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SOME VALUE FUNCTIONS FOR MULTICOMPONENT ISOTOPE SEPARATION - APPLICATION TO A UNIT COST SCALE FOR URANIUM-235, 236, 238 MIXTURES

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ABSTRACT

This report presents a theoretical study of multicomponent isotope separation cascades. A theory is developed which leads to the multicomponent analogue of the two component "ideal cascade". The multicomponent analogue is a "matched abundance ratio cascade". Multicomponent analogues are derived for "value functions","separative work", and various relationships of importance in two component isotope separation cascade theory.

The theory is applied specifically to the derivation of a multicomponent cost formula which could be used to price uranium containing U-236. (This cost formula is derived merely as an illustration of the theory and no recognition or commitment on the part of the U.S.A.E.C. is implied).

The multicomponent matched abundance ratio cascade does not minimize total cascade flow as does the two component ideal cascade. It is found, however, that for uranium isotope separation the total flow in the matched U-235/U-238 abundance ratio cascade exceeds the minimum by an insignificant fraction for a wide range of U-236 concentrations.

SOME VALUE FUNCTIONS FOR MULTICOMPONENT ISOTOPE SEPARATION - APPLICATION TO A UNIT COST SCALE FOR URANIUM-235, 236, 238 MIXTURES

INTRODUCTION

A large part of the theory of isotope separation cascades for isotopic mixtures which contain only two components involves three mutually related concepts - "separative work," the "value function," and the "ideal cascade." This is especially true when the separation process is an irreversible discrete stage process with a small stage separation factor, as is the case for the gaseous diffusion process for the separation of uranium isotopes. Reference is made to [1,2] . So far there have been few developments in multicomponent isotope cascade theory, and there are no multicomponent analogues to the two component separative work, value function, and ideal cascade formulas to apply to this situation. The broad objective of this report is to extend the theory of multicomponent isotope separation in cascades. Multicomponent analogues to the two component separative work, value function, and ideal cascade formulas are presented.

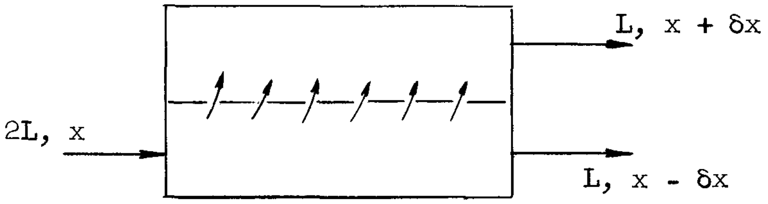
This report also has a more specific and immediate objective. When the feed to a gaseous diffusion cascade is natural uranium, it is permissible in most situations of interest to treat U-234 concentrations as being negligibly small in comparison to the U-235 and U-238 concentrations and thus regard uranium as a mixture of two isotopes. When this is done, the application of cascade theory and the three concepts mentioned above lead to simplified formulas for estimating cascade design requirements such as power, barrier area, and the number and sizes of individual stages. Similar considerations lead to estimates of the unit cost for the production of U-235 at any concentration. As is well known, the published U.S.A.E.C. price schedule can be fit very precisely with a cost curve based upon a unit cost of separative work. Reference is made to [1,3] . In certain situations where a gaseous diffusion cascade receives as feed uranium which has been discharged from a nuclear reactor, a fourth isotope, U-236, may be present in a concentration sufficiently high so that it affects appreciably the separative work requirements, and hence the unit costs for the production of enriched U-235, as shown in [4] . The more specific and immediate objective of this report then is the derivation of a multicomponent unit cost formula which may be used to price uranium containing U-236. It is to be emphasized that this multicomponent unit cost formula is here presented solely to illustrate application of the developed theory, and in no way is it implied that the U.S.A.E.C. recognizes this formula as the basis for a price schedule for uranium containing U-236.

In view of the second objective of the paper, the development of the theory and presentation of results will be done in the setting of the separation of uranium isotopes by gaseous diffusion. It is to be understood, however, that the results are not limited either to these particular isotopes or separation method. The results are applicable to any separation process in which the separation factor is small and independent of composition.

BRIEF REVIEW OF PERTINENT TWO COMPONENT SEPARATION THEORY

To fix ideas, some pertinent features of two component separation theory are first briefly reviewed. Consider for this purpose the separation of the U-235, U-238 two component mixture (as UF_6) by the gaseous diffusion process. By way of nomenclature, a U-235 concentration (mol fraction) is denoted by x with appropriate subscripts where required to denote feed, product, and waste concentrations of a cascade. The symbols for the cascade feed, product, and waste rates are F, P, and W. The interstage flow rate (upflow through the barrier) at stage n is denoted by L, which is understood to be a function of stage number. The stage separation factor for the U-235, U-238 separation is denoted by ψ_1.

Consider then an isolated stage, operating at a "cut" of one half, as shown in Figure 1.



Stage Increment:
$2\delta x = \psi_1 x(1 - x)$

Schematic of Stage Processing a Two Component Mixture

Figure 1

The separative work done by the stage is defined to be $L\,\psi_1^{\,2}/4$, and is, of course, independent of concentration. The value function is then obtained by associating with uranium at concentration x a value $\mathcal{V}(x)$, say per mol, and requiring that the net change in value effected by the stage equal the separative work of the stage. Thus, for the stage shown in Figure 1,

$$L\,\mathcal{V}(x + \delta x) + L\,\mathcal{V}(x - \delta x) - 2L\mathcal{V}(x) = \frac{1}{4}\,L\,\psi_1^{\,2}\ , \tag{1}$$

where $\delta x = \psi_1\, x(1 - x)/2$. Since the concentration change δx is small, $\mathcal{V}(x + \delta x)$ and $\mathcal{V}(x - \delta x)$ are expanded about $\mathcal{V}(x)$ in a Taylor expansion, and from equation (1), one then obtains the ordinary differential equation

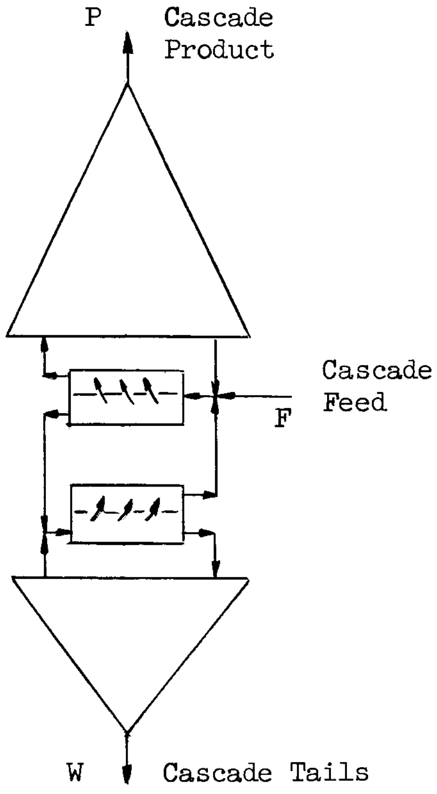
$$\left[x(1 - x)\right]^2\, d^2\mathcal{V}/dx^2 = 1\ , \tag{2}$$

which is correct to terms of order $\psi_1^{\,2}$. The general solution of equation (2) is

$$\mathcal{V}(x) = c_o + c_1 x + (2x - 1)\ \ln\ \left[x/(1 - x)\right]\ , \tag{3}$$

where c_o and c_1 are arbitrary constants. Thus, the two component value function $\mathcal{V}(x)$ has been obtained.

Application of $\mathcal{V}(x)$ to cascade theory follows immediately. Consider the cascade shown in Figure 2.



Defining Cascade Schematic

Figure 2

In an <u>ideal cascade</u> for two component separation, stages are linked together to form a cascade and feeds are introduced to the cascade so that there are no losses of separative work anywhere in the cascade. This is simply accomplished by bringing streams together at stage links and feed points only if the concentrations of the mixed streams are equal. Consequently, the separative work of the ideal cascade equals the sum of the separative work of the stages; thus:

$$\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2} = \sum_{\text{cascade}} \left[L\,\mathcal{V}(x + \delta x) + L\,\mathcal{V}(x - \delta x) - 2L\,\mathcal{V}(x)\right] \quad . \tag{4}$$

Furthermore, since an output L from one stage is an input to another, all the interstage flows cancel in the cascade summation. Hence, for the cascade of Figure 2 - supposing the cascade to be ideal - one obtains from equation (4), the following important relation:

$$\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2} = PV(x_P) + WV(x_W) - FV(x_F), \tag{5}$$

where $V(x) = (2x - 1)\ \ln\ [x/(1 - x)]$. (6)

In an equation, such as (5), the linear terms of the general value function $\mathcal{V}(x)$ vanish by material balance, so that henceforth, for convenience in these applications, the <u>elementary value function</u> V(x) is used and the linear terms are added when required.

It may be seen from equation (5) that by means of the value function the separative work of an ideal cascade may be expressed in terms of the flows and concentrations of the external cascade streams. This value function expression for the separative work of an ideal cascade leads to remarkably simple formulas for estimating cascade performance. Thus, for the cascade of Figure 2 - supposing the cascade to be ideal - one may write the "productivity equations" below, these being the availability of separative work and the two material balances:

$$\begin{aligned} \frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2} &= PV(x_P) + WV(x_W) - FV(x_F)\ , \\ 0 &= Px_P + Wx_W - Fx_F, \\ 0 &= P + W - F\ . \end{aligned} \tag{7}$$

It may be seen how value function terms and material balance terms are added to the equation (7) for application to an ideal cascade with side feeds and/or side withdrawals. These productivity equations are basic, and their uses are many. As stated in [1] , given an ideal cascade at specified feed and product conditions so that the cascade separative work is known, the best possible performance of the cascade at another set of conditions can be calculated from the applicable productivity equations by treating the cascade separative work as a constant property of the cascade. Such a calculation is valid provided the stages are re-arranged or the interstage flows are adjusted, if need be, so that under the changed conditions the mixing of streams of different concentrations is avoided. The following are cited in [1] as examples of cascade problems which may be solved by this means:

1. The effect of change in product rate on product concentration.
2. The effect of change in feed rate on product rate at constant product concentration.
3. The effect of incremental feeds of different concentrations on product rate.
4. The effect of withdrawing partially enriched product on product rate.

Other examples may be cited, but the above suffice to show the usefulness of the productivity equations.

In the gaseous diffusion process, the cascade operating costs are very nearly proportional to the cascade separative work. This fact quickly leads to a unit cost system for pricing uranium. Let $D(x)$ be the unit cost in dollars per kilogram of uranium whose U-235 concentration is x, and let K be the unit cost of separative work per kilogram (of uranium)*.

Then,

$$D(x) = K\,[a_0 + a_1 x + V(x)] \quad , \tag{8}$$

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*Kilograms and weight fraction, rather than mols and mol fraction, are the accepted units for uranium costs. The ratio of the atomic weights of the uranium isotopes is so close to unity that the difference between mol fraction and weight fraction is negligible.

where a_o and a_1 are constants, is a unit cost scale which assures that the operating costs of the (ideal) cascade are accounted for by the material charges and credits; thus, for the cascade of Figure 2:

$$\text{Cascade Operating Costs} = K\left(\frac{1}{4} \sum_{\text{cascade}} L\, \psi_1^{\,2}\right),$$

$$= PD(x_P) + WD(x_W) - FD(x_F) \;. \qquad (9)$$

The constants a_o and a_1 in the unit cost expression (8) are determined so that $D(x)$ has the correct value for natural uranium and is equal to zero at a concentration x_o called the "concentration of zero value." This concentration x_o may be regarded as that of depleted uranium which can be used as feed at no cost in a cascade to produce a product at the natural uranium concentration x_N and at the same unit cost $D(x_N)$ as natural uranium. When the constants are thus evaluated, one obtains

$$D(x) = K\left[V(x) - V(x_o) - (x - x_o)\, V'(x_o)\right] \;. \qquad (10)$$

It has been mentioned previously that the U.S.A.E.C. price schedule can be fit very precisely by a formula based on a unit cost for separative work; references [1] and [3] should be consulted for further discussion. For the price schedule in the form (10), the constants given in [1] are:

$$K = \$37.286/\text{kg. U},$$
$$D(x_N) = D(0.007115) = \$39.27/\text{kg. U},$$
$$x_o = 0.0022138 \text{ weight fraction U-235.}$$

From this brief review of two component cascade theory, one can well see the relations between the concepts of separative work, value function, and ideal cascade, and their possible wide application to cascade design, evaluation of cascade performance, and unit cost systems. These results are particularly useful in application to gaseous diffusion plants because the ideal cascade minimizes the required cascade total flows, and hence, leads to minimum power gaseous diffusion plants.

STATEMENT OF SPECIFIC PROBLEMS AND OUTLINE OF SOLUTION FOR A MULTICOMPONENT VALUE FUNCTION

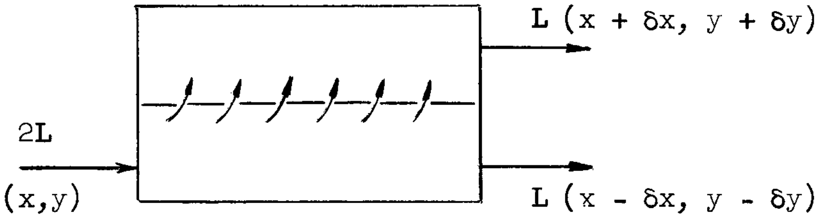
The problems in multicomponent separation now considered are those of finding multicomponent analogues to the separative work, value function, and ideal cascade which have been reviewed for two components. For convenience, the problems and results are presented in the three component setting of U-235, 236, and 238. Extension to additional isotopes will be noted where required.

First, new nomenclature is introduced as follows: U-235 and U-236 concentrations are denoted by x and y, respectively, with appropriate subscripts where required to denote the cascade feed, product, and waste concentrations. The stage separation factor for the two component U-235 and U-238 separation is denoted by ψ_1, and the corresponding factor for the U-236 and U-238 separation is denoted by ψ_2. An additional symbol k is defined by

$$k = \psi_2/\psi_1 \ , \qquad (11)$$

which has the value 2/3 for the uranium problem under discussion. Other nomenclature is as before.

Consider an isolated stage at a cut of a half and handling the three component mixture as shown in Figure 3.



Stage Increments:

$$2\ \delta x = g(x,y) = \psi_1\ x(1 - x - y) + (\psi_1 - \psi_2)\ xy,$$
$$2\ \delta y = h(x,y) = \psi_2\ y(1 - x - y) - (\psi_1 - \psi_2)\ xy.$$

Schematic of Stage Processing a Three-Component Mixture

Figure 3

By analogy with the two component results, the separative work done by the stage is defined to be L $\psi_1^2/4$, and the three component value function is obtained by associating a value $\mathcal{U}(x,y)$ with material at concentrations (x,y) such that the net change in value effected by the stage equals the separative work of the stage; thus:

$$L\mathcal{U}(x + \delta x,\ y + \delta y) + L\mathcal{U}(x - \delta x,\ y - \delta y) - 2L\mathcal{U}(x,y) = \frac{1}{4}\, L\, \psi_1^{\,2} \ , \qquad (12)$$

where:

$$\delta x = \frac{1}{2}\, g(x,y) = \frac{1}{2}\,\left[\psi_1\, x(1 - x - y) + (\psi_1 - \psi_2)\, xy\right] \quad ,$$

$$\delta y = \frac{1}{2}\, h(x,y) = \frac{1}{2}\,\left[\psi_2\, y(1 - x - y) - (\psi_1 - \psi_2)\, xy\right] \quad .$$

Again carrying out the indicated Taylor expansion as with two components, one obtains the partial differential equation

$$g^2(\partial^2 \mathcal{V}/\partial x^2) + 2gh(\partial^2 \mathcal{V}/\partial x \partial y) + h^2(\partial^2 \mathcal{V}/\partial y^2) = \psi_1^{\,2} \ , \qquad (13)$$

which is the three component analogue of the ordinary differential equation (2) for two components. There are many functions of x and y which satisfy this partial differential equation. From the possible solutions, one must be chosen which has certain desirable properties discussed below.

In the situation of the uranium isotopes U-235, 236, and 238, it must be realized that not all concentration differences are effected at the expense of separative work. Thus, the addition (or depletion) of U-236 in a reactor in no way involves the expenditure of separative work, and accordingly, it should be possible, for example, to feed material containing U-236 to a cascade initially free of U-236 without incurring a mixing loss in the sense of separative work losses. The three component $\mathcal{V}(x,y)$ should then have the property that it permits the mixing of materials of some appreciably different concentrations without a loss in separative work, i.e., the value of the unmixed materials equals the value of the mix. Furthermore, this $\mathcal{V}(x,y)$ should permit the linking of stages together in a cascade and the introduction of feed materials to the cascade so that separative work is conserved everywhere in the cascade. In such a cascade, at every location where streams are brought together and mixed, the value of the materials before mixing must equal the value of the resulting mix. The separative work of the cascade then equals the sum of the separative work of the stages; hence:

$$\frac{1}{4} \sum_{\text{cascade}} L\, \psi_1^{\,2} = \sum_{\text{cascade}} L\mathcal{V}(x + \delta x,\ y + \delta y) + L\mathcal{V}(x - \delta x,\ y - \delta y) - 2L\mathcal{V}(x,y) \quad , \qquad (14)$$

and furthermore, as in the two component situation,

$$\frac{1}{4} \sum_{\text{cascade}} L\, \psi_1^{\,2} = P\mathcal{V}(x_P, y_P) + W\mathcal{V}(x_W, y_W) - F\mathcal{V}(x_F, y_F). \qquad (15)$$

It may be seen from equation (15) that, as in the two component case, such a value function $\mathcal{V}(x,y)$ can then be the basis for a unit cost system for pricing three component isotopic mixtures. Of course, such a unit cost system will be strictly applicable only to some reference cascade - in the same way that the two component cost system is strictly applicable to the ideal cascade.

It has thus been seen that for the purpose of pricing U-235, 236, 238 mixtures, one requires a function $\mathcal{V}(x,y)$ which satisfies the partial differential equation (13) and furthermore has the property that it permits the mixing of some materials with appreciably different isotopic concentrations so that the value of the unmixed materials is equal to the value of the resulting mix. Given a material with concentrations (x_1,y_1), one must then be able to establish whether another material with concentrations (x_2,y_2) can be mixed with the first with value being conserved. For this purpose, consider introducing a function $M(x,y)$, called a "match function," with the property that if two materials have distinct concentrations (x_1,y_1) and (x_2,y_2) such that $M(x_1,y_1)$ equals $M(x_2,y_2)$, then these two materials can be mixed with value being conserved. Once the match function is known, stages can be linked together to form a cascade and feeds can be introduced to the cascade so that separative work is conserved everywhere in the cascade. Such a cascade is formed by matching streams which come together, i.e., the match function for streams coming together has the same value.

For application to operations with U-235, 236, and 238 mixtures, the applicable value function $\mathcal{V}(x,y)$ and associated match function $M(x,y)$ should also satisfy the boundary condition that they reduce to the usual two component formulas when U-236 is not present, that is, when y equals zero. Hence, it is required that $\mathcal{V}(x,0)$ equal $\mathcal{V}(x)$ in (3), and since in the absence of U-236, materials are mixed with value being conserved only when the materials have the same U-235 concentration x, it is required that $M(x,0)$ equal x.

The above discussion on the desired properties of the value function $\mathcal{V}(x,y)$ is now summarized in concise mathematical language.

It is desired to find a value function $\mathcal{V}(x,y)$ and an associated match function $M(x,y)$ with the following properties:

1. $$g^2(\partial^2\mathcal{V}/\partial x^2) + 2gh(\partial^2\mathcal{V}/\partial x\partial y) + h^2(\partial^2\mathcal{V}/\partial y^2) = \psi_1{}^2 \,, \qquad (16)$$

where:

$$g = \psi_1 \, x(1 - x - y) + (\psi_1 - \psi_2) \, xy,$$
$$h = \psi_2 \, y(1 - x - y) - (\psi_1 - \psi_2) \, xy,$$
$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad x + y \leqslant 1.$$

2. $$w_1 \mathcal{V}(x_1,y_1) + w_2 \mathcal{V}(x_2,y_2) = (w_1 + w_2) \, \mathcal{V}(x_3,y_3), \tag{17}$$

where:

(x_1,y_1) and (x_2,y_2) are distinct and such that $M(x_1,y_1) = M(x_2,y_2)$,

and

$$x_3 = (w_1 \, x_1 + w_2 \, x_2)/(w_1 + w_2) \,,$$
$$y_3 = (w_1 \, y_1 + w_2 \, y_2)/(w_1 + w_2),$$

$w_1 \geqslant 0$, $w_2 \geqslant 0$, i.e., w_1 and w_2 are material quantities.

3. $$\mathcal{V}(x,0) = c_o + c_1 \, x + (2x - 1) \ln \left[x/(1 - x)\right] \,, \tag{18}$$

where c_o and c_1 are arbitrary constants.

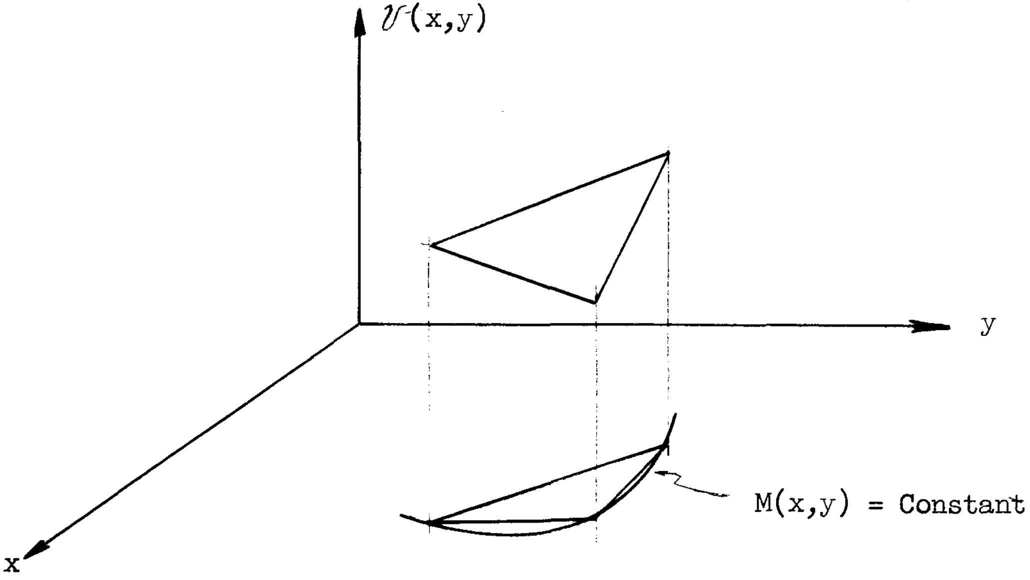
4. $$M(x,0) = x. \tag{19}$$

Finding $\mathcal{V}(x,y)$ and the associated $M(x,y)$ is a formidable problem. The rigorous mathematical analysis and solution are presented in Appendix I. Unfortunately, the analysis there presented has resisted all attempts by the writers at simplification. Several observations, based on physical and geometrical considerations, do permit a heuristic development of the solution. Such a development follows.

Note first that equation (17) states that if two matched materials are mixed, value is conserved. Considering that when two materials are mixed, a portion of one may first be mixed with a portion of the other, and again, sub-portions may be mixed, etc., until finally the two original materials are mixed, one may expect that if any number of matched materials are mixed, value is conserved. It may then be further concluded that the curve in the (x,y) plane described by the relation

$$M(x,y) = \text{constant} \tag{20}$$

must be a straight line. The argument, based on Figure 4, is as follows:



M(x,y) Constant on a Curve

Figure 4

As shown in Figure 4, if M(x,y) is constant on a curve not a straight line, due to the above extension of (17), $\mathcal{V}(x,y)$ is a plane for concentrations (x_i,y_i) resulting from the mixing of three or more materials with distinct concentrations on the curve M(x,y) = constant; thus:

$$\mathcal{V}(x,y) = a + bx + cy, \text{ where a, b, and c are constants.}$$

But then $\mathcal{V}(x,y)$ cannot satisfy the differential equation (16). Hence the locus of (20) must be a straight line. Consider now that such lines may be described by

$$x + y\,B(M) = C(M) \,, \tag{21}$$

where B(M) and C(M) are functions of M only. Since (19) demands that M(x,0) = x, it then follows from (21) that

$$x + y\,B(M) = M. \tag{22}$$

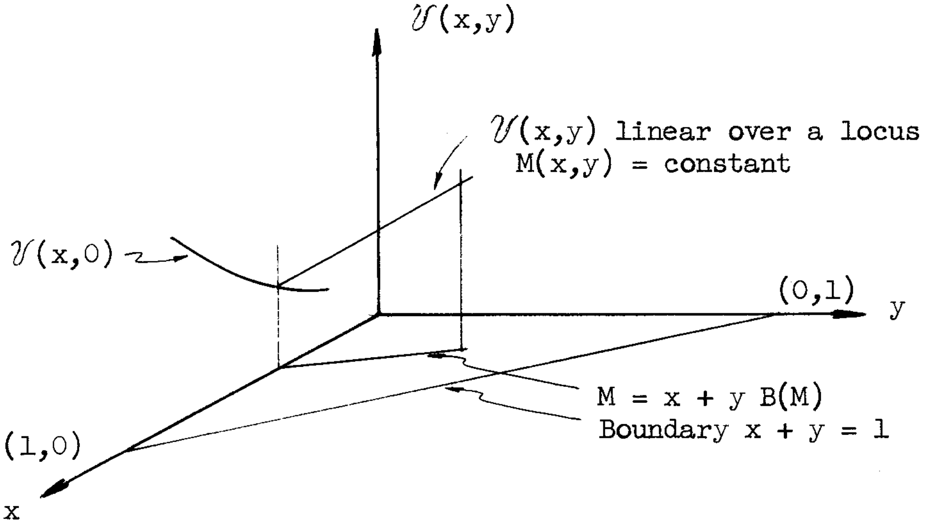
It further follows, as a consequence of (17), that over a line, such as (22), $\mathcal{V}(x,y)$ is a line, and hence,

$$\mathcal{V}(x,y) = U(M,y) = \alpha(M) + y\,\beta(M), \tag{23}$$

where $\alpha(M)$ and $\beta(M)$ are functions of M only. Since $\mathcal{V}(x,0)$ is given by (18) and since from (19), $M(x,0) = 0$, it follows that $\alpha(M)$ in (23) is known; thus:

$$\alpha(M) = c_0 + c_1 M + (2M - 1) \ln \left[M/(1 - M)\right] \quad . \tag{24}$$

The picture showing these relations between $\mathcal{V}(x,y)$ and $M(x,y)$ is shown in Figure 5.



Relations Between $\mathcal{V}(x,y)$ and $M(x,y)$

Figure 5

Consider now that due to (18), which states that $\mathcal{V}(x,y)$ reduces to the usual two component value function in the absence of U-236, $\mathcal{V}(x,y)$ must recognize the expenditure of separative work for the separation of U-235 and U-238. In particular, note that

$$\mathcal{V}(0) = \infty \,, \quad \text{and} \quad \mathcal{V}(1) = \infty \quad ,$$

which imply the expenditure of infinite separative work for the complete removal of U-235 or U-238 in the absence of U-236. One may then expect that

$$\mathcal{V}(0,y) = \infty \text{ , and } \mathcal{V}(x,\ 1 - x) = \infty \text{ ,}$$

which imply the expenditure of infinite separative work for the complete removal of U-235 or U-238 in the presence of U-236. It may then be seen that $\mathcal{V}(x,y)$ should be infinite on the boundaries $x = 0$ and $x + y = 1$ in Figure 5. Note further that again from separative work considerations, $\mathcal{V}(x,y)$ should be finite for all concentration points (x_i,y_i) in the interior of the triangle formed by the three boundaries $x = 0$, $y = 0$, and $x + y = 1$. It then follows from the above considerations that lines of constant M cannot densely intersect either of the boundaries $x = 0$ and $x + y = 1$, for then $\mathcal{V}(x,y)$ over such lines would be infinite. It further follows that through every interior point (x_i,y_i), the line of constant M passing through (x_i,y_i) must intersect the boundary $y = 0$ in the interval $0 < x < 1$, for if there are interior points whose lines of constant M do not intersect the boundary $y = 0$ in this interval, then such lines must densely intersect at least one of the boundaries $x = 0$ and $x + y = 1$. In summary, lines of constant M must originate on the boundary $y = 0$ in the interval $0 < x < 1$, such lines must pass through every interior point of the triangle formed by the three boundaries, and such lines cannot densely intersect the boundaries $x = 0$ and $x + y = 1$. Accordingly, either no lines of constant M may be drawn, in which case the desired $\mathcal{V}(x,y)$ does not exist, or all such lines intersect at the corner point (0,1), in which case $\mathcal{V}(x,y)$ may be expected to have some irregularity at this one point. In the latter case,

$$M = x/(1 - y) \ . \tag{25}$$

It remains to find out whether with the above M, a value function $\mathcal{V}(x,y)$ is permitted by the partial differential equation (16). This is simply found out by using (25) to eliminate x in (16). Thus, one arrives at a partial differential equation in M and y coordinates in which (23) may be substituted. The only unknown in (23) is $\beta(M)$, and the differential equation in M and y coordinates must be solved for $\beta(M)$, provided a solution is permissible. On carrying out the necessary algebraic simplification when the substitution of (23) is made, one finds that the partial differential equation in M and y coordinates reduces to an ordinary differential equation for $\beta(M)$, the y variable vanishing. Hence, a solution for $\beta(M)$ is permissible, and on solving for $\beta(M)$, one has found the value function $\mathcal{V}(x,y) = U(M,y)$, given by (23), with the associated match function $M(x,y)$, given by (25). The details of this part of the development are presented in the addendum to Appendix I. The results are that:

for $2k = 1$,

$$\mathcal{V}(x,y) = c_0 + c_1x + c_2y + c_3y \ln R + \left[2x + \frac{(\ln R)}{2}\, y - 1\right] \ln R, \tag{26}$$

for $2k \neq 1$,

$$(x,y) = c_o + c_1 x + c_2 y + c_3 y\, R^{-(2k-1)} + \left(2x + \frac{2k}{2k-1}\, y - 1\right) \ln R, \qquad (27)$$

where:

$$R = M/(1 - M) = x/(1 - x - y), \qquad (28)$$

and c_o, c_1, c_2, and c_3 are arbitrary constants.
This completes the development of the value function $\mathcal{V}(x,y)$ and its associated match function $M(x,y)$.

THE MATCHED ABUNDANCE RATIO CASCADE

It has been seen that the value function $\mathcal{V}(x,y)$ and the associated match function $M(x,y)$ developed in the previous section permit the mixing of two materials with value being conserved provided the concentrations (x_1,y_1) and (x_2,y_2) are such that $M(x_1,y_1) = M(x_2,y_2)$, i.e., the materials are matched. A cascade with the property that separative work is conserved everywhere can then be formed by matching streams wherever they come together at stage links and feed points.

Consider now that when materials are matched with $M = x/(1 - y)$, the materials are also matched with the abundance ratio $R = x/(1 - x - y)$, and conversely. Thus, there is no difference whether M or R is used as the match function. Since for good reasons the abundance ratio is already prevalent in isotope separation work, the abundance ratio R will be used as the match function associated with the developed value function $\mathcal{V}(x,y)$. The cascade formed by matching R will be called a matched abundance ratio cascade, or briefly, a matched R cascade.

In this section, formulas for the concentration gradients and interstage flow rates and the cascade productivity equations will be developed for the matched R cascade. The matched R cascade has very much the same role in three component separation as the ideal cascade has in two component separation.

(a) Concentration Gradients and Interstage Flows:

First, consider the concentration gradients in a three component cascade without as yet specifying how streams are matched in the cascade. From the usual material balances and the concentration differences effected by a stage, one readily obtains for stages in the enricher of the cascade in Figure 2 the relations

$$dx/dn = g - \left[P(x_P - x)/L\right] \quad , \qquad dy/dn = h - \left[P(y_P - y)/L\right] \quad . \tag{29}$$

A matched abundance ratio cascade is to be considered, and hence, one may expect abundance ratios rather than mol fractions to be the more convenient variables. Accordingly, introduce

$$R = x/(1 - x - y) \text{ and } S = y/(1 - x - y). \tag{30}$$

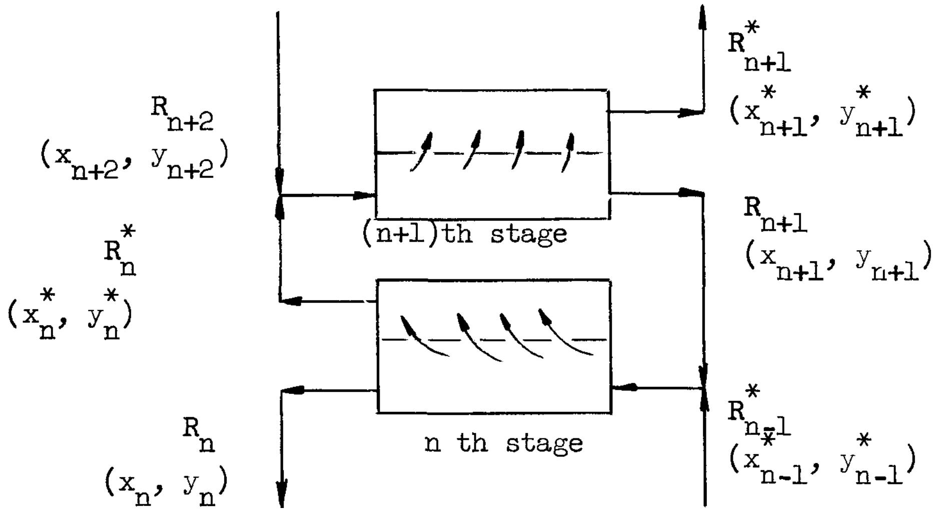
Carrying out the necessary substitutions for the indicated transformation of variables in (29), one then obtains

$$dR/dn = R\,\psi_1 - \left[P(1 + R + S)\,(R_P - R)/L(1 + R_P + S_P)\right], \tag{31.1}$$

$$dS/dn = S\,\psi_2 - \left[P(1 + R + S)\,(S_P - S)/L(1 + R_P + S_P)\right]. \tag{31.2}$$

These are the gradient equations in terms of abundance ratios in <u>any</u> three component cascade*.

Consider now stages in a cascade as shown in Figure 6.



<u>Schematic of Stages in a Cascade</u>

Figure 6

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*Note how easily the total reflux gradients are obtained from these equations.

Suppose the cascade in Figure 6 to be a matched R cascade. Then,

$$R_{n+2} = R_n^* . \qquad (32)$$

From the concentration differences effected by the n-th stage, it follows that

$$R_n^* - R_n = \left[g(\partial R/\partial x) + h(\partial S/\partial y)\right]_n = R_n \psi_1 ,$$

so that from (32) and the above, one obtains

$$R_{n+2} = R_n(1 + \psi_1). \qquad (33)$$

In the present treatment of the stage number n as a continuous variable, one has that

$$R_{n+2} - R_n = 2(dR/dn).$$

and hence, (33) becomes

$$dR/dn = R\, \psi_1/2 , \qquad (34)$$

which is the gradient equation for the R abundance ratio in a matched R cascade.

The interstage flows in the matched R cascade are now readily obtained. One substitutes (34) in (31.1), solves for L, and finds that

$$L = \frac{2}{\psi_1} P \frac{1 + R + S}{1 + R_P + S_P} \cdot \frac{R_P - R}{R} , \qquad (35)$$

which is applicable to the enricher of the cascade shown in Figure 2.

The relation between the R and S gradients in a matched R cascade also follows at once. Substituting (35) in (31.2), dividing the result by (34), results in

$$dS/dR = 2k(S/R) - [(S_P - S)/(R_P - R)],$$

which may be reduced to

$$dS(R_P - R)\, R^{-2k}/dR = - S_P\, R^{-2k} . \qquad (36)$$

As in the development of the value function, integration is here seen to present two cases, 2k = 1 and 2k ≠ 1. For the case 2k ≠ 1, which is applicable to the U-235, 236, and 238 situation, one then obtains for for the enricher of the cascade shown in Figure 2,

$$\frac{S}{S_P} = - \frac{1}{2k - 1} \, \frac{(R_P/R)^{-(2k-1)} - 1}{(R_P/R) - 1} \, . \tag{37}$$

At the feed point, $R = R_F$, the R abundance ratio of the feed, since in the matched R cascade, the R abundance ratio of the feed is matched to the cascade gradient value. The S abundance ratio S_{Fi} at the feed point is then obtained from (37) with $R = R_F$. It is to be emphasized that S_{Fi} is the cascade gradient value for the S abundance ratio at the feed point, and this value S_{Fi} is <u>not</u> necessarily equal to S_F, the S abundance ratio of the feed.

From the above examples, it is clear how similar formulas may be developed for gradients and interstage flows of the stripper of the cascade in Figure 2, as well as for more complicated cascade situations.

(b) V Balances and H Balances

A set of productivity equations for a matched R cascade may be developed directly from cascade considerations; thus, the cascade separative work may be found by adding interstage flows, given by expressions such as (35), over all the cascade. Another approach, based on value function considerations, is simpler to develop, simpler to apply to more complex cascade situations, and simpler to extend to additional isotopes. For this purpose, the value function, either (26) or (27), is written in the form:

$$\mathcal{V}(x,y) = c_o + c_1 x + c_2 y + c_3 \, H(x,y) + V(x,y). \tag{38}$$

Thus, for the case $2k \neq 1$,

$$H(x,y) = y \, R^{-(2k-1)} \tag{39}$$

$$V(x,y) = \left(2x + \frac{2k}{2k - 1} \, y - 1\right) \ln R. \tag{40}$$

As in the two component situation, $V(x,y)$ is called the <u>elementary value function</u>. The new function $H(x,y)$ is called the <u>homogeneous function</u>.

For the reasons stated in the discussion leading to (15), a balance on a matched R cascade nets to the sum of the separative work of the stages; thus, for the cascade of Figure 2, if operated as a matched R cascade, one has

$$\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^2 = P\mathcal{V}(x_P,y_P) + W\mathcal{V}(x_W,y_W) - F\mathcal{V}(x_F,y_F)\ .$$

More generally, for the same reasons as above, a $\mathcal{V}$ balance on some considered section of a matched R cascade leads to the sum of the separative work of the stages in the considered section. Thus, if E_j, $j = 1,2,\ldots\ldots, J$, are the stream flows, at concentrations (x_j,y_j), cutting across the envelope defining some section of a matched R cascade,

$$\frac{1}{4}\sum_{\text{Section}} L\,\psi_1^2 = \sum_{j=1}^{J} E_j\,\mathcal{V}(x_j,y_j)\ , \qquad (41)$$

where outputs are entered as positive quantities. Since c_0, c_1, c_2, and c_3 in (38) are arbitrary, it follows at once from (41) that

$$\frac{1}{4}\sum_{\text{Section}} L\,\psi_1^2 = \sum_{j=1}^{J} E_j\,V(x_j,y_j),\ \text{called a V balance,} \qquad (42)$$

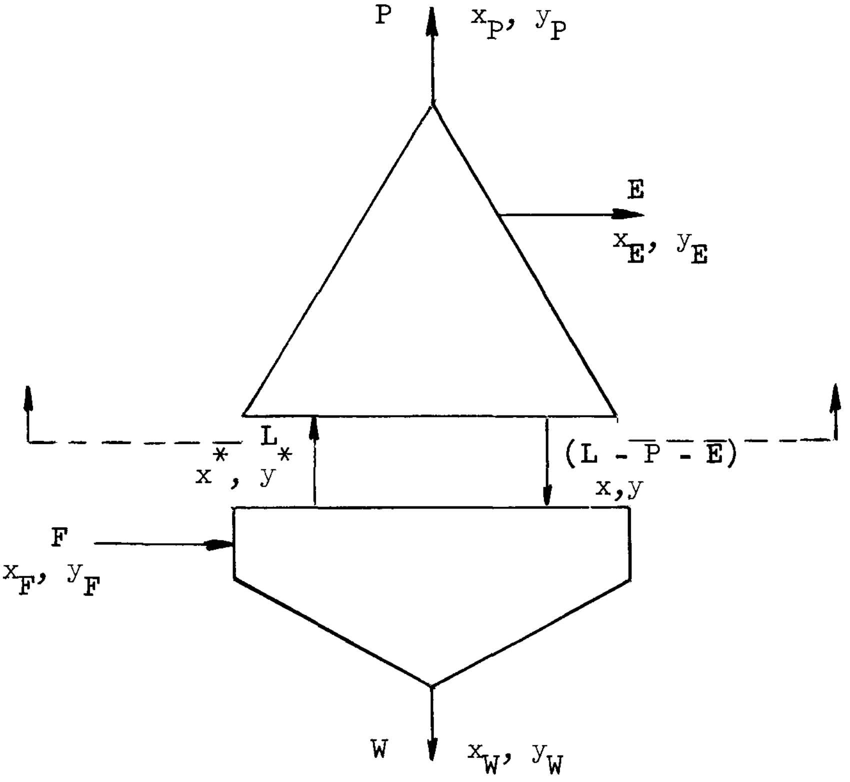
and

$$0 = \sum_{j=1}^{J} E_j\,H(x_j,y_j),\ \text{called an H balance.} \qquad (43)$$

(One also obtains $\Sigma E_j = 0$, $\Sigma E_j x_j = 0$, and $\Sigma E_j y_j = 0$, but these are simply the three independent material balances which are true in <u>any</u> cascade.)

Both V balances and H balances are of use in obtaining productivity equations. In particular, it may be noted that an H balance permits obtaining with ease the (x,y) gradient in the cascade. The use of these balances is illustrated by the following example.

Consider the cascade shown in Figure 7.



Schematic of a Cascade

Figure 7

By means of an H balance on the indicated section of the cascade in Figure 7, an expression for the (x,y) gradient between the E withdrawal point and the F feed point will be obtained. The H balance on the section is

$$PH(x_P,y_P) + EH(x_E,y_E) + (L - P - E)\ H(x,y) - LH(x^*,y^*) = 0,$$

which is re-written

$$P\ [H(x_P,y_P) - H(x,y)] + E\ [H(x_E,y_E) - H(x,y)] - L\ [H(x^*,y^*) - H(x,y)] = 0\ . \qquad (44)$$

Consider now that for most isotope separations, the differences $(x^* - x)$ and $(y^* - y)$ are small, so that one may write

$$H(x^*,y^*) - H(x,y) = (x^* - x)(\partial H/\partial x) + (y^* - y)(\partial H/\partial y).$$

Furthermore, by material balance,

$$L(x^* - x) = P(x_P - x) + E(x_E - x),$$
$$L(y^* - y) = P(y_P - y) + E(y_E - y).$$

Substituting the above expressions in (44), one then obtains

$$P\left[H(x_P,y_P) - H(x,y) - (x_P - x)(\partial H/\partial x) - (y_P - y)(\partial H/\partial y)\right]$$
$$+ E\left[H(x_E,y_E) - H(x,y) - (x_E - x)(\partial H/\partial x) - (y_E - y)(\partial H/\partial y)\right] = 0, \quad (45)$$

which is seen to be an expression for the (x,y) gradient between the E withdrawal point and the F feed point.

From the above example, it is clear how, by means of H balances, relations involving external flows and concentrations may be obtained for other cascade situations.

(c) Productivity Equations

By means of V balances and H balances, a set of productivity equations for any three component cascade, operated as a matched R cascade, is quickly obtained. Thus for the cascade of Figure 7, one immediately has the V balance, the H balance, and the three material balances for the entire cascade:

$$\begin{aligned}
\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^2 &= PV(x_P,y_P) + EV(x_E,y_E) + WV(x_W,y_W) - FV(x_F,y_F)\ , \\
0 &= PH(x_P,y_P) + EH(x_E,y_E) + WH(x_W,y_W) - FH(x_F,y_F)\ , \\
0 &= Px_P + Ex_E + Wx_W - Fx_F\ , \\
0 &= Py_P + Ey_E + Ey_W - Fy_F\ , \\
0 &= P + E + E - F\ .
\end{aligned} \qquad (46)$$

If in Figure 7, one supposes that E is a side feed, the above five equations suffice for the usual productivity calculations. (E is then entered as a negative quantity in the above productivity equations.) Note however, that if E is a side withdrawal, the above five equations are not complete, since both x_E and y_E cannot be specified. One may specify a desired x_E for the withdrawal, but one then must take the y_E found at the point of withdrawal. The section H balance which led to (45) gives the required additional relation to complete the productivity equations in this situation. Since (x_E, y_E) are gradient concentrations, (45) is evaluated at $x = x_E$ and $y = y_E$, and

$$H(x_P, y_P) - H(x_E, y_E) - (x_P - x_E)(\partial H/\partial x_E) - (y_P - y_E)(\partial H/\partial y_E) = 0, \qquad (47)$$

where:

$$(\partial H/\partial x_E) = (\partial H/\partial x) \text{ at } x = x_E,\ y = y_E\ ,$$

$$(\partial H/\partial y_E) = (\partial H/\partial y) \text{ at } x = x_E,\ y = y_E\ .$$

The five equations (46) and equation (47) complete the set of productivity equations for the cascade of Figure 7. Other situations are similarly treated and offer no new difficulties.

Needless to say, the above three component productivity equations have the same uses as those for two component separation. In addition, by means of differential analysis, the three component equations readily lead to evaluating the nuisance effects of a third component of low concentration in the separation of two major components, such as the effects of U-234 on the separation of U-235 and U-238, which can become significant at high U-235 concentrations.

UNIT COST SCALE WITH THE MATCHED ABUNDANCE RATIO CASCADE

The three component value function $\mathcal{V}(x,y)$ which has been developed immediately leads to a unit cost scale for U-235, 236, and 238 mixtures being separated by gaseous diffusion in a matched R cascade. Let D(x,y) be the unit cost in dollars per kilogram of uranium whose U-235 concentration is x and whose U-236 concentration is y, and as before, let K be the unit cost of separative work in dollars per kilogram. Then,

$$D(x,y) = K\left[a_0 + a_1 x + a_2 y + a_3\, H(x,y) + V(x,y)\right]\ , \qquad (48)$$

where the a's are arbitrary constants, and from (39) and (40), H and V for U-235, 236, and 238 mixtures are:

$$H(x,y) = y\left[x/(1 - x - y)\right]^{-1/3}\ , \qquad (49)$$

$$V(x,y) = (2x + 4y - 1)\ \ln\ \left[x/(1 - x - y)\right]\ . \qquad (50)$$

For the reasons already given for the two component cost scale used by the U.S.A.E.C., this three component cost scale assures that the operating costs of a matched R cascade are accounted for by the material charges and credits.

There are four arbitrary constants in (48). These constants are evaluated from the following considerations:

1. The cost scale $D(x,y)$ should have a locus of zero value; thus:

$$D(x_B,y_B) = 0.$$

2. Uranium at concentrations (x_B,y_B) may be used as feed to a matched R cascade to produce some product, say, at concentrations (x_T,y_T). Though this product does not accrue feed costs, it does accrue cascade separative work costs, and consequently, $D(x_T,y_T) > 0$ for concentrations (x_T,y_T) not on the locus of zero value. To assure a non-negative cost scale for the entire range of concentrations, $D(x,y)$ is made a minimum on the locus of zero value; thus:

$$\partial D/\partial x = 0, \quad \partial D/\partial y = 0, \qquad \text{for } x = x_B,\ y = y_B\ .$$

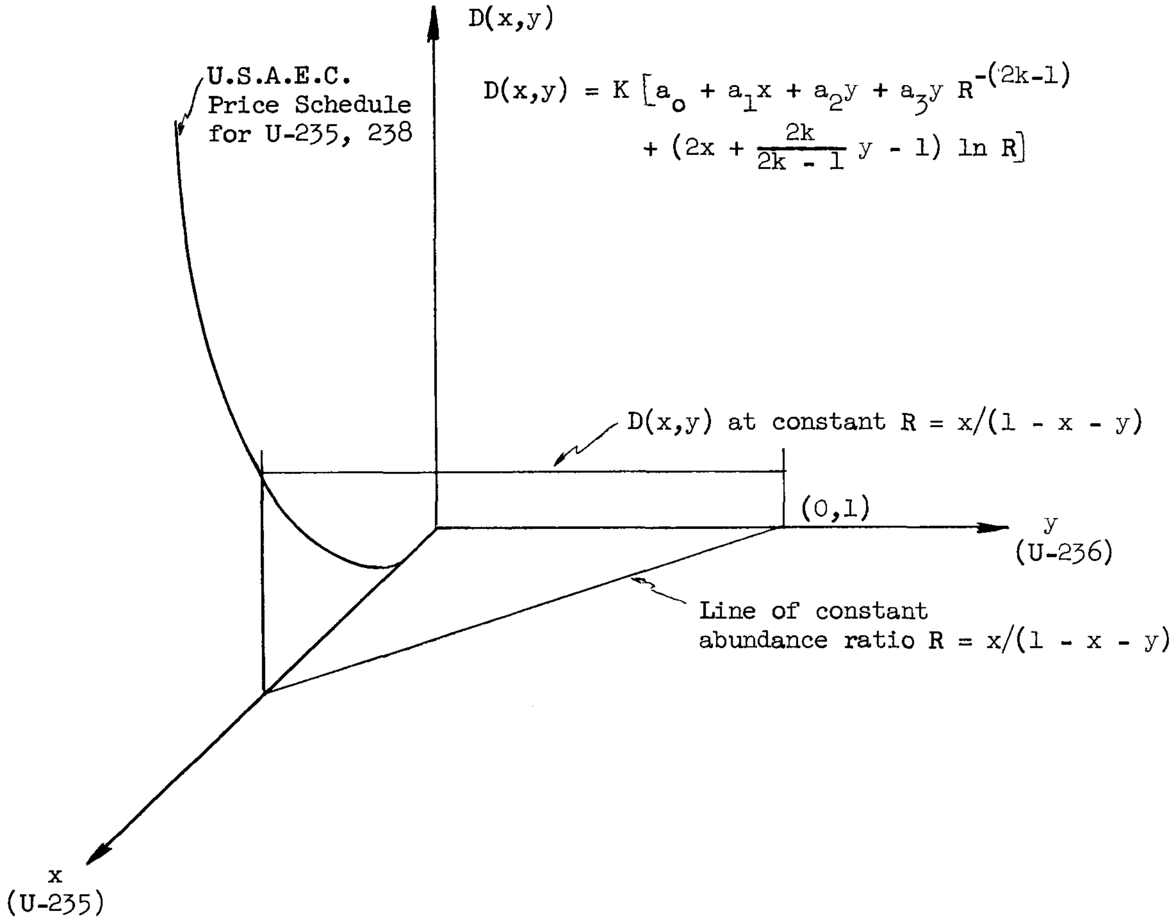
3. In order that the resulting cost scale include the U.S.A.E.C. schedule for U-235 and 238 mixtures, the locus of zero value should include x_o, the zero point of the U.S.A.E.C. price schedule; thus:

$$D(x_o,0) = 0.$$

The details of the required mathematical analysis for the evaluation of the constants are presented in Appendix II. Suffice it to say that in this analysis, it is very convenient to consider the cost scale as a function of the abundance ratio R and the U-236 concentration y rather than a function of x and y. The resulting values of the constants are shown below:

$$\begin{aligned}
a_o &= [(R_o - 1) + \ln R_o] = -7.1086\ , \\
a_1 &= -\left[\frac{(R_o - 1)(R_o + 1)}{R_o} + 2\ln R_o\right] = 462.9338\ , \\
a_2 &= -\left[(R_o - 1) + \frac{1}{(2k-1)^2} + \frac{2k}{2k-1}\ln R_o\right] = 16.4411, \\
a_3 &= \frac{1}{(2k-1)^2}\, R_o^{\,2k-1} = 1.738\ ,
\end{aligned} \tag{51}$$

where $R_o = x_o/(1 - x_o)$. The numerical values given above are obtained with $k = 2/3$, applicable to U-235, 236, and 238 mixtures, and x_o = 0.0022138 weight fraction U-235, the zero value of the U.S.A.E.C. price schedule given in [1] .



The Three Component Unit Cost Scale

Figure 8

Before detailed numerical results are presented, the general aspects of the resulting cost scale D(x,y) are briefly reviewed. As shown in Figure 8, the following are the more important characteristics of the cost scale:

1. The locus of zero value is the line of constant abundance ratio R_o which passes through the two component concentration x_o of zero value.
2. When $y = 0$, the U.S.A.E.C. price schedule for U-235, 238 mixtures is obtained.
3. The cost scale becomes infinite on the axis representing the U-235 and U-236 mixture, as well as that of the U-236 and U-238 mixture.

Some numerical results are now presented. Table I lists D(x,y) so as to show the effect of U-236 on the unit cost of uranium at various U-235 concentrations of interest. The tabulated D(x,y) is obtained with the constants in (48) evaluated as in (51) and a unit cost of separative work K = $37.286/kg U, given in [1]. It is to be noted in Table I how the unit cost of uranium increases as the U-236 concentration increases at a constant U-235 concentration. The rise is small over most of the permissible U-236 range, but it may surprise one that it rises at all. The reason for the rise is that the developed cost scale accounts for cascade separative work, and as the U-236 concentration increases at fixed U-235 concentrations, it takes more separative work per kilogram of uranium for that uranium to be produced from feed material of zero value.

SOME REACTOR CONSIDERATIONS

The increase of the developed unit cost for uranium as the U-236 concentration increases at constant U-235 concentration may seem a most peculiar characteristic of the cost scale to a reactor operator. In particular, since as a rule*, U-236 is produced in a reactor, it would appear that by producing U-236 a reactor can increase the unit cost of material, and thereby conceivably generate power at no cost. Some investigation of what a reactor does in relation to the developed cost scale is then in order.

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*A possible exception is when the U-236 concentration of the reactor charge is considerably greater than the U-235 concentration, and the U-236 production from U-235 neutron capture is less than the U-236 lost to U-237 formation.

TABLE I

UNIT COST SCALE($/kg. U) FOR U-235, 236, 238 MIXTURES

x = Mol Fraction U-235; y = Mol Fraction U-236

y x	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.90	15335.	15362.	15392.	15428.	15479.	15557.*												
0.80	13575.	13594.	13615.	13637.	13660.	13685.	13712.	13744.	13783.	13838.	13926.*							
0.70	11830.	11847.	11864.	11882.	11900.	11919.	11939.	11960.	11982.	12006.	12032.	12292.*						
0.60	10094.	10109.	10124.	10140.	10156.	10172.	10188.	10206.	10223.	10242.	10261.	10376.	10655.*					
0.50	8365.	8379.	8392.	8406.	8420.	8434.	8449.	8464.	8479.	8495.	8511.	8600.	8717.	9014.*				
0.40	6642.	6654.	6666.	6679.	6691.	6704.	6717.	6730.	6743.	6757.	6771.	6846.	6935.	7053.	7366.*			
0.30	4926.	4936.	4947.	4958.	4969.	4980.	4991.	5003.	5014.	5026.	5039.	5103.	5176.	5263.	5381.	5710.*		
0.20	3218.	3227.	3236.	3246.	3255.	3264.	3274.	3284.	3294.	3304.	3314.	3368.	3429.	3498.	3581.	3697.	4040.*	
0.10	1527.	1533.	1540.	1548.	1555.	1562.	1570.	1577.	1585.	1592.	1600.	1642.	1688.	1740.	1801.	1877.	1985.	2335.*
0.08	1192.	1199.	1205.	1212.	1218.	1225.	1232.	1239.	1246.	1253.	1260.	1298.	1341.	1389.	1445.	1514.	1611.	1835.
0.06	860.9	866.5	872.1	877.9	883.7	889.7	895.7	901.9	908.1	914.5	920.9	955.2	993.2	1036.	1087.	1149.	1234.	1398.
0.04	534.4	539.0	543.7	548.5	553.4	558.3	563.4	568.5	573.8	579.1	584.5	613.5	645.9	682.9	726.6	780.5	853.5	981.7
0.03	374.6	378.6	382.7	386.8	391.0	395.3	399.7	404.2	408.7	413.4	418.1	443.5	472.2	505.3	544.5	593.2	659.4	772.7
0.02	219.5	222.6	225.9	229.1	232.5	235.9	239.4	243.0	246.7	250.5	254.3	275.0	298.8	326.6	360.0	402.1	459.9	558.0
0.01	75.46	77.36	79.30	81.30	83.36	85.47	87.65	89.89	92.21	94.59	97.05	110.6	126.7	146.1	170.5	202.3	247.7	326.6
0.009	62.44	64.16	65.93	67.75	69.63	71.56	73.56	75.62	77.74	79.93	82.20	94.73	109.7	128.0	151.0	181.4	225.0	301.4
0.008	49.89	51.42	53.00	54.63	56.31	58.05	59.85	61.70	63.62	65.61	67.66	79.09	92.89	109.9	131.5	160.2	201.9	275.4
0.007	37.93	39.26	40.64	42.06	43.53	45.05	46.63	48.26	49.96	51.71	53.54	63.76	76.24	91.80	111.8	138.7	178.2	248.7
0.006	26.76	27.86	29.01	30.20	31.44	32.73	34.06	35.45	36.90	38.41	39.97	48.84	59.86	73.79	91.97	116.8	153.8	220.8

*At stated U-236 concentration less 0.005.

A reactor takes in material which contains a quantity X kg. of U-235 and Z kg. of U-238. The R abundance ratio of this material then is $R = X/Z$. Suppose an incremental amount δX of U-235 is burned up. At the same time an incremental amount δZ of U-238 is used. One has then that

$$(\delta R/R) = (\delta X/X) - (\delta Z/Z).$$

But since δX is proportional to $X\sigma_5$, and δZ is proportional to $Z\sigma_8$, where σ_5 and σ_8 are the appropriate cross-sections, it follows that

$(\delta R/R) = [1 - (\sigma_8/\sigma_5)] \cdot (\delta X/X) \simeq 0.99(\delta X/X)$ in the thermal neutron range. Note then that a reactor always <u>decreases</u> the R abundance ratio of material because of U-235 burn-up. At the same time, except for the situation already noted, U-236 is produced, so that the U-236 concentration y <u>increases</u>. Now, it may be established from the developed cost scale that

$$\partial D/\partial R > 0, \quad \text{and } \partial D/\partial y < 0,$$

except on the locus of zero value. Hence, it may be seen that in the considered situation, a reactor <u>decreases</u> the unit cost of material on both counts, namely, decreasing R and increasing y. Wrong conclusions should not be drawn from the characteristic of the cost scale of increasing unit costs with increasing U-236 concentration at fixed U-235 concentration.

Perhaps a more natural presentation of the cost scale for the above purposes is a tabulation of the cost scale in coordinates of the abundance ratio R and the U-236 concentration y, as shown in Table II. Here, it may be clearly seen that as a reactor decreases the U-235 relative to the U-238, (that is, decreases R), the unit cost drops, and as U-236 is introduced, the unit cost also drops.

Some burn-up costs as determined with the developed cost scale will now be briefly reviewed. These are shown in Table III. It is here supposed that a reactor at 30% burn-up is originally charged with material at the stated charge concentrations which cover a range of U-235 concentration levels with and without U-236. The discharge concentrations are computed from the assumed burn-up of 30% and from the indicated uranium isotope cross sections in the thermal neutron range. The unit cost of burn-up in dollars per kilogram of U-235 has been computed two ways for comparison. The burn-up cost C_4 has been determined from a four component cost scale (U-234, 235, 236, and 238) developed in the manner exemplified for three components. The other

TABLE II

UNIT COST SCALE ($/kg U) FOR U-235, 236, 238 MIXTURES AS A FUNCTION OF R AND y

R = Mol Fraction U-235/Mol Fraction U-238; y = Mol Fraction U-236

R \ y	0.00	0.02	0.04	0.06	0.08	0.10	0.20	0.40	0.60	0.80	0.95
10.0	15497.	15200.	14902.	14605.	14307.	14010.	12523.	9549.	6574.	3600.	1369.
5.0	14159.	13887.	13615.	13343.	13071.	12799.	11438.	8717.	5996.	3275.	1234.
2.0	11251.	11035.	10819.	10603.	10388.	10172.	9093.	6935.	4776.	2618.	999.8
1.0	8365.	8206.	8046.	7887.	7728.	7568.	6771.	5176.	3581.	1986.	790.4
0.5	5497.	5394.	5290.	5187.	5083.	4980.	4463.	3429.	2394.	1360.	584.2
0.2	2652.	2604.	2555.	2507.	2459.	2411.	2170.	1688.	1206.	724.6	363.2
0.1	1374.	1351.	1327.	1303.	1279.	1255.	1136.	898.5	660.6	422.6	244.2
0.05	658.0	647.4	636.9	626.4	615.9	605.3	552.7	447.4	342.2	236.9	158.0
0.04	509.6	501.7	493.9	486.0	478.1	470.2	430.9	352.1	273.4	194.6	135.6
0.03	360.8	355.5	350.3	345.0	339.7	334.4	308.0	255.1	202.3	149.4	109.8
0.02	213.5	210.7	207.9	205.0	202.2	199.3	185.2	156.8	128.4	100.0	78.72
0.01	74.16	73.39	72.63	71.87	71.10	70.34	66.52	58.89	51.26	43.63	37.91
0.009	61.41	60.82	60.22	59.62	59.02	58.42	55.43	49.44	43.45	37.46	32.97
0.008	49.11	48.66	48.22	47.77	47.32	46.87	44.63	40.16	35.68	31.20	27.84
0.007	37.37	37.06	36.75	36.43	36.12	35.80	34.25	31.12	28.00	24.87	22.53
0.006	26.38	26.18	25.98	25.79	25.59	25.40	24.41	22.45	20.49	18.53	17.06

TABLE III

EXAMPLES OF U-235 BURN-UP COSTS
(30% Burn-Up Assumed)

Charge Concentrations U-235	U-236	Burn-Up Costs ($/Kg. U-235) C_2	C_4
.90	.00	17204	17044
	.05	17204	17072
.80	.00	17148	17018
	.05	17147	17037
.60	.00	17078	16977
	.05	17078	16995
.40	.00	17003	16921
	.05	17002	16944
.20	.00	16846	16785
	.05	16844	16826
.10	.00	16585	16540
	.05	16583	16615
.05	.00	16105	16072
	.05	16101	16207
.02	.00	14723	14705
	.05	14717	14990
.01	.00	12465	12456
	.05	12457	12931

C_2 is based on the U.S.A.E.C. price schedule for U-235, 238 mixtures.

C_4 is based on a four-component unit cost scale applicable to a matched-R cascade processing U-234, U-235, U-236, and U-238 mixtures.

burn-up cost C_2 has been determined from the U.S.A.E.C. price schedule for U-235, 238. It is to be remembered that the four component cost scale includes the U.S.A.E.C. schedule as a special case. The comparison between C_4 and C_2 then indicates the effects of the nuisance isotopes on some reactor economics. In particular, note how the unit cost of burn-up increases with the presence of U-236 in the reactor charge; at least initially, this rate of increase is seen to be unexpectedly small. Detailed studies on the effects of the developed cost scale on reactor economics similar to those presented in [4] are nevertheless indicated and will be the subject of another report.

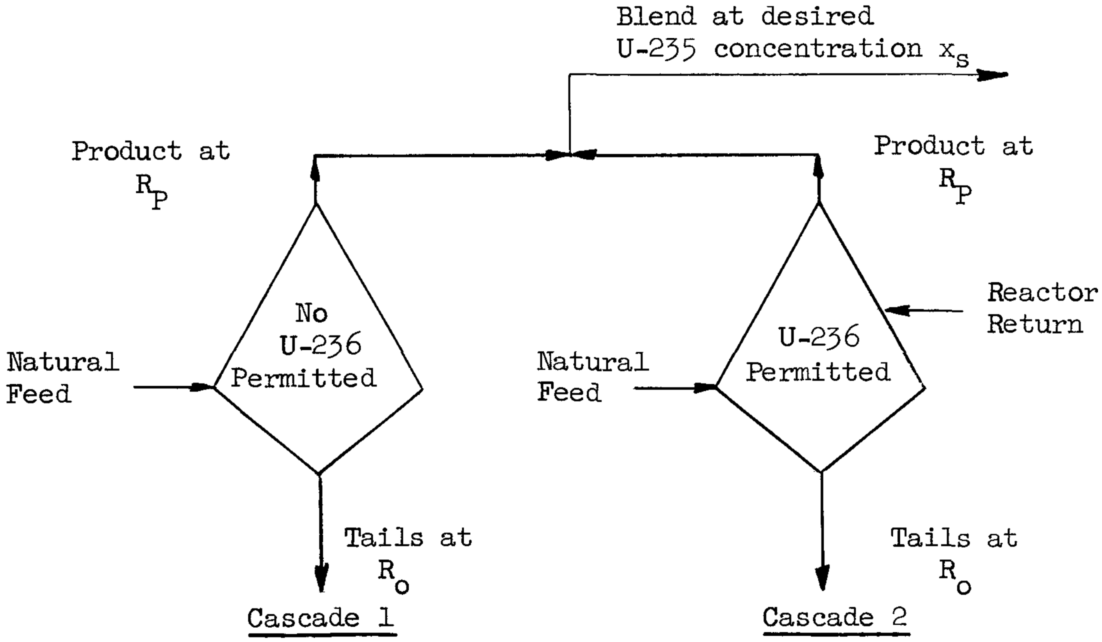
EXAMPLE OF PERMISSIBLE CASCADE OPERATIONS

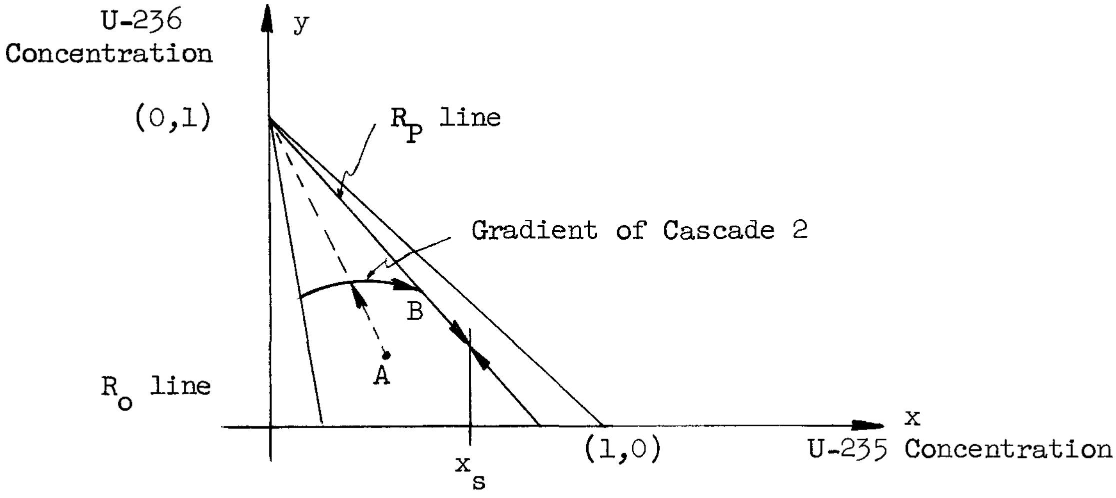
It has been previously stated that the developed cost scale has the property that materials having the same R abundance ratio may be mixed without incurring a mixing loss. This property permits some mixing operations of interest both in reactor and cascade operations, and an example of particular interest in cascade operations will be briefly discussed.

Consider two cascades as shown in Figure 9. Here, Cascade 1 is kept free of U-236, and Cascade 2 receives uranium with U-236. Both cascades are matched R cascades, and both span the same R range, namely, from R_o to R_p. (Furthermore, since both span the same R range, both have the same number of stages.) The mixing operations possible are best exemplified on the (x,y) plot in Figure 9. In the considered situation, a reactor returns material at point A and requests material with a U-235 concentration x_s. The reactor return can then be fed to Cascade 2, enriched up to point B with abundance ratio R_p, and product from Cascade 2 can be blended with product from Cascade 1 also at R_p to make a blend at the desired U-235 concentration x_s. The possibility of such blending operations is of interest particularly with mixtures containing U-234 and U-236 at relatively high U-235 concentrations. In such a situation, there is the possibility of finding oneself very short of stages if further U-235 enriching is to be done in a single cascade.

INVESTIGATION OF PROPERTIES OF THE MATCHED R CASCADE

It has been seen that the matched R cascade plays a central role in the application of the value function and unit cost scales which have been developed. It has been previously mentioned that the two component ideal cascade has the very desirable property of being a minimum power



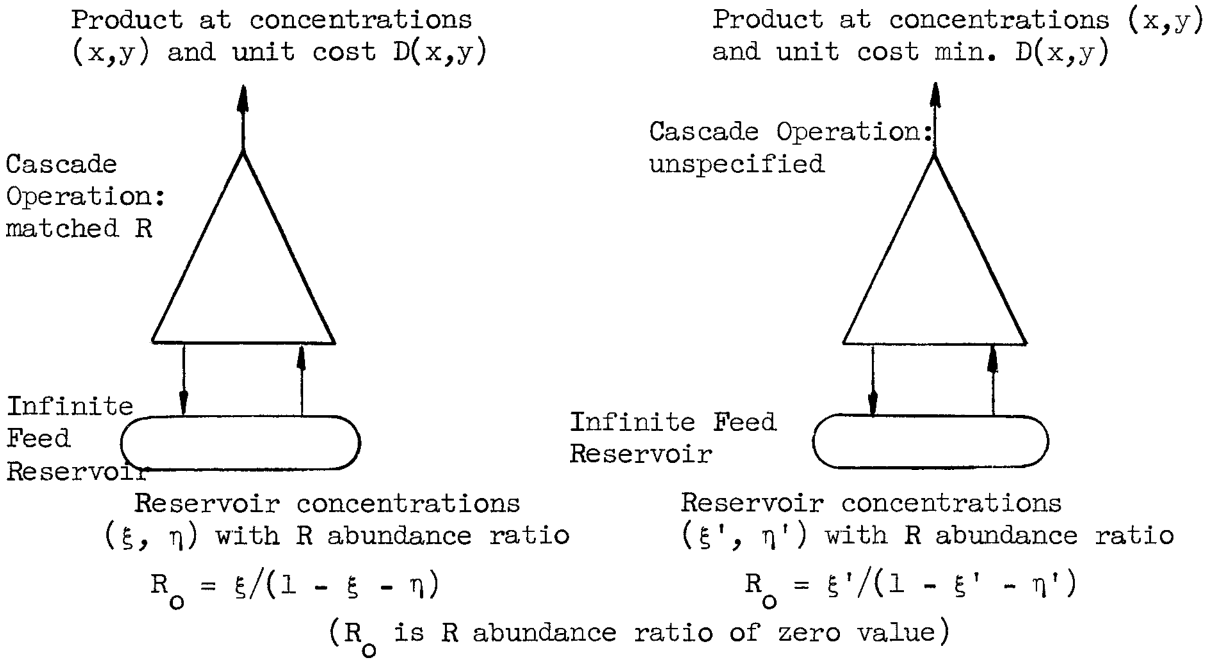


Example of Permissible Blending Operations with Matched R Cascades

Figure 9

cascade; and, of course, a similar property for the matched R cascade would also be desirable. The writers have not been able to establish such a property for the matched R cascade. Some cost properties of the matched R cascade have been established and these are discussed below.

As shown in Appendix III the unit cost D(x,y), developed for U-235, 236, 238 mixtures, may be interpreted as the unit cost of material at concentrations (x,y) which has been produced in a matched R cascade from material of zero value with abundance ratio R_o. A pertinent question to ask is whether with some other mode of cascade operation it is possible to produce at lower cost material at the same concentrations (x,y), the feed of zero value still having abundance ratio R_o. The situation is as shown in Figure 10.



Comparison of a Matched R Cascade Operation with a Minimum Cost Cascade Operation

Figure 10

For the situation shown in Figure 10, it has been possible to establish a lower bound for min. D(x,y). For the development of this lower bound, reference is made to Appendix III; it is there shown that:

$$\text{min. } D(x,y) \geqslant (1 - y)\, D(x^*,0) + y\, I(x,y) \;, \qquad (52)$$

where:

$$x^* = x/(1 - y),$$

$$I(x,y) = K \ln \left(\left[\frac{k(R - R_0) + R_0}{R_0} \right]^{1/k} \left[\frac{k(R - R_0) + R_0}{R} \right]^{1/(1-k)} \right) ,$$

and other notation is as before. The above inequality is applicable to the situation of Figure 10 for the case $0 < k < 1$, which includes the U-235, 236, 238 mixture with $k = 2/3$. With the lower bound given by (52), it is then possible to estimate how much cheaper material of specified concentrations (x,y) can be produced from material of zero value with some other mode of cascade operation than the matched R mode. Since here the feed is at zero value, the difference between D(x,y) and min. D(x,y) is due to a difference in separative work requirements, and hence, this difference also indicates how far from minimum power a matched R gaseous diffusion cascade may be. For the purposes of these evaluations, Table IV lists the ratio D(x,y)/min. D(x,y), with min. D(x,y) estimated by (52), for concentration ranges of interest. It may be seen that the tabulated ratios indicate a very favorable comparison of the matched R cascade with some other mode of cascade operation for a large range of U-235 and U-236 concentrations. Incidentally, the fact that the matched R cascade is not a minimum power cascade implies that it is possible to mix some materials with a gain in value in the sense of the developed $\mathcal{V}(x,y)$. This opens some blending possibilities to both the cascade and reactor operators. Thus, for example, it may be advantageous to a reactor operator to blend two materials, before returning these to the cascade, and to ask the cascade operator for credit on the blend. Table IV indicates such possibilities to be small, but nevertheless, they will be considered in the previously mentioned study on the effects of the developed cost scale on reactor economics.

Another very desirable property of the two component cascade is that considerable changes from ideality can be made without large changes from the ideal separative work requirements. Indeed, it is this stationary property which permits application of ideal cascade theory to actual non-ideal cascades. This stationary property of the ideal cascade has been established analytically; reference is made to [2]. It has not been possible to establish a similar property for the

TABLE IV

COMPARISON OF D(x,y) AND MIN. D(x,y)

The Tabulated Numbers are $100 \{ [D(x,y)/\text{min. } D(x,y)] - 1 \}$.

x \ y	0.00	0.02	0.04	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.90	0	0.03	0.06	0.16	0.23*												
0.80	0	0.03	0.06	0.12	0.16	0.20	0.24	0.30	0.38	0.52*							
0.60	0	0.03	0.06	0.11	0.15	0.18	0.21	0.25	0.29	0.33	0.58	1.32*					
0.40	0	0.03	0.06	0.13	0.16	0.20	0.23	0.27	0.31	0.35	0.57	0.86	1.29	2.73*			
0.20	0	0.04	0.08	0.17	0.22	0.26	0.31	0.36	0.41	0.46	0.74	1.06	1.47	2.01	2.88	6.13*	
0.10	0	0.05	0.11	0.22	0.28	0.34	0.41	0.47	0.53	0.60	0.97	1.40	1.92	2.57	3.48	4.98	10.97*
0.05	0	0.07	0.13	0.28	0.35	0.42	0.50	0.58	0.66	0.75	1.21	1.76	2.42	3.27	4.40	6.12	9.79
0.01	0	0.06	0.12	0.26	0.33	0.40	0.48	0.56	0.65	0.74	1.25	1.91	2.76	3.89	5.50	7.97	12.64
0.006	0	0.04	0.08	0.17	0.22	0.28	0.34	0.40	0.46	0.53	0.94	1.51	2.27	3.35	4.93	7.46	12.35

---

*At stated U-236 concentration y less 0.005.

three component matched R cascade, but numerical studies do indicate that matched R cascades have a desired stability. Examples of some numerical studies are shown in Figure 11. Here a matched x cascade has been completely determined numerically by an iterative computing scheme and the cascade separative work has been determined by summing L $\psi_1{}^2/4$ over all the stages. With the external stream flows and concentrations of the matched x cascade, a cascade value balance has been made with the developed unit cost scale so as to obtain an estimate of the cost of cascade separative work K ΣL $\psi_1{}^2/4$, whence the cascade separative work is estimated by dividing by K, the unit cost of separative work used for the cost scale. A comparison of the actual and estimated separative work is tabulated in Figure 11 for three different cascade situations. It may be seen that the comparisons are favorable, and this would indicate that some freedom may be taken in applying the value function for matched R cascades to other modes of cascade operation in the same manner that the two component value function for ideal cascades is applied to non-ideal cascades. Analytical investigation along these lines is nevertheless needed; results already available are presented in Appendix IV.

GENERALIZATION TO MORE COMPONENTS

The three component value function $\mathcal{V}(x,y)$ and the three component matched R cascade readily generalize to more components. Consider, for example a mixture of the four components C_1, C_2, C_3, and C_4. Let x,y and z denote the concentrations of C_1, C_2, and C_3. Let ψ_1 denote the stage separation factor for the two component C_1 and C_4 separation; and similarly, let ψ_2 and ψ_4 denote the corresponding factors for the C_2 and C_4 separation and the C_3 and C_4 separation. Further, let

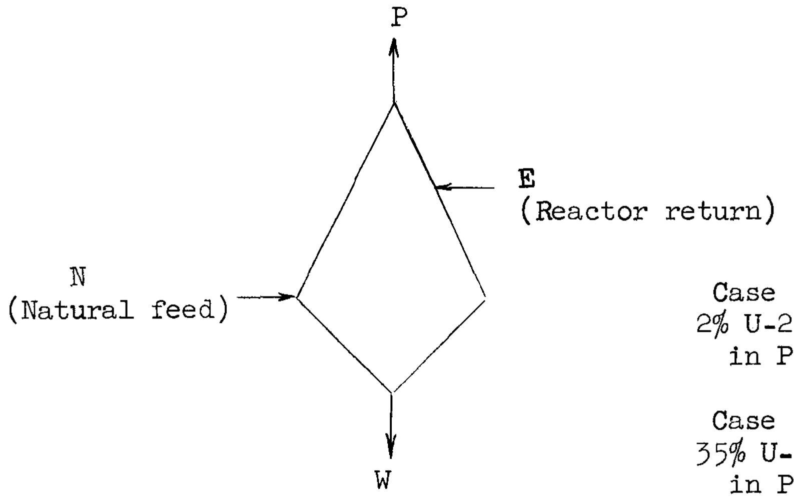
$$k_{2,1} = \psi_2/\psi_1 ,$$

$$k_{3,1} = \psi_3/\psi_1 .$$

Then, a four component value function $\mathcal{V}(x,y,z)$, applicable to the case $2k_{2,1} \neq 1$ and $2k_{3,1} \neq 1$, is:

$$(x,y,z) = c_o + c_1 + c_2 y + c_3 z + c_4\ H_{2,1}(x,y,z) + c_5\ H_{3,1}(x,y,z) + V(x,y,z) ,$$

where the c's are arbitrary constants, and



V_{c1} = Separative work required by matched-x cascade.

V_{c2} = Separative work given by D(x,y) balance on streams and stream concentrations of matched-x cascade.

		Quantity	Concentration x(U-235)	Concentration y(U-236)	
Case 1 2% U-235 in P	P:	100.00	.0200	.00556	V_{c1}/V_{c2} = 1.00014
	E:	99.11	.0121	.00675	
	N:	159.22	.0072	.00000	
Case 2 35% U-235 in P	P:	100.00	.3500	.3264	V_{c1}/V_{c2} = 1.0080
	E:	87.49	.2412	.3908	
	N:	2780.14	.0072	.0000	
Case 3 80% U-235 in P	P:	100.00	.8000	.1536	V_{c1}/V_{c2} = 1.0029
	E:	86.35	.7427	.1811	
	N:	3174.37	.0072	.0000	

x_W = .0022 for all cases.

Application of D(x,y) for Matched R Cascade to a Matched x Cascade

Figure 11

$$H_{2,1} = y\, R^{-(2k_{2,1} - 1)} ,$$

$$H_{3,1} = z\, R^{-(2k_{3,1} - 1)} ,$$

$$V = \left(2x + \frac{2k_{2,1}}{2k_{2,1} - 1}\, y + \frac{2k_{3,1}}{2k_{3,1} - 1}\, z - 1\right) \ln R ,$$

$$R = x/(1 - x - y - z) .$$

The cascade associated with the above four component value function is, of course, the matched R cascade, R now denoting $x/(1 - x - y - z)$. The two homogeneous functions, $H_{2,1}$ and $H_{3,1}$, and the elementary value function V play the same role as before.

From the above example of a four component value function, it is readily seen how other four component value functions are obtained for different separation factor ratios, and in general, how value functions for more components are obtained.

OTHER SUBJECTS

A few brief remarks will now be made on more general subjects.

Consider first the match function which has been here introduced to isotope separation in cascades. The match function played a significant role in the development of the three component value function. The match function here prescribes how to link stages together to form a cascade and how to introduce feeds to the cascade so that separative work is conserved wherever streams are mixed. Now, most of the mixing in a cascade is done at the stage links, and the concentrations mixed at these links differ by very little in contrast to the large differences in concentration which may occur at a feed point. Hence, it makes sense to consider match functions on a purely local basis and ask for a match function which makes it possible to conserve separative work at the stage links to order ψ^2, which after all, is the same order of correctness of the partial differential equation defining the value function. Suffice it then to say that it is in fact possible to develop such local match functions in situations where some mixing is permissible. The development of local match functions for three component mixtures is presented in Appendix IV.

From a more general viewpoint, one can forget about an association between match functions and value functions and simply consider a match function as a recipe for linking stages together to do a certain separation job. For example, one may consider processing some mixture of isotopes in a cascade when the interest is not in enriching a particular isotope, but rather, in enhancing some average property of the mixture, such as an average cross-section. In such a situation, there is intuitive appeal in investigating a match function which is a linear form in the concentrations of the various isotopes. From this more general viewpoint, much remains to be investigated about match functions.

Finally, a word about value functions. The three component value function here developed permits the absence of one of the components. This value function is applicable to operations with U-235, 236, and 238 mixtures which may or may not contain U-236. Suppose, however, that the isotopic mixture always includes the three components. A value function applicable to the processing of such a mixture in a cascade must then of necessity become infinite when any one of the components is absent. Furthermore, all mixing should result in a loss of separative work. A value function of particular interest in this situation is:

$$V_1(x,y) = \frac{1}{2}\psi_1^{\,2} \left\{ \psi_1^{-2} (2x + y - 1) \ln[x/(1 - x - y)] \right.$$
$$\left. + \psi_2^{-2} (x + 2y - 1) \ln[y/(1 - x - y)] + (\psi_1 - \psi_2)^{-2} (x - y) \ln(x/y) \right\}.$$

The function $V_1(x,y)$ is a solution of the partial differential equation given previously. It is symmetric in the concentrations and has a minimum at the point (1/3, 1/3), and these are properties analogous to those of the elementary value function for two components. The single minimum can be moved by means of the evaluation of arbitrary constants in an additive linear form in x and y to the concentrations of the natural mixture, that is, an infinite reservoir of zero value material. This function also has the property that any mixing whatsoever results in loss of value, as would be expected when all concentration changes are effected by expending separative work. The application of this function to isotope separation remains to be exploited.

In conclusion, as seen from the above remarks and others previously made, there remain many unsolved problems in the theory of multicomponent isotope separation in cascades.

APPENDIX I

MATHEMATICAL DEVELOPMENT OF THE VALUE FUNCTION

A. Statement of Problem

We have seen in the text that for our purposes, we wish to find a value function $\mathcal{V}(x,y)$ and a match function $M(x,y)$ with the following properties:

1. $$g^2 \mathcal{V}_{xx} + 2gh\mathcal{V}_{xy} + h^2\mathcal{V}_{yy} = \psi_1^2 , \qquad (I\text{-}1)$$

where

$g = \psi_1 x(1 - x) - \psi_2 xy,$
$h = \psi_2 y(1 - y) - \psi_1 xy,$
$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad x + y \leqslant 1.$

In this appendix the partial derivatives are denoted by the appropriate subscripts.

2. $$w_1 \mathcal{V}(x_1,y_1) + w_2 \mathcal{V}(x_2,y_2) = (w_1 + w_2) \mathcal{V}(x_3,y_3), \qquad (I\text{-}2)$$

where

(x_1,y_1) and (x_2,y_2) are distinct and such that
$M(x_1,y_1) = M(x_2,y_2),$
$x_3 = (w_1 x_1 + w_2 x_2)/(w_1 + w_2),$
$y_3 = (w_1 y_1 + w_2 y_2)/(w_1 + w_2),$
$w_1 \geqslant 0, \quad w_2 \geqslant 0.$

3. $$\mathcal{V}(x,0) = c_o + c_1 x + (2x - 1) \ln \frac{x}{1 - x}, \qquad (I\text{-}3)$$

where c_o and c_1 are arbitrary constants.

$$M(x,0) = x. \qquad (I\text{-}4)$$

It is to be noticed that with (I-3), we imply that (I-1) holds on the boundary $y = 0$.

Subsequently, we will, for convenience, denote $\mathcal{V}(x,y)$ and $M(x,y)$ by $\mathcal{V}$ and M, respectively, and we will follow this practice with other functions we introduce.

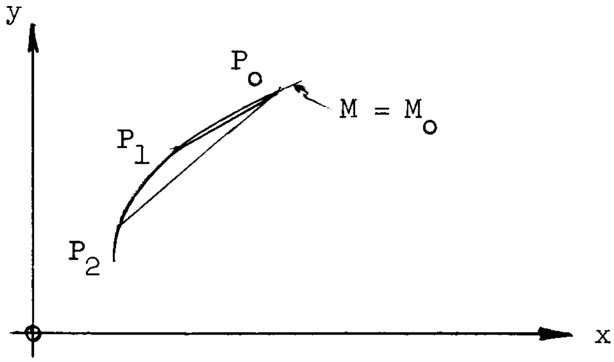
B. A Necessary Form of M

We first show that the locus on the (x,y) plane of

$$M = M_o, \text{ a constant,} \tag{I-5}$$

must be a straight line.

Suppose $M = M_o$ on a curve not a straight line, specifically, a curve such that there is one point $P_o = (x_o, y_o)$ on the curve in whose neighborhood d^2y/dx^2 (or d^2x/dy^2) is not zero. Let $P_1 = (x_o + \epsilon_1, y_o + \eta_2)$ be points on the curve in this neighborhood, and as shown in Figure I-1, let P_o, P_1, and P_2 be distinct.



M Constant on a Curve

Figure I-1

Consider first the secant (P_o, P_1). Since $M(P_o) = M(P_1) = M_o$, it follows from (I-2) that for (x,y) on (P_o, P_1), $\mathcal{V}$ is linear in x and y. Hence,

$$d^2\mathcal{V} = \mathcal{V}_{xx}\, d^2x + 2\, \mathcal{V}_{xy}\, dxdy + \mathcal{V}_{yy}\, d^2y = 0$$

for (x,y) on (P_o, P_1) and dx and dy such that $dy/dx = \sigma_1$, the slope of (P_o, P_1). Therefore,

$$\mathcal{V}_{xx} + 2\sigma_1\, \mathcal{V}_{xy} + \sigma_1^2\, \mathcal{V}_{yy} = 0 \text{ at } (x,y) = (x_o, y_o). \tag{I-6}$$

By considering the secant (P_o, P_2), we similarly obtain

$$\mathcal{V}_{xx} + 2\sigma_2\, \mathcal{V}_{xy} + \sigma_2^2\, \mathcal{V}_{yy} = 0 \text{ at } (x,y) = (x_o, y_o), \tag{I-7}$$

where σ_2 is the slope of (P_o, P_2).

Consider now (I-2). Let $w_1 = w_2 = 1$, and let $x_1 = x_o$, $y_1 = y_o$, $x_2 = x_o + 2\delta x$, $y_2 = y_o + 2\delta y$. (I-2) then becomes

$$\mathcal{V}(x_o,y_o) - 2\mathcal{V}(x_o + \delta x,\ y_o + \delta y) + \mathcal{V}(x_o + 2\ \delta x,\ y_o + 2\delta y) = 0, \quad \text{(I-8)}$$

where $M(x_o,y_o) = M(x_o + 2\delta x,\ y_o + 2\delta y)$. (I-9)

Let $\delta x \to 0$. Since from (I-9), it follows that $\delta y/\delta x \to -M_x/M_y$, it then follows from (I-8) that

$$\mathcal{V}_{xx} + 2\sigma_o\,\mathcal{V}_{xy} + \sigma_o^{\,2}\,\mathcal{V}_{yy} = 0 \text{ at } (x,y) = (x_o,y_o), \quad \text{(I-10)}$$

where $\sigma_o = -M_x/M_y$, the slope of the tangent to the curve $M = M_o$ at the point (x_o,y_o).

Consider now that it has been supposed that in the neighborhood of P_o, the derivative d^2y/dx^2 is not zero. It follows that it is always possible to find P_1 and P_2 such that

$$\sigma_o \neq \sigma_1, \quad \sigma_1 \neq \sigma_2, \quad \sigma_o \neq \sigma_2 .$$

Consequently, (I-6), (I-7), and (I-10) can hold only if $\mathcal{V}_{xx}$, $\mathcal{V}_{xy}$, and $\mathcal{V}_{yy}$ are zero at (x_o,y_o). But then (I-1) cannot be satisfied. Hence, we have shown that $M = M_o$ cannot have a neighborhood where d^2y/dx^2 is not zero, and therefore, loci of constant M must be straight lines in the (x,y) plane.

Consider further that due to (I-4), lines of constant M must intersect the x-axis, and hence, such lines may be described by

$$x + y\,B(M) = C(M), \quad \text{(I-11)}$$

where B(M) and C(M) are functions of M only. Since (I-4) demands that $M(x,0) = x$ at $y = 0$, it follows from (I-11) that

$$x + By = M. \quad \text{(I-12)}$$

We have thus established a simple and useful form of M.

C. A Necessary Form of $\mathcal{V}$

With (I-12) we have that

$$\mathcal{V}(x,y) = \mathcal{V}(M - By,\ y),$$

and hence, we may express the value function as a function of M and y, i.e.,

$$U(M,y) = \mathcal{V}(x,y). \tag{I-13}$$

We proceed to develop a necessary form of U.

With direct differentiation of (I-13), we obtain

$$U_{yy} = (M_y/M_x)^2 \mathcal{V}_{xx} - 2(M_y/M_x) \mathcal{V}_{xy} + \mathcal{V}_{yy} - (M_y/M_x)_y \mathcal{V}_x. \tag{I-14}$$

From (I-10) we see that the term in brackets in (I-14) is zero, and from (I-12), we obtain directly that

$$M_y/M_x) = B(M),$$

from which it follows that at constant M,

$$(M_y/M_x)_y = 0.$$

Hence, we have established that

$$U_{yy} = 0,$$

and therefore, that

$$U = \alpha(M) + y\beta(M), \tag{I-15}$$

where $\alpha(M)$ and $\beta(M)$ are functions of M only. Furthermore, we see from (I-12) that $\mathcal{V}(x,0) = U(M,0)$. It then follows that $U(M,0) = \mathcal{V}(M,0)$, and therefore, from (I-3) and (I-15) we have that

$$\alpha(M) = c_0 + c_1 M + (2M - 1) \ln \left[M/(1 - M)\right] . \tag{I-16}$$

Thus, α is known.

At this point of our development of the value function and match function, we see from (I-12), (I-15), and (I-16) that we have two unknown functions, namely, B(M) and $\beta(M)$. We proceed to eliminate β.

With direct differentiation of (I-13), we find that

$$\mathcal{V}_y = U_M M_y + U_y , \tag{I-17}$$

and from (I-12) and (I-15), we find directly that

$$M_y = B/(1 - B'y); \quad U_M = \alpha' + \beta' y; \quad U_y = \beta. \qquad \text{(I-18)}$$

We now let

$$f(x) = \mathcal{V}_y \text{ evaluated at } y = 0 \qquad \text{(I-19)}$$

and we then have from (I-17) and (I-18) that

$$\beta = f - \alpha' B. \qquad \text{(I-20)}$$

Furthermore, we see from (I-12) and (I-19) that

$$f(x) = f(M), \qquad \text{(I-21)}$$

and with this and (I-20), we may express U(M,y), given by (I-15), as

$$U = \alpha + (f - \alpha' B)y, \qquad \text{(I-22)}$$

and we now need to find f.

Consider that from (I-1) we have that

$$\frac{\partial}{\partial y}(g^2 \mathcal{V}_{xx} + 2gh \mathcal{V}_{xy} + h^2 \mathcal{V}_{yy}) = 0,$$

and hence,

$$\begin{aligned} 2g\, g_y \mathcal{V}_{xx} + g^2 \mathcal{V}_{yxx} + 2h(g \mathcal{V}_{xy})_y + 2\, g\, h_y \mathcal{V}_{yx} \\ + 2h\, h_y \mathcal{V}_{yy} + h^2 \mathcal{V}_{yyy} = 0. \end{aligned} \qquad \text{(I-23)}$$

In particular, at $y = 0$, we have that

$$\left.\begin{aligned} &\text{from (I-1), } g = \psi_1 x(1 - x), \\ &\text{from (I-1), } g_y = -\psi_2 x, \\ &\text{from (I-1), } h = 0, \\ &\text{from (I-1), } h_y = \psi_2 - \psi_1 x, \\ &\text{from (I-3), } \mathcal{V}_{xx} = 1/[x(1 - x)]^2, \\ &\text{from (I-19), } \mathcal{V}_{yx} = f', \text{ and } \mathcal{V}_{yxx} = f''. \end{aligned}\right. \qquad \text{(I-24)}$$

We substitute (I-24) in (I-23), carry out indicated algebraic reduction, and we obtain

$$f'' + 2\frac{k - x}{x(1 - x)} f' = \frac{2k}{x^2(1 - x)^3}, \qquad \text{(I-25)}$$

where $k = \psi_2/\psi_1$. Thus, we now have an ordinary differential equation for f. Carrying out both integrations, we obtain

$$f = c_2 + c_3\, \phi_1 + \phi_2\ , \tag{I-26}$$

where c_2 and c_3 are arbitrary constants, and

$$\begin{aligned} \phi_1 &= \int x^{-2k}\ (1-x)^{-2(1-k)}\ dx, \\ \phi_2 &= 2k \int \phi_1{}'\Big(\int x^{-2(1-k)}\ (1-x)^{-(2k+1)}\ dx\Big)\ dx. \end{aligned} \tag{I-27}$$

We now substitute (I-26) and (I-16) in (I-15), and on carrying out indicated algebraic reduction, we obtain

$$\begin{aligned} U(M,y) = c_0 + c_1(M - By) + c_2\, y + c_3 y\, \phi_1 + \Big[\phi_2 - \frac{2M-1}{M(1-M)}\, B\Big]\ y \\ + \ \big[2(M - By) - 1\big]\ \ \ln \frac{M}{1-M}\ , \end{aligned} \tag{I-28}$$

where:

from (I-21), $\phi_1(x) = \phi_1(M)$, and $\phi_2(x) = \phi_2(M)$,
from (I-12), $M = x + By$.

Thus, we have succeeded in expressing the value function and the match function in terms of only one unknown function, $B = B(M)$. We proceed to find the necessary B.

D. The Necessary B

Consider first the function

$$J(M,y) = y\phi_1 \tag{I-29}$$

in (I-28). Since the product of $J(M,y)$ and an arbitrary constant is a term of $U(M,y)$, it follows that if $\mathcal{U}(x,y) = U(M,y)$ satisfies the inhomogeneous equation (I-1), then $H(x,y) = J(M,y)$ must satisfy the homogeneous equation

$$g^2\ H_{xx} + 2\ gh\ H_{xy} + h^2\ H_{yy} = 0, \tag{I-30}$$

and similarly, if $\mathcal{U}(x,y)$ satisfies (I-10), $H(x,y)$ must also satisfy (I-10), and hence,

$$M^2_{\ y}\ H_{xx} - 2\ M_x\ M_y\ H_{xy} + M^2_{\ x}\ H_{yy} = 0. \tag{I-31}$$

We now combine (I-30) and (I-31) and we obtain

$$(gM_x + h\ M_y)\ (g\ M_y\ H_{xx} + h\ M_x\ H_{yy}) = 0. \tag{I-32}$$

Note that the quantity

$$\mu = g\ M_x + h\ M_y \tag{I-33}$$

cannot be identically zero; in particular, at $y = 0$, $\mu = \psi_1\ x(1 - x)$. Hence, from (I-33) we conclude that

$$g\ M_y\ H_{xx} + h\ M_x\ H_{yy} = 0. \tag{I-34}$$

Using only the relations $J_{yy} = 0$ and $J_M = y\ J_{My}$, which are readily obtained from (I-29), we now reduce (I-34) to

$$\left[\left(g\ \frac{M_{xx}}{M_x} + h\ \frac{M_{yy}}{M_y}\right) + 2\ \frac{h}{y}\right] \frac{yJ_{My}}{J_{MM}} + \mu = 0. \tag{I-35}$$

We eliminate the indicated derivatives of J and M. From (I-27) and (I-29), we obtain

$$yJ_{My}/J_{MM} = M(1 - M)/2(M - k),$$

and from (I-12),

$$\begin{aligned} M_x &= 1/(1 - B'y)\ ; & M_y &= B\ M_x\ ; \\ M_{xx} &= B''\ M_x^{\ 3}y\ ; & M_{yy} &= B^2\ M_{xx} + 2\ BB'\ M_x^{\ 2}\ . \end{aligned} \tag{I-36}$$

Substituting these expressions in (I-35), we reduce it to

$$G + \frac{M(1 - M)}{2(M - k)}\ \mu\ B'' = 0, \tag{I-37}$$

where

$$G = \frac{1}{y}\left[\frac{h}{y}\ \frac{M(1 - M)}{M - k} + g + Bh\right]\ .$$

With substitution from (I-12), we now write h and g as functions of M and y; we obtain

$$\begin{aligned} g &= \psi_1\ (M - By)\ \left[(1 - M) + (B - k)\ y\right]\ , \\ h &= \psi_1\ y\ \left[(B - k)y - (M - k)\right]\ . \end{aligned} \tag{I-38}$$

With these expressions and indicated algebraic reduction, we find that

$$G = \psi_1 k(1 - k) (B - M)/(M - k),$$

and hence, (I-36) becomes

$$(B - M) + \frac{M(1 - M)}{2k(1 - k)} \frac{\mu}{\psi_1} B'' = 0. \tag{I-39}$$

We see at once from (I-39) that since (B - M) is a function of M only, one of the following cases must hold:

(i) $B'' = 0$,

(ii) $\mu = \mu(M)$, a function of M only.

We examine these cases individually.

Suppose case (i) holds. From (I-39) we have then that B = M.

Suppose case (ii) holds. Consider first that in any case we have from (I-33) and (I-36) that

$$\frac{\mu}{\psi_1} = \frac{M(1 - M) + [M(B - k) - B(1 - k)] y}{1 - B'y} . \tag{I-40}$$

Carrying out algebraic reduction, we find that

$$(B - M)' = (1 - k - M) (B - M)/M(1 - M),$$

from which it follows that

$$B'' = k(1 - k) (B - M)/[M(1 - M)]^2 . \tag{I-41}$$

When in (I-39) we set $\mu = \psi_1 M(1 - M)$ and B'' as given by (I-41), we readily see that B = M.

We thus see that both case (i) and case (ii) result in

$$B = M \tag{I-42}$$

and hence, we have established the necessary B.

Substituting (I-42) in (I-28), we obtain

$$U(M,y) = c_o + c_1 M(1 - y) + c_2 y + c_3 y \phi_1 + (\phi_2 - \frac{2M - 1}{1 - M})y + [2M(1 - y) - 1] \ln \frac{M}{1 - M} , \tag{I-43}$$

and

$$M(x,y) = x/(1 - y). \tag{I-44}$$

Thus, we have succeeded in specifying completely the value function and the match function, and furthermore, both functions are uniquely specified.

E. Proof that Specified $\mathcal{V}(x,y)$ is Solution

We have established that $\mathcal{V}(x,y) = U(M,y)$ must be given by (I-43), but we have not shown that $\mathcal{V}(x,y)$ satisfies the partial differential equation (I-1). This we proceed to do.

Again we find it more convenient to consider U(M,y). We first transform (I-1) to an equation in U(M,y); we obtain directly:

$$(g^2 M_{xx} + 2g\,h\,M_{xy} + h^2 M_{yy})\,U_M + \mu^2 U_{MM} + 2\mu h\,U_{My} + h^2 U_{yy} = \psi_1^2\,, \tag{I-45}$$

where, as before,

$$\mu = g\,M_x + h\,M_y.$$

Our specified U(M,y) given by (I-43) is:

$$U = \alpha + \beta y,$$

where:

$$\alpha'' = 1/[M(1 - M)]^2\,,$$

$$\beta = f - \alpha' M, \text{ (from (I-20))},$$

$f = f(M)$ satisfies the differential equation

$$f'' + 2\,\frac{k - M}{M(1 - M)}\,f' = \frac{2k}{M^2(1 - M)^3}\,, \text{ (from (I-21) and (I-25))},$$

$$M(x,y) = x/(1 - y).$$

We must show that this specified U satisfies (I-45).

We first obtain directly that

$$\begin{aligned} M_x &= 1/(1 - y); & M_{xy} &= M_x^2\,; \\ M_{xx} &= 0\;; & M_{yy} &= 2M_x M_y\,. \end{aligned} \tag{I-46}$$

Substituting the above in (I-45), we note that the factor of U_M collapses to

$$2h\mu M_x,$$

and hence, (I-45) may be re-written;

$$\left[2h(M_x U_M + U_{My}) + \mu U_{MM}\right]\mu + h^2 U_{yy} = \psi_1^2 . \qquad \text{(I-47)}$$

We may now verify that:

from (I-38), $h = \psi_1 (k - M) y(1 - y)$,
from (I-46), $hM_x = \psi_1 (k - M)y$,
from (I-40), $\mu = \psi_1 M(1 - M)$,

and from direct differentiation of $U = \alpha + \beta y$:

$$U_M = \alpha' + \beta' y ; \qquad U_{My} = \beta' ;$$
$$U_{MM} = \alpha'' + \beta'' y ; \qquad U_{yy} = 0.$$

When we substitute the above in (I-47), we obtain

$$2(k - M)(\alpha' + \beta') + M(1 - M)\beta'' = 0. \qquad \text{(I-48)}$$

We recall that our specified U requires

$$\alpha'' = 1/[M(1 - M)]^2 \text{ and } \beta = f - \alpha' M,$$

whence

$$\alpha''' = -2(1 - 2M)/[M(1 - M)]^3 ,$$
$$\beta' = f' - \alpha'' M - \alpha', \text{ and } \beta'' = f'' - \alpha''' M - 2\alpha''.$$

We substitute these expressions in (I-48), carry out indicated algebraic reduction, and we obtain

$$f'' + 2\,\frac{k - M}{M(1 - M)}\quad f' = \frac{2k}{M^2(1 - M)^3} ,$$

which is in fact the differential equation f satisfies. Hence, our specified U(M,y) satisfies (I-45) and $\mathcal{V}(x,y)$ satisfies (I-1). Furthermore, it is easily verified that our specified U also satisfies the other solution requirements (I-2), (I-3), and (I-4).

F. Explicit Expressions for $\mathscr{V}(x,y)$

To obtain explicit expressions for $\mathscr{V}(x,y)$ from (I-43), we must carry out the necessary integrations for ϕ_1 and ϕ_2 in (I-27), remembering that here $\phi_1(x) = \phi_1(M)$ and $\phi_2(x) = \phi_2(M)$. We find that there are two cases for the required integration, $2k = 1$, and $2k \neq 1$. Carrying out the integration, we then find:

for $2k = 1$:

$$\begin{aligned} \phi_1 &= \ln \frac{M}{1 - M} , \\ \phi_2 &= \frac{2M - 1}{1 - M} + \left(\ln \frac{M}{1 - M}\right) + \frac{1}{2}\left(\ln \frac{M}{1 - M}\right)^2 . \end{aligned} \tag{I-49}$$

for $2k \neq 1$:

$$\begin{aligned} \phi_1 &= \left(\frac{M}{1 - M}\right)^{-(2k-1)} , \\ \phi_2 &= \frac{2M - 1}{1 - M} + \frac{2k}{2k - 1} \ln \frac{M}{1 - M} . \end{aligned} \tag{I-50}$$

Substituting (I-49) and (I-50) in (I-43), we obtain $U(M,y)$; thus:

for $2k = 1$:

$$\begin{aligned} U(M,y) = c_o &+ c_1 M(1 - y) + c_2 y + c_3 y \ln \frac{M}{1 - M} \\ &+ \left[2M(1 - y) + \frac{y}{2}\left(\ln \frac{M}{1 - M}\right) - 1\right] \ln \frac{M}{1 - M} , \end{aligned} \tag{I-51}$$

for $2k \neq 1$:

$$\begin{aligned} U(M,y) = c_o &+ c_1 M(1 - y) + c_2 y + c_3 y\left(\frac{M}{1 - M}\right)^{-(2k-1)} \\ &+ \left[2M(1 - y) + \frac{2k}{2k - 1} y - 1\right] \ln \frac{M}{1 - M} . \end{aligned} \tag{I-52}$$

To obtain $\mathscr{V}(x,y)$, we substitute (I-44), $M = x/(1 - y)$; thus:

for $2k = 1$:

$$\mathscr{V}(x,y) = c_o + c_1 x + c_2 y + c_3 y \ln R + \left[2x + \frac{y}{2}(\ln R) - 1\right] \ln R, \tag{I-53}$$

for $2k \neq 1$:

$$(x,y) = c_o + c_1 x + c_2 y + c_3 y\, R^{-(2k-1)} + \left(2x + \frac{2k}{2k - 1} y - 1\right) \ln R, \tag{I-54}$$

where

$$R = x/(1 - x - y).$$

APPENDIX I - ADDENDUM

DEVELOPMENT OF THE VALUE FUNCTION, GIVEN THE MATCH FUNCTION

Given that

$$M = x/(1 - y) \tag{IA-1}$$

is the necessary match function, it remains to find out whether a value function $\mathcal{V}(x,y)$ is permitted by the partial differential equation (16) in the text. This is done below.

Text equation (16) is first transformed to coordinates of M and y; thus:

$$\left(g^2 \frac{\partial^2 M}{\partial x^2} + 2gh \frac{\partial^2 M}{\partial x \partial y} + h^2 \frac{\partial^2 M}{\partial y^2}\right) \frac{\partial U}{\partial M} + \mu^2 \frac{\partial^2 U}{\partial M^2} + 2\mu h \frac{\partial^2 U}{\partial M \partial y} + h^2 \frac{\partial^2 U}{\partial y^2} = \psi_1{}^2, \tag{IA-2}$$

$$\text{where } \mu = g \frac{\partial M}{\partial x} + h \frac{\partial M}{\partial y}, \tag{IA-3}$$

and U(M,y) is given by text (23). From text (25), one finds directly that

$$\partial M/\partial x = 1/(1 - y), \quad \partial M/\partial y = M(\partial M/\partial x),$$

$$\partial^2 M/\partial x^2 = 0, \quad \partial^2 M/\partial y^2 = 2(\partial M/\partial x)(\partial M/\partial y), \quad \partial^2 M/\partial x \partial y = (\partial M/\partial x)^2. \tag{IA-4}$$

Substituting from (IA-4) in (IA-2), one first notes that the factor of $\partial U/\partial M$ collapses to

$$2\, h\, \mu(\partial M/\partial x),$$

and hence, (IA-2) may be re-written as

$$\left[2h\left(\frac{\partial M}{\partial x} \frac{\partial U}{\partial M} + \frac{\partial^2 U}{\partial M \partial y}\right) + \mu \frac{\partial^2 U}{\partial M^2}\right] \mu + h^2 \frac{\partial^2 U}{\partial y^2} = \psi_1{}^2. \tag{IA-5}$$

From (IA-1, 3, 4), and the expressions for g and h, it may be verified that:

$$\begin{aligned} h &= \psi_1(k - M)y\,(1 - y), \\ h(\partial M/\partial x) &= \psi_1(k - M)y, \\ \mu &= \psi_1\, M(1 - M), \end{aligned} \tag{IA-6}$$

and from direct differentiation of text (23),

$$\partial U/\partial M = \alpha' + \beta' y, \quad \partial^2 U/\partial M \partial y = \beta' , \qquad \text{(IA-7)}$$
$$\partial^2 U/\partial M^2 = \alpha'' + \beta'' y, \quad \partial^2 U/\partial y^2 = 0 ,$$

and furthermore, from text (24),

$$\alpha'' = 1/[M(1 - M)]^2 . \qquad \text{(IA-8)}$$

One now substitutes from (IA-6, 7, 8) in (IA-5) and finds that

$$M(1 - M)\ (\alpha'' + \beta'') + 2(k - M)\ (\alpha' + \beta') - 1/M(1 - M) = 0, \qquad \text{(IA-9)}$$

the y variable vanishing, and thus, the partial differential equation (IA-2) for the value function $\mathcal{V}(x,y) = U(M,y)$ has been reduced to an ordinary differential equation for $\beta = \beta(M)$, the only unknown function in the expression text (23) for the value function. Hence, it has now been established that there is in fact a value function $\mathcal{V}(x,y)$ with an associated match function $M(x,y)$ given by (IA-1), this $\mathcal{V}(x,y) = U(M,y)$ being given by text (23) with $\alpha = \alpha(M)$ given by text (24), and $\beta = B(M)$ being the solution of (IA-9).

Equation (IA-9) is now solved. It is easier to solve for $(\alpha + \beta)$. In carrying out the indicated integrations, one finds that two cases for the integrations are necessary; namely: $2k = 1$, and $2k \neq 1$. One then obtains:

for $2k = 1$:

$$\alpha + \beta = C_o + C_1 \ln(\frac{M}{1 - M}) + \frac{1}{2} [\ln(\frac{M}{1 - M})]^2 , \qquad \text{(IA-10)}$$

for $2k \neq 1$:

$$\alpha + \beta = C_o + C_1 (\frac{M}{1 - M})^{-(2k-1)} + \frac{1}{2k - 1} \ln(\frac{M}{1 - M}), \qquad \text{(IA-11)}$$

where C_o and C_1 are arbitrary constants.

From (23), one has that

$$U(M,y) = (1 - y)\ \alpha + y(\alpha + \beta),$$

and hence, with text (24) and (IA-10) and (IA-11), the value function may be expressed as a function $U(M,y)$ of M and y for both integration cases. To obtain $\mathcal{V}(x,y)$, one now uses (IA-1) to eliminate M from $U(M,y)$, and one obtains:

for $2k = 1$:

$$\mathcal{V}(x,y) = c_o + c_1 x + c_2 y + c_3 y \ln R + [2x + \frac{(\ln R)}{2} y - 1] \ln R, \qquad \text{(IA-12)}$$

for $2k \neq 1$:

$$\mathcal{V}(x,y) = c_o + c_1 x + c_2 y + c_3 y R^{-(2k-1)} + (2x + \frac{2k}{2k - 1} y - 1) \ln R, \qquad \text{(IA-13)}$$

where: $R = x/(1 - x - y)$, (IA-14)

and c_o, c_1, c_2, and c_3 are arbitrary constants.

APPENDIX II

DETERMINATION OF CONSTANTS FOR THE U-235, 236, 238 UNIT COST SCALE

A. INTRODUCTION

We have seen in the text that the unit cost scale applicable to a three component material produced in a matched R cascade is

$$D(x,y) = K\left[a_0 + a_1x + a_2y + a_3H(x,y) + V(x,y)\right], \qquad \text{(II-1)}$$

where:

$D(x,y)$ is the unit cost of material at concentrations x and y,
K is the unit cost of separative work,

$$H(x,y) = y\left(\frac{x}{1 - x - y}\right)^{-(2k-1)},$$

$$V(x,y) = \left(2x + \frac{2k}{2k - 1}\, y - 1\right) \ln \frac{x}{1 - x - y},$$

and the a's are arbitrary constants.
(We here suppose the case $2k \neq 1$.)

In this appendix, we determine values of the arbitrary constants applicable to the pricing of U-235, 236, and 238 mixtures. We let x denote the U-235 concentration and y the U-236 concentration. To evaluate the constants, we impose the following conditions on the cost scale:

1. The cost scale $D(x,y)$ should have a locus of zero value.
2. On the locus of zero value, $D(x,y)$ should be a minimum, thus assuring a positive cost scale for the entire range of concentrations.
3. In order that the resulting cost scale include the usual U-235, 238 cost scale, the locus of zero value should include x_o, the U-235 concentration of zero value of the U-235, 238 cost scale.

We proceed to develop a $D(x,y)$ satisfying the above conditions.

B. Mathematical Development

We first note from (II-1) that for a constant

$$R = x/(1 - x - y), \tag{II-2}$$

$D(x,y)$ is linear in x and y. Since at constant R, x and y are linearly related, we may then write $D(x,y)$ as

$$D^*(R,y) = (1 - y)\, D^*(R,0) + y\, D^*(R,1), \tag{II-3}$$

where:

$$\begin{aligned} D^*(R,0) &= D(x,0)\ , \\ D^*(R,1) &= D(0,1)\ . \end{aligned} \tag{II-4}$$

From (II-1), we have that at constant R,

$$\begin{aligned} D(x,0) &= K\left[a_o + a_1 x + (2x - 1) \ln R\right]\ , \\ D(0,1) &= K\left[a_o + a_2 + a_3 R^{-(2k-1)} + \frac{1}{2k - 1} \ln R\right]\ , \end{aligned} \tag{II-5}$$

where $R = x/(1 - x)$.

Hence, from (II-4) and (II-5) we have that

$$D^*(R,0) = K\left[a_o + a_1 \frac{R}{1 + R} + \frac{R - 1}{R + 1} \ln R\right]\ , \tag{II-6.1}$$

$$D^*(R,1) = K\left[a_o + a_2 + a_3 R^{-(2k-1)} + \frac{1}{2k - 1} \ln R\right]\ , \tag{II-6.2}$$

and we thus know $D^*(R,y)$ given by (II-3).

The conditions (1), (2), and (3) can be stated for $D^*(R,y)$ by reading R for x, and we proceed to impose these conditions on $D^*(R,y)$; thus:

(1) $D^*(R_b,\ y_b) = 0,$ (II-7)

i.e., $(R_b,\ y_b)$ is the locus of zero value.

(2) $$\left.\frac{\partial D^*}{\partial R}\right]_{\substack{R = R_b \\ y = y_b}} = 0. \tag{II-8}$$

$$\left.\frac{\partial D^*}{\partial y}\right]_{\substack{R = R_b \\ y = y_b}} = 0. \tag{II-9}$$

(3) $D^*(R_o,0) = 0$, where $R_o = x_o/(1 - x_o)$. (II-10)

We now determine the arbitrary constants in (II-1) with these conditions.

We first have from (II-3) that

$$\frac{\partial D^*}{\partial y} = -D^*(R,0) + D^*(R,1),$$

and hence, from (II-9), it is necessary that

$$-D^*(R_b,0) + D^*(R_b,1) = 0. \qquad \text{(II-11)}$$

In particular, since from (II-10) it is necessary that R_b take the value R_o, and furthermore that

$$D^*(R_o,0) = 0, \qquad \text{(II-12)}$$

it follows from (II-11) that it is necessary that

$$D^*(R_o,1) = 0. \qquad \text{(II-13)}$$

Hence, we see from (II-3), (II-12), and (II-13) that then

$$D^*(R_o,y_b) = (1 - y_b)\ D^*(R_o,0) + y_b\ D^*(R_o,1) = 0 \text{ for all } y_b. \qquad \text{(II-14)}$$

Therefore, $R_b = R_o$ is a locus of zero value. Thus, if we choose the arbitrary constants so that (II-12) and (II-13) are satisfied, the resulting $D^*(R,y)$ satisfies (II-7), (II-9), and (II-10), but (II-8) remains to be met. To satisfy (II-8), it is necessary that

$$\left.\frac{\partial D^*}{\partial R}\right]_{\substack{R = R_o \\ y = y_b}} = 0. \qquad \text{(II-15)}$$

From (II-3), we have that

$$\frac{\partial D^*}{\partial R} = (1 - y)\ \frac{dD^*(R,o)}{dR} + y\ \frac{dD^*(R,1)}{dR}\ ,$$

and hence, from (II-15), it is necessary that

$$(1 - y_b)\ \left.\frac{dD^*(R,o)}{dR}\right]_{R = R_o} + y_b\ \left.\frac{dD^*(R,1)}{dR}\right]_{R = R_o} = 0 \text{ for all } y_b, \qquad \text{(II-16)}$$

from which it follows at once that

$$\left.\frac{dD^*(R,o)}{dR}\right]_{R = R_o} = 0, \qquad \text{(II-17)}$$

$$\left.\frac{dD^*(R,1)}{dR}\right]_{R = R_o} = 0, \qquad \text{(II-18)}$$

and satisfying these two equations assures that (II-8) is satisfied.

In summary, for our cost scale $D^*(R,y)$ to satisfy conditions (1), (2), and (3), the arbitrary constants must be chosen so that

$$D^*(R_o,0) = 0, \qquad \text{(II-19)}$$

$$\left.\frac{dD^*(R,0)}{dR}\right]_{R = R_o} = 0, \qquad \text{(II-20)}$$

and

$$D^*(R_o,1) = 0, \qquad \text{(II-21)}$$

$$\left.\frac{dD^*(R,1)}{dR}\right]_{R = R_o} = 0. \qquad \text{(II-22)}$$

We now proceed to determine the constants a_o, a_1, a_2, and a_3 from (II-19, 20, 21, and 22). We first determine a_o and a_1. From (II-6.1) and (II-20), we have that

$$a_1 = - \left.\frac{d \, \frac{R - 1}{R + 1} \ln R \,/dR}{d(\frac{R}{R + 1})/dR}\right]_{R = R_o} ,$$

whence,

$$a_1 = - \left[2(\ln R_o) + \frac{(R_o - 1) \, (R_o + 1)}{R_o}\right] . \qquad \text{(II-23)}$$

From (II-6.1) and (II-19), we have that

$$a_o = - a_1 \frac{R_o}{R_o + 1} - \frac{R_o - 1}{R_o + 1} \ln R_o ,$$

and with (II-23), we find that

$$a_o = \left[(\ln R_o) + (R_o - 1)\right] . \qquad \text{(II-24)}$$

We now determine a_2 and a_3. From (II-6.2) and (II-22), we have that

$$a_3 = - \frac{1}{2k - 1} \left. \frac{d\ln R/dR}{dR^{-(2k-1)}/dR} \right]_{R = R_o} ,$$

whence,

$$a_3 = \frac{R_o^{2k-1}}{(2k - 1)^2} . \qquad \text{(II-25)}$$

From (II-6.2) and (II-21), we now have that

$$a_2 = - a_o - a_3 R_o^{-(2k-1)} - \frac{1}{2k - 1} \ln R_o ,$$

and with substitution from (II-24) and (II-25), we obtain

$$a_2 = - \left[\frac{2k}{2k - 1} (\ln R_o) + \frac{1}{(2k - 1)^2} + (R_o - 1)\right] . \qquad \text{(II-26)}$$

This completes the determination of the constants. In summary, we have:

$$a_o = \left[(R_o - 1) + \ln R_o\right] , \qquad \text{(II-27)}$$

$$a_1 = - \left[\frac{(R_o - 1)(R_o + 1)}{R_o} + 2 \ln R_o\right] , \qquad \text{(II-28)}$$

$$a_2 = - \left[(R_o - 1) + \frac{1}{(2k - 1)^2} + \frac{2k}{2k - 1} \ln R_o\right] , \qquad \text{(II-29)}$$

$$a_3 = \frac{R_o^{2k-1}}{(2k - 1)^2} , \qquad \text{(II-30)}$$

where $R_o = x_o/(1 - x_o)$,
x_o = the concentration of zero value when $y = 0$.

C. Numerical Evaluation of Constants for U-235, 236, and 238 Unit Cost Scale

Benedict and Pigford in reference [1] give the constants for the U.S.A.E.C. price schedule ($/kg. U) for U-235 and 238 mixtures in the form of UF_6 as follows:

x_o = 0.0022138 weight fraction U-235,

K = \$37.286/kg. U separative work.

With these constants and our formulas, we may then compute a unit cost scale for U-235, 236, and 238 mixtures. We have that

$R_o = x_o/(1 - x_o) = 0.0022187$,

k = 2/3 for the ratio of the separation factor of U-236 from U-238 to that of U-235 from U-238.

With equations (II-27, 28, 29, and 30) we then compute

$$a_o = -7.1086,$$
$$a_1 = 462.9338,$$
$$a_2 = 16.4411,$$
$$a_3 = 1.1738.$$

Our unit cost scale for U-235, 236, and 238 mixtures then is given by (II-1) with the above values for the constants. Thus, we have

$$D(x,y) = 37.286 \left[-7.1086 + 462.9338\, x + 16.4411\, y + 1.1738\, y\left(\frac{x}{1 - x - y}\right)^{-1/3} + (2x + 4y - 1) \ln\left(\frac{x}{1 - x - y}\right)\right], \quad \text{(II-31)}$$

which gives the unit cost ($/kg. U) of uranium in the form of UF_6 having isotopic concentrations x U-235 weight fraction and y U-236 weight fraction.

The unit cost scale D(x,y) resulting with the above constants is tabulated in text Table I. The behavior of D(x,y) for relatively high U-235 concentration x and high U-236 concentration y is of particular interest. The cost formula shows that as y approaches (1 - x), D(x,y) goes to infinity. Numerical evaluation shows, however, that this approach of D(x,y) to infinity is remarkably slow as shown in Table II-1 below.

TABLE II-1

D(0.85,y) VERSUS y

y	D(0.85,y)
0.00	14452.
0.04	14498.
0.08	14552.
0.12	14629.
0.14	14703.
0.145	14742.
0.1495	14859.
0.1499	14937.
0.149999	15161.

APPENDIX III

A LOWER BOUND FOR A MINIMUM UNIT COST

A. INTRODUCTION

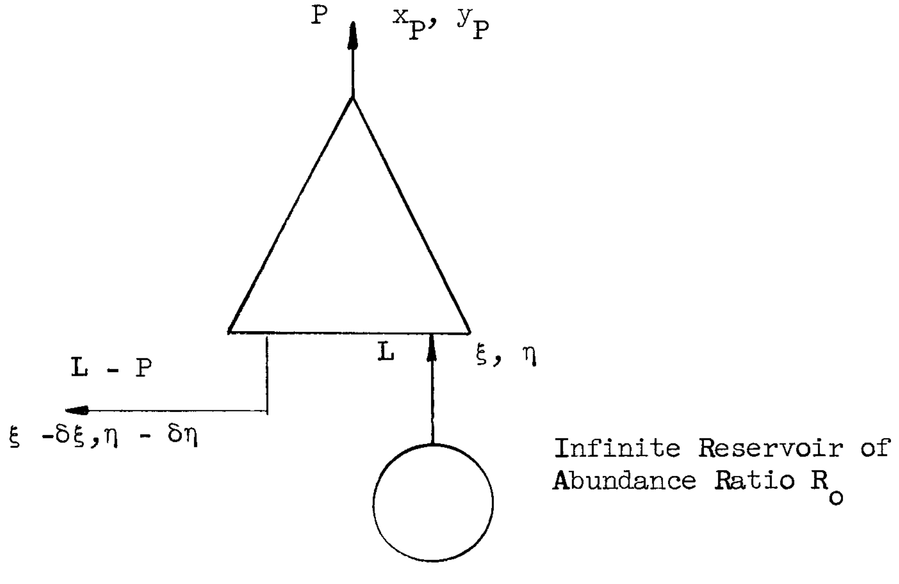
In Appendix II, we developed a unit cost scale $D(x,y)$ applicable to a matched R cascade processing U-235, 236, and 238. The cost scale has the following two properties:

1. Material with an abundance ratio R_o has zero value. Thus, $D(\xi,\eta) = 0$ for a material with U-235 concentration ξ and a U-236 concentration η such that $R_o = \xi/(1 - \xi - \eta)$.
2. The unit cost scale $D(x,y)$ is at a minimum for material with an abundance ratio R_o. Thus,

$$\frac{\partial D}{\partial x}\Bigg]_{\substack{x=\xi\\y=\eta}} = 0 \quad \text{and} \quad \frac{\partial D}{\partial y}\Bigg]_{\substack{x=\xi\\y=\eta}} = 0 \; ;$$

where ξ and η are as above.

As a result of these two properties, our cost scale $D(x,y)$ may be interpreted as the separative work cost per unit of material of concentrations (x,y) which is produced in a matched R cascade from material of abundance ratio R_o available in infinite quantities. This may be seen from Figure III-1 as follows:



Production From Material of Zero Value

Figure III-1

Since our cost scale is of the form

$$D(x,y) = K[a_0 + a_1x + a_2y + a_3H + V] ,$$

with the cascade in Figure III-1 operated as a matched R cascade, we have by means of V and H balances and the three indicated material balances, as illustrated in the text, that

$$P\,[D(x_P,y_P) - D(\xi,\eta) - (x_P - \xi)(\partial D/\partial\xi) - (y_P - \eta)(\partial D/\partial\eta)] = K(\tfrac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2}).$$

Since our cost scale has the property that $D(\xi,\eta) = 0$, $\partial D/\partial\xi = 0$, and $\partial D/\partial\eta = 0$, we then see that

$$D(x_P,y_P) = K(\tfrac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2})/P,$$

and thus, $D(x_P,y_P)$ is the cost of separative work per unit product, a not unexpected result.

Consider now that in the situation of Figure III-1, the cascade need not be operated as a matched R cascade. If we can prescribe some mode of cascade operation which results in less separative work requirements per unit of product than the matched R cascade, the unit cost of product will then be less than that given by our cost scale which is applicable to the matched R cascade. Hence, a pertinent question to ask is: for how much less than the $D(x_P,y_P)$ of the matched R cascade may we produce material of concentrations (x_P,y_P) from material of abundance ratio R_0 at zero value? In this Appendix, we give a partial answer to this question. We develop a lower bound for the unit cost of production at concentrations (x_P,y_P) from material of abundance ratio R_0 of zero value for the case $0 < k < 1$, which is applicable to U-235, 236, 238 mixtures.

B. Some Preliminary Formulas

A matched R cascade is operated so that

$$dR/dn = R\psi_1/2.$$

More generally, we now let

$$dR/dn = \rho\, R\psi_1 , \qquad \text{(III-1)}$$

where $\rho = \rho(R)$, a function of R. We proceed to find an expression for the separative work of the cascade in Figure III-1 with this general condition. From text equations (31.1) and (31.2), we have that

$$\frac{dR}{dn} = R\psi_1 - \frac{P}{L}\,\frac{1 + R + S}{1 + R_P + S_P}\,(R_P - R), \tag{III-2a}$$

$$\frac{dS}{dn} = S\psi_2 - \frac{P}{L}\,\frac{1 + R + S}{1 + R_P + S_P}\,(S_P - S). \tag{III-2b}$$

We now substitute (III-1) in (III-2a) and we obtain the following expression for the inter-stage flow:

$$L = \frac{P}{(1 - \rho)\,\psi_1}\,\frac{1 + R + S}{1 + R_P + S_P}\,\frac{R_P - R}{R}. \tag{III-3}$$

From (III-3) we obtain

$$L\,dn = \frac{P}{(1 - \rho)\,\psi_1}\,\frac{1 + R + S}{1 + R_P + S_P}\,\frac{R_P - R}{R}\,\frac{dR}{(dR/dn)}.$$

We eliminate dR/dn with (III-1), and with some re-arranging, we obtain

$$\frac{1}{4}\,L\psi_1^{\,2}\,dn/P = \frac{1}{4\rho(1 - \rho)}\,\frac{1 + R + S}{1 + R_P + S_P}\,\frac{R_P - R}{R^2}\,dR. \tag{III-4}$$

The separative work per unit product for the cascade shown in Figure III-1 then is

$$(\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^{\,2})/P = \int_{R_o}^{R_P} \frac{1}{4\rho(1 - \rho)}\,\frac{1 + R + S}{1 + R_P + S_P}\,\frac{R_P - R}{R^2}\,dR. \tag{III-5}$$

We may notice from both (III-4) and (III-5) that

$$0 < \rho < 1 . \tag{III-6}$$

We now need a relation between R and S. We substitute (III-3) in (III-2b), divide the result by (III-1), and we obtain

$$\frac{dS}{dR} = \frac{k}{\rho}\,\frac{S}{R} - \frac{1 - \rho}{\rho}\,\frac{S_P - S}{R_P - R}. \tag{III-7}$$

To determine $(\frac{1}{4} \sum_{\text{cascade}} L\, \psi_1^{\,2})/P$ for a given $\rho = \rho(R)$, we need to solve (III-7) for S, substitute this result in (III-5), and carry out the indicated integration.

C. Development of Lower Bound on Unit Cost

With the results of the previous section, we see that for the situation of Figure III-1, the minimum unit cost of production at specified concentrations (x_P, y_P) from material of abundance ratio R_o is

$$\text{min. } D(x_P, y_P) = \text{min. } K \int_{R_o}^{R_P} \frac{1}{4\rho(1-\rho)}\, \frac{1 + R + S}{1 + R_P + S_P}\, \frac{R_P - R}{R^2}\, dR, \qquad \text{(III-8)}$$

where

$$R_P = x_P/(1 - x_P - y_P), \quad S_P = y_P/(1 - x_P - y_P),$$

and S satisfies (III-7), here re-written:

$$\frac{dS}{dR} = S' = \frac{1}{\rho}\left(k\,\frac{S}{R} - \frac{S_P - S}{R_P - R}\right) + \frac{S_P - S}{R_P - R}\,. \qquad \text{(III-9)}$$

To determine the min. $D(x_P, y_P)$ we must find the function $\rho = \rho(R)$ which minimizes the integral in (III-8) constrained by the differential equation (III-9). Determining this minimum is seen to be a difficult task. Determining a lower bound for the minimum is not too difficult, and this we proceed to do.

We first note from (III-8) that

$$\text{min. } D(x_P, y_P) \geqslant K \int_{R_o}^{R_P} \frac{1}{[\text{max. } 4\rho(1-\rho)]}\, \frac{1 + R + [\text{min. } S]}{1 + R_P + S_P}\, \frac{R_P - R}{R^2}\, dR. \qquad \text{(III-10)}$$

Since from (III-6), we have that

$$0 < \rho(R) < 1,$$

it follows that

$$\text{max. } 4\rho(1-\rho) = 1 \text{ for all } R.$$

Hence,

$$\text{min. } D(x_P,y_P) > K \int_{R_o}^{R_P} \frac{1 + R + [\text{min. } S]}{1 + R_P + S_P} \frac{R_P - R}{R^2} \, dR, \tag{III-11}$$

and we must find a lower bound for S.

We first introduce the function $S_1 = S_1(R)$ defined by

$$k \frac{S_1}{R} - \frac{S_P - S_1}{R_P - R_1} = 0, \tag{III-12}$$

from which we find directly that

$$S_1 = \frac{S_P \, R}{k(R_P - R) + R}, \tag{III-13}$$

and

$$\frac{dS_1}{dR} = S'_1 = \frac{k \, S_P \, R_P}{[k(R_P - R) + R]^2}. \tag{III-14}$$

We now note that with (III-9) and (III-12), we may write

$$S' - S'_1 = \frac{1}{\rho} \left[\left(k\frac{S}{R} - \frac{S_P - S}{R_P - R} - \left(k \frac{S_1}{R} - \frac{S_P - S_1}{R_P - R}\right)\right] - \frac{S - S_1}{R_P - R} + k \frac{S_1}{R} - S'_1. \tag{III-15}$$

Let

$$\Delta(R) = S - S_1, \tag{III-16}$$

whence, $\Delta' = S' - S'_1$,
and we re-write (III-15) as

$$\Delta' - \left(\frac{k}{\rho} \frac{1}{R} + \frac{1 - \rho}{\rho} \frac{1}{R_P - R}\right) \Delta = \frac{k \, S_1}{R} - S'_1. \tag{III-17}$$

We now substitute from (III-13) and (III-14), and we obtain

$$\Delta' - \left(\frac{k}{\rho} \frac{1}{R} + \frac{1 - \rho}{\rho} \frac{1}{R_P - R}\right) \Delta = \frac{-k(1 - k) \, S_P(R_P - R)}{[k(R_P - R) + R]^2}. \tag{III-18}$$

The integrating factor for (III-18) is

$$G(R) = \exp - \int \left(\frac{k}{\rho}\frac{1}{R} + \frac{1-\rho}{\rho}\ \frac{1}{R_P - R}\right) dR, \tag{III-19}$$

and hence, we have

$$\Delta \cdot G \Big]_R^{R_P} = -k(1-k)\ S_P \int_R^{R_P} \frac{(R_P - t)\ G(t)}{[k(R_P - t) + t]^2}\ dt. \tag{III-20}$$

Note now from (III-13) that

$$S_1(R_P) = S_P\ ,$$

and hence,

$$\Delta(R_P) = S(R_P) - S_1(R_P) = 0.$$

We then have from (III-20) that

$$S - S_1 = \frac{k(1-k)\ S_P}{G(R)} \int_R^{R_P} \frac{(R_P - t)\ G(t)}{[k(R_P - t) + t]^2}\ dt. \tag{III-21}$$

We may now conclude that if

$$0 < k < 1,$$

then

$$S - S_1 \geqslant 0 \text{ for } R \leqslant R_P\ ,$$

and hence, we have established* that S_1 is a lower bound for S. With this result and (III-11), we obtain the desired lower bound on the unit cost, thus:

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*It should be mentioned here that Dr. J. W. Neuberger of the University of Tennessee has shown that S_1 is the greatest lower bound for S.

$$\text{min. } D(x_P,y_P) \geqslant K \int_{R_o}^{R_P} \frac{1 + R + S_1}{1 + R_P + S_P} \; \frac{R_P - R}{R^2} \, dR \, , \qquad \text{(III-22)}$$

where from (III-13),

$$S_1 = \frac{S_P \, R}{k(R_P - R) + R} \, .$$

This lower bound holds provided $0 < k < 1$.

D. Evaluation of Lower Bound on Unit Cost

In this section, we develop an expression for our lower bound on $D(x_P,y_P)$ suitable for numerical evaluation. We first note that (III-22) may be written:

$$\text{min. } D(x_P,y_P) \geqslant I_1 + I_2 \, , \qquad \text{(III-23)}$$

where:

$$I_1 = K \frac{1 + R_P}{1 + R_P + S_P} \int_{R_o}^{R_P} \frac{1 + R}{1 + R_P} \; \frac{R_P - R}{R^2} \, dR, \qquad \text{(III-24)}$$

$$I_2 = \frac{K}{1 + R_P + S_P} \int_{R_o}^{R_P} S_1 \; \frac{R_P - R}{R^2} \, dR. \qquad \text{(III-25)}$$

We proceed to evaluate I_1 and I_2.

We note first from (III-24) and (III-5) that

$$\int_{R_o}^{R_P} \frac{1 + R}{1 + R_P} \; \frac{R_P - R}{R^2} \, dR$$

is the separative work per unit product in a cascade in Figure III-1 with $\rho \equiv 1/2$ and $S_P = 0$. A cascade with $\rho \equiv 1/2$ is a matched R cascade, and hence,

$$K \int_{R_o}^{R_P} \frac{1 + R}{1 + R_P} \frac{R_P - R}{R^2} \, dR = D(x_P^*, 0), \qquad \text{(III-26)}$$

where $D(x^*, 0)$ is our unit cost applicable to a matched R cascade for material at concentration $y = 0$ and concentration $x = x^*$ such that

$$x_P^*/(1 - x_P^*) = x_P/(1 - x_P - y_P) = R_P \, .$$

We readily find that

$$x_P^* = x_P/(1 - y_P) \, , \qquad \text{(III-27)}$$

and hence, we obtain from (III-24, 26, 27) that

$$I_1 = \frac{1 + R_P}{1 + R_P + S_P} \, D(x_P^*, 0) = (1 - y_P) \, D(x_P^*, 0). \qquad \text{(III-28)}$$

Thus, I_1 has been evaluated.

We now evaluate I_2. We first substitute for S_1 from (III-13), and with some algebraic reduction, we obtain:

$$I_2 = y_P \, K \int_{R_o}^{R_P} \frac{1}{k(R_P - R) + R} \, \frac{R_P - R}{R} \, dR. \qquad \text{(III-29)}$$

We now make the substitution $q = R/(R_P - R)$, and we obtain directly:

$$\int \frac{R_P - R}{k(R_P - R) + R} \, \frac{dR}{R} = \int \frac{dq}{q(1 + q)(k + q)} = \frac{1}{k} \ln\left(\frac{q}{k + q}\right)$$

$$+ \frac{1}{1 - k} \ln\left(\frac{1 + q}{k + q}\right).$$

Reverting back to R, we now obtain

$$\int_{R_o}^{R_P} \frac{R_P - R}{k(R_P - R) + R} \cdot \frac{dR}{R} = \left[\frac{1}{k} \ln \frac{R}{k(R_P - R) + R} + \frac{1}{1 - k} \ln \frac{R_P}{k(R_P - R) + R}\right]_{R_o}^{R_P} .$$

Carrying out the indicated end-point evaluation, we find that

$$I_2 = y_P\, K \left\{ \ln \left[\frac{k(R_P - R_o) + R_o}{R_o}\right]^{1/k} \left[\frac{k(R_P - R_o) + R_o}{R_P}\right]^{1/(1 - k)} \right\}, \qquad \text{(III-30)}$$

and hence, I_2 has been evaluated.

In conclusion, with (III-28) and (III-30), we have from (III-23) that

$$\begin{aligned} \text{min. } D(x_P, y_P) \geqslant{} & (1 - y_P)\, D(x_P^*, 0) \\ & + y_P\, K \ln \left[\frac{k(R_P \; 0 \; R_o)}{R_o}\right]^{1/k} \left[\frac{k(R_P - R_o) + R_o}{R_P}\right]^{1/(1 - k)}, \end{aligned}$$

where:

$x_P^* = x_P/(1 - y_P)$,

$R_P = x_P/(1 - x_P - y_P)$,

which may be used to compute the lower bound for $D(x_P, y_P)$.

APPENDIX IV

THE CONJUGATE MATCHED ABUNDANCE RATIO CASCADE

A. Introduction

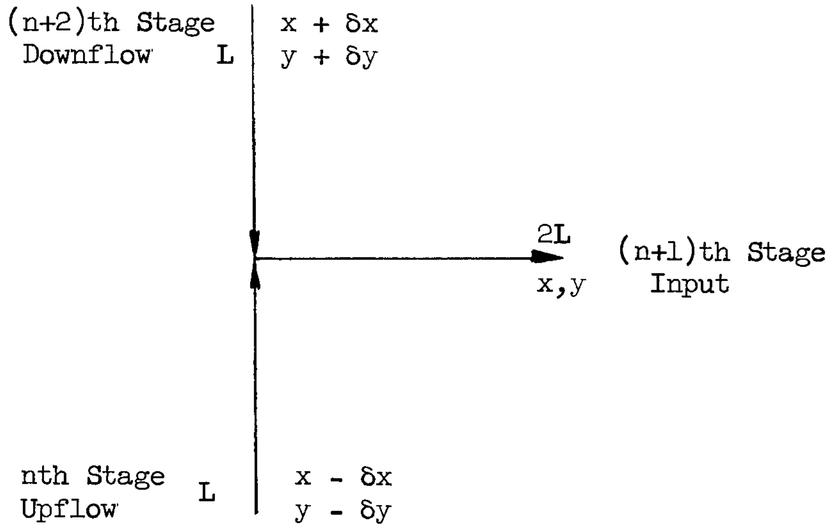
In Appendix I, a three component value function $\mathcal{V}(x,y)$ was developed with the property that if materials of the same abundance ratio R are mixed, value is conserved. This property of $\mathcal{V}(x,y)$ immediately led to the matched R cascade discussed in the text. In the matched R cascade, only streams of the same R abundance ratio are mixed, and consequently, separative work is everywhere conserved. For a cascade such as that of text Figure 2 - supposing it to be a matched R cascade - we then have that

$$\sum_{\text{cascade}} L\,\psi_1^{\,2}/4 = P\mathcal{V}(x_P,y_P) + W\mathcal{V}(x_W,y_W) - F\mathcal{V}(x_F,y_F)\ . \qquad \text{(IV-1)}$$

In this appendix, we will show that the matched R cascade is not the only cascade with the properties that separative work is everywhere conserved and that a value balance on the cascade with the developed value function $\mathcal{V}(x,y)$ nets to the cascade separative work. We do this by developing a *local* match function $m(x,y)$ which may be used to match interstage flows in the formation of the cascade. Feed streams to the cascade, however, must still be matched with the abundance ratio R. We call the resulting cascade a *conjugate matched R cascade*.

B. Mixing of Two Equal Streams Having Small Concentration Differences

A stage link consists of the n-th stage up-flow meeting with the (n+2)-th stage down-flow to form the input to the (n+1)-th stage. If we neglect the *net* up-flow with respect to the inter-stage flow, a stage link can be shown as follows:



Schematic of a Stage Link

Figure IV-1

We want to know how to make the above stage link so that there is no mixing loss in the sense of the developed value function $\mathcal{V}(x,y)$. Thus, we want

$$L\mathcal{V}(x + \delta x,\ y + \delta y) + L\mathcal{V}(x - \delta x,\ y - \delta y) - 2L\mathcal{V}(x,y) = 0. \tag{IV-2}$$

We now remember that the partial differential equation satisfied by $\mathcal{V}(x,y)$ is a stage value balance correct to second order terms in g and h, which are the concentration differences effected by the stage. Let us then content ourselves to have (IV-2) correct to second order terms in δx and δy which are of the same order of magnitude as g and h. Expanding $\mathcal{V}(x + \delta x,\ y + \delta y)$ and $\mathcal{V}(x - \delta x,\ y - \delta y)$ about $\mathcal{V}(x,y)$, we then obtain in place of (IV-2)

$$(\delta x)^2\,\mathcal{V}_{xx} + 2\delta x\delta y\,\mathcal{V}_{xy} + (\delta y)^2\,\mathcal{V}_{yy} = 0. \tag{IV-3}$$

Let us now write

$$\mathcal{V}_{xx} + 2(dy/dx)\,\mathcal{V}_{xy} + (dy/dx)^2\,\mathcal{V}_{yy} = 0. \tag{IV-4}$$

Since $\mathcal{V}(x,y)$ is given, we consider (IV-4) as an ordinary differential equation. We may then solve for (dy/dx), and on integration, we obtain two solutions, corresponding to the two roots, $(dy/dx)_1$ and $(dy/dx)_2$, of the indicated quadratic. Let these two solutions be:

$$\begin{aligned} \mathcal{M}_1(x,y) &= C_1 \ , \\ \mathcal{M}_2(x,y) &= C_2 \ , \end{aligned} \qquad \text{(IV-5)}$$

where C_1 and C_2 are constants. We then have that

$$\begin{aligned} (dy/dx)_1 &= -\mathcal{M}_{1x}/\mathcal{M}_{1y} \ , \\ (dy/dx)_2 &= -\mathcal{M}_{2x}/\mathcal{M}_{2y} \ . \end{aligned} \qquad \text{(IV-6)}$$

Suppose for the moment that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are <u>real</u> functions of x and y. It is then seen that $\mathcal{M}_1$ and $\mathcal{M}_2$ may be considered to have <u>local</u> matching properties. If in Figure IV-1, we join the two streams at the stage link such that, say,

$$\mathcal{M}_1(x + \delta x,\ y + \delta y) = \mathcal{M}_1(x - \delta x,\ y - \delta y),$$

we then have that

$$\delta y/\delta x = -\ \mathcal{M}_{1x}/\mathcal{M}_{1y} \ , \qquad \text{(IV-7)}$$

provided δx and δy are small. On comparing (IV-6) and (IV-7), we see that δy/δx then satisfies (IV-4) and, hence, (IV-2). Therefore, when we match at the stage links with either $\mathcal{M}_1$ or $\mathcal{M}_2$, we conserve value in the sense of the developed value function $\mathcal{V}(x,y)$ to order ψ_1^2.

We supposed above that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are real functions, and this is now easily established.

Consider that at constant abundance ratio

$$R = x/(1 - x - y),$$

our developed value function $\mathcal{V}(x,y)$ is a linear form in x and y, and hence, for

$$R(x + \delta x,\ y + \delta y) = R(x - \delta x,\ y - \delta y),$$

(IV-2) is exactly satisfied. Therefore,

$$(dy/dx)_1 = - R_x/R_y \, , \qquad \text{(IV-8)}$$

is a root of the quadratic (IV-4). Since $(dy/dx)_1$ is real, $(dy/dx)_2$ must then also be real. Since $(dy/dx)_1$ is known, $(dy/dx)_2$ is immediately obtained from the relation that the product of the two roots of a quadratic equals the constant term. Hence, we have from (IV-4) and (IV-8) that

$$(dy/dx)_2 = -(R_y/R_x)(\mathcal{V}_{xx}/\mathcal{V}_{yy}). \qquad \text{(IV-9)}$$

That the above $(dy/dx)_2$ is not always equal to $(dy/dx)_1$, corresponding to the R abundance ratio, may be verified directly from (IV-8) and (IV-9), but this conclusion follows from a theorem in differential geometry on developable surfaces.

We have now found two match functions:

$$\mathcal{M}_1(x,y) = R,$$

and

$$\mathcal{M}_2(x,y) = m(x,y), \text{ the solution of (IV-9).}$$

The abundance ratio R, as we have seen before, has the property that streams may be matched with value being conserved regardless of whether the concentrations of the streams have small differences or not. On the other hand, m(x,y) may be used to match streams only when the concentrations of the streams have small differences; when inter-stage flows are thus matched, value is conserved to order ψ_1^2, which is acceptable accuracy. For these reasons we call R a broad match function and m a local match function. We now proceed to show how the local match function may be used in forming a cascade in which separative work is everywhere conserved.

C. The Conjugate Matched R Cascade

Consider stages in a cascade as shown in text Figure 6, and suppose the stages to be linked with our local match function; thus:

$$m(x_{n+2}, y_{n+2}) = m(x_n^*, y_n^*) \, . \qquad \text{(IV-10)}$$

Since the concentration differences are small, we have that

$$\begin{aligned} m(x_n^*, y_n^*) &= m(x_n, y_n) + (x_n^* - x_n)\, m_x + (y_n^* - y_n)\, m_y\,, \\ m(x_{n+2}, y_{n+2}) &= m(x_n, y_n) + (x_{n+2} - x_n)\, m_x + (y_{n+2} - y_n)\, m_y, \end{aligned} \tag{IV-11}$$

and

$$\begin{aligned} x_{n+2} - x_n &= 2(dx/dn), \\ y_{n+2} - y_n &= 2(dy/dn). \end{aligned} \tag{IV-12}$$

Since the n-th stage concentration increments are

$$\begin{aligned} x_n^* - x_n &= g, \\ y_n^* - y_n &= h, \end{aligned}$$

with g and h as before, we obtain from (IV-10, 11, 12),

$$\left[(dx/dn) - g/2\right] m_x + \left[(dy/dn) - h/2\right] m_y = 0. \tag{IV-13}$$

Supposing the considered stages to be in a simple enricher with a single product withdrawal P at concentrations (x_P, y_P), we have from text (29) that

$$\begin{aligned} dx/dn &= g - \left[P(x_P - x)/L\right]\,, \\ dy/dn &= h - \left[P(y_P - y)/L\right]\,. \end{aligned} \tag{IV-14}$$

From (IV-13) and (IV-14), we may then solve for L; thus:

$$L = 2P\,\frac{(x_P - x)\, m_x + (y_P - y)\, m_y}{g\, m_x + h\, m_y}\,. \tag{IV-15}$$

Returning to (IV-9), we see that for the local match function

$$(m_x/m_y) = (R_y/R_x)\,(\mathcal{V}_{xx}/\mathcal{V}_{yy}). \tag{IV-16}$$

Substituting (IV-16) in (IV-15), we obtain

$$L = 2P\,\frac{(x_P - x)\, R_y \mathcal{V}_{xx} + (y_P - y)\, R_x \mathcal{V}_{yy}}{g\, R_y \mathcal{V}_{xx} + h\, R_x \mathcal{V}_{yy}}\,. \tag{IV-17}$$

With (IV-14) and (IV-17), and similar equations for a stripper, we are then able to compute by the usual "run-down" calculations the x and y concentration gradients and the interstage flows. *For separative work to be conserved everywhere in the cascade, the feed F is matched to the cascade with the R abundance ratio.* The resulting cascade we call the conjugate matched R cascade. It has the property that

$$\frac{1}{4}\sum_{\text{cascade}} L\,\psi_1^2 = P\mathcal{V}(x_P,y_P) + W\mathcal{V}(x_W,y_W) - F\mathcal{V}(x_F,y_F),$$

correct to order ψ_1^2.

The conjugate matched R cascade is of particular interest because it shows that cascade value balances with our value function net to the cascade separative work for other modes of cascade operation than the matched R one. Indeed, for these purposes it is applicable to any cascade whose stages are linked with either m and/or R, and not all stages must be linked with the same match function. Feeds, however, must always be matched with R. It is also of importance to note that since our developed unit cost scale is of the form

$$D(x,y) = K\mathcal{V}(x,y),$$

our cost scale is also applicable to a variety of cascade operations - which is certainly a desirable property for a cost scale.

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