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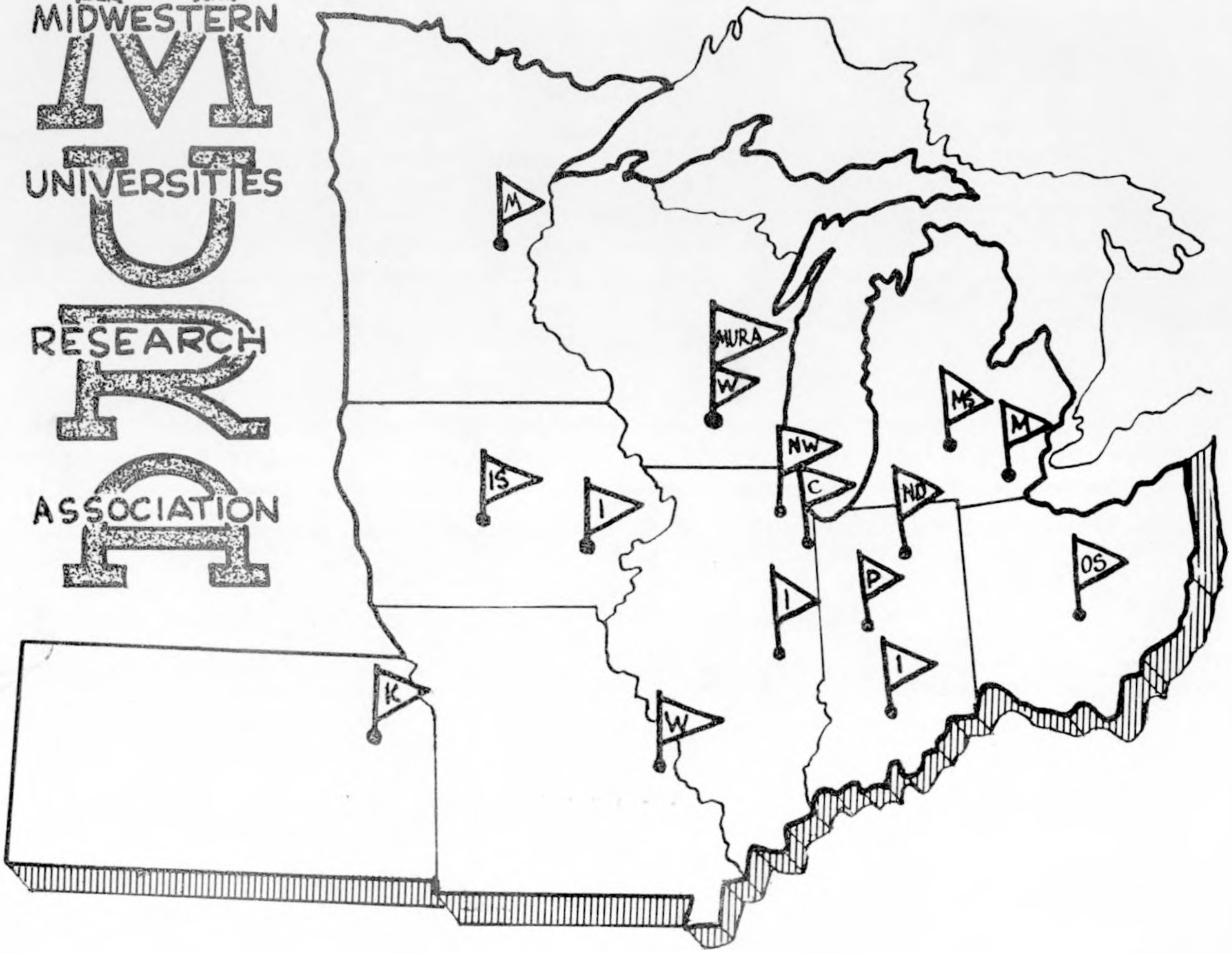

PRELIMINARY ANALYTIC WORK ON NON-SCALING SPIRAL SECTOR FFAG ACCELERATORS

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ABSTRACT

Approximate analytical results are given for non-scaling spiral sector FFAG Accelerators. The equilibrium orbit scalloping is calculated, as well as the betatron oscillation frequencies in the smooth approximation. The effects of derivatives of k, $1/w$ and flutter appear to be small; ν_x and ν_y are given fairly accurately by the local value of these parameters. The transition energy is determined essentially by the local value of k.

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** On leave from State University of Iowa.

I. INTRODUCTION

There has been an increase of interest in non-scaling accelerators in the recent past.[1] Apparently the large injection aperture desirable for high intensities can be achieved in a non-scaling accelerator without the increased circumference factor which is present in scaling machines. We envisage a spiral sector accelerator with large flutter and loose spiral at injection and small flutter and light spiral at output. It seems clear that it is necessary to change the spiral angle as the flutter is varied in order to keep the betatron oscillation frequencies constant. Otherwise, resonances will be crossed and beam lost.

This report records some preliminary considerations on non-scaling spiral sector accelerators so that they will be available for reference. A good part of this work is application of the analytic work of Parzen[2], with some notational changes. One may discuss betatron oscillations either in terms of derivatives of Fourier components of the fields, as Parzen has done, or in terms of k, $1/w$, flutter and their derivatives. The present report uses the latter quantities in an attempt to exploit the familiarity with scaling machines.

We remark to give a general idea of the range of parameters which is of interest that we have in mind a flutter which varies from approximately 2 at injection to approximately 0.5 at output. For a 15 Bev machine with 200 Mev injection, we might have $N \cong 30$, $k \cong 50$, $\nu_x \cong 7$, $\nu_y \cong 5$. If the machine is all similar to a scaling machine over any small radial span, that is, if the effect of derivatives of k, $1/w$ and flutter on ν_x and ν_y is small, the spiral

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II. MAGNETIC FIELD EXPANSIONS

The median plane magnetic field is customarily written for discussion of scaling accelerators in the form

$$B_r = B_\theta = 0$$
$$B_z = -B_o \left(\frac{r}{r_o}\right)^k \sum_{n=0}^{\infty} \left\{ g_n \cos n\Phi + f_n \sin n\Phi \right\} \qquad (2.1)$$
$$\Phi = K\, \ln\left(\frac{r}{r_o}\right) - N\theta ,$$

where k, $K = 1/w$ and the g_n and f_n are constants and r_o is an arbitrary reference radius. We can also write (2.1) as

$$B_z = -B_o \sum_{n=-\infty}^{\infty} \lambda_n (1+x)^{k_n} e^{inN\theta} , \qquad (2.2)$$

where we have used the relative variable $x = (r-r_o)/r_o$ and have defined

$$\left\{ \begin{array}{ll} \lambda_n = \frac{1}{2}(g_n + i f_n) , & n > 0 \\ \quad = g_o , & n = 0 \\ \quad = \frac{1}{2}(g_n - i f_n) ; & n < 0 \\ k_n = k - inK & . \end{array} \right. \qquad (2.3)$$

Then, in a scaling accelerator, λ_n and k_n are independent of radius. A non-scaling accelerator can be described by giving λ_n and k_n as functions of radius. The same radial variation can be described by either λ_n or k_n, so that the description is not unique. One can see immediately an equivalence between k_n and variation of the λ_n with radius, since one can write

$$\lambda_n(r) = \lambda_n(r_o)\, e^{\ln \frac{\lambda_n(r)}{\lambda_n(r_o)}}$$

and can define a new exponent

$$"k_n" = k_n + \frac{\ln\left(\frac{\lambda_n(r)}{\lambda_n(r_o)}\right)}{\ln \frac{r}{r_o}} .$$

This equivalence is just the flare focusing of Roberts[3].

For an analytic treatment it is necessary to expand the median plane field in powers of x. We shall write such a field as

$$B_z = -B_o \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} z_{m,n} \, e^{inN\theta} \, x^m. \tag{2.4}$$

z_{mn} is just the m^{th} derivative of Parzen's H_n evaluated at $r = r_o$. We can also express the z_{mn} in terms of the scaling quantities by comparing (2.4) and (2.2). Then

$$\begin{aligned} z_{o,n} &= \lambda_n \\ z_{1,n} &= r_o \frac{d\lambda_n}{dr} + \lambda_n k_n \\ z_{2,n} &= \frac{1}{2} r_o^2 \frac{d^2\lambda_n}{dr^2} + r_o \frac{d}{dr_o} (\lambda_n k_n) + \frac{1}{2} \lambda_n k_n (k_n - 1) . \end{aligned} \tag{2.5}$$

III. EQUILIBRIUM ORBITS

The equation of motion of a particle in the median plane is

$$\left(\frac{x'}{X}\right)' = \frac{1+x}{X} - \alpha (1+x) \sum_{m,n} z_{m,n} x^m e^{inN\theta} , \tag{3.1}$$

where primes denote total derivatives with respect to θ and

$$\left\{ \begin{aligned} X &= \sqrt{(1+x)^2 + x'^2} \\ \alpha &= \frac{e r_o B_o}{cp} \end{aligned} \right. . \tag{3.2}$$

We expand (3.1) in powers of x and x' by expanding the Lagrangian, so that the approximate equation of motion is still Hamiltonian. Through second order, the equation of motion is

$$\begin{aligned} x'' - xx'' = 1 + \frac{1}{2} x'^2 - \alpha \Big\{ & \sum_n z_{o,n} e^{inN\theta} + x \sum_n (z_{1,n} + z_{o,n}) e^{inN\theta} \\ & + x^2 \sum_n (z_{2,n} + z_{1,n}) e^{inN\theta} + \dots \Big\} . \end{aligned} \tag{3.3}$$

The equilibrium orbit x_e has the period of the magnet so that it can be expanded in a Fourier series

$$x_e = \sum_{n=-\infty}^{\infty} x_n e^{inN\theta} \quad , \tag{3.4}$$

We substitute (3.4) into (3.3) and equate terms of the same frequency (a method known more elegantly as "harmonic balance"). Then

$$\begin{aligned} -n^2N^2x_n = \delta_{no} - \alpha z_{o,n} - \alpha \sum_m (z_{1,m} + z_{0,m})\ x_{n-m} \\ - \alpha \sum_{m,p} (z_{2,m} + z_{1,m})\ x_m x_{n-m-p} \\ - \frac{1}{2} N^2 \sum_m m\ (n+m)\ x_m x_{n-m} + \dots \ , \end{aligned} \tag{3.5}$$

where the last term combines xx'' and x'^2.

(3.5) can be solved by an approximation method which assumes that the terms involving x on the right hand side are small compared to those independent of x. This amounts to assuming that the change of field across the equilibrium orbit is small compared to the field. We calculate $x_n^{(p)}$, the p^{th} approximation, by substituting $x_n^{(p-1)}$ on the right hand side of (3.5). There is a difficulty with x_o, whose size depends on the reference radius r_o chosen. r_o is fixed (for a given field strength and momentum) by α. We can circumvent the x_o difficulty by choosing α such that r_o is the average radius of the equilibrium orbit; then $x_o = 0$ and the $n = 0$ equation gives a value for α.

Our assumption is then

$$x_n^{(0)} = 0$$

Then, by substituting this in the r. h. s of (3. 5),

$$
\begin{cases}
x_n^{(1)} = \dfrac{\alpha z_{0,n}}{n^2/N^2} \\[2ex]
x_n^{(2)} = \dfrac{\alpha}{n^2 N^2}\left\{ z_{0,n} + \dfrac{\alpha}{N^2} \sum_{m \neq n}{}' \dfrac{(z_{1,m} + z_{0,m})\, z_{0,-m}}{(n-m)^2} \right. \\[2ex]
\qquad\qquad \left. + \dfrac{\alpha}{2N^2} \sum_{m \neq 0,n}{}' \dfrac{(m+n)\, z_{0,m}\, z_{0,n-m}}{m(n-m)^2} \right\}.
\end{cases}
\tag{3. 6}
$$

for $n \neq 0$. α satisfies the $n = 0$ equation with x_n substituted from (3. 6). Correct through terms quadratic in , this equation is

$$
1 - \alpha z_{0,0} - \alpha^2 \left\{ \sum_{m \neq 0}{}' \frac{(z_{1,m} + \frac{3}{2} z_{0,m})\, z_{0,-m}}{m^2 N^2} \right\} = 0 . \tag{3.7}
$$

In practice $x_n^{(2)}$ agrees with computer experiments to a few percent, while $x_n^{(1)}$ differs from $x_n^{(2)}$ by 10-20%. The method of solution seems a posteriori to be justified. For discussion of motion about the equilibrium orbit, $x_n^{(1)}$ seems adequately accurate.

From its definition (3. 2), is a relation between field strength, radius and momentum. Given the field as a function of radius, the value of calculated from (3. 7) gives the average radius of the equilibrium orbit as a function of momentum. The term of (3. 7) linear in α describes the bending of the equilibrium orbit due to the average field. The term quadratic in describes the additional bending due to the fact that the scalloping of the equilibrium orbit carries the particle into regions of different field. In radial sector accelerators, this term is important; in fact, it is responsible for all of the orbit bending in a two-way accelerator where $z_{0,0} = 0$. In a spiral sector accelerator with a flutter of about unity, this term decreases by about 5%, since $k/N^2 \sim 0.05$ and spiraling effects can be shown to cancel. Since this term is so small, an accurate solution of (3.7) is

$$\alpha = \frac{1}{\mathfrak{z}_{0,0}} \left\{ 1 - \sum_{m \neq 0}' \frac{(\mathfrak{z}_{1,m} + \frac{3}{2}\mathfrak{z}_{0,m})\mathfrak{z}_{0,-m}}{\mathfrak{z}_{0,0}^2 \quad m^2 N^2} \right\}. \tag{3.8}$$

IV. LINEAR MOTION ABOUT THE EQUILIBRIUM ORBIT

We use a coordinate system based on the equilibrium orbit. All lengths are measured in units of R_o, the equilibrium orbit length devided by 2π. $R_o\psi$ and $R_o\eta$ are the displacements perpendicular to the equilibrium orbit in the median plane and perpendicular to it, respectively. We use as independent variable

$$\vartheta = \int_0^s \frac{ds}{R_o} , \tag{4.1}$$

where ds is the element of arc length along the equilibrium orbit.

For most purposes we can neglect the difference between ϑ and θ. From (4.1),

$$\vartheta = \frac{r_o}{R_o} \int_0^\theta X \, d\theta = \frac{r_o}{R_o} \int_0^\theta \left\{ 1 + x + \frac{1}{2} x'^2 \right\} d\theta$$

$$= \frac{r_o}{R_o} \left\{ \theta \left[1 + \frac{N^2}{2} \sum_{m \neq 0}' m^2 x_m x_{-m} + \dots \right] \right.$$

$$\left. + \sum_{n \neq 0} \frac{e^{inN\theta} - 1}{inN} \left[x_n - \frac{N^2}{2} \sum_{m \neq 0, n}' m(n-m) \, x_m x_{n-m} \right] \right\}$$

The periodic terms are of order $1/N^3$ compared to unity and are thus very small for N 10. The term linear in θ differs from unity by terms which are of order $1/N^2$. These terms just give R_o in terms of r_o, since $\vartheta = 2$ when $\theta = 2$. Then

$$R_0 = r_0 \left\{ 1 + \frac{1}{2}\alpha^2 \sum_{m \neq 0}{}' \frac{z_{0,m}\, z_{0,-m}}{m^2 N^2} \right\} . \tag{4.2}$$

The difference between R_0 and r_0 is then small for N   20.

The linear equations of motion about the equilibrium orbit are

$$\begin{cases} \dfrac{d^2\psi}{d\vartheta^2} + \lambda\left[\eta_1 + \lambda\eta_0^2\right]\psi = 0 \\ \dfrac{d^2\eta}{d\vartheta^2} - \lambda\,\eta_1\,\eta = 0 , \end{cases} \tag{4.3}$$

where

$$\begin{cases} \lambda = \dfrac{e r_0 B_0}{cp} \\ \eta_m = -\dfrac{1}{B_0}\left(\dfrac{\partial^m B_z}{\partial\psi^m}\right)_{\psi=\eta=0} . \end{cases} \tag{4.4}$$

and   are equal whenever we make the approximation that $R_0 = r_0$. The field derivatives are to be evaluated on the equilibrium orbit and are with respect to   , which differs from the radial direction r because of the scalloping. Then

$$\frac{\partial B_z}{\partial\psi} = \frac{\partial B_z}{\partial r}\frac{\partial r}{\partial\psi} + \frac{\partial B_z}{\partial\theta}\frac{\partial\theta}{\partial\psi} .$$

Define the angle   by

$$\tan\phi = -\frac{x_e'}{1+x_e} , \tag{4.5}$$

so that   is the angle between the   and r directions. Then a little geometrical exercise gives

$$\begin{cases} \dfrac{\partial r}{\partial\psi} = R_0 \cos \\ \dfrac{\partial\theta}{\partial\psi} = \dfrac{R_0}{r_0}\,\dfrac{\sin}{1 + x_e} \end{cases}$$

and

$$\eta_1 = -\frac{1}{B_o}\frac{1}{\sqrt{(1+x_e)^2 + x_e'^{\,2}}}\left\{\frac{\partial B_z}{\partial x}(1+x_e) - \frac{\partial B_z}{\partial \theta}\frac{x_e'}{1+x_e}\right\}, \qquad (4.6)$$

where we have neglected the difference between R_o and r_o. We now neglect in (4.6) x_e and $x_e'^{\,2}$ compared to unity, since they are of order $1/N^2$. When we substitute $x_n^{(1)}$ from (3.6) and evaluate on the equilibrium orbit, we find

$$\left\{\begin{aligned} \eta_0 &= \sum_n{}' e^{inN\theta}\left\{\mathfrak{z}_{0,n} + \alpha\sum_{m\neq n}{}'\frac{\mathfrak{z}_{1,m}\,\mathfrak{z}_{0,n-m}}{(n-m)^2N^2}\right\} \\ \eta_1 &= \sum_n{}' e^{inN\theta}\left\{\mathfrak{z}_{1,n} + 2\alpha\sum_{m\neq n}{}'\frac{(\mathfrak{z}_{2,m}+\frac{1}{2}\mathfrak{z}_{1,m})\mathfrak{z}_{0,n-m}}{(n-m)^2N^2} + \frac{1}{2}\,\frac{m\,\mathfrak{z}_{0,m}\,\mathfrak{z}_{0,n-m}}{(n-m)N}\right\} \end{aligned}\right. \qquad (4.7)$$

The second terms of η_0 and η_1 are of order $\alpha k F/N^2$ or $\alpha K F/N^2$ relative to the first terms, where F is the flutter. These quantities are usually smaller than about 0.2 in either radial or spiral sector accelerators.

We shall estimate the betatron oscillation frequencies with the smooth approximation which we take in the following form. When applied to a Hill equation

$$u'' + \left(\sum_n{}' a_n e^{inN\theta}\right)u = 0,$$

the smooth approximation gives for the frequency

$$\nu^2 = a_0 + \sum_{m\neq 0}{}'\frac{a_m\,a_{-m}}{m^2N^2}. \qquad (4.8)$$

When we evaluate the sum in (4.8), we need to take only the leading terms of (4.7), since these terms when summed give terms of order $\alpha k F/N^2$ or $\alpha K F/N^2$. Then we find

$$\left\{\begin{aligned} \nu_x^2 &= \alpha(\mathfrak{z}_{1,0} + \alpha\mathfrak{z}_{0,0}^2) + \alpha^2\sum_{m\neq 0}{}'\left[\frac{(2\mathfrak{z}_{2,m}+\mathfrak{z}_{1,m})\mathfrak{z}_{0,-m} + \mathfrak{z}_{1,m}\,\mathfrak{z}_{1,-m}}{m^2N^2}\right] \\ \nu_y^2 &= -\alpha\mathfrak{z}_{10} - \alpha^2\sum_{m\neq 0}{}'\left[\frac{(2\mathfrak{z}_{2,m}+\mathfrak{z}_{1,m})\mathfrak{z}_{0,-m} - \mathfrak{z}_{1,m}\,\mathfrak{z}_{1,-m}}{m^2N^2} - \mathfrak{z}_{0,m}\,\mathfrak{z}_{0,-m}\right] \end{aligned}\right. \qquad (4.9)$$

It is interesting for the interpretation of non-scaling terms to rewrite (4.9) in terms of k, K, the Fourier coefficients g_n and f_n and their derivatives. We resolve the ambiguity in the description of the field variation by choosing $g_o = 1$. We also use the value of given by (3.8). Then, neglecting unity compared to k,

$$\left\{\begin{aligned} \nu_x^2 &= k + 1 + k^2 G_o + 2r_o k' G_o + 3r_o k G_1 + r_o^2 (G_2 + G_3) \\ \nu_y^2 &= -k + k^2 G_o + F^2 \left(\frac{K^2}{N^2} + \frac{1}{2}\right) + 2r_o k' G_o + r_o k G_1 + r_o^2 (G_2 - G_3) \\ &\quad - 2\,\frac{r_o K H_1}{N}, \end{aligned}\right. \tag{4.10}$$

where primes now denote derivatives with respect to r_o and

$$\left\{\begin{aligned} G_o &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{f_m^2 + g_m^2}{m^2 N^2} \\ F^2 &= \sum_{m=1}^{\infty} \left(f_m^2 + g_m^2\right) \\ G_1 &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{g_m g_m' + f_m f_m'}{m^2 N^2} \\ G_2 &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{g_m'^2 + f_m'^2}{m^2 N^2} \\ G_3 &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{g_m g_m'' + f_m f_m''}{m^2 N^2} \\ H &= \sum_{m=1}^{\infty} \frac{f_m g_m' - g_m f_m'}{m N} \end{aligned}\right. \tag{4.11}$$

The results for a scaling machine are just a special case of (4.10) with all derivative terms zero, i.e. $k' = 0$ and only G_o and F are different from zero. It is interesting to note that all terms depending on K' have cancelled. The terms in k' are corrections to the average field which arise from the scalloping of the equilibrium orbit. The terms in G_1, G_2 and G_3 are corrections to the alternating gradient focusing which come from the change of flutter with radius.

We can estimate the size of these terms by considering a field with only a single harmonic, say $g_1 = 2 - 20x$, which gives the variation described in the introduction. Then $r_o\ G_1 = 20\ G_o$ and $r_o\ k\ G_1 \sim k^2 G_o$. Similarly, $r_o^2 G_2 = 400\ G_o$ and $r_o^2 G_2 \sim k^2 G_o$. But $k^2 G_o \ \frac{1}{5} k$, so that all of these terms are small. Since they are small, ν_x will be constant if k is approximately constant and the term $r_o k' G_1$ will be small. Since $\frac{dG_o}{dr_o}$ is negative, we would expect that k' must be positive to compensate for the decrease of the $k^2 G_o$ term. To keep ν_y constant, K should vary so that FK is constant.

One may regard these results as encouraging. Evidently the effects of all derivatives are small and a non-scaling machine may be thought of roughly as a succession of scaling machines.

V. TRANSITION ENERGY

We would expect the transition energy to be raised for two reasons. First, k increases as a function of radius, so that the local value of k is higher at the transition radius than at the injection radius and second, the orbit scalloping (and therefore the orbit length) decreases as a function of radius. Of these, the first effect would seem to be much more important because the effect of orbit scalloping on orbit length is small (as we can see from (4. 2)). The transition energy is given approximately by ν_x^2, whose dominant term is just k .

A detailed calculation bears out this guess. We begin with the well-known relation for the change of revolution frequency f with momentum:

$$\frac{p}{f}\frac{df}{dp} = \frac{1}{\gamma^2} - \frac{p}{R_o}\frac{dR_o}{dp} \tag{5.1}$$

where $\quad = E/E_o$ and $2 \quad R_o$ is the orbit length. From (4.2),

$$R_o = r_o \left(1 + \frac{1}{2} \quad {}^2 G_o \right)$$

and from (3.8) with $z_{o,o} = 1$,

$$\quad = 1 - \quad (k G_o + r_o G_1 \quad)$$

Differentiation gives

$$\frac{dR_o}{dp} = \frac{dr_o}{dp} \Big\{ 1 + \frac{1}{2} \quad (G_o + r_o G_1) - 2r_o^{\,2} G_1^{\,2} - k^2 G_o^{\,2}$$

$$+ r_o^{\,2} G_o (G_2 + G_3) - r_o G_o G_1 - r_o \, k' G_o^{\,2} \Big\} \tag{5.2}$$

We calculate $\dfrac{dr_o}{dp}$ from (3.2), the relation defining

$$p = \frac{e}{c} \frac{r_o B_o (r_o)}{\quad (r_o)}$$

and note that for a given field, the constant B_o varies with the reference radius r_o as $r_o^{\,k(r_o)}$. Then some calculation gives

$$\frac{p}{r_o}\frac{dr_o}{dp} = \frac{1}{k + 1 + \dfrac{2r_o k G_1 + r_o^{\,2} (G_2 + G_3) + r_o k' G_o}{1 - (k G_o + r_o G_1)}} \tag{5.3}$$

Then finally,

$$\frac{p}{R_o}\frac{dR_o}{dp} = \frac{1 + \frac{1}{2}(G_o + r_o G_1) - 2r_o^{\,2} G_1^{\,2} - k^2 G_o^{\,2} + r_o^{\,2} G_o (G_2 + G_3) - r_o G_o G_1 - r_o k' G_o^{\,2}}{\left(1 + \frac{1}{2} G_o - \quad G_o^{\,2} - r_o G_o G_1\right)\left(k + 1 + \dfrac{2r_o k G_1 + r_o^{\,2} (G_2 + G_3) + r_o k' G_o}{1 - (k G_o + r_o G_1)}\right)} \tag{5.4}$$

The estimates of Section IV can be applied to (5.4). When terms less than 1% of the leading terms are neglected, the transition energy is given by

$$\gamma_T^{\,2} = k + 1 + 2r_o k G_1 + r_o^{\,2}(G_2 + G_3) \qquad (5.5)$$

The correction terms given in (5.5) are of order 2%, so that essentially the local value of k determines the transition energy; that is, it is as difficult to avoid the transition energy in a non-scaling as in a scaling accelerator.

REFERENCES

1. See MURA-456, especially p. 24.

2. MURA-397 and MURA-451.

3. Proceedings of the CERN Symposium on Accelerators (1956) Vol. I, p. 59.