma(-r+\frac{1}{2})}$$

$$+\rho^{0}\frac{\sqrt{\pi}}{4}\kappa\sum_{r=1}^{\infty}(4r-1)\frac{P_{2r-1}(-\frac{1}{\kappa})\,P_{2r-1}(\mu)}{r!\,\Gamma(-r+\frac{3}{2})} \tag{121}$$

$$+\rho\frac{\sqrt{\pi}}{8}\kappa^{2}\sum_{r=1}^{\infty}(4r-3)\frac{P_{2r-2}(-\frac{1}{\kappa})\,P_{2r-2}(\mu)}{r!\,\Gamma(-r+\frac{5}{2})}$$

$$+\rho^{2}\frac{\sqrt{\pi}}{16}\kappa^{3}\sum_{r=2}^{\infty}(4r-5)\frac{P_{2r-3}(-\frac{1}{\kappa})\,P_{2r-3}(\mu)}{r!\,\Gamma(-r+\frac{7}{2})}+\ldots\,.$$

After the omission of all terms with negative powers of κ in (121) these four terms should be equal to the corresponding four terms in the development of that part of the new solution (40) which is <u>even in κ and does not contain c</u>. This part in (40) is

$$\frac{1}{\pi}e^{-\mu\rho}\left\{\frac{\cos(\rho\sqrt{1-\mu^2})}{\rho\sqrt{1-\mu^2}}+\kappa\int_0^1 dv\,\frac{\cos\left(\rho\sqrt{(1-\mu^2)(1-v^2)}\right)}{\sqrt{1-v^2}}\sinh\left(\kappa\rho\sqrt{1-\mu^2}\,v\right)\right\}$$

$$=\rho^{-1}\cdot\frac{1}{\pi\sqrt{1-\mu^2}}+\rho^{0}\frac{-\mu}{\pi\sqrt{1-\mu^2}}+\rho\cdot\frac{1}{\pi}\left(\frac{1}{2\sqrt{1-\mu^2}}-(1-\kappa^2)\sqrt{1-\mu^2}\right) \tag{122}$$

$$+\rho^{2}\frac{1}{\pi}\left[\frac{-\mu}{6\sqrt{1-\mu^2}}+\left(\frac{2}{3}-\kappa^2\right)\mu\sqrt{1-\mu^2}\right]+\ldots\,.$$

G. Bauer[11] has derived three formulas which may be used advantageously for the comparison of equations (121) and (122):

$$\frac{2}{\pi\sqrt{1-\mu^2}}=P_0(\mu)+5\cdot\left(\frac{1}{2}\right)^2P_2(\mu)+9\left(\frac{1\cdot3}{2\cdot4}\right)^2P_4(\mu)+13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2P_6(\mu)+\ldots \tag{123}$$

---

[11] G. Bauer, Journal fuer die reine und angewandte Mathematik, Berlin, Vol. 56, p. 101 (1859). See also E. W. Hobson, the Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press, 2nd printing, p. 49 (1955).

$$\frac{2}{\pi}\sqrt{1-\mu^2} = \frac{1}{2}P_0(\mu) - 5\cdot\frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 P_2(\mu) - 9\cdot\frac{3}{6}\left(\frac{1}{2\cdot 4}\right)^2 P_4(\mu) - 13\cdot\frac{5}{8}\left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 P_6(\mu) - \ldots \tag{124}$$

$$\frac{2}{\pi}\frac{\mu}{\sqrt{1-\mu^2}} = 3\cdot\frac{1}{2}P_1(\mu) + 7\cdot\frac{3}{4}\left(\frac{1}{2}\right)^2 P_3(\mu) + 11\cdot\frac{5}{6}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 P_5(\mu) + 15\cdot\frac{7}{8}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 P_7(\mu) + \ldots . \tag{125}$$

Applying the recursion formula for Legendre polynomials

$$\mu\, P_m(\mu) = \frac{1}{2m+1}\left[(m+1)\, P_{m+1}(\mu) + m\, P_{m-1}(\mu)\right] ,$$

these formulas may be extended to arbitrary powers μ^n as factors of $(1-\mu^2)^{\frac{1}{2}}$ or $(1-\mu^2)^{-\frac{1}{2}}$, for instance to

$$\frac{2}{\pi}\mu\sqrt{1-\mu^2} = \frac{3}{8}P_1(\mu) - \frac{7}{32}P_3(\mu) - \frac{5\cdot 11}{1024}P_5(\mu) - \frac{3\cdot 5\cdot 7}{4096}P_7(\mu) - \ldots . \tag{126}$$

Therefore our comparison of the coefficients of (121) and (122) could be continued to arbitrary large powers of ρ.

(a) We start with the comparison of the coefficient of ρ^{-1} in the equations (121) and (122). We obtain from (121)

$$\frac{\sqrt{\pi}}{2}\sum_{r=0}^{\infty}(4r+1)\,\frac{P_{2r}(-\frac{1}{\kappa})\,P_{2r}(\mu)}{r!\,\Gamma(-r+\frac{1}{2})}$$

$$= \frac{1}{2}\left\{P_0(\mu) + 5\cdot\left(\frac{1}{2}\right)^2\left(1-\frac{3}{\kappa^2}\right)P_2(\mu) + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1-\frac{10}{\kappa^2}+\frac{35}{\kappa^4}\right)P_4(\mu) + \right.$$

$$+ 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(1-\frac{21}{\kappa^2}+\frac{63}{\kappa^4}-\frac{231}{5\kappa^6}\right)P_6(\mu)$$

$$\left. + 17\left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\left(1-\frac{36}{\kappa^2}+\frac{198}{\kappa^4}-\frac{1716}{5\kappa^6}+\frac{1287}{7\kappa^8}\right)P_8(\mu) + \ldots\right\}$$

$$= \frac{1}{\pi\sqrt{1-\mu^2}} - \frac{1}{\kappa^2}\left[\ldots\right] + \frac{1}{\kappa^4}\left[\ldots\right] - + \ldots \tag{127}$$

using the formula (123). Hence we see that the coefficient of ρ^{-1} in (121) is identical with the corresponding coefficient in (122) after the omission of all terms with negative powers of κ as factors.

(b) We compare now the corresponding coefficient of ρ^0. We obtain from (121)

$$\frac{\sqrt{\pi}}{4}\kappa \sum_{r=1}^{\infty} (4r-1) \frac{P_{2r-1}\left(-\frac{1}{\kappa}\right) P_{2r-1}(\mu)}{r!\,\Gamma\left(-r+\frac{3}{2}\right)}$$

$$= -\frac{1}{2}\left\{3 \cdot \frac{1}{2} P_1(\mu) + 7 \cdot \frac{3}{4}\left(\frac{1}{2}\right)^2\left(1-\frac{5}{3\kappa^2}\right) P_3(\mu) + 11 \cdot \frac{5}{6}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\left(1 - \frac{14}{3\kappa^2} + \frac{21}{5\kappa^4}\right) P_5(\mu)\right.$$

$$\left. + 15 \cdot \frac{7}{8}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2\left(1 - \frac{9}{\kappa^2} + \frac{99}{5\kappa^4} - \frac{429}{35\kappa^6}\right) P_7(\mu) + \ldots \right\}$$

$$= \frac{-\mu}{\pi\sqrt{1-\mu^2}} + \frac{1}{\kappa^2}\left[\ldots\right] - \frac{1}{\kappa^4}\left[\ldots\right] + \ldots \qquad (128)$$

using the formula (125). After the omission of all terms with negative powers of κ the coefficients of ρ^0 in (121) and (122) are equal.

(c) We compare now the corresponding coefficients of ρ. We obtain from (121)

$$\frac{\sqrt{\pi}}{8}\kappa^2 \sum_{r=1}^{\infty} (4r-3) \frac{P_{2r-2}\left(-\frac{1}{\kappa}\right) P_{2r-2}(\mu)}{r!\,\Gamma\left(-r+\frac{5}{2}\right)}$$

$$= \frac{1}{8}\left\{2\kappa^2 P_0(\mu) - \frac{5}{4}(\kappa^2-3) P_2(\mu) - \frac{9}{4 \cdot 8}\left(\kappa^2 - 10 + \frac{35}{3\kappa^2}\right) P_4(\mu)\right.$$

$$- \frac{13 \cdot 5}{8 \cdot 64}\left(\kappa^2 - 21 + \frac{63}{\kappa^2} - \frac{231}{5\kappa^4}\right) P_6(\mu)$$

$$\left. - \frac{17 \cdot 35}{64 \cdot 128}\left(\kappa^2 - 36 + \frac{198}{-\kappa^2} - \frac{1716}{5\kappa^4} + \frac{1287}{7\kappa^6}\right) P_8(\mu) - - \ldots\right\}$$

$$= \frac{1}{4}\Big[P_0(\mu) + 5 \cdot \left(\frac{1}{2}\right)^2 P_2(\mu) + 9\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 P_4(\mu) + 13\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 P_6(\mu) + 17\left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2$$

$$P_8(\mu) + \ldots\Big]$$

$$- \frac{1}{2}(1-\kappa^2)\left[\frac{1}{2} P_0 - 5 \cdot \frac{1}{4}\left(\frac{1}{2}\right)^2 P_2(\mu) - 9 \cdot \frac{3}{6}\left(\frac{1}{2 \cdot 4}\right)^2 P_4(\mu) - 13 \cdot \frac{5}{8}\left(\frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 P_6(\mu)\right.$$

$$\left. - 17 \cdot \frac{7}{10}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2 P_8(\mu) - - \ldots\right] - \frac{1}{\kappa^2}\left[\cdots\right] + \frac{1}{\kappa^4}\left[\cdots\right] - + \cdots \quad (129)$$

$$= \frac{1}{\pi}\left\{\frac{1}{2\sqrt{1 - \mu^2}} - (1-\kappa^2)\sqrt{1 - \mu^2}\right\} - \frac{1}{\kappa^2}\left[\cdots\right] + \frac{1}{\kappa^4}\left[\cdots\right] - + \cdots \quad (129)$$

using the formulas (123) and (124). After omission of all terms with negative powers of κ the coefficients of ρ in (121) and (122) are equal.

(d) Finally we compare the corresponding coefficients of ρ^2. We obtain from (121)

$$\frac{\sqrt{\pi}}{16}\kappa^3 \sum_{r=2}^{\infty} (4r-5) \frac{P_{2r-3}(-\frac{1}{\kappa})\, P_{2r-3}(\mu)}{r!\,\Gamma(-r+\frac{7}{2})}$$

$$= \frac{1}{16}\left\{- 3\kappa^2 P_1(\mu) + \frac{7}{12}(3\kappa^2-5) P_3(\mu) + \frac{11}{384}(15\kappa^2-70+\frac{63}{\kappa^2}) P_5(\mu)\right.$$

$$\left. + \frac{3}{512}(35\kappa^2-315+\frac{693}{\kappa^2}-\frac{429}{\kappa^4}) P_7(\mu) + \cdots\right\}$$

$$= (\frac{2}{3}-\kappa^2) \cdot \frac{1}{2}\left[\frac{3}{8} P_1(\mu) - \frac{7}{32} P_3(\mu) - \frac{5 \cdot 11}{1024} P_5(\mu) - \frac{3 \cdot 5 \cdot 7}{4096} P_7(\mu) - \cdots\right]$$

$$- \frac{1}{12}\left[3 \cdot \frac{1}{2} P_1(\mu) + 7 \cdot \frac{3}{4}\left(\frac{1}{2}\right)^2 P_3(\mu) + 11 \cdot \frac{5}{6}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 P_5(\mu) + 15 \cdot \frac{7}{8}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 P_7(\mu) + \cdots\right]$$

$$+ \frac{1}{\kappa^2}\left[\cdots\right] - \frac{1}{\kappa^4}\left[\cdots\right] + - \cdots$$

$$= \frac{1}{\pi}\left[\left(\frac{2}{3} - \kappa^2\right)\mu\sqrt{1-\mu^2} - \frac{\mu}{6\sqrt{1-\mu^2}}\right] + \frac{1}{\kappa^2}\left[\cdots\right] - \frac{1}{\kappa^4}\left[\cdots\right] + \cdots \tag{130}$$

using the formulas (125) and (126). After the omission of all terms with negative powers of κ the coefficients of ρ^2 in (121) and (122) are equal.

We turn now to the part of (119) which has the factor c:

$$\frac{c}{2}\sqrt{\frac{\pi}{2}}\sum_{\ell=1}^{\infty}(2\ell+1)\,W_{\ell-1}\left(-\frac{1}{\kappa}\right)\frac{I_{-(\ell+\frac{1}{2})}(\kappa\rho)}{\sqrt{\kappa\rho}}\,P_\ell(\mu) \tag{131}$$

$$= \frac{c}{4}\sqrt{\pi}\sum_{\ell=1}^{\infty}{\sum_{r=0}^{\infty}}'\frac{(2\ell+1)\,W_{\ell-1}\left(-\frac{1}{\kappa}\right)P_\ell(\mu)\left(\frac{\kappa\rho}{2}\right)^{-\ell-1+2r}}{r!\,\Gamma(-\ell+r+\frac{1}{2})}$$

= (terms with negative powers of κ only) +

$$+\frac{c}{2}\left\{\rho^0\left[\frac{3}{1\cdot 2}P_1(\mu) + \frac{7}{3\cdot 4}P_3(\mu) + \frac{11}{5\cdot 6}P_5(\mu) + \cdots\right]\right.$$

$$+\rho\left[-\frac{5\cdot 3}{16}P_2(\mu) - \frac{5\cdot 11}{128}P_4(\mu) - \cdots\right] \tag{132}$$

$$+\rho^2\left[\left(\frac{7\cdot 5}{96}P_3(\mu) + \frac{11\cdot 49}{24\cdot 128}P_5(\mu) + \cdots\right)\right.$$

$$\left.\left.+\kappa^2\left(\frac{3}{8}P_1(\mu) - \frac{7}{72}P_3(\mu) - \frac{11}{720}P_5(\mu) - \cdots\right)\right] + \cdots\right\}.$$

G. Bauer[12] gives the formula

$$\log(1+\mu) = (\log 2) - 1 + \frac{3}{1\cdot 2}P_1(\mu) - \frac{5}{2\cdot 3}P_2(\mu) + \frac{7}{3\cdot 4}P_3(\mu) - + \cdots \tag{133}$$

from which one derives the extensions

[12] Loc. cit.

$$\log\sqrt{\frac{1+\mu}{1-\mu}} = \sum_{n=0}^{\infty} \frac{(4n+3)\,P_{2n+1}(\mu)}{(2n+1)(2n+2)} \tag{134}$$

$$\mu \log\sqrt{\frac{1+\mu}{1-\mu}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(4n+1)\,P_{2n}(\mu)}{(n+1)(2n-1)}\right] \tag{135}$$

$$\mu^2 \log\sqrt{\frac{1+\mu}{1-\mu}} = \mu + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n(2n+3)(4n+3)\,P_{2n+1}(\mu)}{(n+1)(n+2)(2n-1)(2n+1)} \tag{136}$$

and so on, if one wants to proceed with further steps of the comparison. The part of the new singular solution (40), which is even in κ and contains the factor c, is

$$\frac{c}{2} e^{-\mu\rho}\left\{\sum_{m=1}^{\infty} \frac{(-1)^m h_m}{(m!)^2}\left(\frac{h}{2}\right)^{2m} + J_0(h)\log\sqrt{\frac{1+\mu}{1-\mu}}\right.$$

$$+ \frac{1}{\pi}\int_0^1 ds\, \frac{\cos(d_0\sqrt{1-s^2})}{\sqrt{1-s^2}}\left\{2(\log 2s)\cosh(\kappa d_0 s)\right.$$

$$\left. - e^{-d_0 s}\sum_{m=1}^{\infty}\frac{h_m}{m!}\left[(1+\kappa)^m + (1-\kappa)^m\right] d_0^m s^m\right)$$

$$\left. + \frac{1}{2}\sum_{n=1}^{\infty}\frac{J_n(h)}{n}\left[\left(\frac{1+\kappa}{1-\kappa}\right)^{\frac{n}{2}} + \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{n}{2}}\right]\left[\left(\frac{1+\mu}{1-\mu}\right)^{\frac{n}{2}} - (-1)^n\left(\frac{1-\mu}{1+\mu}\right)^{\frac{n}{2}}\right]\right\} \tag{137a}$$

$$= \frac{c}{2}\left\{\rho^0\left(\log\sqrt{\frac{1+\mu}{1-\mu}} + \rho\left[1 - \frac{2}{\pi}\sqrt{1-\mu^2} - \mu\log\sqrt{\frac{1+\mu}{1-\mu}}\right]\right.\right.$$

$$+ \rho^2\left[\left(-\frac{3\mu}{4} + \frac{2}{\pi}\mu\sqrt{1-\mu^2} - \frac{1-3\mu^2}{4}\log\sqrt{\frac{1+\mu}{1-\mu}}\right)\right.$$

$$\left.\left.\left. + \kappa^2\left(\frac{\mu}{4} + \frac{1-\mu^2}{4}\log\sqrt{\frac{1+\mu}{1-\mu}}\right)\right] + \cdots\right\} . \tag{137b}$$

Applying the aid formulas (124), (126), (134-136) to (137b) one finds (132) if one omits the negative powers of κ in (132). This procedure could be continued to higher powers of ρ, but never completed in this manner. If there exists any justice, however, the following relation between the singular solution in spherical harmonics (111) and the new form (40) should hold:

Omitting all terms with negative powers of κ in the spherical harmonics solution, (111) and (40) become equal.

Remark. Presumably the same method could be applied to the cylindrical case for the construction of a converging singular solution from the corresponding spherical harmonics series.

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