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A STUDY OF THE STABILITY OF A RELATIVISTIC PARTICLE BEAM PASSING THROUGH A PLASMA

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A STUDY OF THE STABILITY OF A RELATIVISTIC PARTICLE BEAM PASSING THROUGH A PLASMA

by

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ABSTRACT

The dispersion law is derived for small amplitude disturbances of the spatially non-uniform steady state configuration of a relativistic particle beam of finite cross section and infinite length passing through a low temperature dense plasma. First, a macroscopic analysis is given in which Maxwell's equations are supplemented by fluid equations for the beam and plasma effects are accounted for by means of a scalar conductivity. A more realistic treatment of the plasma is then obtained by introducing a variable tensor conductivity and appropriate boundary conditions, permitting the effects of Hall currents, density and temperature gradients, and metal walls to be assessed. Finally, the analysis is refined by treating the beam particles by means of

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the collisionless Boltzmann equation while maintaining the scalar conductivity description of the plasma. Use of the orbit integral technique for solving the Boltzmann equation permits the perturbed beam current to be expressed as an integral over the perturbed field variables, and the relativistic dynamics and the geometry of the configuration greatly increase the tractability of the expressions. Introduction of appropriate Hankel transforms of the field variables leads to an integral form for Maxwell's equations and to the expression of the stability problem as a set of three linear, coupled integral equations. A formal solution of these equations is given, and the dispersion relation is seen to appear as a solvability condition for the equations. Asymptotic evaluations of the formal expressions are given for the case of low frequency, long wavelength disturbances and high frequency, highly localized disturbances.

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Chapter 1

INTRODUCTION

Discussion of the Stability Problem

The extensive published literature on the stability of plasma-field configurations [1,2] contains relatively few topics that have been completely analyzed. The subject is more difficult, both experimentally and theoretically, than hydrodynamic stability problems and has been less fully explored. One consequence of this is that the major theoretical progress in plasma stability problems has come through improved formalisms which facilitate the posing of questions simple enough to be treated analytically, rather than through more elaborate numerical analyses aimed at a complete description of perturbed flows. The principal achievements have been to analyze the small amplitude behavior of perturbations via normal mode (Laplace transform) techniques [3] and complex variable theory, [4-7] to find variational principles suitable for the simpler problem of whether or not a configuration is stable, [8-12] and to use asymptotic methods, both as a means of simplifying complicated equations [11,13,14] and of extracting the approximate content of formal solutions to equations. [3,15]

One additional theoretical difficulty is that the presence of instabilities may alter the form of the governing equations. [16]

However, unstable modes which develop more rapidly than particle collision rates should be well described, at least in the linear regime, by the coupled set of Maxwell's equations and collisionless Boltzmann equations. That is, the particles are described by a distribution function f_a for each species, the fields are described by $\underset{\sim}{E}$ and $\underset{\sim}{B}$, and the system satisfies the equations

$$\frac{\partial}{\partial t} f_a + \underset{\sim}{v} \cdot \nabla f_a + \frac{d}{dt}\underset{\sim}{p} \cdot \nabla_p f_a = 0,$$

$$\nabla \cdot \underset{\sim}{E} = 4\pi\rho, \qquad \nabla \cdot \underset{\sim}{B} = 0,$$

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{B} \quad \text{and} \quad \nabla \times \underset{\sim}{B} = 4\pi \underset{\sim}{j} + \frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{E},$$

where

$$\frac{d}{dt}\underset{\sim}{p} = e_a\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{B}\right)$$

$$\rho = \sum_a e_a \int d^3p\, f_a$$

and

$$\underset{\sim}{j} = \sum_a \frac{e_a}{c} \int d^3p\, \underset{\sim}{v}\, f_a$$

In principle, these equations are easy to solve. The distribution functions are constant along particle trajectories, so initial values suffice to determine the functions at later times. With the functions known, the charge density and current density may be computed and Maxwell's equations solved. The difficulty is that the computed fields must agree with the fields used to find f_a. Thus the task is to find non-linear, self-consistent solutions of the set of equations. In general, not enough is known about particle orbits to permit many solutions to be exhibited.

For stability analyses the problem is simplified by solving the full set of equations for only the undisturbed state of the system and using linearized equations to describe departures from that state. The perturbed distribution functions are then obtained as linear functionals of the perturbed fields, specifically as time integrals of the perturbed fields which are evaluated along the unperturbed particle orbits, and Maxwell's equations may be formulated as a set of linear integro-differential equations or as a set of purely integral equations. The solution of Maxwell's equations then determines the stability of the system. Of course, lack of knowledge of particle orbits still severely limits what can be done with this formalism, but, once the unperturbed particle orbits are known, no additional orbits need be found for the stability analysis. The major problems for the linear stability theory are to find solutions for the unperturbed configuration which are sufficiently close to physical situations of interest and, with these solutions in hand, to analyze Maxwell's equations for the perturbed fields.

At present it has been possible to complete these tasks only for infinite, uniform plasma geometries, with and without an external magnetic field present. This work will examine a simple, non-uniform geometry produced by a particle beam passing through a plasma. The configuration forms a natural generalization of the extensively studied [15,17-20] case of a plasma in a uniform external magnetic field. The beam will be assumed to be highly relativistic, so that an approximate form of the relativistic equations of motion may be used to simplify the particle dynamics, and composed of electrons, although other particle species can be treated with nominal modifications in the formalism. As noted above, the beam particles will be described by a collisionless Boltzmann equation. However, the plasma - electrons, ions, and neutrals - will be described by simple macroscopic fluid equations. The problem is simply too complex when both the plasma and the beam are treated microscopically. Moreover, some of the most interesting modes occur at frequencies which are much larger than the collision rates for beam particles but much smaller than plasma collision rates. These modes are driven by plasma collisions and cannot be described by collisionless plasma equations.

The equilibrium configuration is assumed to have symmetry and be invariant to translations in the z direction, the direction of beam motion. The plasma is taken to have an arbitrary degree of ionization but to be sufficiently dense to neutralize the beam without appreciable disturbance to its homogeneity. The beam particles are assumed to satisfy the approximate relativistic equations of motion

$$\frac{d}{dt}\underline{v}_\perp = -\frac{e}{m\gamma_0}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)_\perp$$

and

$$\frac{d}{dt}v_z = -\frac{e}{m\gamma_0^3}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)_z$$

where
$$\gamma_0 = \left(1 - u^2/c^2\right)^{-1/2}$$

and u is the average beam velocity. During most of the work, it will be assumed that the z velocity of the beam particles is constant, and it is this approximation which permits a simple description of the particle orbits.

A substantial body of this work is devoted to a purely macroscopic analysis of low frequency, long wavelength disturbances in which fluid equations rather than Boltzmann's equation are used to describe the beam particles. This provides an appropriate and simple level of description for these modes, since they are essentially macroscopic in nature. It also provides a limiting case for the microscopic analysis. Finally, it provides a tractable model for the assessment of tensor effects in the plasma conductivity. These are shown to be small, justifying to some degree the use of a scalar plasma conductivity throughout the microscopic analysis. However, it is desired to treat a wide range of frequencies, so an arbitrary ratio of conduction to polarization current is permitted and a model is given relating this ratio to plasma conditions.

In the unperturbed configuration, the self-magnetic field of the beam constrains the beam particles to move in betatron orbits, and finite density changes occur over one orbit diameter. In consequence, particle motion must be treated very carefully during the analysis of disturbances. This analysis, culminating in a formal solution of Maxwell's equations and a formal expression for the dispersion relation, constitutes the principal result of this work. Unfortunately, it has not been possible to extract the explicit content of these expressions for all modes of disturbance. Asymptotic methods are therefore used to discuss limiting cases.

For low frequency, long wavelength disturbances, the equations yield the macroscopic results plus small correction terms. The modes are driven by the finite plasma conductivity and have the characteristic behavior $\omega \sim \eta$, where η is the plasma resistivity, of the non-localized, finite conductivity instabilities found by Furth, Rosenbluth, and Killeen.[54] For high frequency, highly localized disturbances, essentially electrostatic instabilities are found. However, the self-consistent treatment of particle orbits yields corrections to analyses treating the beam motion as straight line orbits.[21-24,55] In particular, certain low frequency instabilities predicted by Watson, Bludman, and Rosenbluth[24] are shown not to occur, and a more stringent condition for the validity of the analysis is found. Not as much information on the relation between WKB solutions for highly localized disturbances and the exact solutions has been found here as in the work of Frieman, Goldberger, Watson, Weinberg, and Rosenbluth [55] on purely straight line orbits. The extension of their analysis to the present self-consistent treatment of particle orbits still forms a highly worthwhile problem.

Survey of Relativistic Streaming Phenomena

Under normal circumstances plasmas occurring in nature have particle distribution functions which are approximately Maxwellian in form. This form is maintained by collisional processes and has a natural generalization when relativistic thermal energies are involved. In particular, anisotropies in the distribution of particle velocities are inhibited by collisions. The present work deals with the streaming of relativistic electrons through a plasma of moderate temperature - a highly anisotropic configuration. The study of this configuration is suggested by a review of plasma stability theory, but the relation of this problem to natural phenomena remains to be discussed. For this purpose a brief description will be given of several situations in which relativistic streaming of particles does occur and in which the persistence or stability of the streaming motion is of importance to the observed phenomena.

When relativistic particle streaming occurs in nature, some mechanism is required to accelerate the particles. Theoretical attempts to account for the observed distribution of cosmic ray particles have put forward a number of mechanisms in which the acceleration of particles occurs as a result of field configurations set up by large scale events. For an observer in general position, any local anisotropies produced by such acceleration mechanisms average out, but the earth is not in general position. It records substantial increases in cosmic ray intensity during periods of solar activity. The theory of discharges at neutral points [25] seems able to explain the connection between solar flares and production of cosmic rays but provides too many particles if they remain localized in a beam. Presumably, the particles are scattered at the solar surface via instabilities or in interplanetary space by turbulent magnetic fields.

The process of electron "runaway" seems to be of common occurrence in 100 ev - 1 Kev plasmas[26-29] and is typically a by-product of time dependent magnetic fields. The induced electric field accelerates electrons along B lines, while Coulomb collisions tend to randomize this directed energy. An analysis [30] predicts that some electrons will diffuse to a region of velocity space where Coulomb encounters are no longer significant and then will "run away". When the electric field strength exceeds a certain "critical" value, the runaway occurs at once and is called "strong runaway". The detection of 100 Kev to 1 Mev x rays indicates that electrons can reach weakly relativistic energies before escaping from the experimental chamber. The runaway electrons may be aided in their escape by instabilities. Unpublished calculations suggest that velocity space instabilities should be produced by the anisotropic electron distribution.

The runaway process can also be used deliberately, and can form the basis of the design of a plasma betatron.[31,32] Such a machine has been run by the CERN group[31] but apparently has not yet operated in the strong runaway regime necessary for its efficient use as a betatron. These and other high-current plasma accelerator concepts[33,34] deal with neutralized, self-constricted beams of relativistic electrons and lead naturally to studies of the formation, equilibrium properties, and stability of such configurations.[34-39] Definitive theoretical analyses have not been given, but instabilities appear likely to occur while the configuration is being formed, as well as afterward in the steady state. Although the present work is concerned with an electron beam passing through a plasma and not with a beam neutralized by ions only, certain localized high-frequency disturbances which are analyzed here also describe behavior of the latter configuration and supplement previous stability analyses.[34,36,39]

The Astron concept of a thermonuclear reactor[40] is another configuration involving the streaming of relativistic electrons. Here the electrons are injected into a magnetic mirror geometry, forming a current sheet or E-layer which gives rise to a system of closed B lines in the experimental chamber. A cold gas is then brought into the chamber. It is then heated to thermonuclear temperatures by collisions with the E-layer and confined by the magnetic field. Experiments are being designed to test the concept but have not been completed. A considerable amount of theoretical work on the steady state configuration and its stability properties has been done, however. The work encounters severe mathematical problems but simplified treatments have given encouraging results.

An experiment which approximates the geometry of the present work has been proposed for the Astron electron accelerator.[41] The electron beam would be extracted from the accelerator, focused through self-magnetic forces, and passed through a large experimental chamber filled with plasma. If an approximately steady state configuration could be attained in the chamber, its stability properties would be observed. The experiment should facilitate comparison of observed and calculated instability growth rates and give observations on the non-linear behavior of disturbances which are inaccessible to theory. It is to be expected that the simplest models of plasma and beam dynamics supplemented by boundary conditions should suffice for the calculation of growth rates of many modes of disturbance from the steady state, since this result is found in the present work for the case of an infinite plasma.

In general, a theoretical analysis of the stability of relativistic streaming phenomena occurring in nature is too difficult to carry out in detail. The present work which analyzes disturbances from a steady state contains many features of the proposed

Astron accelerator experiment, yet it considers an idealized geometry and can give no description of the experimentally important processes of the self-constriction of the beam and the entry of the beam into the plasma. Thus this work cannot aim at giving a complete description of any experiment. Its utility lies in its contribution to plasma stability theory of a model which can be rather fully worked out and which permits the evaluation of the effects of individual particle trajectories on a wide class of modes of disturbance.

Chapter 2

LOW FREQUENCY BEAM INSTABILITIES

Plasma Conductivity Law

The formulation of the present stability problem as an explicit set of equations for the perturbed field components requires that the perturbed beam current and the plasma current be expressed in terms of the perturbed fields and steady state parameters. The object of this section is to derive such expressions for the plasma current.

The derivation is based on a simple model of the plasma dynamics and is carried out in the rest frame of the steady state plasma. This frame, which is also the laboratory frame of reference, is used throughout the stability analysis. The plasma is composed of electrons with mass m, singly charged ions with mass M, and neutral particles with mass M_g. Gradients in the density of these species are required to maintain electrical neutrality in the steady state, but are ignored in the present discussion. This approximation restricts the admissible ranges of steady state particle densities. The neutral particle density N may be of arbitrary magnitude, but the electron or ion density $\bar{n}$ must be much greater than the beam density

n_o.

Perturbed motions of the plasma species are described by means of linearized, pressureless hydrodynamic equations in which momentum transfer between distinct species is accounted for by phenomenological collision terms. Flexibility is given to the model by three arbitrary parameters which permit arbitrary collision frequencies between particles of distinct frequencies. When all magnetic forces are ignored, the model gives a complex scalar conductivity as the ratio of the plasma current to the perturbed electric field.

All time dependences are taken to be of the form $e^{i\omega t}$ and perturbed quantities are denoted by

$\underline{v}_e$ (electron velocity) $\underline{v}_g$ (neutral velocity)

$\underline{v}_i$ (ion velocity) $\underline{E}$ (electric field).

All magnetic forces are neglected and the linearized momentum transfer equations become

$$
\begin{aligned}
i\omega\bar{n}m\underline{v}_e &= -e\bar{n}\underline{E} - \alpha(\underline{v}_e - \underline{v}_g) - \beta(\underline{v}_e - \underline{v}_i) \\
i\omega\bar{n}M\underline{v}_i &= e\bar{n}\underline{E} - \gamma(\underline{v}_i - \underline{v}_g) - \beta(\underline{v}_i - \underline{v}_e) \\
i\omega N M_g \underline{v}_g &= \alpha(\underline{v}_e - \underline{v}_g) + \gamma(\underline{v}_i - \underline{v}_g),
\end{aligned}
\tag{1}
$$

where α, β, and γ are taken to be constants characteristic of the equilibrium plasma and are proportional to the electron-neutral, electron-ion, and ion-neutral collision frequencies.

It is useful to introduce the definitions

$$\begin{aligned} F &= \alpha + \beta + i\omega\bar{n}m \\ G &= \gamma + \beta + i\omega\bar{n}M \\ H &= \alpha + \gamma + i\omega N M_g . \end{aligned} \qquad (2)$$

Then Eq. (1) is equivalent to

$$\begin{aligned} &H\mathbf{v}_g = \alpha\mathbf{v}_e + \gamma\mathbf{v}_i \\ &FH\mathbf{v}_e - \beta H\mathbf{v}_i = \alpha H\mathbf{v}_g - e\bar{n}H\mathbf{E} \\ &-\beta H\mathbf{v}_e + GH\mathbf{v}_i = \gamma H\mathbf{v}_g + e\bar{n}H\mathbf{E} , \end{aligned} \qquad (3)$$

or

$$\begin{aligned} &(FH-\alpha^2)\mathbf{v}_e - (\beta H + \alpha\gamma)\mathbf{v}_i = -e\bar{n}\mathbf{E} \\ &-(\beta H + \alpha\gamma)\mathbf{v}_e + (GH - \gamma^2)\mathbf{v}_i = e\bar{n}\mathbf{E} . \end{aligned} \qquad (4)$$

Equation (4) may be solved for $\mathbf{v}_e$ and $\mathbf{v}_i$, giving

$$\begin{aligned} \mathbf{v}_i &= e\bar{n}\mathbf{E}\left[\frac{FH - \beta H - \alpha(\alpha+\gamma)}{FGH - (\alpha^2 G + \beta^2 H + \gamma^2 F + 2\alpha\beta\gamma)}\right] \\ \mathbf{v}_e &= -e\bar{n}\mathbf{E}\left[\frac{GH - \beta H - \gamma(\alpha+\gamma)}{FGH - (\alpha^2 G + \beta^2 H + \gamma^2 F + 2\alpha\beta\gamma)}\right] . \end{aligned} \qquad (5)$$

The plasma current density

$$\mathbf{j} = (e\bar{n}/c)(\mathbf{v}_i - \mathbf{v}_e) \qquad (6)$$

may be obtained by substitution from Eq. (5), giving

$$\mathbf{j} = \left(\frac{e^2\bar{n}^2}{c}\right)\left[\frac{H(F+G-2\beta) - (\alpha+\gamma)^2}{FGH - (\alpha^2 G + \beta^2 H + \gamma^2 F + 2\alpha\beta\gamma)}\right]\mathbf{E} . \qquad (7)$$

The conductivity law

$$\mathbf{j} = \sigma\mathbf{E} \qquad (8)$$

may be combined with Eqs. (2) and (7) to yield

$$\sigma = \left[\frac{e^2\bar{n}^2}{c}\right]\left[\frac{(\alpha+\gamma)(\bar{n}[m+M] + NM_g) + i\omega\bar{n}[m+M]NM_g}{(\alpha\beta+\alpha\gamma+\beta\gamma)(\bar{n}[m+M] + NM_g - \omega^2\bar{n}m\bar{n}MNM_g + i\omega(\bar{n}m\bar{n}M[\alpha+\gamma] + \bar{n}mNM_g[\beta+\gamma] + \bar{n}MNM_g[\alpha+\beta])}\right] . \qquad (9)$$

Equation (9) is valid for any non-negative values of α, β, γ, $\bar{n}$, N, M, and M_g, but for later use it is convenient to give several limiting values of this expression which are simpler in form.

Case I: $\alpha = \beta = \gamma = 0$ (collisionless plasma)

Eq. (9) becomes

$$\sigma = \left(\frac{e^2\bar{n}}{mc}\right)\left(\frac{m+M}{M}\right)\frac{1}{i\omega}. \tag{10}$$

Case II: $\alpha = \gamma = N = 0$ (fully ionized plasma)

Eq. (9) becomes

$$\sigma = \left(\frac{e^2\bar{n}}{mc}\right)\left(\frac{m+M}{M}\right)\frac{1}{i\omega+\nu} \tag{11}$$

where

$$\nu = \frac{(m+M)}{\bar{n}mM}\beta.$$

In this case the phase of σ is approximately 0 when $\nu \gg \omega$ and is approximately $-(\pi/2)$ when $\omega \gg \nu$.

Case III: $\gamma \to \infty$ (neutrals move with ions)

Eq. (9) becomes

$$\sigma = \left(\frac{e^2\bar{n}}{mc}\right)\left(\frac{\bar{n}m+\bar{n}M+NM_g}{\bar{n}M+NM_g}\right)\frac{1}{i\omega+\nu} \tag{12}$$

where

$$\nu = \frac{(\bar{n}[m+M]+NM_g)}{\bar{n}m(\bar{n}M+NM_g)}(\alpha+\beta).$$

Here again the phase of σ is approximately 0 when $\nu \gg \omega$ and is approximately $-(\pi/2)$ when $\omega \gg \nu$.

For any positive values of the parameters α, β, and γ Eq. (9) gives $\mathrm{Re}\,\sigma \geq 0$ and $\mathrm{Im}\,\sigma \leq 0$, or

$$-(\pi/2) \leq \text{phase of } \sigma \leq 0. \tag{13}$$

Hence the plasma is resistive and inductive in general.

The effect of the steady state magnetic field, $\underline{B}_0$, on the conductivity tensor will be investigated when $\nu \gg |\omega|$

in Case III. The treatment will be based on the facts that the ions and neutrals are approximately motionless and that $|\omega|$ is small. Thus the current is given by

$$\underset{\sim}{j} = -\frac{e\bar{n}}{c}\underset{\sim}{v}_e, \tag{14}$$

the conductivity is given by

$$\sigma = \left(\frac{e^2\bar{n}}{mc}\right)\frac{1}{\alpha+\beta}, \tag{15}$$

and the equations of motion become

$$0 = -(e\bar{n})\left[\underset{\sim}{E} + (\underset{\sim}{v}_e/c)\times\underset{\sim}{B}_0\right] - (\alpha+\beta)\underset{\sim}{v}_e. \tag{16}$$

Combination of these equations gives

$$\underset{\sim}{j} = \sigma\left[\underset{\sim}{E} - (\underset{\sim}{j}/e\bar{n})\times\underset{\sim}{B}_0\right], \tag{17}$$

which may be solved for $\underset{\sim}{j}$. This is done in cylindrical coordinates for which

$$\underset{\sim}{B}_0 = (0, B_0(r), 0). \tag{18}$$

The solution of Eq. (17) for the components of $\underset{\sim}{j}$ gives for this case

$$\begin{aligned} j_r &= \sigma\frac{(E_r+\mu E_z)}{(1+\mu^2)} \\ j_\theta &= \sigma E_\theta \\ j_z &= \sigma\frac{(E_z-\mu E_r)}{(1+\mu^2)} \end{aligned} \tag{19}$$

where

$$\mu = \left[\sigma B_0(r)/e\bar{n}\right].$$

The description of the plasma current given by Eqs. (9), (10), (11), (12), and (19) is sufficient for the present stability problem. Attention will now be directed to the beam dynamics.

Macroscopic Beam Equations

A macroscopic description of the beam dynamics is used here and in Chapter 3 to study the $m = 1$ disturbances of low frequency and long wavelength. This comparatively simple analysis discusses the instability mechanism of such disturbances and the effect of various plasma conditions on growth rates. The restriction to $m = 1$ is made solely for simplicity; other m values may be treated by similar means. In contrast, the restriction to low frequencies and long wavelengths is essential to the macroscopic treatment. Orbit effects are important for other types of disturbances and are best treated by a collisionless Boltzmann equation.

The model used here is one in which the beam is taken to be a perfect, incompressible fluid subject to electromagnetic body forces, while plasma effects are accounted for by a scalar conductivity. Explicit low frequency approximations are made in Maxwell's equations. The model is used to derive the dispersion law for the disturbances and to obtain a detailed description of the perturbed configuration. The dispersion law will also be derived by methods which do not depend on a specific beam model but which give comparatively little information about the perturbed configuration.

In this section general field equations and jump conditions will be derived for the field variables. A particular solution will be adopted as the steady state solution, and the equations will be linearized about the steady state. All equations will be evaluated in cylindrical coordinates unless otherwise noted, and Gaussian units will be used throughout. The several sets of field variables will be distinguished as follows: general field variables will have primes as superscripts, and steady state variables will have zeros as subscripts. The variables giving the deviations from steady state values will have no labels.

It is convenient to denote the field variables by

$\underset{\sim}{E}'$ = electric field

$\underset{\sim}{B}'$ = magnetic field

$(\underset{\sim}{j})'_p$ = plasma current density

$\underset{\sim}{j}'$ = beam current density

$\underset{\sim}{K}'$ = surface current

$\underset{\sim}{p}'$ = beam momentum

$\underset{\sim}{v}'$ = beam velocity

$\underset{\sim}{n}'$ = surface normal

p' = beam pressure

n' = beam particle density.

The beam is assumed to have a sharp surface so that $\underset{\sim}{n}'$ and $\underset{\sim}{K}'$ are well defined quantities. All steady state current comes from the macroscopic velocity of the beam. Surface currents appear in the perturbed equations because of the small amplitude motion of the beam surface.

The relativistic velocity of the beam causes only minor changes in the form of the hydrodynamic equations. A covariant formalism is not needed and is not used. The usual vector equations need only be supplemented by the relativistic rule connecting momentum and velocity. The charge and rest mass of the electron will be denoted by e and m respectively. The incompressibility condition becomes

$$\nabla \cdot \underset{\sim}{v}' = 0, \tag{20}$$

while the momentum equation becomes

$$n' \frac{d}{dt} \underset{\sim}{p}' = -\nabla p' + \underset{\sim}{j}' \times \underset{\sim}{B}' - en'\underset{\sim}{E}', \tag{21}$$

where

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \underset{\sim}{v}' \cdot \nabla \right) \tag{22}$$

and

$$\underset{\sim}{v}' = c\left[m^2c^2 + (\underset{\sim}{p}' \cdot \underset{\sim}{p}')\right]^{-1/2} \underset{\sim}{p}'. \tag{23}$$

It should be emphasized that these equations contain conditions not likely to be met in practice. Electron-electron collisions will not be sufficiently numerous to keep the pressure tensor in scalar form, and no forces will constrain the beam motion to be strictly incompressible. In fact, the condition of incompressibility depends on the instantaneous transmission of forces and cannot occur in a basic theory of relativistic fluids. The adequacy of these equations is shown by the later microscopic analysis, which is free from these objections.

Charge neutrality will be assumed in Maxwell's equations and displacement currents will be ignored. The equations become

$$\nabla \cdot \underline{E}' = 0 \qquad \nabla \cdot \underline{B}' = 0$$
$$\nabla \times \underline{E}' = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B}' \qquad \nabla \times \underline{B}' - 4\pi(\underline{j})'_p = 4\pi \underline{j}' \tag{24}$$

Both approximations should be valid when the frequencies involved are much smaller than the plasma frequency. The plasma will always be much denser than the beam so that the plasma electrons can maintain charge neutrality without appreciably changing particle densities.

To complete the equations it is necessary to give the relations between particle currents and other field quantities. The plasma is taken to have uniform density and temperature. Magnetic plasma forces will be ignored. Thus the plasma current is given by

$$(\underline{j})'_p = \sigma \underline{E}', \tag{25}$$

where σ is a constant whose value depends on plasma parameters. The beam current is given by

$$\underline{j}' = -\left(\frac{en'}{c}\right)\underline{v}', \tag{26}$$

and the surface current is related to the perturbed velocity at the beam surface. However, it is most convenient to give this relation after the linearized equations have been derived.

At the beam surface the field variables undergo finite discontinuities which are constrained by jump conditions. The jump conditions are obtained as in standard treatments of hydromagnetic equations [42, 43] by integrating the macroscopic equations across the surface. Finite contributions to the integrals come only from the normal component of the gradient operator. In particular, integrals of field variables and their hydrodynamic time derivatives contribute nothing. This implies that relativistic velocities will not alter the jump conditions. It is convenient to label field variables which are evaluated just outside and just inside the surface by the subscripts "out" and "in" and to denote their difference by a bracket. For example, the jump in the magnetic field is denoted by $\underline{B}'_{out} - \underline{B}'_{in} = [\underline{B}']$. The jump conditions become

$$\underset{\sim}{n}' \cdot [\underset{\sim}{E}'] = \underset{\sim}{n}' \cdot [\underset{\sim}{B}'] = \underset{\sim}{n}' \cdot \underset{\sim}{v}'_{in} = 0$$

$$\underset{\sim}{n}' \times [\underset{\sim}{E}'] = 0 \qquad (27)$$

$$\underset{\sim}{n}' \times [\underset{\sim}{B}'] = 4\pi \underset{\sim}{K}'$$

and

$$8\pi p'_{in} \underset{\sim}{n}' + 4\pi \underset{\sim}{K}' \times (\underset{\sim}{B}'_{out} + \underset{\sim}{B}'_{in}) = 0.$$

A final surface condition is given by the equation [43]

$$\frac{d}{dt} \underset{\sim}{n}' = \underset{\sim}{n}' \times [\underset{\sim}{n}' \times (\nabla \underset{\sim}{v}') \cdot \underset{\sim}{n}'] \qquad (28)$$

for the time development of the surface normal.

The steady state configuration is described by an exact solution of the macroscopic equations in which the beam is taken to be an infinite circular cylinder of radius r_o which has a uniform density n_o and a velocity $u \simeq c$ in the z direction. A self-magnetic force acts on the beam and, for steady state conditions, is balanced by a radial pressure gradient. For $r \leq r_o$ the solution is given by

$$\underset{\sim}{E}_o = 0$$

$$\underset{\sim}{B}_o = 2\pi j_o r\,(0, 1, 0)$$

$$(\underset{\sim}{j})_{po} = 0$$

$$
\begin{aligned}
\mathbf{j}_0 &= j_0\,(0, 0, 1) \\
\mathbf{K}_0 &= 0 \\
\mathbf{p}_0 &= m\gamma_0 u\,(0, 0, 1) \\
\mathbf{v}_0 &= u\,(0, 0, 1) \\
\mathbf{n}_0 &= (1, 0, 0) \\
P_0 &= \pi j_0^2 (r_0^2 - r^2)
\end{aligned}
\tag{29}
$$

and

$$n_0 = n_0,$$

where

$$j_0 = -(e n_0 u / c)$$

and

$$\gamma_0 = [1 - (u^2/c^2)]^{-1/2}. \tag{30}$$

When $r > r_0$ the solution is even simpler. The only non-zero field variable is given by

$$\mathbf{B}_0 = (2\pi j_0 r_0^2 / r)(0, 1, 0). \tag{31}$$

The macroscopic equations must be linearized in the deviations of field variables from steady state values. Since the perturbed velocity $\mathbf{v}$ will be small, it is useful to express the perturbed momentum $\mathbf{p}$ in terms of $\mathbf{v}$. Sufficient accuracy is given by a first order expansion of the exact relations

$$\mathbf{p} = m[1 - (w^2/c^2)]^{-1/2}\,\mathbf{w} - \mathbf{p}_0$$

and (32)

$$\underline{w} = \underline{v}_0 + \underline{v}.$$

The bracket is evaluated as

$$[1-(w^2/c^2)]^{-1/2} = \gamma_0 + \gamma_0^3 (u v_z / c^2) \tag{33}$$

and $\underline{p}$ is given by

$$\underline{p} = m\gamma_0 (v_r, v_\theta, \gamma_0^2 v_z). \tag{34}$$

It is also convenient to replace the linearized Eq. (26) by the condition

$$\underline{j} = 0. \tag{35}$$

That is, beam volume currents are taken to be much smaller than the surface current. This approximation will be shown to be valid for long wavelengths. The precise criterion for its validity is that the wavelengths must be much longer than the beam radius. The macroscopic nature of the analysis has already required that the wavelengths must be much longer than the betatron length, and for most cases this restriction is stronger.

The remainder of the linearization is straightforward. Maxwell's equations become

$$\begin{aligned} &\nabla \cdot \underline{E} = 0 &&\nabla \cdot \underline{B} = 0 \\ &\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial}{\partial t}\underline{B} &&\nabla \times \underline{B} - 4\pi\sigma \underline{E} = 0 \end{aligned} \tag{36}$$

The beam normal is constrained by

$$\frac{d}{dt}\underset{\sim}{n} = \underset{\sim}{n}_0 \times \left[\underset{\sim}{n}_0 \times (\nabla \underset{\sim}{v}) \cdot \underset{\sim}{n}_0\right], \tag{37}$$

and the hydrodynamic equations become

$$\nabla \cdot \underset{\sim}{v} = 0$$

$$m\nu_0 \frac{d}{dt} v_r = \frac{\partial}{\partial r} p + (\underset{\sim}{j}_0 \times \underset{\sim}{B})_r - en_0 E_r$$

$$m\nu_0 \frac{d}{dt} v_\theta = -\frac{1}{r}\frac{\partial}{\partial \theta} p + (\underset{\sim}{j}_0 \times \underset{\sim}{B})_\theta - en_0 E_\theta \tag{38}$$

and

$$m\nu_0^3 \frac{d}{dt} v_z = -\frac{\partial}{\partial z} p - en_0 E_z ,$$

where

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial z}\right) .$$

Similarly, the jump conditions are given by

$$\underset{\sim}{n}_0 \cdot [\underset{\sim}{E}] = \underset{\sim}{n}_0 \cdot [\underset{\sim}{B}] = 0$$

$$\underset{\sim}{n}_0 \times [\underset{\sim}{E}] = 0$$

$$\underset{\sim}{v}_{0\,in} \cdot \underset{\sim}{n} + \underset{\sim}{v}_{in} \cdot \underset{\sim}{n}_0 = 0 \tag{39}$$

$$\underset{\sim}{n}_0 \times [\underset{\sim}{B}] = 4\pi \underset{\sim}{K}$$

and

$$8\pi p_{in} \underset{\sim}{n}_0 + 8\pi \underset{\sim}{K} \times \underset{\sim}{B}_{0\,in} = 0.$$

The constraint on the perturbed surface current $\underset{\sim}{K}$ will now be derived. It will be useful to express the constraint in terms of $\underset{\sim}{a}$, the local surface displacement vector. The form of Eqs. (36)-(39) insures that the field variables may be taken to be complex valued functions whose z, θ , and t dependences are contained in the factor

$$\psi = \exp i\left[\omega t + kz + \theta\right]. \tag{40}$$

Either k or ω may be chosen arbitrarily, and the other quantity will be fixed by the dispersion law. As Eq. (40) indicates, only m = 1 disturbances will be analyzed. Physical variables are obtained from the real part of the complex field variables.

Since surface currents are due to the perturbed motion of the beam, the vector $\underset{\sim}{K}$ has the form

$$\underset{\sim}{K} = j_0 a \psi\, (0, 0, 1), \tag{41}$$

and the corresponding value of $\underset{\sim}{a}$ will be deduced. The physical variable Re $\underset{\sim}{K}$ is given by

$$\begin{aligned} \mathrm{Re}\, K_z &= e^{-\mathrm{Im}[\omega t + kz]} j_0 a \cos\left(\theta + \mathrm{Re}[\omega t + kz]\right) \\ &= e^{-\mathrm{Im}[\omega t + kz]} j_0 a \begin{bmatrix} \cos \mathrm{Re}[\omega t + kz] \cos\theta \\ -\sin \mathrm{Re}[\omega t + kz] \sin\theta \end{bmatrix}. \end{aligned} \tag{42}$$

By comparison, displacements of amplitude a in the x and y directions would give

$$\mathrm{Re}\, K_z = j_0 a \cos\theta$$

and (43)

$$\mathrm{Re}\, K_z = j_0 a \sin\theta$$

respectively. Thus in rectangular coordinates Re $\underset{\sim}{a}$ is given by

$$\mathrm{Re}\,\underset{\sim}{a} = a e^{-\mathrm{Im}[\omega t + kz]}(\cos \mathrm{Re}[\omega t + kz], -\sin \mathrm{Re}[\omega t + kz], 0) \tag{44}$$

and $\underset{\sim}{a}$ by

$$\underset{\sim}{a} = a e^{i[\omega t + kz]}(1, i, 0). \tag{45}$$

In cylindrical coordinates Eq. (45) becomes

$$\underset{\sim}{a} = a \psi (1, i, 0), \tag{46}$$

since the relevant unit vectors satisfy

$$[\hat{x} + i\hat{y}] = e^{i\theta}[\hat{r} + i\hat{\theta}]. \tag{47}$$

The beam surface undergoes a helical displacement $\underset{\sim}{a}$ which leaves its cross-section invariant. For real k the displacement travels in the z direction and has a time dependent amplitude.

The beam must move with its surface, giving the constraint

$$\underset{\sim}{v}_{in} = \frac{d}{dt}\underset{\sim}{a}. \tag{48}$$

Eqs. (36)-(39), (41), (46), and (48) provide a complete macroscopic description of the perturbed beam motion. They will be analyzed in subsequent sections.

The Solution of Maxwell's Equations

Long wavelength solutions of the form (40) will be obtained here for Maxwell's equations and the corresponding jump conditions. It is convenient to incorporate the surface current (41) into the curl equations, obtaining

$$-\left(\frac{i\omega}{c}\right)\underset{\sim}{B} = \nabla\times\underset{\sim}{E}$$

and (49)

$$\nabla\times\underset{\sim}{B} - 4\pi\sigma\underset{\sim}{E} = 4\pi j_0 a\psi\,\delta(r-r_0)(0,0,1).$$

These equations may be combined to give

$$\nabla\times(\nabla\times\underset{\sim}{E}) + \left(\frac{4\pi i\omega\sigma}{c}\right)\underset{\sim}{E} = -\left(\frac{4\pi i\omega j_0 a}{c}\right)\psi\,\delta(r-r_0)(0,0,1). \quad (50)$$

When Eq. (50) is expressed in terms of the auxiliary vector $\underset{\sim}{f}(r)$ defined by

$$\underset{\sim}{E} = \psi\,\underset{\sim}{f}(r), \quad (51)$$

it becomes a set of ordinary differential equations. Its z component is

$$\left(\begin{array}{c} -\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - h^2\right] f_z \\ + ik\left[\frac{1}{r}\frac{d}{dr} r f_r(r) + \frac{i}{r} f_\theta\right] \end{array}\right) = -\left(\frac{4\pi i\omega j_0 a}{c}\right)\delta(r-r_0), \quad (52)$$

where the parameter h is defined by

$$h^2 = \left(\frac{4\pi i\omega\sigma}{c}\right)$$

and (53)

$$\mathrm{Re}\, h > 0.$$

For long wavelengths (small k) Eq. (52) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - h^2\right] f_z(r) = \left(\frac{4\pi i\omega j_0 r_0 a}{c}\right)\frac{1}{r_0}\,\delta(r-r_0). \quad (54)$$

Examination of the components of Eq. (50) shows that this approximation is valid for wavelengths much longer than the beam radius r_o. Hence no new restriction on wavelengths is imposed by the transition from Eq. (52) to Eq. (54).

The set of equations

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-h^2\right]g_n(r,r_o)=\frac{1}{r_o}\,\delta(r-r_o), \tag{55}$$

where $n=0, 1, 2, \ldots,$ are treated in detail in Appendix I. This information may be used to determine $f_z(r)$, since

$$f_z(r)=\left(\frac{4\pi i\omega j_o r_o a}{c}\right)g_1(r,r_o). \tag{56}$$

For later use it suffices to give $f_z(r)$ in the region $r\leq r_o$ occupied by the unperturbed plasma. For this region Eq. (A-12) of Appendix I gives

$$g_1(r,r_o)=-(i\pi/2)\,H_1^1(ihr_o)\,J_1(ihr) \tag{57}$$

and Eq. (56) gives

$$f_z(r)=\pi r_o(2\pi\omega j_o a/c)\,H_1^1(ihr_o)\,J_1(ihr). \tag{58}$$

The same small k approximation gives for the r and θ components of Eq. (50)

$$\frac{i}{r}\left[\frac{1}{r}\frac{d}{dr}rf_\theta(r)-\frac{i}{r}f_r(r)\right]+h^2f_r(r)=0$$

and (59)

$$-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}rf_\theta(r)-\frac{i}{r}f_r(r)\right]+h^2f_\theta(r)=0.$$

The only admissible solution of these equations is

$$f_r(r)=0$$

and (60)

$$f_\theta(r)=0$$

for all r.

The perturbed electromagnetic fields are given in terms of $f_z(r)$ by

$$\underset{\sim}{E} = \psi\left(0, 0, f_z(r)\right)$$

and (61)

$$\underset{\sim}{B} = \left(\frac{c}{i\omega}\right)\psi\left(-\frac{i}{r}f_z(r), \frac{d}{dr}f_z(r), 0\right).$$

More explicitly, for $r \leq r_0$ the electromagnetic fields are given by

$$\underset{\sim}{E} = \left(\frac{\pi\omega r_0}{c}\right)\left(2\pi j_0 a\right)H_1'\left(ihr_0\right)\psi\left(0, 0, J_1(ihr)\right) \quad (62)$$

and

$$\underset{\sim}{B} = -i\pi r_0\left(2\pi j_0 a\right)H_1'\left(ihr_0\right)\psi\left(-\frac{i}{r}J_1(ihr), \frac{d}{dr}J_1(ihr), 0\right), \quad (63)$$

while the surface current is given by

$$\underset{\sim}{K} = j_0 a \psi\left(0, 0, 1\right). \quad (64)$$

This completes the derivation of the perturbed electromagnetic fields.

Solution of the Hydrodynamic Equations

This section continues the analysis of low frequency, long wavelength disturbances. Approximate solutions of the hydrodynamic equations are obtained here. The r dependence of each field variable is denoted by the symbol for the variable itself, while the z, θ, and t dependences are contained in the factor ψ. For example, $\underset{\sim}{v} = \psi \underset{\sim}{v}(r)$. Since the currents and the electromagnetic fields are known, the hydrodynamic equations contain only $\underset{\sim}{v}(r)$ and p(r) as variables.

The electromagnetic body forces are given by

$$\underset{\sim}{j_0} \times \underset{\sim}{B} = \left(\frac{ic}{\omega} j_0\right) \gamma \left(\frac{d}{dr} f_z(r), \frac{i}{r} f_z(r), 0\right)$$

and (65)

$$-en_0 \underset{\sim}{E} = -(en_0)\gamma\left(0, 0, f_z(r)\right),$$

where $f_z(r)$ is determined by Eq. (58). Thus the momentum equations become

$$i(\omega + ku)n_0 m \gamma_0 v_r(r) = -\frac{d}{dr} p(r) + \left(\frac{ic}{\omega} j_0\right)\frac{d}{dr} f_z(r),$$

$$i(\omega + ku)n_0 m \gamma_0 v_\theta(r) = -\frac{i}{r} p(r) + \left(\frac{ic}{\omega} j_0\right)\frac{i}{r} f_z(r), \qquad (66)$$

and

$$i(\omega + ku)n_0 m \gamma_0^3 v_z(r) = -ikp(r) - (en_0) f_z(r). \qquad (67)$$

For these equations the inequalities

$$|\omega r_0| << u$$

and (68)

$$|kr_0| << 1$$

define the low frequency, long wavelength region. In this region the velocity $v_z(r)$ is much smaller than the transverse velocity and its precise behavior is not important. It is therefore permissible to replace Eq. (67) by the more convenient equation

$$i(\omega+ku)\,n_0 m\gamma_0 v_z(r) = -ikp(r) + \left(\frac{ic}{\omega} j_0\right) ikf_z(r), \tag{69}$$

which also gives small $v_z(r)$.

Eqs. (66) and (69) become in vector notation

$$i(\omega+ku)\,n_0 m\gamma_0 \underset{\sim}{v} = -\nabla_{\perp} p(r) + \nabla\left(\frac{ic}{\omega} j_0\right)_{\perp} f_z(r). \tag{70}$$

This equation may be written more simply as

$$\underset{\sim}{v} = \nabla_{\perp} \phi(r), \tag{71}$$

where the auxiliary variable $\phi(r)$ is defined by

$$\phi(r) = \left[n_0 m\gamma_0(\omega+ku)\right]^{-1}\left[ip(r) + \left(\frac{c}{\omega} j_0\right) f_z(r)\right]. \tag{72}$$

Eq. (72) relates the functions $\phi(r)$ and $p(r)$, but either one may be chosen arbitrarily. No additional constraint is provided by Eq. (71), which is satisfied for all $\phi(r)$. Much of this arbitrariness is removed by the incompressibility condition

$$\nabla^2_{\perp} \phi(r) = 0. \tag{73}$$

This equation has two solutions, but only one of them,

$$\phi = AJ_1(ikr), \tag{74}$$

satisfies the necessary regularity conditions at $r=0$. Since $|kr_0| \ll 1$, an adequate approximation to Eq. (74) is given by

$$\phi(r) = Cr, \tag{75}$$

and $\underset{\sim}{v}$ is given adequately by

$$\underset{\sim}{v} = c\psi(1, i, 0). \tag{76}$$

The constant $\hat{A}$ may be determined from the boundary conditions (46) and (48). Equation (48) gives

$$c\psi(1, i, 0) = i(\omega + ku)a\psi(1, i, 0), \tag{77}$$

or

$$c = i(\omega + ku)a. \tag{78}$$

The velocity field takes the simple form

$$\begin{aligned} \underset{\sim}{v} &= i(\omega + ku)a\psi(1, i, 0) \\ &= \frac{d}{dt}\underset{-}{a}, \end{aligned} \tag{79}$$

corresponding to a rigid helical displacement of the entire beam, and the pressure is given by

$$p = a\psi\left[n_0 m \gamma_0 (\omega + ku)^2 r + (2\pi j_0^2 r_0) i\pi H_1^1(ihr_0) J_1(ihr)\right]. \tag{80}$$

Thus the hydrodynamic variables and the electromagnetic fields have been obtained without using the full set of linearized equations. The remaining equations furnish constraints which must be examined for consistency.

Equation (37) for $\underset{\sim}{n}$ may be written as

$$i(\omega + ku)\underset{\sim}{n} = \psi\left(0, \frac{1}{r} v_\theta(r) - \frac{i}{r} v_r(r), -ik v_r(r)\right), \tag{81}$$

while the remaining two jump conditions become

$$u n_z(r_0) + v_r(r_0) = 0 \tag{82}$$

and

$$p(r_0) - a(2\pi j_0^{\,2} r_0) = 0. \tag{83}$$

These equations complete the set of linearized macroscopic equations. Eq. (81) gives

$$\underset{\sim}{n} = -\left[\frac{k v_r(r)}{(\omega + ku)}\right] \psi(0, 0, 1), \tag{84}$$

so that Eq. (82) becomes

$$-\left[\frac{ku}{(\omega + ku)}\right] v_r(r_0) + v_r(r_0) = 0. \tag{85}$$

This equation is consistent only if

$$|\omega| << |ku|, \tag{86}$$

and provides a more severe restriction than the earlier condition

$$|\omega r_0| << u. \tag{87}$$

Thus Eq. (86) will be used to define the low frequency region.

Eq. (83) gives the dispersion law for the disturbances. The equation takes the form

$$\left\{\begin{array}{l} a\psi\left[n_0 m \gamma_0 (\omega + ku)^2 r_0 + (2\pi j_0^{\,2} r_0)\, i\pi H_1'(ihr_0) J_1(ihr_0)\right] \\ -a\psi(2\pi j_0^{\,2} r_0) \end{array}\right\} = 0, \tag{88}$$

which may be rewritten as

$$(n_0 \pi r_0^2) m \gamma_0 (\omega + ku)^2 = (2\pi^2 j_0^2 r_0^2)\left[1 - i\pi H_1^1(ihr_0) J_1(ihr_0)\right]. \tag{89}$$

Eq. (89) is the dispersion relation. It may be simplified to

$$(\omega + ku)^2 = \omega_B^2\left[1 - i\pi H_1^1(ihr_0) J_1(ihr_0)\right], \tag{90}$$

where the betatron frequency ω_B is defined by

$$\omega_B^2 = \left(\frac{2\pi e^2 n_0 u^2}{m \gamma_0 c^2}\right). \tag{91}$$

It is useful to compare the various low frequency, long wavelength restrictions that have been made during the analysis. The use of a macroscopic analysis requires that each beam particle should be influenced by quasi-static electromagnetic fields during the course of one betatron oscillation. Since the beam has a large macroscopic velocity, this requirement is most conveniently expressed as

$$|\omega + ku| << \omega_B. \tag{92}$$

The approximation of charge neutrality implies the two restrictions

$$|\omega| << \omega_p$$

and

$$\omega_B << \omega_p, \tag{93}$$

where ω_p is the plasma frequency, while the hydrodynamic equations require that

$$|\omega| << |ku|. \tag{94}$$

Finally, the long wavelength approximation may be stated as

$$|kr_0| << 1. \tag{95}$$

Equations (92)-(95) are not independent; they are equivalent to the inequalities

$$|ku| << \omega_B,$$
$$|\omega| << |ku|,$$
and
$$|kr_0| << 1. \tag{96}$$

Under most circumstances the third inequality of Eq. (96) is not needed.

It has been assumed that the perturbed beam current is primarily surface current. The consistency of this assumption will now be shown. The magnitude of the surface current flow is given roughly by

$$I_s = (2\pi j_0 r_0) a. \tag{97}$$

Similarly, the volume current flow is given by

$$I_v \simeq \pi r_0^2 \left|\frac{en_0 v_r}{c}\right|. \tag{98}$$

Equations (79) and (96) show that

$$|v_r| \simeq |ku| a, \tag{99}$$

giving

$$I_v \simeq |kr_0|(\pi j_0 r_0) a. \tag{100}$$

Thus the flow of volume current is much smaller than the flow of surface current.

For most disturbances

$$|i\pi H_1^1(ihr_0) J_1(ihr_0)| \simeq 1, \tag{101}$$

which implies that the rate of change of momentum is much smaller than the pressure gradient. It is then reasonable to ask whether the dispersion law (89) is due to the artificial form chosen for the pressure tensor. This question is dealt with in the next section.

A Second Derivation of the Dispersion Law

Here Eq. (89) will be derived from the assumption that the primary motion of the beam is a rigid helical ($m = 1$) displacement and that the pressure tensor satisfies a minor restriction. Secondary eddying motions of the beam are not excluded, but it is assumed that these motions account for a small fraction of the momentum flow. Under these circumstances the beam volume currents may be neglected in comparison with surface currents, and Eqs. (62) and (63) for the electromagnetic fields are valid. The dispersion relation is obtained by integrating the $\underline{a}$ component of the momentum equation over a beam cross-section.

The integrand is given by the scalar product of the momentum equation and $\underset{\sim}{a}^*$, the complex conjugate of $\underset{\sim}{a}$. As before, field variables are taken to be complex-valued functions whose z, θ, and t dependences are contained in the factor ψ.

To good approximation the hydrodynamic equations become

$$-n_0 m \gamma_0 (\omega + ku)^2 \underset{\sim}{a} = -\nabla \cdot P - en_0 \underset{\sim}{E} + \underset{\sim}{j}_0 \times \underset{\sim}{B} + \underset{\sim}{j} \times \underset{\sim}{B}_0, \tag{102}$$

where P is a general pressure tensor, and the currents and electromagnetic fields are given by previous equations. Equation (102) includes the effects of surface currents and makes use of the approximation of rigid beam motion

$$\underset{\sim}{v} = i(\omega + ku)\underset{\sim}{a}, \tag{103}$$

where $\underset{\sim}{a}$ is given by Eq. (46). The dispersion relation is then derived from the expression

$$\begin{aligned} &-2\pi n_0 m \gamma_0 (\omega+ku)^2 \int_0^{r_0} r\,dr\,(\underset{\sim}{a} \cdot \underset{\sim}{a}^*) \\ &= -2\pi \int_0^{r_0} r\,dr\,(\nabla \cdot P) \cdot \underset{\sim}{a}^* \\ &\quad + 2\pi \int_0^{r_0} r\,dr\,(\underset{\sim}{j}_0 \times \underset{\sim}{B} + \underset{\sim}{j} \times \underset{\sim}{B}_0) \cdot \underset{\sim}{a}^*, \end{aligned} \tag{104}$$

in which the integrals are all evaluated from $r = 0$ to $r = r_0^+$ to include surface effects.

The pressure term of Eq. (104) will be examined first. The components of $(\nabla \cdot P)$ are given by

$$(\nabla\cdot P)_r = \gamma\left(\frac{1}{r}\frac{d}{dr}[rP_{rr}(r)] + \frac{i}{r}P_{r\theta}(r) - \frac{1}{r}P_{\theta\theta}(r) + ikP_{rz}(r)\right)$$

and (105)

$$(\nabla\cdot P)_\theta = \gamma\left(\frac{1}{r}\frac{d}{dr}[rP_{r\theta}(r)] + \frac{1}{r}P_{r\theta}(r) + \frac{i}{r}P_{\theta\theta}(r) + ikP_{\theta z}(r)\right).$$

Thus the pressure term becomes

$$-2\pi\int_0^{r_0} r\,dr\,(\nabla\cdot P)\cdot \underset{\sim}{a}^*$$

$$= -2\pi a\ \exp-\left(2\,\mathrm{Im}[\omega t + kz]\right)\int_0^{r_0} dr\,\frac{d}{dr}\,r\left(P_{rr}(r) - iP_{r\theta}(r)\right) \qquad (106)$$

$$-2\pi a\ \exp-\left(2\,\mathrm{Im}[\omega t + kz]\right)\int_0^{r_0} r\,dr\,ik\left(P_{rz}(r) - iP_{\theta z}(r)\right).$$

The total derivative term vanishes, since there is no fluid at $r = r_0^+$, and Eq. (106) becomes

$$-2\pi\int_0^{r_0} r\,dr\,(\nabla\cdot P)\cdot \underset{\sim}{a}^* = 0, \qquad (107)$$

provided that

$$\int_0^{r_0} r\,dr\left(P_{rz}(r) - iP_{\theta z}(r)\right) = 0. \qquad (108)$$

The weak restriction on the pressure tensor given by Eq. (108) is assumed to hold for all electron beams. This assumption should be valid for highly relativistic beams, since cross terms in the pressure tensor are due to variations in the particle velocities, and a large longitudinal mass inhibits variations in the z velocity. For non-relativistic beams the assumption may be less satisfactory. A similar analysis of the pressure tensor has been given by Rosenbluth, [23] by means of a volume integration over one wavelength of the beam.

Since the pressure term vanishes, substitution into Eq. (104) gives

$$\begin{aligned} &-2a^2 \exp-(2\,\mathrm{Im}[\omega t+kz])(n_0\pi r_0^2 m\gamma_0)(\omega+ku)^2 \\ &= 2\pi a^2 \exp-(2\,\mathrm{Im}[\omega t+kz])\int_0^{r_0} dr\,\frac{ic}{\omega a}\frac{d}{dr}[r f_z(r)] \\ &\quad -2\pi a^2 \exp-(2\,\mathrm{Im}[\omega t+kz])[2\pi j_0^2 r_0^2]. \end{aligned} \tag{109}$$

Simplification yields

$$\begin{aligned} &(n_0\pi r_0^2)m\gamma_0(\omega+ku)^2 \\ &=(2\pi j_0^2 r_0^2)(1-i\pi H_1'(ihr_0)J_1(ihr_0)), \end{aligned} \tag{110}$$

which is just Eq. (89). Thus the assumption that the beam motion is essentially rigid and the restriction (108) on the form of the pressure tensor suffice to derive the dispersion law.

Analysis of the Dispersion Relation

It is convenient to make use of the definitions

$$h^2=\left(\frac{4\pi\sigma i\omega}{c}\right),\ \mathrm{Re}\,h>0$$

$$
\begin{aligned}
\omega_B^2 &= \left(\frac{2\pi e^2 n_0 u^2}{m\gamma_0 c^2}\right) \\
\omega_p^2 &= \left(\frac{4\pi e^2 \bar{n}}{m}\right) \\
z &= \left(\frac{\omega}{\omega_B}\right) \\
s &= \left(\frac{ku}{\omega_B}\right)
\end{aligned}
\tag{111}
$$

and

$$\Omega = \left(\frac{\omega + ku}{\omega_B}\right)$$

in re-expressing the dispersion relation as

$$\Omega^2 = 1 - i\pi H_1^1(ihr_0)\, J_1(ihr_0). \tag{112}$$

A complete numerical analysis of Eq. (112) would be straightforward but lengthy. Instead, a number of limiting cases will be treated by analytic means.

The conductivity σ is given in general by Eq. (9), but this equation is more complicated than is convenient for the present analysis. The behavior of σ as a function of ω is that for small ω, σ is positive, while for large real ω, σ has small magnitude, and its phase approaches $-\frac{\pi}{2}$. This behavior is retained in the two limiting cases (a fully ionized plasma and a plasma in which the ions move with the neutrals) which are examined here. In both cases it is possible to define a collision frequency ν such that σ is well approximated by

$$\sigma = \left(\frac{en}{mc}\right)\frac{1}{\nu + i\omega}. \tag{113}$$

Eq. (113) will be taken as the defining equation for σ in the subsequent analysis. It is also convenient to define a normalized collision frequency w by

$$w = \left(\frac{\nu}{\omega_B}\right). \tag{114}$$

Then h is defined by

$$h^2 r_o^2 = \left(\frac{\omega_p^2 r_o^2}{c^2}\right)\left(\frac{iz}{w+iz}\right). \tag{115}$$

Case I: $w \simeq 0$

In this approximation

$$hr_o = \left(\frac{\omega_p r_o}{c}\right) > 0. \tag{116}$$

The function

$$g(x) = i\pi H_1'(ix) J_1(ix) \tag{117}$$

is a monotonic decreasing function of x for $x > 0$ such that

$$g(0) = 1$$

and

$$g(\infty) = 0. \tag{118}$$

Thus the dispersion relation may be written

$$\Omega^2 = 1 - i\pi H_1'(ihr_o) J_1(ihr_o) > 0, \tag{119}$$

and the major content of the dispersion relation may be summarized by the equation

$$\text{Im}(\omega + ku) = 0. \qquad (120)$$

The usual instability problem is an initial value problem in which a disturbance is imposed on the system at some specified time, and it is required to find whether any Fourier components of the disturbance are amplified or damped as time progresses. This amounts to specifying real k and searching the dispersion relation for complex ω. In this case Eq. (120) gives immediately

$$\text{Im}\,\omega = 0. \qquad (121)$$

That is, the system is purely oscillatory - neither damping nor growth of disturbances is possible.

It is also possible to impose a periodic disturbance on the system at some given beam cross-section. Then it is required to find whether any range of frequencies gives rise to amplified or attenuated disturbances downstream. This problem is solved by imposing real ω and searching the dispersion relation for complex k. In the present case Eq. (120) gives

$$\text{Im}\,k = 0, \qquad (122)$$

and neither amplification nor attenuation is possible. In this approximation the system is completely stable.

Case II: $w \gg |z|$ and $|hr_0| \ll 1$.

For this case the defining equation for h becomes

$$h^2 r_0^2 = \left(\frac{\omega_p^2 r_0^2}{c^2}\right)\left(\frac{iz}{w}\right). \tag{123}$$

Small argument expansions are used to evaluate the Bessel functions. An adequate approximation is given by

$$J_1(ihr_0) = \left(\frac{ihr_0}{2}\right),$$

$$H_1^1(ihr_0) = -\left(\frac{2}{\pi hr_0}\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} hr_0\right|\right), \tag{124}$$

and

$$i\pi H_1^1(ihr_0) J_1(ihr_0) = 1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} hr_0\right|.$$

The dispersion equation may be rewritten as

$$\Omega^2 = \left(\frac{\omega_p^2 r_0^2}{2c^2 w}\right)\left(-\ln\left|\frac{\gamma}{2} hr_0\right|\right) iz. \tag{125}$$

In the above expressions $\ln\gamma$ refers to the Euler-Mascheroni constant and has nothing to do with previous usage of γ and γ_0.

To solve Eq. (125) it is helpful to make use of the fact that the quantity

$$R = \left(\frac{\omega_p^2 r_0^2}{2c^2 w}\right)\left(-\ln\left|\frac{\gamma}{2} hr_0\right|\right) \tag{126}$$

is a very weak function of z. Thus Ω and z are determined as functions of R and s from the equations

$$\Omega = z + s \tag{127}$$

and

$$\Omega^2 - iR\Omega + iRs = 0. \tag{128}$$

Then the condition $|hr_0| \ll 1$ must be imposed as a restriction on the admissible values of R and s. Finally, for admissible solutions the logarithmic term in R is evaluated and the corresponding value of $(\omega_p^2 r_0^2/c^2 w)$ is determined.

First the initial value problem will be solved. The parameter s is taken to be real and R is given by

$$\Omega = \frac{1}{2}\left(iR \pm \sqrt{-R^2 - 4iRs}\right). \tag{129}$$

When $|s| << R$ this becomes

$$\Omega = \left(\frac{iR}{2}\right)\left(1 \pm \left[1 + \frac{2is}{R} + \frac{2s^2}{R^2} - \frac{4is^3}{R^3}\right]\right). \tag{130}$$

Growth occurs when Im $\Omega < 0$, or when

$$z = -\left(\frac{is^2}{R}\right) - \left(\frac{2s^3}{R^2}\right). \tag{131}$$

Thus ω is given by

$$\omega = -\left(\frac{2u^3k^3}{\omega_B^2 R^2}\right) - i\left(\frac{u^2k^2}{\omega_B R}\right), \tag{132}$$

the phase velocity v_p is given by

$$v_p = -\left(\frac{2u^3k^2}{\omega_B^2 R^2}\right), \tag{133}$$

and the group velocity v_g is given by

$$v_g = -\left(\frac{6u^3k^2}{\omega_B^2 R^2}\right), \tag{134}$$

or

$$v_g = -\left(\frac{6s^2}{R^2}\right)u. \tag{135}$$

From Eq. (132) the growth rate α of these disturbances may be written as

$$\alpha = \left(\frac{s^2}{R}\right)\omega_B. \tag{136}$$

Thus the waves propagate a distance $L_a = (|v_g|/\alpha)$ downstream in the typical amplification time $(\alpha)^{-1}$. This distance L_a is given by

$$L_a = \left(\frac{b}{R}\right)\left(\frac{u}{\omega_B}\right), \tag{137}$$

and is nearly independent of the wavelength of the disturbance. If $R \gg 1$ this length can be much smaller than the betatron length $L_B = (u/\omega_B)$.

When $|s| \gg R$, Ω is given by

$$\Omega = \pm\sqrt{Rs}\, e^{-\frac{i\pi}{4}} \tag{138}$$

For growing waves this becomes

$$\omega = -\left(uk - \sqrt{\frac{Ru\omega_B k}{2}}\right) - i\sqrt{\frac{Ru\omega_B k}{2}}. \tag{139}$$

Then v_p and v_g are given by

$$v_p = -\left(u - u\sqrt{\frac{R\omega_B}{2uk}}\right) \tag{140}$$

and

$$v_g = -\left(u - u\sqrt{\frac{R\omega_B}{8uk}}\right). \tag{141}$$

It is now necessary to investigate the restrictions on R and s imposed by the condition

$$|h^2 r_0^2| \ll 1. \tag{142}$$

This condition may be adequately restated as

$$R\frac{|\omega|}{\omega_B} \ll 1. \tag{143}$$

Here R will be taken to be a free parameter corresponding to an arbitrary choice of steady state beam and plasma parameters, and Eq. (143) will be used to give restrictions on the range of s allowed in the present treatment. When $R \gg |s|$, Eq. (143) has the approximate form

$$s^2 \ll 1, \tag{144}$$

while for $|s| \gg R$ the corresponding approximation gives

$$R|s| \ll 1. \tag{145}$$

For all cases a useful minimum wavelength is derived from the condition

$$|s|_{max} = 1, \tag{146}$$

or

$$\lambda_{min} = \left(\frac{2\pi u}{\omega_B}\right). \tag{147}$$

For disturbances of shorter wavelength it is unjustified to neglect the details of the internal motion of the beam particles. This estimate may be used to derive maximum values for the growth rates of this type of instability. (As $k \to \infty$ in Eqs. (132) and (139) the growth rates diverge.) When $R \gg 1$ the maximum becomes

$$(-\omega_i)_{max} = \frac{\omega_B}{R}, \tag{148}$$

and for $R \ll 1$ it becomes

$$(-\omega_i)_{max} = \omega_B \sqrt{\frac{R}{2}}. \tag{149}$$

The case of a periodic disturbance can be treated even more simply. The dispersion relation

$$(s+z)^2 = iRz \tag{150}$$

is to be solved for real z and complex s. This gives

$$s = -z \pm \sqrt{Rz}\, e^{i\frac{\pi}{4}}, \tag{151}$$

and amplification occurs when

$$s = -z + \sqrt{\frac{Rz}{2}} - i\sqrt{\frac{Rz}{2}}. \tag{152}$$

The requirement $|hr_o|^2 << 1$ may be written as

$$|Rz| << 1. \tag{153}$$

Thus k is given by

$$k = \left(-\frac{\omega}{u} + \frac{1}{u}\sqrt{\frac{R\omega\omega_B}{2}}\right) - i\frac{1}{u}\sqrt{R\omega\omega_B}, \tag{154}$$

and an amplification length L may be defined by

$$(-\mathrm{Im}\,k)L = 1. \tag{155}$$

This gives

$$L = \left(\frac{u}{\omega_B}\right)\sqrt{\frac{2}{Rz}}, \tag{156}$$

and a minimum length L_{min} is given by the betatron length L_B.

The slow wave occurring in the initial value problem when R>>1 and R>>s would probably be the easiest of the above instabilities to observe experimentally. Although such disturbances would have wavelengths longer than L_B they would amplify many times before propagating downstream a distance L_B. The other disturbances are fast waves which require a much greater distance for amplification.

Case III: w arbitrary and $|hr_0| \gg 1$

Here the dispersion relation is evaluated through the use of asymptotic expansions for the Bessel functions. Sufficient accuracy is given by

$$J_1(ihr_0) = \frac{i}{\sqrt{2\pi hr_0}} \exp hr_0, \tag{157}$$

$$H_1^1(ihr_0) = -\sqrt{\frac{2}{\pi hr_0}} \exp -hr_0,$$

and

$$i\pi H_1^1(ihr_0) J_1(ihr_0) = \left(\frac{1}{hr_0}\right). \tag{158}$$

The dispersion relation then becomes

$$(z+s)^2 = 1 - \left(\frac{1}{hr_0}\right), \tag{159}$$

where

$$h^2 r_0^2 = \left(\frac{\omega_p^2 r_0^2}{c^2}\right)\left(\frac{z}{z-iw}\right), \tag{160}$$

and is to be solved subject to the condition

$$|h^2 r_0^2| \gg 1. \tag{161}$$

A first approximation to a solution of Eq. (159) is provided by

$$\left(\frac{1}{hr_0}\right) = 0, \tag{162}$$

and

$$(z+s) = \pm 1. \tag{163}$$

To this approximation the system is completely stable. A second approximation is obtained by using Eqs. (163) and (160) to evaluate $(hr_0)^{-1}$. If this value of $(hr_0)^{-1}$

satisfies condition (161), substitution into Eq. (159) gives a small correction to the value of $(z+s)$ given by Eq. (163). This procedure gives a fairly accurate solution of Eq. (159), and the second approximation to $(z+s)$ exhibits growing and damped oscillations. However, Eq. (163) is a warning that Eq. (159) probably does not contain all the essential information for this range of parameters. The beam particle orbits are likely to contribute important effects in this regime, and even the macroscopic theory must be treated more carefully when s is large.

The corrections to Eq. (163) have roughly the same form whether s or z is real, and only the case of real s will be treated here. Only positive s will be considered, since this simplifies the analysis without loss of generality. The roots of Eq. (163) will be denoted by $z_0^{\pm}$ and are given by

$$z_0^{\pm} = -s \pm 1. \tag{164}$$

Each root gives a corrected value for $(hr_0)^{-1}$, which is

$$\left(\frac{1}{hr_0}\right)^{\pm} = \left(\frac{c}{\omega_p r_0}\right)\sqrt{\frac{-s\pm 1-iw}{-s\pm 1}} \tag{165}$$

The first order corrections $z_1^{\pm}$ to the roots are then given by the equation

$$\pm 2z_1^{\pm} = -\left(\frac{1}{hr_0}\right)^{\pm}. \tag{166}$$

Thus z_1^{-} is given by

$$z_1^- = \left(\frac{c}{2\omega_B r_0}\right)\sqrt{\frac{s+1+iw}{s+1}}, \tag{167}$$

and the root $z^- = z_0^- + z_1^- + \ldots$ is stable for all permissible parameters. When $s < 1$, z_1^+ is given by

$$z_1^+ = -\left(\frac{c}{2\omega_B r_0}\right)\sqrt{\frac{(1-s)-iw}{(1-s)}} \tag{168}$$

and $z^+ = z_0^+ + z_1^+ + \ldots$ is a stable root. But when $s > 1$

$$z_1^+ = -\left(\frac{c}{2\omega_B r_0}\right)\sqrt{\frac{(s-1)+iw}{(s-1)}}, \tag{169}$$

and z^+ is unstable whenever Eq. (161) is satisfied; that is, whenever

$$\left|\frac{s-1+iw}{s-1}\right| << \frac{\omega_p^2 r_0^2}{c^2}. \tag{170}$$

No instabilities occur if

$$\left(\frac{\omega_p^2 r_0^2}{c^2}\right) \leq 1, \tag{171}$$

but if

$$\left(\frac{\omega_p^2 r_0^2}{c^2}\right) >> 1, \tag{172}$$

instabilities occur for sufficiently short wavelengths. Such instabilities are fast waves with rather small growth rates.

Case IV: w arbitrary and $|hr_0| << 1$

In this case the dispersion relation becomes

$$(z+s)^2 = P\left(\frac{z}{z-iw}\right), \tag{173}$$

where

$$P = \left(\frac{\omega_p^2 r_0^2}{2c^2}\right)\left(-\ln\left|\frac{\gamma}{2} h r_0\right|\right). \tag{174}$$

This equation is to be solved subject to the condition

$$P\left|\frac{z}{z-iw}\right| << 1, \tag{175}$$

but the restrictions imposed by Eq. (175) will not be discussed in detail here.

For the initial value problem P and s are parameters and Eq. (173) becomes a cubic equation

$$z^3+(2s-iw)z^2+(s^2-P-2iws)z-iws^2=0 \tag{176}$$

for z. This equation has been solved numerically for a set of values of P and s. The behavior of solutions may be summarized as follows: for small s, $|z|$ is also small, and the unstable solution behaves as in Case II when $R=\frac{P}{w}>>s$. Both $|z|$ and the growth rate are increasing functions of s, and the instability is a slow wave. When $|z|\simeq w$ the instability changes from a slow wave to a fast wave, the growth rate becomes a decreasing function of s, and $|z|$ remains an increasing function of s. As s increases further, the system tends to the stability predicted in Case I when $|z|>>w$. For all s only one root of Eq. (176) is unstable.

For periodic disturbances P and z are positive and Eq. (173) becomes

$$s^2+2zs+\left(z^2-\frac{Pz}{z-iw}\right)=0, \tag{177}$$

which has the amplified root

$$s = -z - \sqrt{\frac{pz}{z - iw}}\,. \tag{178}$$

Again, the instabilities come from the finite collision rate w, and by Eq. (175) require many betatron lengths for amplification. This case has also been analyzed by Cooper and Raether.[44]

Summary of the Chapter

A macroscopic analysis has been given of the m=1 disturbances of a uniform relativistic electron beam which is imbedded in a dense, uniform plasma. Such disturbances correspond to a gross motion of the beam and may be expected to give the worst instabilities. The electromagnetic effects of the plasma have been accounted for by a scalar conductivity whose phase is a measure of the relative importance of collisional and inertial forces in the plasma. A simple hydrodynamic model has been used for the beam dynamics.

The analysis is valid only for low frequencies and long wavelengths. The region of validity is established by examination of the macroscopic equations. The beam disturbance consists primarily of a rigid helical displacement which travels in the z direction with a time dependent amplitude. Approximate analysis shows that eddying motions in the beam are of secondary importance. The major electromagnetic effects of this displacement are due to the corresponding surface currents. The plasma attempts to screen out these currents through a flow of volume current, and the resultant electromagnetic fields give rise to a drag force on the beam. The phase of this velocity-dependent force is such as to give instabilities when collision rates are large and to give completely oscillatory behavior under collisionless conditions. This is in marked contrast to the case analyzed by Finkelstein and Sturrock [39] of a relativistic electron beam neutralized by an ion beam. In that case maximum instability rates occur under collisionless conditions. For the modes discussed here $\omega \sim \eta$, the resistivity, as in the non-localized finite conductivity instabilities found by Furth, Rosenbluth, and Killeen. [54]

The dispersion law for these disturbances is obtained by summing the electromagnetic and the inertial beam forces over the beam cross-section. A non-zero contribution comes from the component in the direction of the beam displacement, and this gives the dispersion relation. Under quite general conditions the pressure tensor does not contribute to the dispersion relation. However, in the non-relativistic regime a more detailed calculation may be necessary. The dispersion relation has also been obtained by an analysis of the specific hydrodynamic model assumed for the beam dynamics.

Chapter 3

FURTHER ANALYSIS OF THE LOW FREQUENCY INSTABILITY

A Treatment of Hall Currents

When the self-magnetic field of the beam is large, the plasma conductivity tensor becomes highly non-diagonal and the analysis of $m = 1$ disturbances in Chapter 2 is not valid. This defect is remedied here by introducing a tensor conductivity into the electromagnetic field equations but otherwise using the same hydrodynamic description of the beam motion. As before, the dispersion law is found most easily by integrating the $\underset{\sim}{a}$ component of the momentum equations, but here the determination of the perturbed electromagnetic fields is more difficult. The dispersion law is similar in form to the one obtained earlier and will not be analyzed in detail. The discussion given here and in the following section is based on the work of Enoch, Longmire, and Mjolsness.[22]

The treatment is again confined to low frequency, long wavelength disturbances, but the analysis of these disturbances is considerably more intricate. The equations governing the perturbed electric fields become coupled ordinary differential equations whose form implies that all components of $\underset{\sim}{f}$ are non-zero. Thus a more complex electrostatic body force acts on the beam and gives rise to an additional term in the dispersion law. Although the perturbed magnetic field

$$\underline{B} = \left(\frac{ic}{\omega}\right)\psi\left(\frac{i}{r}f_z(r), -\frac{d}{dr}f_z(r), \frac{1}{r}\frac{d}{dr}rf_\theta(r) - \frac{i}{r}f_r(r)\right) \tag{1}$$

has an additional term, the magnetic forces

$$\underline{j}_0 \times \underline{B} = \left(\frac{ic}{\omega}j_0\right)\psi\left(\frac{d}{dr}f_z(r), \frac{i}{r}f_z(r), 0\right)$$

and (2)

$$\underline{j} \times \underline{B}_0 = \left(2\pi j_0^2 ar\right)\delta(r-r_0)\psi(-1, 0, 0)$$

are unaltered in form and differ only through changes in the values of $f_z(r)$.

It is convenient to reduce the problem to the solution of Maxwell's equations by obtaining the dispersion relation in terms of an arbitrary vector $\underline{f}(r)$. This is most conveniently done, as in the second method of Chapter 2, by integrating the scalar product of $\underline{a}^*$ and the approximate momentum equation

$$-(\omega + ku)^2 n_0 m \gamma_0 \underline{a} = -(\nabla \cdot P) + (\underline{j}_0 \times \underline{B} + \underline{j} \times \underline{B}_0) - en_0\underline{E} \tag{3}$$

over a beam cross-section. The pressure term is again assumed to vanish, and the result is

$$-2 a^2 \exp -\left(2\,\mathrm{Im}[\omega + ku]\right) n_0 \pi r_0^2 m v_0 (\omega + ku)^2$$

$$= a^2 \exp -\left(2\,\mathrm{Im}[\omega + ku]\right) \left[\begin{array}{l} -(2\pi j_0 r_0)^2 \\ + \left(\frac{2\pi i c}{\omega a} j_0\right) \int_0^{r_0} dr \frac{d}{dr} r f_z(r) \\ - \left(\frac{2\pi e n_0}{a}\right) \int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right) \end{array} \right] . \quad (4)$$

A simple rearrangement gives

$$n_0 \pi r_0^2 m v_0 (\omega + ku)^2$$

$$= \left(2\pi^2 j_0^2 r_0^2\right) - \left(\frac{i\pi r_0 c}{\omega a} j_0\right) f_z(r_0) \quad (5)$$

$$+ \left(\frac{\pi e n_0}{a}\right) \int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right),$$

which may be evaluated explicitly, once $\underset{\sim}{f}(r)$ has been determined. The last term in Eq. (5) represents the contribution of the electrostatic body force and did not appear in Chapter 2.

The coupling of Maxwell's equations is provided by the plasma current, which takes the form

$$(j)_{pr} = \sigma\left(\frac{E_r + \mu E_z}{1+\mu^2}\right)$$

$$(j)_{p\theta} = \sigma E_\theta \qquad (6)$$

$$(j)_{pz} = \sigma\left(\frac{E_z - \mu E_r}{1+\mu^2}\right),$$

where $\mu = \left(\frac{\sigma B_0(r)}{e\bar{n}}\right)$,

when the self-magnetic field effects are included. Unless otherwise noted, the analysis will be restricted to collision dominated plasmas for which σ is positive. The volume Maxwell equations may be summarized by

$$\nabla \times (\nabla \times \underset{\sim}{E}) + \left(\frac{4\pi i\omega}{c}\right)(\underset{\sim}{j})_p = 0, \qquad (7)$$

where $(\underset{\sim}{j})_p$ is given by Eq. (6). These equations must be supplemented by regularity conditions at $r=0$ and $r=\infty$ and by the surface conditions

$$f_r(r_0)_{out} - f_r(r_0)_{in} = 0$$

$$f_\theta(r_0)_{out} - f_\theta(r_0)_{in} = 0$$

$$f_z(r_0)_{out} - f_z(r_0)_{in} = 0 \tag{8}$$

$$\frac{d}{dr_0} f_r(r_0)_{out} - \frac{d}{dr_0} f_r(r_0)_{in} = 0$$

$$\frac{d}{dr_0} f_\theta(r_0)_{out} - \frac{d}{dr_0} f_r(r_0)_{in} = 0$$

and

$$\frac{d}{dr_0} f_z(r_0)_{out} - \frac{d}{dr_0} f_z(r_0)_{in} = \left(\frac{4\pi i \omega a}{c} j_0\right)$$

at $r = r_0$.

For long wavelengths the components of Eq. (7) take the form

$$\frac{i}{r}\left[\frac{1}{r}\frac{d}{dr} r f_\theta(r) - \frac{1}{r} f_r(r)\right] + h^2\left[\frac{f_r(r) + \mu f_z(r)}{1+\mu^2}\right] = 0, \tag{9}$$

$$-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr} r f_\theta(r) - \frac{i}{r} f_r(r)\right] + h^2 f_\theta(r) = 0,$$

and

$$-\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right] f_z(r) + h^2\left[\frac{f_z(r) - \mu f_r(r)}{1+\mu^2}\right] = 0,$$

where h is defined as before. The definitions

$$H = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - h^2 \right)$$

and $$N = \left(\frac{\mu}{r} \right) \tag{10}$$

are introduced in order to facilitate the reduction of Eq. (9) to a pair of coupled differential equations for $f_z(r)$ and the auxiliary field variable $\lambda(r)$, where $\lambda(r)$ is defined by

$$\lambda(r) = \left[\frac{1}{r} f_r(r) + \frac{i}{r} \frac{d}{dr} r f_\theta(r) \right]. \tag{11}$$

The first two components of Eq. (9) may be rewritten as

$$\lambda = -r h^2 \frac{(f_r + \mu f_z)}{(1+\mu^2)}$$

and $$\frac{d\lambda}{dr} = i h^2 f_\theta , \tag{12}$$

and may be used to eliminate the explicit appearance of $f_r(r)$ and $f_\theta(r)$. Thus successive substitutions yield

$$\begin{aligned} \frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr} \lambda \right) &= h^2 \left[\frac{i}{r} \frac{d}{dr} r f_\theta \right] \\ &= h^2 \lambda - h^2 \frac{f_r}{r} \\ &= h^2 \lambda + \frac{(1+\mu^2)}{r^2} \lambda + h^2 \frac{\mu}{r} f_z , \end{aligned} \tag{13}$$

which becomes, after rearrangement,

$$H\lambda = N^{\nu}\lambda + h^{\nu}Nf_z . \tag{14}$$

Further simplification eliminates the explicit appearance of $f_r(r)$ from the third component of Eq. (9). This component may be rewritten as

$$Hf_z = -h^{\nu}f_z + h^{\nu}\frac{(f_z - \mu f_r)}{(1+\mu^{\nu})} , \tag{15}$$

and may be reduced to the form

$$Hf_z = N\lambda . \tag{16}$$

Equations (14) and (16) will replace Eq. (9) in the subsequent analysis. Boundary conditions on $\lambda(r)$ may be obtained from Eq. (8) and are given by

$$\lambda(r_0)_{out} - \lambda(r_0)_{in} = 0$$

and

$$\frac{d}{dr_0}\lambda(r_0)_{out} - \frac{d}{dr_0}\lambda(r_0)_{in} = 0. \tag{17}$$

Exact solutions of these equations may be found for $r \leq r_0$ where

$$N = N_0 = (2\pi j_0 \sigma / e\bar{n}). \tag{18}$$

For $r > r_0$ N is given by

$$N = \frac{N_0 r_0^{\nu}}{r^{\nu}} = \frac{N_1}{r^{\nu}} , \tag{19}$$

and the equations become much more difficult. Two approximate solutions are presented below as Case I and Case II.

Case I: an underestimate of the magnetic coupling

No approximations are needed to solve Eqs. (14) and (16) when $r < r_0$. Solutions will be obtained in the form

$$\lambda(r) = b f_z(r) \tag{20}$$

where b is a constant. The equations then become

$$\left(H - N_0^2 - \frac{h^2 N_0}{b}\right) f_z(r) = 0$$

and

$$(H - N_0 b) f_z(r) = 0, \tag{21}$$

and consistency requires that

$$N_0 b^2 - N_0^2 b - h^2 N_0 = 0. \tag{22}$$

The two roots

$$b^{\pm} = \frac{1}{2}\left(N_0 \pm \sqrt{N_0^2 + 4h^2}\right) \tag{23}$$

of Eq. (22) insure that a complete solution of Eqs. (14) and (16) may be given in the form (20). However, the requirement that the fields be regular at $r=0$ restricts the admissible solutions to

$$f_z(r) = A^+ J_1(i\delta^+ r) + A^- J_1(i\delta^- r)$$

and

$$\lambda(r) = b^+ A^+ J_1(i\delta^+ r) + b^- A^- J_1(i\delta^- r) \tag{24}$$

where $\delta^{\pm}$ is defined by

$$\delta^{\pm} = \sqrt{h^2 + N_0 b^{\pm}} \text{ and } \mathrm{Re}\,\delta^{\pm} > 0,$$

and the quantities $A^{\pm}$ are arbitrary constants.

The determination of $A^{\pm}$ is necessary for the explicit evaluation of the dispersion law (5) and requires that the field variables be known for $r > r_o$. This knowledge is not easily obtained, since the exact equations (14) and (16) are too difficult to solve analytically when $r > r_o$. In this section approximate values of the field variables will be obtained analytically from Eq. (16) and a modification of Eq. (14) which leaves the equation unaltered at $r = r_o$ and generates correct asymptotic behavior when r is large.

The appropriate modification is derived most readily from Eq. (12). In this equation the denominator $(1+\mu^2)^{-1}$ is a measure of the magnetic coupling due to Hall currents. This coupling will be systematically underestimated by replacing the denominator by its value $(1+\mu_o^2)^{-1}$ at $r = r_o$. The approximation is formally equivalent to the assumption that the radial conductivity σ_r varies as

$$\sigma_r = \sigma\left[\frac{1+\mu^2}{1+\mu_o^2}\right], \tag{25}$$

while the theta conductivity remains fixed. With this modification the steps leading to Eq. (14) yield instead

$$H\lambda = N_o\frac{N_1}{r^2}\lambda + h^2\frac{N_1}{r^2}f_z. \tag{26}$$

Equations (16) and (26) form the basis of the following discussion.

Solutions to these equations are again sought in the form (20), and the equations become

$$Hf_z = N_1\left(\frac{N_0}{r^2} + \frac{h^2}{b r^2}\right) f_z$$

and (27)

$$Hf_z = N_1\left(\frac{b}{r^2}\right) f_z .$$

Consistency requires that b must satisfy the equation

$$N_1\left(b^2 - N_0 b - h^2\right) = 0, \tag{28}$$

whose roots $b^{\pm}$ are given as before by Eq. (23). Again, a complete set of solutions may be found in the form (20), but here regularity conditions at $r = \infty$ restrict the admissible solutions to

$$f_z(r) = C^+ H^1_{p^+}(ihr) + C^- H^1_{p^-}(ihr)$$

and (29)

$$\lambda(r) = b^+ C^+ H^1_{p^+}(ihr) + b^- C^- H^1_{p^-}(ihr),$$

where $p^{\pm}$ is given by

$$p^{\pm} = \left(1 + r_0^2 N_0 b^{\pm}\right)^{\frac{1}{2}},$$

and the quantities $C^{\pm}$ are arbitrary constants.

The constants $A^{\pm}$ and $C^{\pm}$ are determined from the boundary conditions (8) and (17) at $r = r_0$. When $N_0 \simeq 0$ this procedure leads to the results of Chapter 2. When N_0 is large, i.e. when $N_0^2 \gg |h^2|$, the asymptotic expressions

$$b^+ = N_0\left(1 + \frac{h^2}{N_0^2}\right)$$

$$b^- = -\left(\frac{h^2}{N_0}\right)\left(1 - \frac{h^2}{N_0^2}\right)$$

$$\delta^+ = N_0\left(1 + \frac{h^2}{N_0^2}\right) \tag{30}$$

$$\delta^- = \frac{h^2}{N_0}$$

$$\rho^+ = \left(1 + N_0^2 r_0^2 + h^2 r_0^2\right)^{\frac{1}{2}}$$

and

$$\rho^- = 1 - \tfrac{1}{2} h^2 r_0^2$$

may be used to simplify the calculations, and an analysis of this case is given in Appendix II. When $N_o r_o \ll 1$ the fields and the dispersion law are unchanged from Chapter 2. That is, the self-magnetic field must still be considered small; the parameter $N_o r_o$ is a measure of the strength of the self-magnetic field and must reach the value $N_o r_o \simeq 1$ before the Hall currents contribute significant effects. When $N_o r_o \gg 1$ the perturbed electric field is given to good approximation by

$$f_r(r) = \left(\frac{1}{N_0 r_0}\right) f_z(r_0) + \left(\frac{2\pi i\omega a}{c}\, j_0\right) \frac{r J_1(i\delta^+ r)}{J_1(i\delta^+ r_0)}$$

$$f_\theta(r) = -\left(\frac{i}{N_0 r_0}\right) f_z(r_0) \tag{31}$$

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c}\, j_0\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right) r,$$

and the additional term in the dispersion law is given by

$$\left(\frac{\pi e n_0}{a}\right)\int_0^{r_0} dr\; r\left(f_r(r) - i f_\theta(r)\right)$$

$$= \left(\frac{\pi r_0 e n_0}{a N_0}\right)\left[f_z(r_0) + \left(\frac{2\pi i\omega a r_0}{c}\, j_0\right)\right]. \tag{32}$$

Thus the Hall currents do not alter the longitudinal component of the electric field but contribute to the transverse fields. It is convenient to rewrite Eq. (5) as

$$n_0 \pi r_0^2 m \gamma_0 (\omega + ku)^2$$

$$= \left(\frac{-i\pi r_0 c}{a\omega}\, j_0\right)\left[f_z(r_0) + \left(\frac{2\pi i\omega a r_0}{c}\, j_0\right)\right] \tag{33}$$

$$+ \left(\frac{\pi e n_0}{a}\right)\int_0^{r_0} dr\; r\left(f_r(r) - i f_\theta(r)\right).$$

Use of Eq. (32) and the identity

$$\left(\frac{\pi r_0 e n_0}{a N_0}\right) = \left(\frac{n_0}{\bar{n}}\right)\left(\frac{4\pi i\omega\sigma}{c}\right)\left(\frac{e\bar{n} r_0}{\sigma 2\pi j_0 r_0}\right)\left(\frac{1}{2 N_0}\right)\left(\frac{-i\pi r_0 c}{a\omega} j_0\right)$$

$$= \frac{1}{2}\left(\frac{n_0}{\bar{n}}\right)\left(\frac{h^\gamma}{N_0^\gamma}\right)\left(\frac{-i\pi r_0 c}{a\omega} j_0\right) \tag{34}$$

then shows that the corrections to the dispersion law from the transverse fields are negligible. Thus the present approximation indicates that the Hall currents have a small (or negligible) effect on the dispersion law.

Case II: an overestimate of the magnetic coupling

As in Case I Eq. (24) provides an exact solution for $f_z(r)$ and $\lambda(r)$ when $r \leq r_0$, and Eq. (16) is retained even when $r > r_0$, but here the denominator $(1+\mu^\gamma)^{-1}$ of Eq. (12) is replaced by its value 1 at $r = \infty$. This gives an overestimate of the magnetic coupling and is equivalent to allowing the radial conductivity to vary as

$$\sigma_r = \sigma(1+\mu^\gamma) \tag{35}$$

while keeping the theta conductivity fixed. This approximation leads to a second modification of Eq. (14) which is

$$H\lambda = h^\gamma \frac{N_1}{r^2} f_z. \tag{36}$$

The analysis given here is based on Eqs. (16) and (36) for

$r > r_o$ and is complementary to the analysis of Case I. Since the two cases make opposite approximations to Eq. (14), the effect of the approximations may be assessed by comparing the two dispersion laws.

Solutions are again sought in the form (20). The equations then become

$$H f_z = \frac{N_1 b}{r^2} f_z$$

and

$$H f_z = \frac{h^2}{b} \frac{N_1}{r^2} f_z, \tag{37}$$

which requires that

$$b^{\pm} = \pm h. \tag{38}$$

A complete set of solutions may be generated by this technique, but as in Case I the admissible solutions are restricted to be of the form

$$f_z(r) = C^+ H'_{n^+}(ihr) + C^- H'_{n^-}(ihr)$$

and

$$\lambda(r) = hC^+ H'_{n^+}(ihr) - hC^- H'_{n^-}(ihr) \tag{39}$$

where

$$n^{\pm} = \sqrt{1 \pm N_o r_o h r_o}$$

and the quantities $C^{\pm}$ are arbitrary constants. Thus the solution differs from Eq. (29) of Case I in the order of the Hankel functions and in the ratio between $f_z^{\pm}(r)$ and $\lambda^{\pm}(r)$.

The constants $A^{\pm}$ and $C^{\pm}$ are determined from the boundary conditions (8) and (17) but must be re-evaluated for this case. This is done in Appendix III for the regime $N_0^2 \gg |h^2|$ in which the asymptotic expressions (30) are valid. The opposite case $N_0 \simeq 0$ will yield no difference from Chapter 2. The discussion is also restricted to those low frequency, long wavelength disturbances for which $|hr_0| \ll 1$. The results are quite similar to those of Case I. When $N_0 r_0 \ll 1$ the Hall currents do not affect the fields or the dispersion law. When $N_0 r_0 \gg 1$ and $|N_0 r_0 h r_0| \ll 1$ the perturbed electric field is well approximated by

$$\begin{aligned} f_r &= \left(\frac{1}{N_0 r_0}\right) f_z(r_0) + \left(\frac{4\pi i \omega a}{c} j_0\right) r \frac{J_1(i\delta^+ r)}{J_1(i\delta^+ r_0)} \\ f_\theta &= -\left(\frac{i}{N_0 r_0}\right) f_z(r_0) \end{aligned} \tag{40}$$

and

$$f_z = -\left(\frac{2\pi i \omega a}{c} j_0\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right) r,$$

while the electrostatic term of the dispersion law is given by

$$\begin{aligned} &\left(\frac{\pi e n_0}{a}\right) \int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right) \\ &= \left(\frac{\pi r_0 e n_0}{a N_0}\right)\left[f_z(r_0) + \left(\frac{4\pi i \omega a r_0}{c} j_0\right)\right]. \end{aligned} \tag{41}$$

Comparison with Eq. (33) shows that the electrostatic term is again negligible. Thus the longitudinal electric field and the dispersion law are unchanged from Chapter 2.

Neither of the approximate treatments given in this section shows a modification of the dispersion law due to the presence of Hall currents, although these currents do affect the values of the transverse electric fields. Since the two approximations correspond to an underestimate and an overestimate of the magnetic coupling in Eq. (12), it is very likely that if Hall currents do alter the dispersion law, the alteration is quite small. An independent approximate treatment given in the next section supports this conclusion and indicates that the principal effect when $|hr_o| \ll 1$ is to alter the value of the logarithmic term of the dispersion law.

A Second Treatment of Hall Currents

The approximation methods of the previous section were adopted for purely mathematical reasons: the exact equations were too difficult to use directly, but a technique of underestimating and overestimating the difficult term led to solvable sets of equations and provided a check on the errors in the treatment. Here physical arguments are used to suggest a rather different approximation scheme which is then considered in detail. The scheme is based on the observation that when $r \lesssim r_o$ and $|hr_o| << 1$ the values of the perturbed fields depend very weakly on the presence of the plasma; since the skin depth is much larger than r_o, screening currents become effective only at large radial distances, and the main effect is simply the field resulting from the surface displacement of a beam in vacuo. This suggests that the perturbed fields should be well approximated by solving Maxwell's equations with fixed sources. That is, the vacuum fields should be calculated and substituted into the conductivity law (6). The result is then used as the fixed current source in Maxwell's equations. Since this method does not treat the plasma screening effect adequately, it is necessary to join these fields to the correct asymptotic fields at some $r = r_p$. The asymptotic fields of Chapter 2 may be used for the matching if $r_p >> r_o$, since the magnitude of the Hall currents will then be negligible when $r \gtrsim r_p$. A consistent approximation results, provided that the fields depend weakly on the value of r_p. In this section only the component $f_z(r)$ of the perturbed electric field will be derived, since the preceding analysis has shown that it is extremely unlikely that the transverse fields contribute significantly to the dispersion law.

The single non-zero component, $f_z^o(r)$, of the vacuum electric field must satisfy the equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) f_z^{\circ}(r) = \left(\frac{4\pi i\omega a}{c} j_0\right)\delta(r - r_0) \tag{42}$$

and regularity conditions at $r = 0$ and $r = \infty$. From these $f_z^{\circ}(r)$ is uniquely determined to be

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right) r \quad \text{for } r \leq r_0$$

and (43)

$$f_z^{\circ}(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\frac{r_0^2}{r} \quad \text{for } r > r_0 .$$

Substitution into Eq. (6) gives for the plasma current

$$(\underset{\sim}{j})_p = -\left(\frac{2\pi i\omega a}{c} j_0\right)\frac{\sigma}{[1 + N_0^2 r^2]}\left(N_0 r^2, 0, r\right) \quad \text{for } r \leq r_0$$

and (44)

$$(\underset{\sim}{j})_p = -\left(\frac{2\pi i\omega a}{c} j_0\right)\frac{\sigma}{[1 + (N_1^2/r^2)]}\left(\frac{N_1 r_0^2}{r^2}, 0, \frac{r_0^2}{r}\right) \quad \text{for } r > r_0 .$$

Equation (44) is used in Eq. (7), and the z component determining $f_z(r)$ becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) f_z(r) = -h^2\left(\frac{2\pi i\omega a}{cN_0^2} j_0\right)\left(\frac{r}{r^2 + \frac{1}{N_0^2}}\right) \quad \text{for } r \leq r_0$$

and (45)

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)f_z(r)=-h^2\left(\frac{2\pi i\omega a}{c}j_0\right)\left(\frac{r}{r^2+N_1^2}\right) \text{ for } r_0\leq r\leq r_p.$$

For this equation the most general solution which also satisfies the necessary regularity conditions at $r=0$ is

$$f_z(r)=-\left(\frac{2\pi i\omega a}{c}j_0\right)-h^2\left(\frac{\pi i\omega a}{2cN_0^2}j_0\right)\left(r+\frac{1}{rN_0^2}\right)\ln\left(1+r^2N_0^2\right)+Ar \quad \text{for } r\leq r_0$$

and (46)

$$f_z(r)=-\left(\frac{2\pi i\omega a}{c}j_0\right)\frac{r_0^2}{r}-h^2r_0^2\left(\frac{\pi i\omega a}{2c}j_0\right)\left(r+\frac{N_1^2}{r}\right)\ln\left[\delta^2\left(1+\frac{r^2}{N_1^2}\right)\right]+\frac{\hat{A}}{r} \text{ for } r_0\leq r\leq r_p,$$

where A, $\hat{A}$, and δ are arbitrary constants. For convenience, the homogeneous solution $f_z^{\,o}(r)$ has been extracted from the remaining homogeneous terms in Eq. (46). Finally, the asymptotic form of the field has been determined in Chapter 2 to be

$$f_z(r)=CH_1'(ihr) \quad \text{for } r>r_p, \tag{47}$$

with C an arbitrary constant, provided that $r_p \gg r_o$.

The constants are determined from the usual boundary conditions at $r=r_o$

$$f_z(r_0)_{out}-f_z(r_0)_{in}=0$$

and (48)

$$\frac{d}{dr_0} f_z(r_0)_{out} - \frac{d}{dr_0} f_z(r_0)_{in} = \left(\frac{4\pi i\omega a}{c} j_0\right),$$

and from the field matching conditions at $r = r_p$

$$f_z(r_p)_{out} - f_z(r_p)_{in} = 0$$

and (49)

$$\frac{d}{dr_p} f_z(r_p)_{out} - \frac{d}{dr_p} f_z(r_p)_{in} = 0.$$

Equations (48) and (49) suffice to determine the four constants A, $\hat{A}$, δ and C uniquely. However, the requirement that the solution should depend weakly on r_p imposes the further condition

$$|hr_p| << 1 \qquad (50)$$

on the choice of r_p. This permits the expansions

$$H_1'(ihr) = -\left(\frac{2}{\pi hr}\right)\left(1 + \frac{1}{2} h^2 r^2 \ln\left|\frac{\gamma}{2} hr\right|\right)$$

and (51)

$$\frac{d}{dr} H_1'(ihr) = \left(\frac{2}{\pi hr^2}\right)\left(1 - \frac{1}{2} h^2 r^2 \ln\left|\frac{\gamma}{2} hr\right|\right)$$

to be used in Eq. (49), and only under these circumstances will the equations have acceptable solutions.

The solution of Eqs. (48) and (49) is given in Appendix IV. As before, when $N_0 r_0 << 1$ the fields and the dispersion law are unchanged from Chapter 2. When $N_o r_o \gg 1$ and $r_p^2 \gg N_o^2 r_o^4$ (and, in consequence, $|N_o r_o h r_o| \ll 1$) the equations possess solutions which depend weakly on r_p, and $f_z(r)$ is well approximated by

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left[\left|\frac{\gamma}{2} h r_0\right| \sqrt{1 + N_0^2 r_0^2}\right]\right). \qquad (52)$$

This is a modification of the field

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right) \qquad (53)$$

of Chapter 2 and gives rise to the modified dispersion law

$$(\omega + ku)^2 = \omega_B^2\left(-\ln\left[\left|\frac{\gamma}{2} h r_0\right| \sqrt{1 + N_0^2 r_0^2}\right]\right)\left(\frac{h^2 r_0^2}{2}\right). \qquad (54)$$

The modification amounts to an increase in the effective beam radius in the logarithmic term, corresponding to the fact that the Hall currents inhibit the flow of plasma current near the beam so that the screening currents must flow at larger radii.

Clearly, the analysis as presented offers no decisive way to choose between the conclusion of the present section (modification of the value of the logarithmic term) and of the previous section (no change in the dispersion law). The treatment of the present section, however, has a much stronger physical rationale and is probably to be preferred.

Neither analysis makes specific use of the condition that σ be positive. Thus the treatments should be valid for arbitrary σ.

Effects Due to Plasma Temperature Gradients

The previous sections have shown that the analysis of Chapter 2 adequately describes those low frequency, long wavelength disturbances for which $|hr_o| \ll 1$ provided that the plasma steady state is uniform in space. However, the assumption of spatial uniformity is not likely to be a good approximation to many experimental situations. For this reason various methods for relaxing the assumption will be investigated here and in following sections. No attempt is made to follow the complex hydrodynamic behavior of the plasma. Instead, it is assumed that the electromagnetic properties of the plasma are adequately represented by a tensor conductivity whose coefficients may depend on position.

In this section the analysis is confined to plasmas for which $\sigma > 0$ and disturbances for which $|hr_o| \ll 1$. It is assumed that the plasma temperature is essentially constant for $r < r_o$ and decreases monotonically as r increases when $r > r_o$. It is also assumed either that $N_o r_o \ll 1$ holds or that the plasma density is essentially constant. Under these circumstances tensor conductivity effects may be ignored, and the scalar conductivity depends on position through its dependence on plasma temperature. These assumptions do not give a fully self-consistent account of the complicated plasma collisional processes, but they do provide a tractable model of the processes which approximates experimental situations in which an appreciable amount of the plasma is created by ionization from the beam.

A specific model

$$\sigma'(r) = \sigma \quad \text{for} \quad r \leq r_0$$

and (42)

$$\sigma'(r) = \sigma\left(\alpha^\nu + \frac{\beta^\nu r_0^\nu}{r^\nu}\right) \quad \text{for} \quad r > r_0,$$

where

$$\alpha^\nu + \beta^\nu = 1,$$

is used for the position (temperature) dependent scalar conductivity $\sigma'(r)$. This simplifies the analysis considerably, but still gives results which are typical of a general monotonic variation of $\sigma'(r)$ which approaches a limiting value after several beam radii. The analysis is quite similar to Chapter 2. The field $f_z(r)$ must satisfy the equations

$$Hf_z = 0 \quad \text{for } r \le r_0$$

and (43)

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - h^2\alpha^2 - \frac{h^2\beta^2 r_0^2}{r^2}\right] f_z = 0 \quad \text{for } r > r_0$$

as well as the usual boundary conditions. The admissible solutions of Eq. (43) are

$$f_z(r) = A J_1(ihr) \quad \text{for } r \le r_0$$

and (44)

$$f_z(r) = \hat{A} H_n^1(i\alpha hr) \quad \text{for } r > r_0,$$

where

$$n = \left[1 + \beta^2 h^2 r_0^2\right] \simeq 1 + \tfrac{1}{2}\beta^2 h^2 r_0^2,$$

and the constant A is determined to be

$$A = \left(\frac{4\pi i\omega a}{c} j_0\right) \frac{H_n^1(i\alpha hr_0)}{\left[J_1 \frac{d}{dr} H_n^1 - H_n^1 \frac{d}{dr} J_1\right]_{r=r_0}} . \qquad (45)$$

Advantage may be taken of the facts that $|hr_0| \ll 1$ and $n \simeq 1$ by evaluating the Bessel functions as in Appendix II. The results are

$$J_1(ihr_0) = \left(\frac{ihr_0}{2}\right),$$

$$\frac{d}{dr_0} J_1(ihr_0) = \left(\frac{ih}{2}\right),$$

$$H_n^1(ihr) = -\left(\frac{2}{\pi\alpha hr}\right)\left(1+\frac{1}{2}h^2(\alpha^2 r^2-\beta^2 r_0^2)\ln\left|\frac{\gamma}{2}hr\right|\right),$$

(46)

$$H_n^1(ihr_0) = -\left(\frac{2}{\pi\alpha hr_0}\right)\left(1+\frac{1}{2}h^2 r_0^2(\alpha^2-\beta^2)\ln\left|\frac{\gamma}{2}hr_0\right|\right),$$

and

$$\frac{d}{dr}H_n^1(ihr)\bigg|_{r=r_0} = \left(\frac{2}{\pi\alpha hr_0^2}\right)\left(1-\frac{1}{2}h^2 r_0^2\ln\left|\frac{\gamma}{2}hr_0\right|\right).$$

Combination yields the denominator

$$\left[J_1\frac{d}{dr}H_n^1 - H_n^1\frac{d}{dr}J_1\right]_{r=r_0} = \left(\frac{2i}{\pi\alpha r_0}\right)\left(1-\frac{1}{2}\beta^2 h^2 r_0^2\ln\left|\frac{\gamma}{2}hr_0\right|\right), \qquad (47)$$

the coefficient

$$A = -\left(\frac{2\pi i\omega a r_0}{c}j_0\right)\left(1+\frac{1}{2}\alpha^2 h^2 r_0^2\ln\left|\frac{\gamma}{2}hr_0\right|\right), \qquad (48)$$

and the field

$$f_z(r) = -\left(\frac{2\pi i\omega a r_0}{c} j_0\right)\left(1+\frac{1}{2}\alpha^2 h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right)\frac{J_1(ihr)}{J_1(ihr_0)} . \qquad (49)$$

Evaluation of the dispersion relation leads to the expression

$$n_0 \pi r_0^2 m \gamma_0 (\omega + ku)^2 = (j_0 \pi r_0^2)^2 \alpha^2 h^2 \left[-\ln\left|\frac{\gamma}{2} h r_0\right|\right] \qquad (50)$$

as compared to the corresponding expression

$$n_0 \pi r_0^2 m \gamma_0 (\omega + ku)^2 = (j_0 \pi r_0^2)^2 h^2 \left[-\ln\left|\frac{\gamma}{2} h r_0\right|\right] \qquad (51)$$

of Chapter 2. Since $|ku|$ dominates $|\omega|$ the decrease in conductivity increases the growth rate by the factor $\alpha^{-2} > 1$ for fixed k. When the plasma is primarily produced by the beam, this increase can be large.

The assertion that Eq. (50) is typical for monotonic variations of $\sigma'(r)$ is supported by the second special case

$$\begin{aligned} \sigma'(r) &= \sigma \quad \text{for } r \leq r_1, \ r_0 < r_1 \\ \sigma'(r) &= \alpha^2 \sigma \quad \text{for } r > r_1 , \end{aligned} \qquad (52)$$

where

$$|hr_1| << 1,$$

which also leads to Eq. (50). Further support comes from the physical consideration that the right hand side of Eq. (52) is produced by plasma screening currents which flow primarily at distances R of the order

$$|hR| \simeq 1. \tag{53}$$

At these distances the conductivity has essentially reached its final value $\alpha^{\gamma}\sigma$, and this final value should appear in the dispersion law.

Effects Due to Plasma Density Gradients

A second idealized model of the plasma collisional processes will be considered in the present section. The model is also restricted to collision dominated plasmas for which $\sigma > 0$ and to disturbances for which $|hr_o| << 1$, but here the plasma temperature is assumed to be constant over all space while the plasma density varies monotonically, being constant for $r < r_o$ and decreasing as r increases for $r > r_o$. Thus the model isolates the effects due to the density gradients which occur, for example, when ionization from the beam produces an appreciable amount of the plasma. Since the scalar part of the conductivity tensor is density independent in this regime, all density gradient effects are due to the parameter μ occurring in the non-diagonal part of the tensor and are negligible when $N_o r_o << 1$. The treatment is therefore restricted to beam parameters for which $N_o r_o >> 1$ in order to exhibit these effects most clearly. For analytical convenience the specific model

$$n'(r) = \bar{n} \quad \text{for} \quad r \leq r_0$$

and (54)

$$n'(r) = \bar{n}\,\frac{r_0^2}{r^2} \quad \text{for} \quad r > r_0$$

is chosen for the position dependent plasma density $n'(r)$. The resulting problem is sufficiently similar to the cases discussed earlier in the chapter that not all details of the present discussion need be given.

The behavior of the fields $f_z(r)$ and $\lambda(r)$ is again described by Eqs. (14) and (16), but in this case $N = N_0$ for all r. Thus the solution furnished by Eq. (24) for $r < r_0$ must be supplemented by the solution

$$f_z(r) = C^+ H_1^1(i\delta^+ r) + C^- H_1^1(i\delta^- r)$$

and (55)

$$\lambda(r) = b^+ C^+ H_1^1(i\delta^+ r) + b^- C^- H_1^1(i\delta^- r),$$

where $b^{\pm}$ and $\delta^{\pm}$ are given by Eq. (30) for $r > r_0$. These solutions have the necessary regularity properties at $r = 0$ and $r = \infty$, and the constants $A^{\pm}$ and $C^{\pm}$ are determined, as usual, from the boundary conditions at $r = r_0$. The calculation is very similar to Case B of Appendix II, and yields

$$A^+ = -\left(\frac{2\pi i \omega a}{c} f_0\right) \frac{h^2}{N_0^2}\left(1 - \frac{3h^2}{N_0^2}\right) i\pi r_0 H_1^1(i\delta^+ r_0)$$

and (56)

$$A^{-} = -\left(\frac{2\pi i\omega}{a} j_0\right)\left(1-\frac{h^2}{N_0^2}\right)\left(1+\frac{1}{2}\left(\frac{h^2}{N_0^2}\right)h^2 r_0^2 \ln\left|\frac{\gamma}{2}hr_0\right|\right)\left(\frac{2i}{\delta^-}\right).$$

As before, the dispersion relation may be evaluated to sufficient accuracy when $f_z(r)$ is known for $r \leq r_0$. From Eq. (56) this field is given by

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1-\frac{h^2}{N_0^2}+\frac{1}{2}\frac{h^2}{N_0^2}h^2 r_0^2 \ln\left|\frac{\gamma}{2}hr_0\right|\right) r$$
$$-\left(\frac{2\pi i\omega a}{c} j_0\right)\left(\frac{h^2}{N_0^2}\right) i\pi r_0 H_1^1\left(i\delta^+ r_0\right) J_1\left(i\delta^+ r\right) \qquad (57)$$

for $r \leq r_0$, and to good approximation

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1-h^2 r_0^2\left[\frac{1}{N_0^2 r_0^2}\right]\right) r \quad \text{for } r \leq r_0. \qquad (58)$$

The dispersion relation becomes

$$n_0 \pi r_0^2 m \gamma_0 (\omega + ku)^2 = \left(j_0 \pi r_0^2\right)^2 h^2 \left[\frac{2}{N_0^2 r_0^2}\right] \qquad (59)$$

instead of Eq. (51), as in Chapter 2. Since $N_0 r_0 \gg 1$ and since $|ku| \gg |\omega|$ Eq. (59) implies that the growth rate for fixed k is larger when a density gradient is present. The reduction (for fixed ω) in the drag force exerted on the beam has a simple physical explanation. A strong self-magnetic field inhibits the flow of screening currents and the density gradient extends this effect to larger radii. Thus the screening currents tend to flow at larger radii, and their reaction on the beam is weaker.

The Effect of Metal Walls on Growth Rates

The stabilizing effect of metal walls on pinched discharges has been known for some time.[45-48] A similar effect occurs in the present configuration and is discussed below. Both the analysis and the dispersion law are somewhat different from the standard results on pinch stability;[46] the analysis makes use of the methods of Chapter 2, although the metal walls impose somewhat different boundary conditions on the problem, while the dispersion law is influenced by the presence of a conducting medium surrounding the beam.

It is assumed, as in Chapter 2, that a scalar conductivity σ gives an adequate account of the electromagnetic properties of the plasma, and attention will be restricted to those disturbances for which $|hr_0| << 1$. No restriction will be placed on the phase of the conductivity. A perfectly conducting wall is assumed to exist at $r = R > r_0$. The field $f_z(r)$ is determined by the same differential equation for $r \leq R$, the usual boundary conditions at $r = r_0$, regularity conditions at $r = 0$, and by the condition

$$f_z(R) = 0. \tag{60}$$

Thus the field takes the form

$$f_z(r) = A J_1(ihr) \quad \text{for } r \leq r_0$$

and (61)

$$f_z(r) = C H_1^1(ihr) + D J_1(ihr) \quad \text{for } r_0 \leq r \leq R,$$

where the constants A, C, and D are determined from the conditions

$$C H_1^1(ihR) + D J_1(ihR) = 0$$

$$C H_1^1(ihr_0) + D J_1(ihr_0) - A J_1(ihr_0) = 0 \tag{62}$$

and

$$C \frac{d}{dr} H_1^1(ihr_0) + D \frac{d}{dr_0} J_1(ihr_0) - A \frac{d}{dr_0} J_1(ihr_0) = \left(\frac{4\pi i \omega a}{c} j_0 \right).$$

Equation (62) readily yields

$$A - D = -\left(\frac{2\pi i \omega a r_0}{c} j_0\right) i\pi H_1'(ihr_0)$$

$$C = -\left(\frac{2\pi i \omega a r_0}{c} j_0\right) i\pi J_1(ihr_0)$$

(63)

and

$$D = -\frac{H_1'(ihR)}{J_1(ihR)} C,$$

so that the constant A is given by

$$A = -\left(\frac{2\pi i \omega a r_0}{c} j_0\right) i\pi \left[H_1'(ihr_0) - \frac{J_1(ihr_0)}{J_1(ihR)} H_1'(ihR)\right], \tag{64}$$

the field $f_z(r)$ is given for $r \leq r_0$ by

$$f_z(r) = -\left(\frac{2\pi i \omega a r_0}{c} j_0\right) i\pi J_1(ihr)\left[H_1'(ihr_0) - \frac{J_1(ihr_0)}{J_1(ihR)} H_1'(ihR)\right], \tag{65}$$

and the dispersion relation becomes

$$(n_0 \pi r_0^2 m v_0^2)(\omega + ku)^2$$
$$= 2(\pi j_0 r_0)^2\left(1 - i\pi J_1(ihr_0)\left[H_1'(ihr_0) - \frac{J_1(ihr_0)}{J_1(ihR)} H_1'(ihR)\right]\right). \tag{66}$$

The betatron frequency ω_B may be used to simplify this expression to

$$\left(\omega+ku\right)^2=\omega_B^2\left(1-i\pi J_1\left(ihr_0\right)\left[H_1^1\left(ihr_0\right)-\frac{J_1(ihr_0)}{J_1(ihR)}H_1^1(ihR)\right]\right). \tag{67}$$

The equation reduces to Eq. (90) of Chapter 2 when $|hR| \gg 1$, since the ratio

$$\frac{H_1^1(ihR)}{J_1(ihR)} \tag{68}$$

becomes exponentially small, and gives an account of the stabilizing effect of the walls when $|hR|$ is smaller. The full analysis of Eq. (67) is rather lengthy and is not carried out here. Instead, the stabilization effect will be exhibited by a treatment of the limiting case $|hR| \ll 1$. Small argument expansions may then be used for all Bessel functions, and the relations

$$|h^2 r_0^2| \ll 1$$

$$|h^2 R^2| \ll 1 \tag{69}$$

and

$$|h^2 r_0^2| \ll \frac{r_0^2}{R^2},$$

where

$$h^2 = \frac{4\pi i\omega\sigma}{c}$$

are useful in estimating the size of the terms involved. The relevant expansions are

$$J_1(ihr_0) \approx \left(\frac{ihr_0}{2}\right)$$

$$J_1(ihR) \approx \left(\frac{ihR}{2}\right)$$

$$H_1^1(ihr_0) \approx -\left(\frac{2}{\pi h r_0}\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right) \tag{70}$$

and

$$H_1^1(ihR) = -\left(\frac{r_0}{R}\right)\left(\frac{2}{\pi h r_0}\right)\left(1 + \frac{1}{2} h^2 R^2 \ln\left|\frac{\gamma}{2} hR\right|\right),$$

and combination gives

$$i\pi J_1(ihr_0) H_1^1(ihr_0) = \left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right)$$

and (71)

$$i\pi J_1(ihr_0) \frac{J_1(ihr_0)}{J_1(ihR)} H_1^1(ihR) = \left(\frac{r_0^2}{R^2} + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} hR\right|\right).$$

Thus the dispersion relation becomes

$$(\omega + ku)^2 = \omega_B^2 \left[\left(\frac{r_0^2}{R^2}\right) + \frac{1}{2} h^2 r_0^2 \ln\left(\frac{R}{r_0}\right)\right]. \tag{72}$$

For fixed k Eq. (72) is simply a quadratic equation in ω and its exact solution is readily obtained. However, it suffices for present purposes to determine ω as an expansion $\omega = \omega_0 + \omega_1 + \ldots$, where ω_0 satisfies the zeroth order equation

$$(\omega_0 + ku)^2 = \omega_B^2 \frac{r_0^2}{R^2} \tag{73}$$

and the correction ω_1 is obtained from

$$2(\omega_0 + ku)\omega_1 = \omega_B^2 \ln\left(\frac{R}{r_0}\right)\frac{2\pi i\sigma}{c}\omega_0 . \tag{74}$$

The two roots of these equations are given by

$$\omega_0^{\pm} = -ku \pm \left(\frac{\omega_B r_0}{R}\right)$$

and (75)

$$\omega_1^{\pm} = i\omega_B\left(\frac{\omega_B r_0}{c}\right)\left(\pi\sigma r_0 \ln\left(\frac{R}{r_0}\right)\right)\left(1 \mp \frac{ukR}{\omega_B r_0}\right),$$

and the instability appears in ω_1. When σ is purely imaginary (a collisionless plasma), the system is stable, as before. When σ has a positive real part, the system is stable for wavelengths for which

$$ku < \left(\frac{\omega_B r_0}{R}\right)$$

or (76)

$$\lambda > 2\pi R\left(\frac{u}{\omega_B r_0}\right)$$

and unstable for shorter wavelengths. For such wavelengths Eq. (75) may be used to compute growth rates provided that

$$\left(\frac{\omega_B R}{c}\right)\left[\pi\sigma r_0 \ln\left(\frac{R}{r_0}\right)\right]\left(\frac{kuR}{\omega_B r_0}\right) << 1. \tag{77}$$

Otherwise, Eq. (72) must be solved more accurately.

This analysis has shown that one effect of metal walls is the suppression of long wavelength instabilities. However, it does not follow that the maximum instability growth rate has been reduced. Growth rates still vary from 0 to about $\dot{\omega}_B$, but in this case they are associated with a finite range of wavelengths.

Arbitrary Beam Density Profiles

The analysis of Chapter 2 is not greatly dependent on the assumption of a uniform, sharp-edged beam. Arbitrary density profiles (in particular, the Bennett profile, which would be expected on statistical grounds) may be treated by similar methods. For disturbances such that $|hr_o| \ll 1$ the treatment leads to a dispersion law similar to Eq. (173) of Chapter 2. The analysis has been given by Rosenbluth [23] and will not be repeated here.

Summary of the Chapter

The low frequency, long wavelength instabilities of Chapter 2 are subject to many influences not studied in that chapter. The present chapter gives a somewhat more realistic treatment of the background plasma and examines the resultant modifications in the dispersion law for the disturbances. Subsequent chapters will study the modifications due to the microscopic beam particle motion.

When the self-magnetic field of the beam is large, the tensor character of the plasma conductivity becomes very pronounced and requires a separate treatment. An approximate analysis of this problem is given in the first portion of the chapter. It shows that while the pattern of plasma current flow is greatly altered by the Hall currents, the dispersion relation is not. The precise alteration of the law has not been established, but it is probable that the dominant effect appears in the logarithmic term of the dispersion law as an apparent increase in the beam diameter.

A crude analysis is given of the effects of plasma density and temperature gradients. It is found that for $\sigma > 0$ and $|hr_o| \ll 1$ the dominant effect in each case is to decrease the drag force for fixed ω and thus to increase instability rates for fixed k. For temperature gradients the scalar conductivity is smaller in the region where screening currents flow, so that the currents extend to larger distances and react back on the plasma less strongly. For density gradients the screening currents also flow at larger distances, but the effect is due to the inhibition of current flow near the beam by the large self-magnetic field.

The suppression of long wavelength instabilities by conducting walls is also illustrated. The effect does not reduce maximum growth rates but it does eliminate those instabilities which would be most effective in disrupting the gross motion of the beam. Finally, the extension of the analysis to arbitrary beam density profiles is indicated but not developed in detail.

Chapter 4

A MICROSCOPIC ANALYSIS OF THE CONFIGURATION

Method of Analysis

In this configuration the perturbing electromagnetic fields affect beam particles and the background plasma in quite different ways. Beam particles have a large z velocity and undergo rapid, large amplitude oscillations. They sample fields in large regions of space, and their response is markedly non-local. In contrast, plasma particles have no systematic motion and, to good approximation, respond locally to fields. Thus the preceding analyses of the configuration have treated the beam dynamics and the plasma dynamics asymmetrically, describing the beam dynamics in greater detail. In the following microscopic analysis, this asymmetry is even more pronounced. A relativistic collisionless Boltzmann equation will be used to describe the beam dynamics, while a scalar conductivity will summarize the relevant plasma dynamics.

This treatment of the plasma fails at very high frequencies, primarily because the beam neutralization is not adequately described. However, such frequencies are much larger than the betatron frequency, and the model is applicable to a broad range of disturbances, including certain well localized, very high frequency disturbances. No attempt will be made to increase the complexity of the plasma model, since severe mathematical problems are encountered with the present model, and since previous work suggests that the neglected effects are small.

An analysis based on the use of a collisionless Boltzmann equation for beam particles has sufficient accuracy to describe disturbances of any frequency and wavelength, provided that the growth rate of these disturbances is larger than the collision rate of the beam particles. The analysis is simplified by using a non-manifestly covariant formalism and by adopting the approximate relativistic beam dynamics developed in Chapter 2. This two-mass approximation is applicable to highly relativistic beams, and the longitudinal mass $m\gamma^3$ is assumed to be infinite, except when very high frequency disturbances are considered. These approximations greatly increase the mathematical tractability of the model without sacrificing much accuracy in the description of the beam dynamics.

The basic equations of the model are first written down in detail. An equilibrium solution having the macroscopic properties specified in Chapter 2 is obtained, and the beam particle orbits are computed. The stability problem is then formulated by linearizing the equations about this equilibrium solution. Maxwell's equations are unchanged in form, but the perturbed distribution function is obtained as a certain integral of the perturbed fields over equilibrium particle orbits. These orbits correspond to betatron oscillations of the beam particles and are responsible for the macroscopic beam pressure. The configuration is not well described by the standard orbit theory approximations, since the orbit size is comparable to the beam diameter.

Appropriate field variables for the stability analysis are suggested by a closer examination of Maxwell's equations. The perturbed current is separated into plasma current, which is incorporated into the homogeneous field equations, and an unspecified second current, which is treated as a driving term. As before, all physical quantities are assumed to have their t, z, and θ dependence contained in the factor

$$\psi = \exp i(\omega t + kz + m\theta), \tag{1}$$

but here m may take on any integer value. This permits Maxwell's equations to be written as a set of coupled ordinary differential equations. By proper choice of variables, the equations are separated and solved in the form of integrals which depend on the second current. These solutions become integral equations when the second current is calculated from the perturbed beam distribution function. The stability problem is thus simplified by a transformation from differential-integral form to purely integral form.

A lengthy evaluation of the perturbed beam current is then given in order to display the integral equations in simple, explicit form. A reduced set of equations is also obtained by assuming that the transverse beam velocity is much smaller than c. Solutions are found for the equations and reduced equations, and in both cases the dispersion relation appears as a condition that the equations be solvable. A detailed examination of the results is given in the next chapter.

Formulation of the Stability Problem

The labeling of field variables will follow the conventions established in Chapter 2, unless otherwise stated. Both rectangular and cylindrical coordinates will be used during the analysis, and conventional notation is adopted for coordinates; e.g.,

$$\underset{\sim}{r} = (x, y, z)$$

and

$$\underset{\sim}{v} = (v_x, v_y, v_z) \qquad (2)$$

denote the position and velocity of beam particles in rectangular coordinates. It will be convenient to denote the parts of vectors which are parallel to and perpendicular to the z axis by the subscripts $\parallel$ and $\perp$ respectively. To eliminate confusion with the beam particle velocities, the symbols $\bar{\mathbf{v}}'$, $\bar{\mathbf{v}}_0$, and $\bar{\mathbf{v}}$ will be used for the macroscopic beam velocities. The general beam particle distribution function is denoted by f'and is chosen to depend on the variables t, $\mathbf{r}$, and $\mathbf{v}$. Other definitions will be introduced as needed.

Maxwell's equations and the plasma conductivity law - - Eqs. (24) and (25) of Chapter 2 - - must be supplemented by the Boltzmann equation and by the relation

$$\mathbf{j}' = -(e/c)\int d^3v\,\mathbf{v}\,f' \tag{3}$$

between f' and the beam current. As before, it is assumed that the beam particles are electrons. The collisionless Boltzmann equation has the general form

$$\frac{\partial}{\partial t} f' + \left(\frac{d}{dt}\mathbf{r}\cdot\nabla\right) f' + \left(\frac{d}{dt}\mathbf{v}\cdot\nabla_{\mathbf{v}}\right) f' = 0 \tag{4}$$

and states that f' is constant along particle trajectories. The time derivatives $(d/dt)\mathbf{r}$ and $(d/dt)\mathbf{v}$ are determined by the two-mass approximation to the relativistic particle dynamics and are evaluated as

$$\frac{d}{dt}\mathbf{r} = \mathbf{v},$$

$$\frac{d}{dt}\underset{\sim}{v}_\perp = -\left(\frac{e}{m\gamma_0}\right)\left(\underset{\sim}{E}' + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{B}'\right)_\perp ,$$

and $$\frac{d}{dt}\underset{\sim}{v}_\parallel = -\left(\frac{e}{m\gamma_0^3}\right)\left(\underset{\sim}{E}' + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{B}'\right)_\parallel . \qquad (5)$$

In this chapter the last part of Eq. (5) is replaced by the simpler condition

$$\frac{d}{dt}\underset{\sim}{v}_\parallel = 0 \qquad (6)$$

(i.e., the longitudinal mass $m\gamma_0^3$ is assumed to be infinite). The Boltzmann equation then becomes

$$\left(\frac{d}{dt} + \underset{\sim}{v}\cdot\nabla\right)f' - \left(\frac{e}{m\gamma_0}\right)\left(\left[\underset{\sim}{E}' + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{B}'\right]_\perp \cdot \nabla_{\underset{\sim}{v}_\perp}\right)f' = 0. \qquad (7)$$

The relativistic invariance implicit in Eq. (4) has been lost in Eq. (7), but a very good approximation to the dynamics of highly relativistic, low temperature electron beams has been retained.

When $\underset{\sim}{E}'$ and $\underset{\sim}{B}'$ are known, first integrals of Eqs. (5) and (6) may be found. Any function of such constants of the motion is a solution of Eq. (7). Solutions corresponding to the equilibrium configuration of Chapter 2 are obtained by finding constants of the motion associated with the equilibrium electromagnetic fields

$$\underset{\sim}{E}_0 = \underset{\sim}{0}$$

and $$\underset{\sim}{B}_0 = \nabla \times \underset{\sim}{A},$$

where

$$\underset{\sim}{A} = A(r)(0, 0, 1),$$

$$A(r) = \left(\frac{\pi e n_0 u}{c}\right) r^2 \quad \text{for } r \leq r_0, \tag{8}$$

and

$$A(r) = \left(\frac{\pi e n_0 u}{c}\right) r_0^{\,2} + \ln(r/r_0) \quad \text{for } r > r_0.$$

and by finding a function of these constants which has the macroscopic properties necessary for self-consistency. That is, the beam must be entirely confined to the region $r \leq r_0$ and must have a uniform density n_0 and a macroscopic velocity $\underset{\sim}{v}_0 = (0, 0, u)$ in this region. It is assumed, as usual, that the plasma neutralizes the beam without appreciably affecting its own uniformity. In rectangular coordinates the magnetic field takes the form

$$\underset{\sim}{B}_0 = \left(\frac{2\pi e n_0 u}{c}\right)(y, -x, 0), \tag{9}$$

while the equations of motion become

$$\ddot{x} + \omega_0^2 x = 0,$$

$$\ddot{y} + \omega_0^2 y = 0,$$

and

$$\dot{v}_z = 0, \tag{10}$$

where

$$\omega_0^2 = \left(\frac{2\pi e^2 n_0}{m\gamma_0}\right)\left(\frac{u v_z}{c^2}\right).$$

Constants of the motion

$$\alpha_1 = v_z ,$$

$$\alpha_2 = v_x^2 + \omega_0^2 x^2 , \tag{11}$$

and

$$\alpha_3 = v_y^2 + \omega_0^2 y^2$$

are obtained by inspection, and the combinations

$$\beta_1 = \alpha_1$$

and

$$\beta_2 = \alpha_2 + \alpha_3 = v_\perp^2 + \omega_0^2 r^2 \tag{12}$$

have the symmetry properties required by the configuration.

The constants β_1 and β_2 are used to construct solutions of Eq. (7) having the required macroscopic properties. This is done by finding solutions f_0 of the form

$$f_0\left(r, v_\perp^2, v_z\right) = n_0 h_0(\beta_1) g_0(\beta_2) , \tag{13}$$

where h_0 satisfies the conditions

$$h_0 \geq 0,$$

$$\int dv_z\, h_0(v_z) = 1, \tag{14}$$

and

$$\int dv_z\, v_z\, h_0(v_z) = u,$$

and g_o satisfies

$$g_o \geq 0$$

and (15)

$$\int dv_x dv_y g_o(\beta_r) = 1 \quad \text{for } r \leq r_o$$
$$= 0 \quad \text{for } r > r_o .$$

The function h_o is not uniquely determined by Eq. (14). For simplicity, the solution

$$h_o = \delta(v_z - u) \tag{16}$$

will be adopted, corresponding to a beam with no longitudinal temperature. Other admissible choices for h_o will not be considered explicitly, since they would not change the instability analysis appreciably. The form of g_o is much more strongly limited by Eq. (15). The integral condition may be rewritten as

$$\int_{\omega_o^2 r^2}^{\infty} d\beta_r g_o(\beta_r) = \frac{1}{\pi} \quad \text{for } r \leq r_o$$
$$= 0 \quad \text{for } r > r_o , \tag{17}$$

which, together with the positivity of g_o, forces g_o to have the form of a delta function centered about $\beta_r = \omega_o^2 r_o^2$. The simplest such solution,

$$g_o = \frac{1}{\pi} \delta\left(v_\perp^2 + \omega_o^2 \left[r^2 - r_o^2 \right]\right) , \tag{18}$$

is the one adopted. The beam is thus forced by Eq. (15) to have no transverse temperature. The form chosen for h_o permits v_z to be eliminated from g_o by means of the identity

$$g_0 = \frac{1}{\pi}\,\delta\left(v_\perp^2 - \omega_B^2\,\varepsilon^2\right), \tag{19}$$

where ω_B, the betatron frequency, is defined by

$$\omega_B^2 = \left(\frac{2\pi e^2 n_0\, u^2}{m\gamma_0\, c^2}\right)$$

and $\mathcal{E}$ is given by (20)

$$\varepsilon^2 = r_0^2 - r^2 .$$

The absence of thermal effects does not imply the absence of beam pressure (i.e., of momentum transfer). Momentum is carried by the macroscopic beam motion; but the individual betatron oscillations of beam particles also transfer momentum, and this effect is quite similar to a thermal pressure. A comparison between the pressure tensor

$$\left(P^0_{ij}\right) = \pi j_0^2\,\varepsilon^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

obtained in Chapter 2 and the tensor

$$\left(P^0_{ij}\right) = n_0 m\gamma_0 \int dv_x\, dv_y\, g_0\left(v_\perp^2 - \omega_B^2\,\varepsilon^2\right) \begin{pmatrix} v_x^2 & v_x v_y & 0 \\ v_y v_x & v_y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

$$= \pi j_0^2\,\varepsilon^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

computed from f_0 verifies again that the present equilibrium configuration reproduces the macroscopic features of the previous configuration. It also indicates that the pressure is due to microscopic particle motions of a non-thermal nature. The discrepancy in the values of P°_{zz} is of no great importance, since this component is not needed for the analysis of low frequency, long wavelength disturbances.

The particle orbits responsible for the beam pressure are obtained from the equations

$$\ddot{x} + \omega_B^2 x = 0,$$

$$\ddot{y} + \omega_B^2 y = 0,$$

and

$$\ddot{z} = 0. \qquad (23)$$

The particle motions are composed of a uniform velocity in the z direction and oscillations (betatron oscillations) at the betatron frequency in the x and y directions. Solutions for time $\hat{t} = t + t'$ which have the form

$$\hat{x} = x(\hat{t}) = x \cos \omega_B t' + (v_x/\omega_B) \sin \omega_B t'$$

$$\hat{y} = y(\hat{t}) = y \cos \omega_B t' + (v_y/\omega_B) \sin \omega_B t' \qquad (24)$$

and

$$\hat{z} = z(\hat{t}) = z + v_z t',$$

where

$$r^2 = x^2 + y^2 \leq r_0^2$$

and

$$v_x^2 + v_y^2 = \omega_B^2 [r_0^2 - r^2],$$

take on the initial conditions $\underset{\sim}{r}$ and $\underset{\sim}{v}$ at time t and describe all possible equilibrium orbits of the beam particles.

The stability problem is formulated, as usual, by giving the linear equations which govern small departures from the equilibrium configuration. The equations governing the plasma current and the electromagnetic fields are already in linear form and are unchanged for the stability analysis, but the Boltzmann equation becomes

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla - \left(\frac{e}{m\gamma_0 c}\right)\left(\underset{\sim}{v} \times \underset{\sim}{B}_0\right)_\perp \cdot \nabla_{\underset{\sim}{v}\perp}\right] f$$
$$= \left(\frac{e}{m\gamma_0}\right)\left[\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v} \times \underset{\sim}{B}\right)_\perp \cdot \nabla_{\underset{\sim}{v}\perp}\right] f_0 \, . \tag{25}$$

This equation states that when the perturbed electromagnetic fields are known, the time derivative of the perturbed distribution function f, evaluated along the equilibrium particle orbits, is a known function. If, in addition, f is known at some instant of time, it may be found at any other instant by integrating Eq. (25) along the equilibrium orbits given by Eq. (24). This technique may always be used in the search for instabilities; since solutions with exponentially growing time dependences are sought, f at time $-\infty$ is necessarily zero. Thus f at time t is given by

$$f(t, \underset{\sim}{r}, \underset{\sim}{v}) = \left(\frac{e}{m\gamma}\right)\int_{-\infty}^{0} dt' \left[\left(\hat{\underset{\sim}{E}} + \frac{1}{c}\hat{\underset{\sim}{v}} \times \hat{\underset{\sim}{B}}\right)_{\perp} \cdot \nabla_{\hat{\underset{\sim}{v}}_{\perp}}\right]\hat{f}_{0}, \tag{26}$$

where all quantities in the integrand are evaluated at time $\hat{t} = t + t'$; that is, the quantities t, $\underset{\sim}{r}$, and $\underset{\sim}{v}$ are replaced by the values $\hat{t}$, $\hat{\underset{\sim}{r}}$, and $\hat{\underset{\sim}{v}}$ specified in Eq. (24) and its derivative. The integral may be simplified to the form

$$f = \left(\frac{2en_0}{m\gamma_0}\right) h_0 g_0' \int_{-\infty}^{0} dt' \hat{\underset{\sim}{v}}_{\perp} \cdot \left(\hat{\underset{\sim}{E}} + \frac{1}{c}\hat{\underset{\sim}{v}}_{\parallel} \times \hat{\underset{\sim}{B}}\right), \tag{27}$$

where

$$g_0' = \frac{\partial}{\partial v_{\perp}^2} g_0\left(v_{\perp}^2 - \omega_B^2 \mathcal{E}^2\right),$$

since g_0 and h_0 are constants of the motion. The perturbed current, needed for solving Maxwell's equations, is then given by

$$\underset{\sim}{j} = -\left(\frac{2e^2 n_0}{m\gamma_0 c}\right)\int d^3v \underset{\sim}{v}\, h_0 g_0' \int_{-\infty}^{0} dt' \hat{\underset{\sim}{v}}_{\perp} \cdot \left(\hat{\underset{\sim}{E}} + \frac{1}{c}\hat{\underset{\sim}{v}}_{\parallel} \times \hat{\underset{\sim}{B}}\right). \tag{28}$$

This equation requires an extensive evaluation before it is helpful in obtaining an explicit solution of Maxwell's equations, but before this is done, it is useful to examine the structure of Maxwell's equations more closely.

Integral Form for Maxwell's Equations

The discussion of Maxwell's equations will be carried out using cylindrical coordinates and adopting $\underset{\sim}{f}(r)$, defined by

$$\underset{\sim}{E} = \psi \underset{\sim}{f}(r), \tag{29}$$

for the basic variable. The magnetic field

$$\begin{aligned} \underset{\sim}{B} &= \left(\frac{ic}{\omega}\right) \nabla \times E \\ &= \left(\frac{ic}{\omega}\right) \psi \left(\frac{im}{r} f_z - ikf_\theta, ikf_r - \frac{d}{dr} f_z, \frac{1}{r}\frac{d}{dr} r f_\theta - \frac{im}{r} f_r\right) \end{aligned} \tag{30}$$

is obtained from the curl $\underset{\sim}{E}$ equation, the plasma conductivity law becomes

$$\underset{\sim}{j}_p = \sigma \psi \underset{\sim}{f}, \tag{31}$$

and the second current is denoted by

$$\underset{\sim}{j} = \psi \underset{\sim}{j}(r). \tag{32}$$

The following reduction is based solely on the fact that the t, z, and θ dependences of all field variables are contained in the factor ψ; however, when the stability problem is analyzed further, $\underset{\sim}{j}$ will be taken to be the beam current.

The curl B equation yields the basic equation governing $\underset{\sim}{f}(r)$, namely,

$$\nabla \times (\nabla \times \psi \underset{\sim}{f}) + \left(\frac{4\pi\sigma i\omega}{c} - \frac{\omega^2}{c^2}\right)\psi \underset{\sim}{f} = -\left(\frac{4\pi i\omega}{c}\right)\psi \underset{\sim}{j}(r) . \qquad (33)$$

The divergence equations are needed only as initial conditions, and, for the instability problem, are satisfied identically at time $t = -\infty$. The first term of Eq. (33) gives an awkward coupling of the components of $\underset{\sim}{f}(r)$, as shown by

$$\left[\nabla \times (\nabla \times \psi \underset{\sim}{f})\right]_r = \psi\left(k^2 f_r + \frac{d}{dr} ikf_z + \frac{im}{r^2}\frac{d}{dr} rf_\theta + \frac{m^2}{r^2} f_r\right),$$

$$\left[\nabla \times (\nabla \times \psi \underset{\sim}{f})\right]_\theta = \psi\left(k^2 f_\theta + \frac{im}{r} ikf_z - \frac{d}{dr}\frac{1}{r}\frac{d}{dr} rf_\theta + im\frac{d}{dr}\frac{1}{r}f_r\right), \qquad (34)$$

and

$$\left[\nabla \times (\nabla \times \psi \underset{\sim}{f})\right]_z = \psi\left(\frac{m^2}{r^2} f_z - \frac{1}{r}\frac{d}{dr} r\frac{d}{dr} f_z + ik\left[\frac{im}{r} f_\theta + \frac{1}{r}\frac{d}{dr} rf_r\right]\right).$$

These expressions may be simplified by isolating the terms $k^2\underset{\sim}{f}(r)$ and $(\nabla \cdot \psi \underset{\sim}{f})$, and by making use of the conventions

$$(\nabla \cdot \psi \underset{\sim}{f}(r)) = \psi\left(\frac{1}{r}\frac{d}{dr} rf_r + \frac{im}{r} f_\theta + ikf_z\right)$$

$$= \psi(\nabla_0 \cdot \underset{\sim}{f})$$

and

$$(\nabla \cdot \psi \underset{\sim}{j}(r)) = \psi(\nabla_0 \cdot \underset{\sim}{j}) . \qquad (35)$$

The first result is

$$\left[\nabla\times(\nabla\times\psi\,\underset{\sim}{f})\right]_r = \psi\left(k^2 f_r + \frac{d}{dr}(\nabla_0\cdot\underset{\sim}{f}) + \left[\frac{m^2}{r^2} - \frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\right]f_r + \frac{2im}{r^2}f_\theta\right),$$

$$\left[\nabla\times(\nabla\times\psi\,\underset{\sim}{f})\right]_\theta = \psi\left(k^2 f_\theta + \frac{im}{r}(\nabla_0\cdot\underset{\sim}{f}) + \left[\frac{m^2}{r^2} - \frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\right]f_\theta - \frac{2im}{r^2}f_r\right), \qquad (36)$$

and

$$\left[\nabla\times(\nabla\times\psi\,\underset{\sim}{f})\right]_z = \psi\left(k^2 f_z + ik(\nabla_0\cdot\underset{\sim}{f}) + \left(\frac{m^2}{r^2} - \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)f_z\right).$$

This increasing similarity among the components of $\nabla\times(\nabla\times\psi\,\underset{\sim}{f})$ may be exploited by introducing the definitions

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr},$$

$$h^2 = \left(\frac{4\pi\sigma i\omega}{c}\right) + k^2 - \frac{\omega^2}{c^2},$$

$$Q_n = \nabla^2 - h^2 - \frac{n^2}{r^2},$$

$$\Delta^{\pm} = \frac{d}{dr} \mp \frac{m}{r}, \qquad (37)$$

$$f^{\pm} = f_r(r) \pm i f_\theta(r),$$

$$j^{\pm} = j_r(r) \pm i j_\theta(r)$$

and

$$j_z = j_z(r)$$

and using the operator identity

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r = \nabla^2 - \frac{1}{r^2}. \tag{38}$$

The +, -, and z components of Eq. (33) then yield

$$Q_{m+1}f^+ = \Delta^+(\nabla_0 \cdot f) + \left(\frac{4\pi i\omega}{c}\right) j^+,$$

$$Q_{m-1}f^- = \Delta^-(\nabla_0 \cdot f) + \left(\frac{4\pi i\omega}{c}\right) j^-, \tag{39}$$

and

$$Q_m f_z = ik(\nabla_0 \cdot f) + \left(\frac{4\pi i\omega}{c}\right) j_z.$$

The only complicating feature of Eq. (39) is the coupling furnished by $(\nabla_0 \cdot f)$. This is eliminated by use of the condition

$$(h^2 - k^2)(\nabla_0 \cdot f) = -\left(\frac{4\pi i\omega}{c}\right)(\nabla_0 \cdot j) \tag{40}$$

obtained from the divergence of Eq. (33). The field equations then take the form

$$Q_{m+1} f^+ = \left(\frac{4\pi i\omega}{c}\right)\left(j^+ + \frac{\Delta^+(\nabla_0 \cdot j)}{k^2 - h^2}\right),$$

$$Q_{m-1} f^- = \left(\frac{4\pi i\omega}{c}\right)\left(j^- + \frac{\Delta^-(\nabla_0 \cdot j)}{k^2 - h^2}\right), \tag{41}$$

and

$$Q_m f_z = \left(\frac{4\pi i\omega}{c}\right)\left(j_z + \frac{ik(\nabla_0 \cdot j)}{k^2 - h^2}\right),$$

and their solution is straightforward.

The fields must be finite at $r = 0$ and zero at $r = \infty$ and can thus be obtained by integration from the

Green's functions of Appendix I. For simplicity the results are expressed in terms of the Hankel transforms

$$F^{\pm}(\ell) = \int_0^{\infty} r\,dr\, J_{m\pm 1}(\ell r)\, f^{\pm}(r)$$

and (42)

$$F_z(\ell) = \int_0^{\infty} r\,dr\, J_m(\ell r)\, f_z(r),$$

rather than of the fields themselves. The notation and conventions of Appendix I will be adopted; in particular, the condition

$$\mathrm{Re}\, h > 0 \qquad (43)$$

is imposed, and the solutions may be expressed as

$$F_z(\ell) = \left(\frac{4\pi i\omega}{c}\right)\int_0^{\infty} r'dr'\, G_m(\ell, r')\left[j_z(r') + \frac{ik}{k^2-h^2}(\nabla_0\cdot \underset{\sim}{j})\right]$$

$$= -\left[\frac{4\pi i\omega}{c(\ell^2+h^2)}\right]\int_0^{\infty} r'dr'\, J_m(\ell r')\left[j_z(r') + \frac{ik}{k^2-h^2}(\nabla\cdot \underset{\sim}{j})\right]$$

and (44)

$$F^{\pm}(\ell) = -\left[\frac{4\pi i\omega}{c(\ell^2+h^2)}\right]\int_0^{\infty} r'dr'\, J_{m\pm 1}(\ell r')\left[j^{\pm}(r') + \left(\frac{d}{dr'} \mp \frac{m}{r'}\right)\frac{(\nabla_0\cdot \underset{\sim}{j})}{k^2-h^2}\right].$$

The equations for $F^{\pm}(\ell)$ may be simplified by the identities

$$\int_0^{\infty} r'dr'\, J_{m+1}(\ell r')\left[\frac{d}{dr'} - \frac{m}{r'}\right](\nabla_0\cdot \underset{\sim}{j}) = -\int_0^{\infty} r'dr'(\nabla_0\cdot \underset{\sim}{j})\left[\frac{m+1}{r'} + \frac{d}{dr'}\right]J_{m+1}(\ell r')$$

$$= -\ell\int_0^{\infty} r'dr'\, J_m(\ell r')(\nabla_0\cdot \underset{\sim}{j})$$

and (45)

$$\int_0^{\infty} r'dr'\, J_{m-1}(\ell r')\left[\frac{d}{dr'} + \frac{m}{r'}\right](\nabla_0\cdot \underset{\sim}{j}) = -\int_0^{\infty} r'dr'(\nabla_0\cdot \underset{\sim}{j})\left[\frac{d}{dr'} - \frac{m-1}{r'}\right]J_{m-1}(\ell r')$$

$$= +\ell\int_0^{\infty} r'dr'\, J_m(\ell r')(\nabla_0\cdot \underset{\sim}{j}),$$

provided that the second current vanishes more rapidly than $r^{3/2}$ as $r \to \infty$. Substitution into Eq. (44) then yields

$$F^{+}(\ell) = -\left[\frac{4\pi i\omega}{c(\ell^2+h^2)}\right]\int_0^\infty r'dr'\left[J_{m+1}(\ell r')j^{+}(r') - \frac{\ell}{k^2-h^2}J_m(\ell r')(\nabla\cdot\underset{\sim}{j})\right],$$

$$F^{-}(\ell) = -\left[\frac{4\pi i\omega}{c(\ell^2+h^2)}\right]\int_0^\infty r'dr'\left[J_{m-1}(\ell r')j^{-}(r') + \frac{\ell}{k^2-h^2}J_m(\ell r')(\nabla\cdot\underset{\sim}{j})\right],$$

(46)

and

$$F_z(\ell) = -\left[\frac{4\pi i\omega}{c(\ell^2+h^2)}\right]\int_0^\infty r'dr'\,J_m(\ell r')\left[j_z(r) + \frac{ik}{k^2-h^2}(\nabla\cdot\underset{\sim}{j})\right].$$

This gives a simple, explicit solution for the fields when the second current is known. For the stability problem the current is known only in terms of the perturbed fields, and Eq. (46) leads to a set of integral equations for $F^{\pm}(\ell)$ and $F_z(\ell)$. A single integration of Eq. (46) yields an analogous set of equations for the variables $f^{\pm}(r)$ and $f_z(r)$. The set is less useful for the stability problem, however, and will not be given explicitly.

Evaluation of Perturbed Currents

An extensive reduction of Eq. (28) is needed to obtain explicit formulae expressing the currents $j^{\pm}$, j_z, and $(\nabla\cdot j)$ as integrals over $F^{\pm}(\ell)$ and $F_z(\ell)$. As a first step the current is separated into local and non-local parts. To do this, the orbit integral of Eq. (28) must be examined. The expressions

$$\frac{1}{c}\underset{\sim}{v}_{\parallel}\times\underset{\sim}{B}=\frac{kv_z}{\omega}\underset{\sim}{E}_{\perp}-\frac{v_z}{i\omega}\nabla_{\perp}E_z$$

and (47)

$$\frac{1}{c}\underset{\sim}{v}_{\perp}\cdot\left(\underset{\sim}{v}_{\parallel}\times\underset{\sim}{B}\right)=\frac{kv_z}{\omega}\left(\underset{\sim}{v}_{\perp}\cdot\underset{\sim}{E}_{\perp}\right)-\frac{v_z}{i\omega}\left(\underset{\sim}{v}_{\perp}\cdot\nabla_{\perp}\right)E_z$$

are obtained from Eqs. (1), (29), and (30) for the electromagnetic fields and may be simplified by means of the hydrodynamic time derivative

$$\begin{aligned}\frac{d}{dt}&=\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}+\left(\underset{\sim}{v}_{\perp}\cdot\nabla_{\perp}\right)\\&=i\left(\omega+kv_z\right)+\left(\underset{\sim}{v}_{\perp}\cdot\nabla_{\perp}\right)\end{aligned}\qquad(48)$$

which is valid for the orbit integral. This permits the integrand to be written as

$$v_{\perp}\cdot\left(\underset{\sim}{E}+\frac{1}{c}v_{\parallel}\times\underset{\sim}{B}\right)_{\perp}=\left(\frac{\omega+kv_z}{\omega}\right)\left(\underset{\sim}{v}\cdot\underset{\sim}{E}\right)-\frac{v_z}{i\omega}\frac{d}{dt}E_z\qquad(49)$$

for time t, with a similar expression holding for time $\hat{t}$. The time derivative is easily integrated, since v_z is a constant of the motion and E_z vanishes at time $\hat{t}=-\infty$. The current then takes the form

$$\underset{\sim}{j}=\left(\frac{2e^2n_0}{m\gamma_0 c}\right)\int d^3v\,\underset{\sim}{v}\,h_0 g_0'\left[\frac{v_z}{i\omega}E_z-\left(\frac{\omega+kv_z}{\omega}\right)\int_{-\infty}^{0}dt'\left(\hat{\underset{\sim}{v}}\cdot\hat{\underset{\sim}{E}}\right)\right],\qquad(50)$$

which gives the required decomposition

$$\underset{\sim}{j} = \underset{\sim}{j}_L + \underset{\sim}{j}_N \; . \tag{51}$$

The evaluation of the local current

$$\underset{\sim}{j}_L = \left(\frac{2e^2 n_0 \gamma}{m\gamma_0 ci\omega}\right) f_z(r) \int d^3v \, v \, v_z \underset{\sim}{v} \, h_0 g_0' \tag{52}$$

is quite simple. Symmetry requires that $j_L^{\pm}$ be zero, while j_{zL} is given by

$$\begin{aligned} j_{zL} &= \left(\frac{2e^2 n_0 u^2}{m\gamma_0 ci\omega}\right) f_z(r) \int_0^\infty dv_\perp^2 \frac{d}{dv_\perp^2} \delta\left(v_\perp^2 - \omega_B^2 \mathcal{E}^2\right) \\ &= -\left(\frac{c}{i\pi\omega}\right) f_z(r)\, \omega_B^2 \, \delta\left(\omega_B^2 \mathcal{E}^2\right) \\ &= -\left(\frac{c}{i2\pi\omega}\right)\left(\frac{f_z(r_0)}{r_0}\right) \delta(r - r_0) \, . \end{aligned} \tag{53}$$

The result is a close approximation to the surface current encountered in Chapter 2.

A direct evaluation of the non-local current is quite difficult; $\hat{r}$ and $\hat{\theta}$ are awkward to compute, and the integrand has a very complex form. Thus a subsidiary decomposition of the integrand

$$\underset{\sim}{v} \cdot \underset{\sim}{E} = v_z E_z + \tfrac{1}{2}\left(v^- E^+ + v^+ E^-\right) \tag{54}$$

into plane waves is adopted, permitting the direct use of rectangular coordinates and simplifying the form of the integrand. The decomposition is suggested by the identities

$$E_z = \psi f_z(r)$$
$$= e^{i(kz+\omega t)} \int_0^\infty \ell\, d\ell\, F_z(\ell) e^{im\theta} J_m(\ell r) \tag{55}$$
$$= e^{i(kz+\omega t)} \int_0^\infty \ell\, d\ell \int_0^{2\pi} \frac{d\delta}{2\pi} F_z(\ell) e^{im(\delta+\theta-\frac{\pi}{2})} e^{i\ell r \cos\delta}.$$ [49]

The definitions

$$\beta = \delta + \theta,$$
$$k_1 = \ell \cos\beta$$
$$k_2 = \ell \sin\beta \tag{56}$$

and

$$\bar{P}_n = \int_0^\infty \ell\, d\ell \int_0^{2\pi} \frac{d\beta}{2\pi} e^{in(\beta-\frac{\pi}{2})}$$

imply that

$$\ell r \cos\delta = k_1 x + k_2 y \tag{57}$$

and yield the decomposition

$$E^{\pm} = e^{i(kz+\omega t)} e^{\mp i\theta} \bar{P}_{m\pm 1} F^{\pm}(\ell) e^{i(k_1 x + k_2 y)}$$

and

$$E_z = e^{i(kz+\omega t)} \bar{P}_m F_z(\ell) e^{i(k_1 x + k_2 y)}. \tag{58}$$

The further identity

$$v^{\pm} = v_r \pm i v_\theta$$
$$= \left[v_x \pm i v_y\right] e^{\mp i\theta} \tag{59}$$

and Eq. (58) permit Eq. (54) to be written as

$$(\underset{\sim}{v}\cdot\underset{\sim}{E}) = e^{i(kz+\omega t)}\left[\begin{array}{l} v_z\,\bar{P}_m\,F_z(\ell)\,e^{i(k_1x+k_2y)} \\ +\frac{1}{2}\left([v_x-iv_y]\bar{P}_{m+1}\,F^+(\ell)+[v_x+iv_y]\,\bar{P}_{m-1}F^-(\ell)\right)e^{i(k_1x+k_2y)}\end{array}\right], \tag{60}$$

which depends only on rectangular position and velocity coordinates and is thus readily evaluated at time $\hat{t}$ by means of Eq. (24).

It is also convenient to introduce the variables s, v, α and λ, the frequency $\bar{\Omega}$, and the phase $\bar{\phi}$ by means of the definitions

$$\begin{aligned} s &= \omega_B t' \\ v &= |\underset{\sim}{v}_\perp| \\ v^{\pm} &= ve^{\pm i\alpha} \\ \lambda &= \alpha-\delta = (\alpha+\theta)-\beta \\ \bar{\Omega} &= \left(\frac{\omega+kv_z}{\omega_B}\right) \end{aligned} \tag{61}$$

and $$\bar{\phi} = e^{i(kz+\omega t)}.$$

Use of Eq. (24) then yields

$$(k\hat{z}+\omega\hat{t}) = (kz+\omega t)+\bar{\Omega}s,$$

$$\hat{v}_z = v_z ,$$

$$\left[\hat{v}_x \pm i\hat{v}_y\right] = \left[v_x \pm iv_y\right]\cos s - \omega_B\left[x \pm iy\right]\sin s$$

$$= e^{\pm i\theta}\left[ve^{\pm i\alpha}\cos s - \omega_B r \sin s\right], \tag{62}$$

and

$$\left(k_1\hat{x} + k_2\hat{y}\right) = \left(k_1 x + k_2 y\right)\cos s + \left(\frac{k_1 v_x + k_2 v_y}{\omega_B}\right)\sin s$$

$$= \ell r \cos\delta \cos s + \left(\frac{\ell v}{\omega_B}\right)\cos\lambda \sin s,$$

and $\left(\hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{E}}\right)$ takes the form

$$\left(\hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{E}}\right) = \bar{\phi} e^{i\bar{\Omega} s} \bar{H} e^{i\ell r \cos\delta \cos s} e^{i(\ell v/\omega_B)\cos\lambda \sin s}$$

where (63)

$$\bar{H} = v_z \bar{P}_m F_z(\ell) + \frac{1}{2}\left(ve^{-i\alpha}\cos s - \omega_B r \sin s\right)e^{-i\theta}\bar{P}_{m+1} F^{+}(\ell)$$

$$+ \frac{1}{2}\left(ve^{i\alpha}\cos s - \omega_B r \sin s\right)e^{i\theta}\bar{P}_{m-1} F^{-}(\ell).$$

The integrand may be written in more convenient form by introducing the operator identity

$$\bar{P}_n = e^{in\theta} P_n$$

$$= e^{in\theta}\int_0^\infty \ell\, d\ell \int_0^{2\pi} \frac{d\delta}{2\pi}\, e^{in\left(\delta - \frac{\pi}{2}\right)} \tag{64}$$

and the definitions

$$H = v_z P_m F_z(\ell) + \frac{1}{2}\left(ve^{-i\alpha}\cos s - \omega_B r \sin s\right) P_{m+1} F^{+}(\ell)$$

$$+\frac{1}{2}\left(ve^{i\alpha}\cos s - \omega_B r \sin s\right) P_{m-1} F^{-}(\ell)$$

and (65)

$$\bar{K} = \bar{K}(v, \alpha, v_z, r, s)$$

$$= He^{i\ell r \cos\delta \cos s} e^{i(ev/\omega_B)\cos\lambda \sin s}.$$

This leads to the expressions

$$(\hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{E}}) = \gamma e^{i\bar{\Omega} s}\bar{K} \tag{66}$$

for the integrand and

$$\underset{\sim}{j}_N = -\left(\frac{2e^2 n_0 \gamma}{m\gamma_0 c\omega}\right)\int dv_z h_0 \int_0^{2\pi}\frac{d\alpha}{2\pi}\int_0^{\infty} dv \int_{-\infty}^{0}\bar{\Omega}\, ds\, e^{i\bar{\Omega} s}\underset{\sim}{v}\bar{K}\frac{d}{dv}\delta(v^2-\omega_B^2\epsilon^2) \tag{67}$$

for the non-local current. The perturbed current thus has the form assumed in the reduction of Maxwell's equations, and will be expressed as a set of integrals over the Hankel transforms.

Further reduction is needed to obtain Eq. (67) in explicit form. Fortunately, certain of the integrals are easy to carry out. The components

$$j_N^{\pm} = -\left(\frac{2e^2 n_0}{m\gamma_0 c\omega}\right)\int_{-\infty}^{0} ds\,\Omega e^{i\Omega s}\int_0^{2\pi}\frac{d\alpha}{2\pi}e^{\pm i\alpha}\int_0^{\infty} dv\, v\, K\frac{d}{dv}\delta(v^2-\omega_B^2\epsilon^2)$$

and (68)

$$j_{zN} = -\left(\frac{2e^2 n_0 u}{m r_0^2 c\omega}\right)\int_{-\infty}^{0} ds\,\Omega e^{i\Omega s}\int_0^{2\pi}\frac{d\alpha}{2\pi}\int_0^{\infty} dv\,K\frac{d}{dv}\delta\left(v^2-\omega_B^2\varepsilon^2\right),$$

where

$$\Omega=\bar{\Omega}(u)=\left(\frac{\omega+ku}{\omega_B}\right)'$$

and (69)

$$K=\bar{K}(v,\alpha,u,r,s),$$

may be simplified by an integration by parts which eliminates the derivative of the delta function. Attention will be restricted to the region $r \leq r_0$, since the currents are otherwise zero. The necessary identities are

$$\int_0^{\infty} dv\,v\,K\frac{d}{dv}\delta\left(v^2-\omega_B^2\varepsilon^2\right) = -\int_0^{\infty} dv\,\delta\left(v^2-\omega_B^2\varepsilon^2\right)\left(1+v\frac{\partial}{\partial v}\right)K$$

$$= -\frac{1}{2}\left[\left(\frac{1}{v}+\frac{\partial}{\partial v}\right)K\right]_{v=\omega_B\varepsilon}$$

and (70)

$$\int_0^{\infty} dv\,K\frac{d}{dv}\delta\left(v^2-\omega_B^2\varepsilon^2\right) = -\delta\left(\omega_B^2\varepsilon^2\right)\left[K\right]_{v=0} - \int_0^{\infty} dv\,\delta\left(v^2-\omega_B^2\varepsilon^2\right)\frac{\partial K}{\partial v}$$

$$= -\left(\frac{\delta(r-r_0)}{2\omega_B^2 r_0}\right)\left[K\right]_{v=0} - \frac{1}{2}\left[\frac{1}{v}\frac{\partial}{\partial v}K\right]_{v=\omega_B\varepsilon},$$

where

$$\left[K\right]_{v=0} = R_0 e^{i\ell r\cos\delta\cos s},$$

$$K=\left(R_0+v\left[e^{i\alpha}R^{+}+e^{-i\alpha}R^{-}\right]\right)e^{i\ell r\cos\delta\cos s}\,e^{i(\ell v/\omega_B)\cos\lambda\,\sin s},$$

$$\frac{\partial K}{\partial v} = \left(\frac{i\ell}{\omega_B}\right)\sin s \cos\lambda\, K$$
$$+ \left[e^{i\alpha}R^{+} + e^{-i\alpha}R^{-}\right] e^{i\ell r \cos\delta \cos s}\, e^{i(\ell v/\omega_B)\cos\lambda \sin s},$$

$$R_{\circ} = u P_m F_z(\ell) - \tfrac{1}{2}\omega_B r \sin s \left[P_{m+1} F^{+}(\ell) + P_{m-1} F^{-}(\ell)\right], \tag{71}$$

$$R^{+} = \tfrac{1}{2}\cos s\, P_{m-1} F^{-}(\ell),$$

and

$$R^{-} = \tfrac{1}{2}\cos s\, P_{m+1} F^{+}(\ell).$$

The components become

$$j_N^{\pm} = \left(\frac{e^2 n_\circ}{m\gamma_\circ c\omega}\right)\int_{-\infty}^{\circ} ds\,\Omega e^{i\Omega s}\int_{\circ}^{2\pi}\frac{d\alpha}{2\pi}\, e^{\pm i\alpha}\left[\left(\frac{1}{v} + \frac{\partial}{\partial v}\right)K\right]_{v=\omega_B \epsilon}$$

and (72)

$$j_{zN} = \left(\frac{e^2 n_\circ u}{m\gamma_\circ c\omega}\right)\int_{-\infty}^{\circ} ds\,\Omega e^{i\Omega s}\left(\frac{\delta(r-r_\circ)}{\omega_B^2 r_\circ}[K]_{v=0} + \int_{\circ}^{2\pi}\frac{d\alpha}{2\pi}\left[\frac{1}{v}\frac{\partial}{\partial v}K\right]_{v=\omega_B \epsilon}\right)$$

and are simplified by carrying out the α and δ integrations, interchanging the order by means of the relations

$$\alpha = \lambda + \delta$$

and $$\int_{\circ}^{2\pi} d\alpha \int_{\circ}^{2\pi} d\delta = \int_{\circ}^{2\pi} d\delta \int_{\circ}^{2\pi} d\lambda. \tag{73}$$

The calculation is facilitated by introducing the definitions

$$A = \left(\frac{e^2 n_\circ}{m\gamma_\circ c\omega}\right),$$

$$T=\int_{-\infty}^{\circ} ds\,\Omega e^{i\Omega s},$$

$$L=\int_{0}^{\infty} \ell\, d\ell,$$

$$D_n=\int_{0}^{2\pi}\frac{d\delta}{2\pi}\, e^{in\left(\delta-\frac{\pi}{2}\right)},$$

$$G=\int_{0}^{2\pi}\frac{d\lambda}{2\pi},$$

$$\phi_r=e^{i\ell r\cos\delta\cos s}, \tag{74}$$

and $\phi_\varepsilon = e^{i\ell\varepsilon\cos\lambda\sin s}$.

Using this notation $\dot{j}_N^{\pm}$ becomes

$$\begin{aligned}
\dot{j}_N^{\pm} = &\left(\frac{Au}{\omega_B c}\right) T L F_z(\ell)\left[D_m e^{\pm i\delta}\phi_r\right]\left[G e^{\pm i\lambda}(1+i\ell\varepsilon\sin s\cos\lambda)\phi_\varepsilon\right] \\
&-\left(\frac{Ar}{2\varepsilon}\right) T \sin s\, L F^{+}(\ell)\left[D_{m+1} e^{\pm i\delta}\phi_r\right]\left[G e^{\pm i\lambda}(1+i\ell\varepsilon\sin s\cos\lambda)\phi_\varepsilon\right] \\
&-\left(\frac{Ar}{2\varepsilon}\right) T \sin s\, L F^{-}(\ell)\left[D_{m-1} e^{\pm i\delta}\phi_r\right]\left[G e^{\pm i\lambda}(1+i\ell\varepsilon\sin s\cos\lambda)\phi_\varepsilon\right] \\
&+\left(\frac{A}{2}\right) T \cos s\, L F^{+}(\ell)\left[D_{m+1} e^{-i\delta} e^{\pm i\delta}\phi_r\right]\left[G e^{-i\lambda} e^{\pm i\lambda}(2+i\ell\varepsilon\sin s\cos\lambda)\phi_\varepsilon\right] \\
&+\left(\frac{A}{2}\right) T \cos s\, L F^{-}(\ell)\left[D_{m-1} e^{i\delta} e^{\pm i\delta}\phi_r\right]\left[G e^{i\lambda} e^{\pm i\lambda}(2+i\ell\varepsilon\sin s\cos\lambda)\phi_\varepsilon\right]
\end{aligned} \tag{75}$$

and $\dot{j}_{zN}$ becomes

$$\dot{j}_{zN}=\left(\frac{A\delta(r-r_0)}{\omega_B^{2} r_0}\right) T L\left(u F_z(\ell) D_m\phi_r-\frac{1}{2}\omega_B r\sin s\left[F^{+}(\ell)D_{m+1}\phi_r+F^{-}(\ell)D_{m-1}\phi_r\right]\right)$$

$$+\left(\frac{u^2 A}{\omega_B^2 \epsilon^2}\right) T \sin s \, L F_z(\ell) \left[D_m \phi_r \right] \left[G i \ell \epsilon \cos\lambda \, \phi_\epsilon \right]$$

$$-\left(\frac{rAu}{2 \omega_B \epsilon^2}\right) T \sin^2 s \, L F^+(\ell) \left[D_{m+1} \phi_r \right] \left[G i \ell \epsilon \cos\lambda \, \phi_\epsilon \right]$$

$$-\left(\frac{rAu}{2 \omega_B \epsilon^2}\right) T \sin^2 s \, L F^-(\ell) \left[D_{m-1} \phi_r \right] \left[G i \ell \epsilon \cos\lambda \, \phi_\epsilon \right] \tag{76}$$

$$+\left(\frac{Au}{2 \omega_B \epsilon}\right) T \cos s \, L F^+(\ell) \left[D_{m+1} e^{-i\delta} \phi_r \right] \left[G e^{-i\lambda} (1 + i\ell\epsilon \sin s \cos\lambda) \phi_\epsilon \right]$$

$$+\left(\frac{Au}{2 \omega_B \epsilon}\right) T \cos s \, L F^-(\ell) \left[D_{m-1} e^{i\delta} \phi_r \right] \left[G e^{i\lambda} (1 + i\ell\epsilon \sin s \cos\lambda) \phi_\epsilon \right].$$

The δ and λ integrations are independent, and each one may be carried out. The basic integral

$$\int_0^{2\pi} \frac{dx}{2\pi} e^{ipx} e^{in(x-\frac{\pi}{2})} e^{iy\cos x} = e^{ip\frac{\pi}{2}} J_{n+p}(y) \tag{77}$$

yields

$$D_m e^{\pm i\delta} \phi_r = \pm i J_{m\pm 1}(\ell r \cos s),$$

$$D_{m+1} e^{\pm i\delta} \phi_r = i J_{m+2}(\ell r \cos s) \text{ for } +$$

$$= -i J_m (\ell r \cos s) \text{ for } -,$$

$$D_{m-1} e^{\pm i\delta} \phi_r = i J_m (\ell r \cos s) \text{ for } +$$

$$= -i J_{m-2}(\ell r \cos s) \text{ for } -,$$

$$D_{m+1} e^{-i\delta} e^{\pm i\delta} \phi_r = J_{m+1}(\ell r \cos s) \text{ for } +$$

$$= -J_{m-1}(\ell r \cos s) \text{ for } -,$$

$$D_{m-1} e^{i\delta} e^{\pm i\delta} \phi_r = -J_{m+1}(\ell r \cos s) \text{ for } +$$

$$= J_{m-1}(\ell r \cos s) \text{ for } -,$$

$$G e^{\pm i\lambda} \phi_\varepsilon = i J_1(\ell \varepsilon \sin s), \tag{78}$$

$$G \cos\lambda \, \phi_\varepsilon = i J_1(\ell\varepsilon \sin s),$$

$$G e^{\pm i\lambda} \cos\lambda \, \phi_\varepsilon = \frac{1}{2}\left[-J_2(\ell\varepsilon \sin s) + J_0(\ell\varepsilon \sin s)\right],$$

$$G e^{\pm i\lambda}(1 + i\ell\varepsilon \sin s \cos\lambda)\phi_\varepsilon = i\ell\varepsilon \sin s \, J_0(\ell\varepsilon \sin s),$$

$$G(2 + i\ell\varepsilon \sin s \cos\lambda)\phi_\varepsilon = 2 J_0(\ell\varepsilon \sin s) - \ell\varepsilon \sin s \, J_1(\ell\varepsilon \sin s),$$

$$G e^{\pm 2i\lambda} \phi_\varepsilon = -J_2(\ell\varepsilon \sin s),$$

$$G e^{\pm 2i\lambda} \cos\lambda \, \phi_\varepsilon = \frac{i}{2}\left[-J_3(\ell\varepsilon \sin s) + J_1(\ell\varepsilon \sin s)\right],$$

and

$$G e^{\pm 2i\lambda}(2 + i\ell\varepsilon \sin s \cos\lambda)\phi_\varepsilon = -\left(\frac{\varepsilon\ell \sin s}{2}\right)\left(J_3 - J_1 - \frac{4}{\varepsilon\ell \sin s} J_2\right)$$

$$= \varepsilon\ell \sin s \, J_1(\ell\varepsilon \sin s),$$

which is sufficient for the evaluation of the current. Substitution into Eqs. (75) and (76) then gives

$$j_N^{+} = -\left(\frac{Au}{\omega_B \epsilon}\right) L F_z(\ell) T \ell\epsilon \sin s\, J_0(\ell\epsilon \sin s)\, J_{m+1}(\ell r \cos s)$$

$$+\left(\frac{A}{2}\right) L F^{+}(\ell) T \begin{bmatrix} \ell r \sin^2 s\, J_0(\ell\epsilon \sin s) J_{m+2}(\ell r \cos s) \\ + \cos s\left[2 J_0(\ell\epsilon \sin s) - \ell\epsilon \sin s\, J_1(\ell\epsilon \sin s)\right] J_{m+1}(\ell r \cos s) \end{bmatrix}$$

$$+\left(\frac{A}{2}\right) L F^{-}(\ell) T \begin{bmatrix} \ell r \sin^2 s\, J_0(\ell\epsilon \sin s) J_m(\ell r \cos s) \\ + \ell\epsilon \sin s \cos s\, J_1(\ell\epsilon \sin s) J_{m+1}(\ell r \cos s) \end{bmatrix}, \qquad (79)$$

$$j_N^{-} = \left(\frac{Au}{\omega_B \epsilon}\right) L F_z(\ell) T \ell\epsilon \sin s\, J_0(\ell\epsilon \sin s) J_{m-1}(\ell r \cos s)$$

$$-\left(\frac{A}{2}\right) L F^{+}(\ell) T \begin{bmatrix} \ell r \sin^2 s\, J_0(\ell\epsilon \sin s) J_m(\ell r \cos s) \\ - \ell\epsilon \sin s \cos s\, J_1(\ell\epsilon \sin s) J_{m-1}(\ell r \cos s) \end{bmatrix}$$

$$-\left(\frac{A}{2}\right) L F^{-}(\ell) T \begin{bmatrix} \ell r \sin^2 s\, J_0(\ell\epsilon \sin s) J_{m-2}(\ell r \cos s) \\ - \cos s\left[2 J_0(\ell\epsilon \sin s) - \ell\epsilon \sin s\, J_1(\ell\epsilon \sin s)\right] J_{m-1}(\ell r \cos s) \end{bmatrix},$$

and

$$j_{zN} = \left(\frac{Au\,\delta(r-r_0)}{\omega_B^2 r_0}\right) T L \begin{bmatrix} u F_z(\ell) J_m(\ell r_0 \cos s) - \frac{1}{2}\omega_B r_0 \sin s\, F^{+}(\ell) J_{m+1}(\ell r_0 \cos s) \\ -\frac{1}{2}\omega_B r_0 \sin s\, F^{-}(\ell) J_{m-1}(\ell r_0 \cos s) \end{bmatrix}$$

$$-\left(\frac{Au^2}{\omega_B^2 \epsilon^2}\right) L F_z(\ell) T \ell\epsilon \sin s\, J_1(\ell\epsilon \sin s) J_m(\ell r \cos s)$$

$$+\left(\frac{Au}{2\omega_B \varepsilon}\right) LF^{+}(\ell) T \left[\begin{array}{l} \ell r \sin^2 s\, J_1(\ell\varepsilon \sin s) J_{m+1}(\ell r \cos s) \\ + \ell\varepsilon \sin s \cos s\, J_0(\ell\varepsilon \sin s) J_m(\ell r \cos s) \end{array}\right]$$

$$+\left(\frac{Au}{2\omega_B \varepsilon}\right) LF^{-}(\ell) T \left[\begin{array}{l} \ell r \sin^2 s\, J_1(\ell\varepsilon \sin s) J_{m-1}(\ell r \cos s) \\ - \ell\varepsilon \sin s \cos s\, J_0(\ell\varepsilon \sin s) J_m(\ell r \cos s) \end{array}\right].$$

These expressions may be brought to simpler form by evaluating the brackets occurring after terms of the form $LF^{\pm}(\ell)$. The brackets are numbered consecutively and are given by

$$\begin{aligned} [\ \ ]_1 &= \frac{\partial}{\partial s}\left(\sin s\, J_0 J_{m+1}\right) + \left(\frac{m \sin^2 s + 1}{\cos s}\right) J_0 J_{m+1}, \\ [\ \ ]_2 &= -\frac{\partial}{\partial s}\left(\sin s\, J_0 J_{m+1}\right) + \left(\frac{m \sin^2 s + 1}{\cos s}\right) J_0 J_{m+1}, \\ [\ \ ]_3 &= \frac{\partial}{\partial s}\left(\sin s\, J_0 J_{m-1}\right) + \left(\frac{m \sin^2 s - 1}{\cos s}\right) J_0 J_{m-1}, \\ [\ \ ]_4 &= -\frac{\partial}{\partial s}\left(\sin s\, J_0 J_{m-1}\right) + \left(\frac{m \sin^2 s - 1}{\cos s}\right) J_0 J_{m-1}, \\ [\ \ ]_5 &= \frac{\partial}{\partial s}\left(\sin s\, J_1 J_m\right) + \left(\frac{m \sin^2 s}{\cos s}\right) J_1 J_m, \end{aligned} \qquad (80)$$

and

$$[\ \ ]_6 = -\frac{\partial}{\partial s}\left(\sin s\, J_1 J_m\right) + \left(\frac{m \sin^2 s}{\cos s}\right) J_1 J_m.$$

An integration by parts is used to remove the derivative with respect to s, and the currents become

$$j_N^{\pm} = \pm\left(\frac{\omega_B^2 e}{4\pi u^2 \omega}\right) TL J_0(\ell\epsilon \sin s) J_{m\pm 1}(\ell r \cos s).$$

$$\cdot \left\{ \begin{array}{l} \left(\left(\frac{m \sin^2 s \pm 1}{\cos s}\right)\left(F^+(\ell) + F^-(\ell)\right) - i\Omega \sin s\left(F^+(\ell) - F^-(\ell)\right)\right) \\ -\left(\frac{2u}{\omega_B \epsilon}\right)\ell\epsilon \sin s F_z(\ell) \end{array} \right\}$$

and (81)

$$j_{zN} = \left(\frac{\omega_B e\delta(r - r_0)}{2\pi u \omega}\right) TL \left[\begin{array}{l} \frac{u}{\omega_B r_0} F_z(\ell) J_m(\ell r_0 \cos s) \\ -\frac{1}{2}\sin s\left(F^+(\ell) J_{m+1}(\ell r_0 \cos s) + F^-(\ell) J_{m-1}(\ell r_0 \cos s)\right) \end{array} \right]$$

$$+\left(\frac{\omega_B^2 e}{4\pi u^2 \omega}\right)\left(\frac{u}{\omega_B \epsilon}\right) TL \sin s\, J_1(\ell\epsilon \sin s) J_m(\ell r \cos s).$$

$$\cdot \left\{ \begin{array}{l} \left(\left(\frac{m \sin s}{\cos s}\right)\left(F^+(\ell) + F^-(\ell)\right) - i\Omega\left(F^+(\ell) - F^-(\ell)\right)\right) \\ -\left(\frac{2u}{\omega_B \epsilon}\right)\ell\epsilon F_z(\ell) \end{array} \right\}.$$

The total current is determined from Eqs. (51), (53), and (81) and, to facilitate the stability analysis, is written in the form

$$-\left(\frac{4\pi i\omega}{c}\right) j^{\pm} = \pm\left(\frac{\omega_B^2}{u^2}\right)\int_0^\infty p\,dp\int_{-\infty}^0 ds\, i\Omega\, e^{i\Omega s} J_0(p\epsilon \sin s)\, J_{m\pm 1}(pr\cos s)\cdot \left\{ \begin{array}{l} \left(\frac{2u}{\omega_B r_0}\right) p r_0 \sin s\, F_z(p) + i\Omega \sin s\left[F^+(p) - F^-(p)\right] \\ -\left(\frac{m \sin^2 s \pm 1}{\cos s}\right)\left[F^+(p) + F^-(p)\right] \end{array} \right\} \tag{82}$$

and

$$\begin{aligned} -\left(\frac{4\pi i\omega}{c}\right) j_z = {} & \left(\frac{2\delta(r-r_0)}{r_0}\right)\left[\begin{array}{l} f_z(r_0) - \int_{-\infty}^0 ds\, i\Omega\, e^{i\Omega s} f_z(r_0\cos s) \\ + \left(\frac{\omega_B r_0}{u}\right)\int_{-\infty}^0 ds\, i\Omega\, e^{i\Omega s} \sin s\, f_r(r_0 \cos s) \end{array} \right] \\ & + \left(\frac{\omega_B^2}{u^2}\right)\left(\frac{u}{\omega_B \epsilon}\right)\int_0^\infty p\,dp\int_{-\infty}^0 ds\, i\Omega\, e^{i\Omega s} \sin s\, J_1(p\epsilon \sin s)\, J_m(pr\cos s)\cdot \\ & \cdot\left\{ \begin{array}{l} \left(\frac{2u}{\omega_B r_0}\right) p r_0\, F_z(p) + i\Omega\left[F^+(p) - F^-(p)\right] \\ -\left(\frac{m \sin s}{\cos s}\right)\left[F^+(p) + F^-(p)\right] \end{array} \right\}. \end{aligned} \tag{83}$$

These expressions are used in the evaluation of

$$\begin{aligned} (\nabla_0 \cdot j) &= \frac{1}{r}\frac{d}{dr} r j_r(r) + \frac{im}{r} j_\theta(r) + ik j_z \\ &= ik j_z + \frac{1}{2}\left(\frac{d}{dr} + \frac{(m+1)}{r}\right) j^+ + \frac{1}{2}\left(\frac{d}{dr} - \frac{(m-1)}{r}\right) j^- . \end{aligned} \tag{84}$$

The contributions of $j^{\pm}$ are assessed by means of the identities

$$\left[\frac{d}{dr} \pm \frac{(m\pm 1)}{r}\right] J_{m\pm 1}(pr\cos s) = \pm p\cos s\; J_m(pr\cos s)$$

$$\frac{d}{dr} J_0(p\epsilon \sin s) = \frac{pr}{\epsilon} \sin s\; J_1(p\epsilon \sin s)$$

$$p\epsilon \cos s \sin s\; J_0(p\epsilon \sin s) = \frac{\partial}{\partial s} \sin s\; J_1(p\epsilon \sin s)$$

$$pr \sin s\left[J_{m+1}(pr\cos s) - J_{m-1}(pr\cos s)\right] = 2\frac{\partial}{\partial s} J_m(pr\cos s) \tag{85}$$

$$pr\cos s\left[J_{m+1}(pr\cos s) + J_{m-1}(pr\cos s)\right] = 2m\, J_m(pr\cos s)$$

and $-\left(\frac{m}{\cos^2 s}\right)\left[F^+(p) + F^-(p)\right] = \frac{\partial}{\partial s}\{\ \}$,

where $\{\ \}$ denotes the bracket occurring in Eq. (83).

Substitutions then yield

$$-\left(\frac{4\pi i\omega}{c}\right)\left(\left[\frac{d}{dr} + \frac{(m+1)}{r}\right] j^+ + \left[\frac{d}{dr} - \frac{(m-1)}{r}\right] j^-\right)$$

$$= \left(\frac{\omega_B^2}{u^2}\right)\int_0^\infty p\,dp \int_{-\infty}^0 ds\, i\Omega e^{i\Omega s} \cdot$$

$$\cdot \begin{bmatrix} \frac{2}{\epsilon} J_m(pr\cos s)\, p\epsilon \cos s \sin s\; J_0(p\epsilon \sin s)\{\ \} \\ + \frac{1}{\epsilon} J_1(p\epsilon \sin s)\, pr \sin^2 s\, \{\ \}\left[J_{m+1}(pr\cos s) - J_{m-1}(pr\cos s)\right] \\ + \frac{1}{\epsilon} J_1(p\epsilon \sin s) \sin s\, pr\cos s\left[J_{m+1} + J_{m-1}\right]\left[-\frac{1}{\cos^2 s}\right]\left[F^+ + F^-\right] \end{bmatrix} \tag{86}$$

$$= \left(\frac{\omega_B^2}{u^2}\right)\int_0^\infty p\,dp \int_{-\infty}^0 ds\, i\Omega e^{i\Omega s}\, \frac{2}{\epsilon}\frac{\partial}{\partial s}\left(\sin s\; J_1(p\epsilon \sin s) J_m(pr\cos s)\{\ \}\right),$$

and the derivative with respect to s is again removed by an integration by parts. This permits the divergence to be written in the simple form

$$
-\left(\frac{4\pi i\omega}{c}\right)\left(\nabla_0\cdot \underset{\sim}{j}\right)=\left(\frac{2ik\delta(r-r_0)}{r_0}\right)\left[\begin{aligned}&f_z(r_0)-\int_{-\infty}^{0}ds\, i\Omega e^{i\Omega s} f_z(r_0\cos s)\\ &+\left(\frac{\omega_B r_0}{u}\right)\int_{-\infty}^{0}ds\, i\Omega e^{i\Omega s}\sin s\, f_r(r_0\cos s)\end{aligned}\right]
$$

$$
-\left(\frac{\omega_B^2}{u^2}\right)\left(\frac{i\omega}{\omega_B\epsilon}\right)\int_0^{\infty}p\,dp\int_{-\infty}^{0}ds\, i\Omega e^{i\Omega s}\sin s\, J_1(p\epsilon\sin s)J_m(pr\cos s)\cdot
$$

$$
\cdot\left\{\begin{aligned}&\left(\frac{2u}{\omega_B r_0}\right)pr_0 F_z(p)+i\Omega\left[F^+(p)-F^-(p)\right]\\ &-\left(\frac{m\sin s}{\cos s}\right)\left[F^+(p)+F^-(p)\right]\end{aligned}\right\}. \tag{87}
$$

A lengthy reduction has led to Eqs. (82), (83), and (87) for the perturbed currents. These equations evaluate the currents appearing in Eq. (46) in terms of integrals of $F^{\pm}$ and F_z; substitution gives rise to a set of integral equations for the Hankel transforms. The formulation and solution of these equations completes the stability analysis. When the Hankel transforms $F^{\pm}$ and F_z are known, all other perturbed quantities may be evaluated.

Integral Equations for the Stability Problem

The r' integrations occurring in Eq. (46) actually run from 0 to r_0, since the currents are zero for $r' > r_0$.

This permits the form of the integrands to be simplified by introducing the new variable ϕ , defined by

$$
\begin{aligned}
r' &= r_0 \cos \phi, \\
\xi' &= r_0 \sin \phi, \\
\text{and} \quad 0 &\le \phi \le \frac{\pi}{2},
\end{aligned} \tag{88}
$$

in place of r'. It is also convenient to define the operators

$$
\begin{aligned}
\Gamma_n &= \Gamma_n(\ell, p) \\
&= \int_0^{\frac{\pi}{2}} d\phi \sin \phi \cos \phi \, J_0(pr_0 \sin s \sin \phi) J_n(pr_0 \cos s \cos \phi) J_n(\ell r_0 \cos \phi)
\end{aligned}
$$

and (89)

$$
\begin{aligned}
\Delta_n &= \Delta_n(\ell, p) \\
&= \int_0^{\frac{\pi}{2}} d\phi \cos \phi \, J_1(pr_0 \sin s \sin \phi) J_n(pr_0 \cos s \cos \phi) J_n(\ell r_0 \cos \phi).
\end{aligned}
$$

With these conventions the substitution of Eqs. (82), (83), and (87) into Eq. (46) yields

$$
F^{\pm}(\ell) = \pm \left(\frac{ik\ell 2 J_m(\ell r_0)}{(\ell^2 + h^2)(h^2 - k^2)} \right) \left[f_z(r_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} f_z(r_0 \cos s) + \left(\frac{\omega_B r_0}{u} \right) \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s f_r(r_0 \cos s) \right]
$$

$$\pm\left(\frac{\omega_B r_0}{u}\right)\frac{\int_0^{\infty} p\,dp}{(\ell^2+h^2)}\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \cdot$$

$$\cdot\left[\Gamma_{m\pm1}\left\{\begin{matrix}\lambda p r_0 \sin s\, F_z(p) + i\Omega\left(\frac{\omega_B r_0}{u}\right)\sin s\left[F^+(p)-F^-(p)\right] \\ -\left(\frac{\omega_B r_0}{u}\right)\left(\frac{m\sin^2 s \pm 1}{\cos s}\right)\left[F^+(p)+F^-(p)\right]\end{matrix}\right\} - \frac{i\omega\ell\sin s}{\omega_B r_0(h^2-k^2)}\Delta_m\left\{\begin{matrix}\lambda p r_0 F_z(p) + i\Omega\left(\frac{\omega_B r_0}{u}\right)\left[F^+(p)-F^-(p)\right] \\ -\left(\frac{\omega_B r_0}{u}\right)\left(\frac{m\sin s}{\cos s}\right)\left[F^+(p)+F^-(p)\right]\end{matrix}\right\}\right] \qquad (90)$$

and

$$\dot{F}_z(\ell) = \left(\frac{h^2 \lambda J_m(\ell r_0)}{(\ell^2+h^2)(h^2-k^2)}\right)\left[\begin{matrix} f_z(r_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} f_z(r_0\cos s) \\ +\left(\frac{\omega_B r_0}{u}\right)\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin s\, f_r(r_0\cos s)\end{matrix}\right]$$

$$+\left[\frac{h^2-k^2-(\omega k/u)}{(h^2-k^2)(\ell^2+h^2)}\right]\int_0^{\infty} p\,dp\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin s\,\Delta_m \cdot$$

$$\cdot\left[\begin{matrix}\lambda p r_0 F_z(p) + i\Omega\left(\frac{\omega_B r_0}{u}\right)\left[F^+(p)-F^-(p)\right] \\ -\left(\frac{\omega_B r_0}{u}\right)\left(\frac{m\sin s}{\cos s}\right)\left[F^+(p)+F^-(p)\right]\end{matrix}\right]. \qquad (91)$$

Thus the stability problem has been expressed by Eqs. (90) and (91). These equations are not complete as they stand but are completed by Eq. (89) for the operators Δ_m and $\Gamma_{m\pm 1}$ and by the relations

$$f_z(r_0 \cos s) = \int_0^\infty p\,dp\; J_m(pr_0 \cos s) F_z(p)$$

and (92)

$$f_r(r_0 \cos s) = \frac{1}{2}\int_0^\infty p\,dp\; J_{m+1}(pr_0 \cos s) F^+(p) + \frac{1}{2}\int_0^\infty p\,dp\; J_{m-1}(pr_0 \cos s) F^-(p)$$

for f_r and f_z.

Reduced Integral Equations

An expansion procedure is suggested by the presence in Eqs. (90) and (91) of the small parameter $\epsilon = (\omega_B r_0 / u)$. The equations may be replaced by an infinite number of simpler equations by expanding the operators and field variables in powers of ϵ and equating the coefficients of the resulting power series to zero. The procedure is made definite by using the relations

$$\begin{aligned} &F^\pm \sim F_z \sim 1 \\ &\ell r_0 \sim pr_0 \sim hr_0 \sim kr_0 \sim 1 \\ &\Omega \sim 1 \\ &\omega \sim \omega k \end{aligned} \tag{93}$$

and $\qquad \Gamma_{m\pm 1} \sim \Delta_m \sim 1$

to classify the terms appearing in the equations. A theory of the equations could be developed by estimating the degree of approximation to the true solutions provided by n terms of the series solutions and by solving for the series expansion of $F^{\pm}$ and F_z. Only part of this program will be carried out. Since ϵ is small whenever the approximations made in the relativistic dynamics are valid, the major content of the theory should appear in the zeroth order equations. For this reason attention is first directed to the derivation and solution of these equations. The field variables are expanded in the form

$$F^{\pm}(\ell) = F_0^{\pm}(\ell) + \epsilon F_1^{\pm}(\ell) + \cdots$$

and

$$F_z(\ell) = F_z^{\circ}(\ell) + \epsilon F_z'(\ell) + \cdots, \tag{94}$$

and direct evaluation yields the zeroth order equations

$$F_0^{\pm}(\ell) = \pm\left(\frac{ik\ell 2 J_m(\ell r_0)}{(\ell^2+h^2)(h^2-k^2)}\right)\left[f_z^{\circ}(r_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} f_z^{\circ}(r_0 \cos s)\right] \tag{95}$$

and

$$F_z^{\circ}(\ell) = \left(\frac{h^2 2 J_m(\ell r_0)}{(\ell^2+h^2)(h^2-k^2)}\right)\left[f_z^{\circ}(r_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} f_z^{\circ}(r_0 \cos s)\right]$$

$$+ 2\left[\frac{h^2-k^2-(\omega k/u)}{h^2-k^2}\right] \frac{\int_0^{\infty} p\,dp\,p r_0 F_z^{\circ}(p)}{(\ell^2+h^2)} \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s\, \Delta_m , \tag{96}$$

where, as before,

$$\Delta_m = \int_0^{\frac{\pi}{2}} d\phi \cos\phi \, J_1(pr_0 \sin s \sin\phi) J_m(pr_0 \cos s \cos\phi) J_m(\ell r_0 \cos\phi).$$

Thus the problem has been reduced to the solution of a single integral equation - Eq. (96) - for $F_z^{\,o}(\ell)$. The variables $F_o^{\,\pm}(\ell)$ may then be obtained by integration.

The structure of Eq. (96) becomes clearer when it is written in compact form with the aid of the functions

$$G(\ell) = \frac{J_m(\ell r_0)}{(\ell^2 + h^2)},$$

$$M(p) = \left(\frac{2h^2 p}{h^2 - k^2}\right)\left[J_m(pr_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(pr_0 \cos s)\right], \tag{97}$$

and

$$N(\ell, p) = \left(\frac{2p^2 r_0 [h^2 - k^2 - (\omega k/u)]}{(\ell^2 + h^2)(h^2 - k^2)}\right) \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s \int_0^{\frac{\pi}{2}} d\phi \cos\phi \,\cdot$$

$$\cdot J_1(pr_0 \sin s \sin\phi) J_m(pr_0 \cos s \cos\phi) J_m(\ell r_0 \cos\phi)$$

and the operators

$$\mathcal{M} = \int_0^{\infty} dp\, M(p)$$

and

$$N = \int_0^{\infty} dp\, N(\ell, p). \tag{98}$$

The equation takes the form

$$F_z(\ell) = G(\ell) \int_0^{\infty} dp\, M(p)\, F_z(p) + \int_0^{\infty} dp\, N(\ell, p)\, F_z(p), \tag{99}$$

where, for convenience, the superscript on the field variable has been ignored, and in usual operator notation this becomes

$$F_z = GMF_z + NF_z . \tag{100}$$

The similarity of Eq. (100) to Fredholm's equation of the second kind suggests that a solution be sought in the form

$$F_z = G + NG + NNG + \cdots = \sum_{j=0}^{\infty} N^j G, \tag{101}$$

where, as usual,

$$N^0 G = G(\ell)$$

and

$$N^j G = \int_0^\infty dp_1 \cdots \int_0^\infty dp_j \; N(\ell, p_1) \cdots N(p_{j-1}, p_j) G(p_j) \tag{102}$$

when $j > 0$. Of course, the right hand side of Eq. (101) could just as well be multiplied by a constant factor, since Eq. (100) is linear. Substitution of the trial solution into Eq. (100) yields

$$\sum_{j=0}^{\infty} N^j G = GM \sum_{j=0}^{\infty} N^j G + N \sum_{j=0}^{\infty} N^j G$$

$$= G \sum_{j=0}^{\infty} MN^j G + \sum_{j=1}^{\infty} N^j G, \tag{103}$$

or

$$N^0 G = G = G \sum_{j=0}^{\infty} MN^j G. \tag{104}$$

For arbitrary choice of parameters - in particular, for arbitrary values of k and ω - Eq. (104) will not be satisfied. Hence this condition plays the role of a solvability condition for the problem and is, in fact, the dispersion relation. The formal dispersion relation is thus given by

$$1 - \sum_{j=0}^{\infty} MN^j G, \tag{105}$$

where

$$MN^0 G = \int_0^\infty dp\, M(p)\, G(p)$$

and

$$MN^j G = \int_0^\infty dp \int_0^\infty dp_1 \cdots \int_0^\infty dp_j\, M(p) N(p, p_1) \ldots N(p_{j-1}, p_j) G(p_j)$$

when $j > 0$. This equation is, in a sense, the final result of the analysis, since it gives a method for testing which values of ω and k lead to admissible disturbances. However, the content of the equation is certainly not in explicit form, and explicit results will be obtained in the next chapter.

Solution of the Coupled Integral Equations

The full integral equations - Eqs. (90) and (91) - have a more complex structure, because of the coupling between the various field variables. However, their form is sufficiently similar to Eq. (96) to permit an iteration solution and a dispersion relation to be obtained in an analogous manner. This can be seen more clearly by introducing functions and operators which permit the equations to be written in the more compact form

$$F_z(\ell) = a(\ell)(b_z F_z + b^+ F^+ + b^- F^-) + (K_z F_z + K^+ F^+ + K^- F^-)$$

and (106)

$$F^{\pm}(\ell) = \pm d_1 \ell a(\ell)(b_z F_z + b^+ F^+ + b^- F^-) \pm (Z^{\pm} F_z + P^{\pm} F^+ + M^{\pm} F^-),$$

where the expression

$$(b_z F_z + b^+ F^+ + b^- F^-)$$

is independent of ℓ. This may be done conveniently with the aid of the functions

$$a(\ell) = \frac{2h^2 J_m(\ell r_0)}{(\ell^2 + h^2)(h^2 - k^2)},$$

$$d_1 = \left(\frac{ik}{h^2}\right),$$

$$d_2 = -\frac{i(\omega\ell/u)}{h^2 - k^2 - (\omega k/u)}$$

$$= d_4 \ell,$$

$$b_z(p) = p\left[J_m(pr_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(pr_0 \cos s)\right],$$

$$b^{\pm}(p) = \frac{1}{2}\left(\frac{\omega_B r_0}{u}\right) p \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s\, J_{m\pm 1}(pr_0 \cos s),$$

$$K_z(\ell, p) = \left[\frac{h^2 - k^2 - (\omega k/u)}{(\ell^2+h^2)(h^2-k^2)}\right] p \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s \int_0^{\frac{\pi}{2}} d\phi \cos\phi \,\cdot$$

$$\cdot\, 2pr_0 J_1(pr_0 \sin s \sin\phi) J_m(pr_0 \cos s \cos\phi) J_m(\ell r_0 \cos\phi), \qquad (107)$$

$$K^{\pm}(\ell, p) = \left(\frac{1}{2pr_0}\right)\left(\frac{\omega_B r_0}{u}\right)\left[\pm i\Omega - \frac{m \sin s}{\cos s}\right] K_z(\ell, p),$$

$$Z^{\pm}(\ell, p) = d_2 K_z(\ell, p)$$

$$+ \left(\frac{\omega_B r_0}{u}\right)\left(\frac{p}{\ell^2+h^2}\right) \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s \int_0^{\frac{\pi}{2}} d\phi \sin\phi \cos\phi \,\cdot$$

$$\cdot\, 2pr_0 J_0(pr_0 \sin s \sin\phi) J_{m\pm 1}(pr_0 \cos s \cos\phi) J_{m\pm 1}(\ell r_0 \cos\phi)$$

$$\equiv d_2 K_z(\ell, p) + Q^{\pm} 2pr_0 \sin s,$$

$$P^{\pm}(\ell, p) = d_2 K^{+}(\ell, p) + \left(\frac{\omega_B r_0}{u}\right) Q^{\pm}\left[i\Omega \sin s - \left(\frac{m \sin^2 s \pm 1}{\cos s}\right)\right],$$

and

$$M^{\pm}(\ell, p) = d_2 K^{-}(\ell, p) + \left(\frac{\omega_B r_0}{u}\right) Q^{\pm}\left[-i\Omega \sin s - \left(\frac{m \sin^2 s \pm 1}{\cos s}\right)\right],$$

where the operator $Q^{\pm}$ is defined by the equation for $Z^{\pm}$. The various operator expressions appearing in Eq. (106) are then obtained by multiplying the corresponding functions and field variables and integrating over all positive values of p. For example, typical terms are given by

$$b_z F_z = \int_0^\infty dp\, b_z(p) F_z(p),$$

$$K^+F^+ = \int_0^\infty dp\, K^+(\ell, p) F^+(p), \tag{108}$$

and $$M^\pm F^- = \int_0^\infty dp\, M^\pm(\ell, p) F^-(p).$$

The variable p which appears in these expressions is, of course, a dummy variable and may be relabeled. This freedom will be made use of in the specification of the iteration solution.

This solution is found more readily by writing Eq. (106) in vector form as

$$\underset{\sim}{F}(\ell) = \underset{\sim}{a}(\ell)(\underset{\sim}{b}\cdot\underset{\sim}{F}) + K\underset{\sim}{F}, \tag{109}$$

where

$$\underset{\sim}{F}(\ell) = (F^+(\ell), F^-(\ell), F_z(\ell))^t,$$

$$\underset{\sim}{a}(\ell) = (d_1 \ell a(\ell), -d_1 \ell a(\ell), a(\ell))^t,$$

$$\underset{\sim}{b}\cdot\underset{\sim}{F} = \int_0^\infty dp\, \underset{\sim}{b}(p)\cdot\underset{\sim}{F}(p),$$

$$\underset{\sim}{b}(p) = (b^+(p), b^-(p), b_z(p))^t,$$

$$K\underset{\sim}{F} = \int_0^\infty dp\, K(\ell, p)\underset{\sim}{F}(p),$$

and

$$K(\ell,p) = \begin{pmatrix} P^{+}(\ell,p) & M^{+}(\ell,p) & Z^{+}(\ell,p) \\ -P^{-}(\ell,p) & -M^{-}(\ell,p) & -Z^{-}(\ell,p) \\ K^{+}(\ell,p) & K^{-}(\ell,p) & K_{z}(\ell,p) \end{pmatrix}$$

The linearity of these equations suggests using the vector

$$\underset{\sim}{g}(\ell) = \left(\alpha_1 d_1 \ell a(\ell), -\alpha_2 d_1 \ell a(\ell), \alpha_3 a(\ell)\right)^{t}, \tag{110}$$

where α_1, α_2, α_3 are arbitrary constants, as a first approximation to the solution and then seeking a solution in the form

$$\begin{aligned} \underset{\sim}{F}(\ell) &= \sum_{n=0}^{\infty} K^{n} \underset{\sim}{g} \\ &= \underset{\sim}{g}(\ell) + K\underset{\sim}{g} + KK\underset{\sim}{g} + \cdots, \end{aligned} \tag{111}$$

where for $n > 0$ the terms in the sum are defined by

$$K^{n}\underset{\sim}{g} = \int_0^{\infty} dp_1 \cdots \int_0^{\infty} dp_n\, K(\ell,p_1)\cdots K(p_{n-1},p_n)\underset{\sim}{g}(p_n). \tag{112}$$

Substitution into Eq. (109) yields

$$\sum_{n=0}^{\infty} K^{n}\underset{\sim}{g} = \underset{\sim}{a}(\ell)\sum_{n=0}^{\infty} \underset{\sim}{b}\cdot K^{n}\underset{\sim}{g} + \sum_{n=1}^{\infty} K^{n}\underset{\sim}{g} \tag{113}$$

or

$$\underset{\sim}{g}(\ell) = \underset{\sim}{a}(\ell)\left(\sum_{n=0}^{\infty} \underset{\sim}{b}\cdot K^{n}\underset{\sim}{g}\right), \tag{114}$$

and equating the components of Eq. (114) yields

$$
\begin{aligned}
(1-R_1)\alpha_1 - R_2\alpha_2 - R_3\alpha_3 &= 0 \\
-R_1\alpha_1 (1-R_2)\alpha_2 - R_3\alpha_3 &= 0 \\
-R_1\alpha_1 - R_2\alpha_2 (1-R_3)\alpha_3 &= 0,
\end{aligned}
\tag{115}
$$

where

$$R_j = \sum_{n=0}^{\infty} \underset{\sim}{b} \cdot K^n \underset{\sim}{g}_j \quad \text{for } j = 1, 2, 3,$$

$$\underset{\sim}{g}_1(\ell) = (d_1\, \ell a(\ell),\, 0,\, 0)^t,$$

$$\underset{\sim}{g}_2(\ell) = (0,\, -d_1\, \ell a(\ell),\, 0)^t,$$

and

$$\underset{\sim}{g}_3(\ell) = (0,\, 0,\, a(\ell))^t.$$

Non-zero constants α_1, α_2 and α_3 will satisfy Eq. (115) if and only if

$$
0 = \begin{vmatrix} (1-R_1) & -R_2 & -R_3 \\ -R_1 & (1-R_2) & -R_3 \\ -R_1 & -R_2 & (1-R_3) \end{vmatrix}
\tag{116}
$$

$$= 1 - (R_1 + R_2 + R_3).$$

Thus the condition

$$1 = R_1 + R_2 + R_3, \tag{117}$$

where

$$R_i = \sum_{n=0}^{\infty} \underset{\sim}{b} \cdot K^n \underset{\sim}{g}_i ,$$

$$\underset{\sim}{b} \cdot K^0 \underset{\sim}{g}_i = \int_0^{\infty} dp\, \underset{\sim}{b}(p) \cdot \underset{\sim}{g}_i (p),$$

and, for $n > 0$,

$$\underset{\sim}{b} \cdot K^n \underset{\sim}{g}_i = \int_0^{\infty} dp \int_0^{\infty} dp_1 \cdots \int_0^{\infty} dp_n \underset{\sim}{b}(p) \cdot K(p, p_1) \ldots K(p_{n-1}, p_n) \underset{\sim}{g}_i (p_n),$$

gives the dispersion relation for the full set of integral equations. It is again a scalar relation between ω and k, and its detailed content will be explored in the next chapter. When it is satisfied, the solution is specified by the restriction

$$\alpha_1 = \alpha_2 = \alpha_3 , \tag{118}$$

and the field variables depend on one arbitrary multiplicative constant, as before. It may be seen by inspection that Eq. (118) reduces to Eq. (105) if ϵ is replaced by 0 throughout Eq. (118).

First Order Corrections to the Reduced Integral Equations

The solution of the full integral equations may be used to check the accuracy of the reduced integral equations. In particular, the dispersion relation (117) may be seen to be a direct generalization of Eq. (105). However, it is also convenient to assess the rate of convergence of the perturbation expansion by obtaining the first-order fields and comparing them with the zeroth order results. This is done in the present section.

Direct substitution into Eqs. (90) and (91) yields for the first order equations

$$F_1^{\pm}(\ell) = \pm \frac{ik\ell 2 J_m(\ell r)}{(\ell^2+h^2)(h^2-k^2)} \int_0^{\infty} p\,dp \left[J_m(pr_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(pr_0 \cos s)\right] F_z^1(p)$$

$$\pm \frac{ik\ell J_m(\ell r)}{(\ell^2+h^2)(h^2-k^2)} \int_0^{\infty} p\,dp \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \begin{bmatrix} J_{m+1}(pr_0 \cos s) F_0^{+}(p) \\ + J_{m-1}(pr_0 \cos s) F_0^{-}(p) \end{bmatrix} \quad (119)$$

$$\pm \frac{\int_0^{\infty} p\,dp}{(\ell^2+h^2)} \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s \left[\Gamma_{m\pm 1} - \frac{i\omega \ell \Delta_m}{\omega_B r_0 (h^2-k^2)}\right] 2pr_0 F_z^0(p)$$

and

$$F_z^1(\ell) = G(\ell) \int_0^{\infty} dp\, M(p) F_z^1(p) + \int_0^{\infty} dp\, N(\ell,p) F_z^1(p) + D(\ell), \quad (120)$$

where Δ_n, Γ_n, $G(\ell)$, $M(p)$ and $N(\ell,p)$ are given by Eqs. (89) and (97), and $D(\ell)$ is given by

$$D(\ell) = G(\ell)\left(\frac{h^2}{h^2-k^2}\right) \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \int_0^{\infty} p\,dp \begin{bmatrix} J_{m+1}(pr_0 \cos s) F_0^{+}(p) \\ + J_{m-1}(pr_0 \cos s) F_0^{-}(p) \end{bmatrix}$$

$$+\left[\frac{h^2-k^2-(\omega k/u)}{(h^2-k^2)(\ell^2+h^2)}\right]\int_0^\infty p\,dp\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin s\,\Delta_m \cdot \tag{121}$$

$$\cdot\left(i\Omega\left[F_0^+(p)-F_0^-(p)\right]-\frac{m\sin s}{\cos s}\left[F_0^+(p)+F_0^-(p)\right]\right).$$

Use of Eq. (95) permits $\mathcal{D}(\ell)$ to be evaluated more concisely as

$$\mathcal{D}(\ell)=-G(\ell)\left(\frac{2h^2}{h^2-k^2}\right)\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\int_0^\infty p\,dp\, A(p)G(p)\left[\frac{d}{dx}J_m(x)\right]_{x=pr_0\cos s}$$

$$-\left[\frac{h^2-k^2-(\omega k/u)}{(h^2-k^2)(\ell^2+h^2)}\right]\int_0^\infty p\,dp\int_{-\infty}^{0} ds\,\Omega^2 e^{i\Omega s}\,2\sin s\,A(p)G(p)\,\Delta_m, \tag{122}$$

where

$$A(\ell)=\left(\frac{2ik\ell}{h^2-k^2}\right)\int_0^\infty p\,dp\,F_z^0(p)\left[J_m(pr_0)-\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}J_m(pr_0\cos s)\right]$$

$$=\frac{ik\ell}{h^2}\int_0^\infty p\,dp\;M(p)\,F_z^0(p)$$

$$=\left(\frac{ik\ell}{h^2}\right),$$

or, in obvious notation,

$$D(\ell) = \alpha G(\ell) + H(\ell). \tag{123}$$

Here use has been made of the zeroth order dispersion relation - Eq. (105) - in the simplification of $A(\ell)$. Actually, the dispersion relation has infinitely many higher order terms, but these do not affect the calculation to this order. They must, of course, be accounted for in the complete perturbation solution. This point is discussed further after the corrected dispersion relation has been displayed.

The structure of Eqs. (119) and (120) is quite similar to that of the zeroth order equations; after $F_2^{\,1}(\ell)$ is determined from Eq. (120), $F_1^{\,\pm}(\ell)$ is obtained by integration. However, the form of Eq. (120) requires that the dispersion relation (105) for the zeroth order equations be modified. That is, the relation between ω and k is a function of ϵ, and Eq. (105) provides only a first approximation to the relation. The required modification is readily obtained if Eqs. (100) and (120) are combined into the single equation

$$F_2^{\,\circ} + \epsilon F_2^{\,1} = GM(F_2^{\,\circ} + \epsilon F_2^{\,1}) + NF_2^{\,\circ} + \epsilon\alpha G + \epsilon H + \epsilon N F_2^{\,1} \tag{124}$$

and solutions are sought in the form

$$F_2^{\,\circ} = \sum_{n=0}^{\infty} N^n G$$

and (125)

$$F_z' = \sum_{n=0}^{\infty} N^n H.$$

Thus, the zeroth order solution is unchanged, except for the dispersion relation, and substitution into Eq. (124) yields

$$G + \sum_{n=1}^{\infty} N^n G + \epsilon H + \epsilon \sum_{n=1}^{\infty} N^n H$$

$$= G\left(M \sum_{n=0}^{\infty} N^n G + \epsilon M \sum_{n=0}^{\infty} N^n H + \epsilon \alpha\right) \quad (126)$$

$$+ \sum_{n=1}^{\infty} N^n G + \epsilon H + \epsilon \sum_{n=1}^{\infty} N^n H$$

or

$$G = G\left(\sum_{n=0}^{\infty} MN^n G + \epsilon \sum_{n=0}^{\infty} MN^n H + \epsilon \alpha\right). \quad (127)$$

The corrected dispersion relation is then given by

$$1 = \sum_{n=0}^{\infty} MN^n G + \epsilon \alpha + \epsilon \sum_{n=0}^{\infty} MN^n H, \quad (128)$$

where $G(\ell)$, $M(p)$ and $N(\ell, p)$ are given by Eq. (97)

$$\epsilon = \left(\frac{\omega_B r_o}{u}\right),$$

$$A(\ell) = \left(\frac{ik\ell}{h^2}\right),$$

$$\alpha = -\left(\frac{2h^2}{h^2 - k^2}\right) \int_{\infty}^{0} ds \, i\Omega e^{i\Omega s} \int_0^{\infty} p\,dp \; A(p)\, G(p) \left[\frac{d}{dx} J_m(x)\right]_{x = pr_o \cos s},$$

$$H(\ell) = -\left[\frac{h^2 - k^2 - (\omega k/u)}{(h^2 - k^2)(\ell^2 + h^2)}\right] \int_0^{\infty} p\,dp \int_{-\infty}^{0} ds\,\Omega^2 e^{i\Omega s}\, 2 \sin s\, A(p) G(p) \Delta_m, \qquad (129)$$

$$N^0 = 1,$$

and, for example,

$$MN^nH = \int_0^{\infty} dp \int_0^{\infty} dp_1 \cdots \int_0^{\infty} dp_n\, M(p) N(p, p_1) \cdots N(p_{n-1}, p_n) H(p_n).$$

A direct calculation shows that the right hand side of Eq. (128) consists of the first two terms of the power series expansion in ϵ of the right hand side of Eq. (118). Thus, the perturbation theory builds up the correct dispersion relation, term by term, the n^{th} approximation giving a dispersion relation which is correct up to terms of order ϵ^n.

Summary of the Chapter

The stability problem has been analyzed on a microscopic level by use of Maxwell's equations and the collisionless Boltzmann equation. Certain Hankel transforms of the perturbed electric field were adopted as the basic field variables, and the perturbed beam current was expressed in terms of these variables. The integral form of Maxwell's equations then led to three linear, coupled integral equations which govern the development of disturbances. The equations are solved by expressing the field variables in the form of infinite series whose typical terms are n-fold integrals of fairly complex structure, and a scalar dispersion relation of the same general nature is obtained as a condition that the equations be solvable. The solutions, but not the dispersion relation, depend on one arbitrary multiplicative constant, which may be related to the amplitude of the disturbance.

The equations and the field variables are also expanded in powers of $\epsilon = (\omega_b r_0 / u)$, and the integral equations are replaced by an infinite set of integral equations of simpler structure. The first two sets of equations are written out explicitly, and series solutions for the corresponding field variables are obtained. A dispersion relation is again obtained as a solvability condition, and, as expected, the fields and dispersion relation form the first two terms of series expansions of the exact solution and dispersion relation in powers of ϵ. Since the parameter ϵ is small whenever the analysis of the chapter is valid, it is to be expected that most results of physical importance will be obtained from the first terms of these expansions.

While the reduction of the stability problem to integral equations has been given in detail, only the general structure of the solution and the dispersion relation has been displayed. Knowledge about the detailed behavior of disturbances is also desirable, and this will be obtained by asymptotic methods in the next chapter.

Chapter 5

EXAMINATION OF SOLUTIONS

Evaluation of Iterations

Case I: reduced integral equations

These comparatively simple equations are used to develop the methods needed for an analysis of the general equations, but they are also of interest as a good approximation to the physical situation. Here attention centers on the detailed evaluation of the series solution obtained in the previous chapter. Rather difficult integrals appear in this process, and their exact evaluation has not been feasible. Instead, approximate values and estimates of error are obtained by asymptotic methods.

The mathematical complexity of the analysis becomes clear when the first terms of the solution

$$F_z = \sum_{j=0}^{\infty} N^j G \tag{1}$$

are examined. The first term

$$\begin{aligned} N^0 G &= G(\ell) \\ &= \frac{J_m(\ell r_0)}{(\ell^2 + h^2)} \end{aligned} \tag{2}$$

is simple enough and corresponds to the field obtained from the macroscopic analysis of Chapter 2. However, the next term

$$N'G = \frac{d_3 TR}{(\ell^2 + h^2)} J_m(\ell r_0 \cos\phi) \int_0^\infty dp \frac{2p^2 r_0}{(p^2+h^2)} J_m(pr_0) J_m(pr_0 \cos\phi \cos s) J_1(pr_0 \sin\phi \sin s), \qquad (3)$$

where

$$d_3 = \left(1 - \left[\frac{\omega k/u}{h^2 - k^2}\right]\right)$$

and the operators T and R are given by

$$T = \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s$$

and

$$R = \int_0^{\frac{\pi}{2}} d\phi \cos\phi,$$

involves the internal structure of the beam and illustrates the typical difficulties encountered by an exact treatment. The p integration may be carried out without difficulty by use of formula (50), p. 55 in the book Tables of Integral Transforms, Vol. II, [50] and application of the relations

$$I_m(z) = e^{-im\frac{\pi}{2}} J_m(iz)$$

and

$$K_m(z) = \frac{i\pi}{2} e^{im\frac{\pi}{2}} H_m^1(iz) \qquad (4)$$

connecting Bessel functions and modified Bessel functions in the right half plane. The result is

$$N'G = -\left[\frac{d_3 \pi h r_0 H_m^1(ihr_0)}{(\ell^2 + h^2)}\right] TR\, J_m(\ell r_0 \cos\phi) J_m(ihr_0 \cos\phi \cos s) J_1(ihr_0 \sin\phi \sin s). \qquad (5)$$

The remaining integrals are quite troublesome to carry through, since quite accurate values are needed for the dispersion relation and for use in subsequent iterations. Further analysis will be restricted to asymptotic regimes. In this section only the case $|hr_o| << 1$ will be analyzed. This corresponds to a low frequency, long wavelength regime in which the skin depth is much larger than the beam radius. Under these conditions the internal structure of the beam should be fairly unimportant, providing only minor corrections to the macroscopic analysis.

When $m \neq 0$ the major content of Eq. (5) may be displayed by regrouping powers of hr_o to establish the size of the term and then introducing the limiting process $hr_o \rightarrow 0$ in that part of the term which is of order one. Specifically, Eq. (5) is replaced by

$$N'G = -d_3\left(\frac{h^2 r_0^2}{2}\right) \lim_{hr_0 \to 0} TR \frac{J_m(\ell r_0 \cos\phi)}{\ell^2 + h^2}\, i\pi H_m^1(ihr_0)\, J_m(ihr_0 \cos\phi \cos s)\left(\frac{2}{ihr_0}\right) J_1(ihr_0 \sin\phi \sin s)$$

$$= -d_3\left(\frac{h^2 r_0^2}{2m}\right) T \sin s \cos^m s \int_0^{\frac{\pi}{2}} d\phi \, \sin\phi \cos^{m+1}\phi \, \frac{J_m(\ell r_0 \cos\phi)}{\ell^2} \qquad (6)$$

$$= -d_3\left(\frac{h^2 r_0^2}{2m}\right)\left[\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin^2 s \cos^m s\right] \frac{1}{\ell^2}\left(\frac{J_{m+1}(\ell r_0)}{\ell r_0}\right),$$

where Sonine's first finite integral has been used to effect the ϕ integration. By introducing (for any m)

$$S_m = \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin^2 s \cos^m s, \tag{7}$$

this term may be written more compactly as

$$N^1 G = -\left(\frac{h^2 r_0^2}{2m}\right) d_3\, S_m \left[\frac{J_{m+1}(\ell r_0)}{\ell^3 r_0}\right] \tag{8}$$

and its order of magnitude is readily established. The constant d_3 is of order one, while S_m is of order one when m is even and of order $\Omega^2 << 1$ when m is odd. Thus in any case $N^1 G$ is of order $h^2 r_o^2$ with respect to $N^0 G$, and when m is odd it is smaller. Of course, a full evaluation of $N^1 G$ would yield an infinite series in hr_o, while Eq. (8) yields merely the dominant term for this regime. However, the expression is quite useful for establishing the size of effects due to internal beam structure, while its very simple form permits subsequent iterations of the N operation to be evaluated to the same level of approximation.

The next term $N^2 G$ is obtained by use of (37), p. 53, T.I.T. and of Sonine's first finite integral as

$$N^2 G = -\left(\frac{d_3^2 S_m}{\ell^2 + h^2}\right)\left(\frac{h^2 r_0^2}{2m}\right) TR\, J_m(\ell r_0 \cos\phi) \int_0^{\infty} dp\, 2p^2 r_0 \left(\frac{J_{m+1}(p r_0)}{p^3 r_0}\right) J_m(p r_0 \cos\phi \cos s) J_1(p r_0 \sin\phi \sin s)$$

$$= -\left(\frac{d_3^2 S_m}{\ell^2 + h^2}\right)\left(\frac{h^2 r_0^2}{2m}\right) T \sin s \cos^m s \int_0^{\frac{\pi}{2}} d\phi \sin\phi \cos^{m+1}\phi\, J_m(\ell r_0 \cos\phi) = \tag{9}$$

$$= -\left(\frac{h^2 r_0^2}{2m}\right)\left(\frac{d_3^2 S_m^2}{\ell^2+h^2}\right)\left[\frac{J_{m+1}(\ell r_0)}{\ell r_0}\right].$$

To compare this term with N^1G, the corresponding limiting process $hr_0 \to 0$ should be carried out, even though in this term there is no difficulty in performing the necessary integrations. The result is

$$N^2G = -\left(\frac{h^2 r_0^2}{2m}\right)\lim_{hr_0 \to 0}\left(\frac{d_3^2 S_m^2}{\ell^2+h^2}\right)\left[\frac{J_{m+1}(\ell r_0)}{\ell r_0}\right]$$

$$= -\left(\frac{h^2 r_0^2}{2m}\right) d_3^2 S_m^2 \left[\frac{J_{m+1}(\ell r_0)}{\ell^3 r_0}\right] \tag{10}$$

or

$$N^2G = d_3 S_m N^1 G. \tag{11}$$

This leads immediately to the induction formula

$$N^k G = -\left(\frac{h^2 r_0^2}{2m}\right)\left(d_3 S_m\right)^k \left[\frac{J_{m+1}(\ell r_0)}{\ell^3 r_0}\right] \text{ for } k \text{ and } m = 1, 2, 3 \ldots \tag{12}$$

and completes the evaluation of $F_z(\ell)$. Collecting terms and inserting an arbitrary multiplicative constant yields when $|d_3 S_m| < 1$ and $m \neq 0$

$$F_z(\ell) = A\left(\left[\frac{J_m(\ell r_0)}{\ell^2+h^2}\right] - \left(\frac{h^2 r_0^2}{2m}\right)\left(\frac{d_3 S_m}{1-d_3 S_m}\right)\left[\frac{J_{m+1}(\ell r_0)}{\ell^3 r_0}\right]\right), \tag{13}$$

an expression valid up through terms of order $h^2 r_0^2$. The assumption that $|d_3 S_m| < 1$ is necessary for the convergence of the series and must be verified from the dispersion relation, but it is, in fact, correct for this regime.

The physical effects due to the beam structure are seen more directly by examining the electric field $f_z(r)$, the Hankel transform of Eq. (13). However, the small hr_o approximation implies that $F_z(\ell)$ is determined inaccurately when ℓ is small and that $f_z(r)$ will be determined inaccurately when r is large. Attention is therefore restricted to the interior region $r \leq r_o$ for which $|hr| \ll 1$ is also satisfied. For this region Eq. (13) is accurate enough, and direct integration yields

$$f_z(r) = A\left[\frac{i\pi}{2} H_m^1(ihr_o) J_m(ihr) - \left(\frac{h^2 r_o^2}{8m^2(m+1)}\right)\left(\frac{d_3 S_m}{1 - d_3 S_m}\right)\left(\frac{r}{r_o}\right)^m \left(1 + m\left[1 - \frac{r^2}{r_o^2}\right]\right)\right]. \qquad (14)$$

This equation is somewhat deceptive, though, since the small hr_o approximation has not been introduced into the first term. All meaningful terms in Eq. (14) may be obtained by expanding the Bessel functions and retaining only terms of order $h^2 r_o^2$ or lower. This will be done explicitly for $m = 1$ to facilitate comparison with the macroscopic theory. A simple substitution yields

$$f_z(r) = \frac{1}{2} A\left(\frac{r}{r_o}\right)\left(\left[1 + \frac{1}{2} h^2 r_o^2 \ln\left(\frac{\gamma}{2} hr_o\right) - \frac{1}{8} h^2 r_o^2 \left(2 - \frac{r^2}{r_o^2}\right)\right] - \frac{1}{8} h^2 r_o^2 \left(\frac{d_3 S_1}{1 - d_3 S_1}\right)\left[2 - \frac{r^2}{r_o^2}\right]\right) \qquad (15)$$

and, since $|d_3 S_1| \sim |\Omega^2| \ll 1$, the internal beam motion is seen to induce only minor corrections to the perturbed electric field in this regime.

The dispersion relation, Eq. (105) of the last chapter, is also readily evaluated. Use of Eqs. (13) and (15) with A = 1 yields for m = 1

$$1-\frac{k^2}{h^2} = 1+\frac{1}{2}h^2r_0^2\ln\left(\frac{\gamma}{2}hr_0\right)-\frac{1}{8}h^2r_0^2\left(\frac{1}{1-d_3S_1}\right)$$
$$-\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\cos s\left[1+\frac{1}{2}h^2r_0^2\ln\left(\frac{\gamma}{2}hr_0\right)-\frac{1}{8}h^2r_0^2\left(2-\cos^2 s\right)\right] \qquad (16)$$

or to good approximation

$$0=\left(\frac{\Omega^2}{1-\Omega^2}\right)+\left(\frac{1}{1-\Omega^2}\right)\frac{1}{2}h^2r_0^2\ln\left(\frac{\gamma}{2}hr_0\right)-\frac{1}{8}h^2r_0^2\left(1+d_3S_1+\frac{11}{9}\Omega^2\right)+\frac{k^2}{h^2}, \qquad (17)$$

where $$d_3 \simeq 1-\frac{\omega}{ku}\frac{k^2}{h^2}\simeq 1$$

and $$S_1 \simeq -\frac{2}{9}\Omega^2.$$

The dominant terms of this equation form the dispersion relation of Chapter 2, and the other terms are corrections due to the finite wave number of the perturbation and to the microscopic properties of the beam particles. The corrections to the dispersion law are readily obtained and will not be given explicitly.

More generally, substitution of Eq. (13) with A = 1 into Eq. (105) yields for the dispersion relation when m > 1

$$1-\frac{k^2}{h^2} = i\pi H_m^{1}(ihr_0)\left[J_m(ihr_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(ihr_0 \cos s)\right]$$

$$- \left[\frac{h^2 r_0^2}{4m^2(m+1)}\right]\left[\frac{d_3 S_m}{1-d_3 S_m}\right]\left[1-\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \cos^m s\left(1+m \sin^2 s\right)\right], \tag{18}$$

a result which contains the small hr_0 approximation in the second term only. The meaningful terms are again extracted by making this approximation in the first term, and this yields the final expression

$$1-\frac{k^2}{h^2} = \frac{1}{m}\left(1-\frac{h^2 r_0^2}{2(m^2-1)}\right) - \frac{1}{m}\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \cos^m s\left[1-\frac{h^2 r_0^2}{4}\left(\frac{1}{[m-1]}-\frac{\cos^2 s}{[m+1]}\right)\right]$$

$$- \left[\frac{h^2 r_0^2}{4m^2(m+1)}\right]\left[\frac{d_3 S_m}{1-d_3 S_m}\right]\left[1-\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \cos^m s\left(1+m \sin^2 s\right)\right], \tag{19}$$

which is correct up to order $h^2 r_0^2$. The algebraic consequences of this dispersion relation are readily obtained and will not be given explicitly. One qualitative feature of interest results from the identification of the first line on the right of Eq. (19) as the effect of macroscopic motion and the second line as the effect of internal beam motion. It then follows that the internal contribution is much smaller than the macroscopic contribution when m is odd but is comparable to the macroscopic contribution when m is even.

Case II: coupled integral equations

To facilitate comparison with the preceding analysis, the first terms of the solution

$$\mathbf{F}(\ell) = \sum_{n=0}^{\infty} K^{n} \mathbf{g} \tag{20}$$

are evaluated by similar techniques, and the arbitrary multiplicative constant α_1 is chosen to be

$$\alpha_1 = \frac{1}{2}\left(1 - \frac{k^2}{h^2}\right). \tag{21}$$

The first term

$$K^{0}\mathbf{g} = \left(d_1 \ell G(\ell), -d_1 \ell G(\ell), G(\ell)\right)^{t},$$

where (22)

$$G(\ell) = \frac{J_m(\ell r_0)}{(\ell^2 + h^2)}$$

and $$d_1 = \frac{ik}{h^2}.$$

again corresponds to the macroscopic field, while the higher terms represent corrections due to the internal beam particle motion.

The next term $K^{1}\mathbf{g}$ must again be evaluated by asymptotic methods and, since the expressions are quite complex, the term will be given in component form. The full term is given by

$$\left[K'g\right]_z = \int_0^\infty dp \left[K^+(\ell,p)\, d_1 p\, G(p) - K^-(\ell,p)\, d_1 p\, G(p) + K_z(\ell,p)\, G(p)\right],$$

$$\left[K'g\right]^+ = \int_0^\infty dp \left[P^+(\ell,p)\, d_1 p\, G(p) - M^+(\ell,p)\, d_1 p\, G(p) + Z^+(\ell,p)\, G(p)\right], \qquad (23)$$

and

$$\left[K'g\right]^- = -\int_0^\infty dp \left[P^-(\ell,p)\, d_1 p\, G(p) - M^-(\ell,p)\, d_1 p\, G(p) + Z^-(\ell,p)\, G(p)\right],$$

which uses the definitions established by Eq. (107), Chapter 4. Attention is again restricted to the case $|hr_0| \ll 1$, $m \geq 1$, and the small hr_0 approximation adopted in the previous section will be applied to the evaluation of Eq. (23). The z component is readily obtained by means of previous results as

$$\left[K'g\right]_z = \left[1 + \frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right] \int_0^\infty dp\, K_z(\ell,p)\, G(p)$$

$$= \left[1 + \frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left[N'G\right] \qquad (24)$$

$$= -\left[1 + \frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{h^2 r_0^2}{2m}\right)\left[\frac{h^2 - k^2 - (\omega k/u)}{h^2 - k^2}\right]\left[\int_{-\infty}^0 ds\, i\Omega\, e^{i\Omega s} \sin^2 s \cos^m s\right]\left[\frac{J_{m+1}(\ell r_0)}{\ell^3 r_0}\right],$$

and the parameter

$$\lambda = \frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right) \qquad (25)$$

is seen to be a measure of the coupling provided by the full integral equations. The other two components are somewhat more cumbersome and are given by

$$\left[K'\underset{\sim}{g}\right]^{\pm} = \pm\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left[d_4 \ell\left[N'G\right]+\int_0^\infty dp\, Q^{\pm}(\ell,p)2pr_0 \sin s\, G(p)\right], \tag{26}$$

where as before

$$d_4 = -\frac{i(\omega/u)}{h^2-k^2-(\omega k/u)}$$

and

$$Q^{\pm}(\ell,p) = \left(\frac{\omega_B r_0}{u}\right)\frac{p}{\ell^2+h^2}\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\int_0^{\frac{\pi}{2}} d\phi \cos\phi \sin\phi\, J_0(pr_0 \sin s \sin\phi).$$

$$\cdot J_{m\pm 1}(pr_0 \cos s \cos\phi) J_{m\pm 1}(\ell r_0 \cos\phi).$$

Here the first term may be obtained as before, but the second term requires investigation. Use of (51), p. 56, of T.I.T. yields for the p integration

$$\int_0^\infty dp\, 2p r_0^2 \frac{J_m(pr_0)}{p^2+h^2} J_0(pr_0 \sin\phi \sin s) J_{m\pm 1}(pr_0 \cos\phi \cos s)$$

$$= \mp 2hr_0 K_m(hr_0) J_0(hr_0 \sin\phi \sin s) I_{m\pm 1}(hr_0 \cos\phi \cos s) \tag{27}$$

$$= (ihr_0)\, i\pi H_m^1(ihr_0) J_0(ihr_0 \sin\phi \sin s) J_{m\pm 1}(ihr_0 \cos\phi \cos s)$$

and substitution into Eq. (26) followed by the small $|hr_0|$ approximation yields

$$\int_0^\infty dp\, Q^{\pm}(\ell,p) 2pr_0 \sin s\, G(p)$$

$$= \left[(ihr_0)\, i\pi H_m^1(ihr_0) J_{m\pm 1}(ihr_0)\right]\left(\frac{\omega_B r_0}{u}\right)\left[\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s \cos^{m\pm 1} s\right]\left[\frac{J_{m\pm 1\pm 1}(\ell r_0)}{\ell^3 r_0}\right], \tag{28}$$

where for meaningful results only the dominant terms of the first bracket should be used in final evaluations. This leads to the final expression

$$\left[K_g'\right]^{\pm} = \pm\left[1 + \frac{ik}{h}\,\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left\{\begin{array}{l}\left(\frac{h^2 r_0^2}{2m}\right)\frac{i(\omega \ell/u)}{h^2-k^2}\left[\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin^2 s\cos^m s\right]\left[\frac{J_{m\pm 1}(\ell r_0)}{\ell^3 r_0}\right] \\ -\pi h r_0 H_m^1(ihr_0) J_{m\pm 1}(ihr_0)\left(\frac{\omega_B r_0}{u}\right)\left[\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin s\cos^{m\pm 1} s\right]\left[\frac{J_{m+1\pm 1}(\ell r_0)}{\ell^3 r_0}\right]\end{array}\right\}. \tag{29}$$

Subsequent iterations may be carried out by similar techniques. However, for comparison with the results for the reduced integral equation, it is sufficient to compare the dispersion relation calculated from Eqs. (22), (24), and (29) with the approximate form of Eq. (18)

$$\begin{aligned}1 - \frac{k^2}{h^2} = {} & i\pi H_m^1(ihr_0)\left[J_m(ihr_0) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(ihr_0\cos s)\right] \\ & - \frac{h^2 r_0^2}{4m(m+1)}\left[\frac{h^2 - k^2 - (\omega k/u)}{h^2 - k^2}\right]\left[\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin^2 s\cos^m s\right]\left[1 - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\cos^m s\,(1 + m\sin^2 s)\right]\end{aligned} \tag{30}$$

which results from keeping only the first two iterations in the expression for $F_z(\ell)$.

To facilitate comparison with Eq. (30), it is convenient to write the present dispersion relation in the form

$$1 - \frac{k^2}{h^2} = D_1 + D_2 ,$$

where

$$D_1 = 2\int_0^\infty dp\,p\left[J_m(pr_0) - \int_{-\infty}^0 ds\,i\Omega e^{i\Omega s} J_m(pr_0\cos s)\right]\frac{J_m(pr_0)}{p^2+h^2}$$

$$+\left(\frac{ik}{h^2}\right)\left(\frac{\omega_B r_0}{u}\right)\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin s\int_0^\infty dp\,p^2\left[J_{m+1}(pr_0\cos s) - J_{m-1}(pr_0\cos s)\right]\frac{J_m(pr_0)}{p^2+h^2}$$

$$= i\pi H_m^1(ihr_0)\left[J_m(ihr_0) - \int_{-\infty}^0 ds\,i\Omega e^{i\Omega s} J_m(ihr_0\cos s)\right] \tag{31}$$

$$-\left(\frac{k}{h}\right)\left(\frac{\omega_B r_0}{u}\right)\frac{i\pi}{2}H_m^1(ihr_0)\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin s\left[J_{m+1}(ihr_0\cos s) - J_{m-1}(ihr_0\cos s)\right]$$

and

$$D_2 = -\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{h^2 r_0^2}{2m}\right)\left[\frac{h^2-k^2-(\omega k/u)}{h^2-k^2}\right]\left[\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin^2 s\cos^m s\right]\cdot$$

$$\cdot 2\int_0^\infty dp\,p\left[J_m(pr_0) - \int_{-\infty}^0 ds\,i\Omega e^{i\Omega s} J_m(pr_0\cos s)\right]\frac{J_{m+1}(pr_0)}{p^3 r_0}$$

$$+\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{\omega_B r_0}{u}\right)\left(\frac{h^2 r_0^2}{2m}\right)\left[\frac{i\omega/u}{h^2-k^2}\right]\left[\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin^2 s\cos^m s\right]\cdot$$

$$\cdot\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin s\int_0^\infty dp\,p^2\left[J_{m+1}(pr_0\cos s) - J_{m-1}(pr_0\cos s)\right]\frac{J_{m+1}(pr_0)}{p^3 r_0}$$

$$-\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{\omega_B r_0}{u}\right)^2\pi hr_0 H_m^1(ihr_0)\int_{-\infty}^0 ds\,i\Omega e^{i\Omega s}\sin s\cdot$$

$$\cdot\int_0^\infty dp\,p\left\{\begin{array}{l} J_{m+1}(ihr_0)\left[\int_{-\infty}^0 ds'\,i\Omega e^{i\Omega s'}\sin s'\cos^{m+1} s'\right]J_{m+1}(pr_0\cos s)\dfrac{J_{m+2}(pr_0)}{p^3 r_0} \\ -J_{m-1}(ihr_0)\left[\int_{-\infty}^0 ds'\,i\Omega e^{i\Omega s'}\sin s'\cos^{m-1} s'\right]J_{m-1}(pr_0\cos s)\dfrac{J_m(pr_0)}{p^3 r_0}\end{array}\right\},$$

corresponding again to a separation into macroscopic and internal beam effects. The p integrations in D_2 may be carried out, yielding the simpler expression

$$D_2 = -\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{h^2 r_0^2}{4m^2(m+1)}\right)\left[\frac{h^2-k^2-(\omega k/u)}{h^2-k^2}\right] S_m^2\left[1-S_m^0-mS_m^2\right]$$

$$+\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{\omega_B r_0}{u}\right)\left(\frac{h^2 r_0^2}{4m}\right)\left[\frac{i(\omega/r_0 u)}{h^2-k^2}\right] S_m^2\left[\frac{1}{(m+1)}S^1_{m+1}-S^3_{m-1}\right]$$

$$-\left[1+\frac{ik}{h}\frac{i\Omega}{hr_0}\left(\frac{\omega_B r_0}{u}\right)\right]\left(\frac{\omega_B r_0}{u}\right)^2\frac{\pi h r_0}{4}H_m^1(ihr_0)\left\{\begin{array}{l} J_{m+1}(ihr_0)\dfrac{S^1_{m+1}}{m+1}\left[S^1_{m+1}-\dfrac{m+1}{m+2}S^1_{m+3}\right] \\ -J_{m-1}(ihr_0)\dfrac{S^1_{m-1}}{m-1}\left[S^1_{m-1}-\dfrac{m-1}{m}S^1_{m+1}\right]\end{array}\right\}, \qquad (32)$$

where

$$S^n_{n'} = \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s}\sin^n s\cos^{n'} s.$$

It is to be noted that when m = 1 the factor m-1 in the denominator of the last term on the right is to be replaced by 1.

When the small $|hr_0|$ approximation is used to evaluate D_1 as

$$D_1 = i\pi H_m^1(ihr_0)\left[J_m(ihr_0)\left[1-S_m^0\right]+\frac{1}{2}\left(\frac{k}{h}\right)\left(\frac{\omega_B r_0}{u}\right)J_{m-1}(ihr_0)S^1_{m-1}\right], \qquad (33)$$

it is seen that the parameter

$$\lambda_1 = \frac{k}{h}\frac{1}{hr_0}\left(\frac{\omega_B r_0}{u}\right) \tag{34}$$

is a measure of the additional contribution to the dispersion relation provided by the full integral equations. Of course, λ and λ_1 are very similar quantities and give essentially the same measure of the strength of coupling inherent in the equations. Similarly, examination of the size of the terms on the right of Eq. (32) shows that the parameters λ,

$$\lambda_2 = \left(\frac{\omega}{ku}\right)\left(\frac{k}{h}\right)\frac{1}{hr_0}\left(\frac{\omega_B r_0}{u}\right)$$

and (35)

$$\lambda_3 = \frac{1}{h^2 r_0^2}\left(\frac{\omega_B r_0}{u}\right)^2$$

serve to indicate the size of the corrections to the internal term of Eq. (30). However, λ_2 need not be considered explicitly, since the previous dispersion relation analyses show that $|\lambda_2| < |\lambda_1|$. The value of $|\lambda_3|$ is not so simply related to $|\lambda_1|$ - it is larger by a factor $|h^2/k^2|$ than $|\lambda_1|^2$ - and λ_3 must be kept as an independent measure of the magnitude of corrections to Eq. (30). For the most unstable disturbances, $|\lambda_3|$ is small, but for long wavelength disturbances of slow growth rate, $|\lambda_3|$ may reach appreciable size.

The analysis of the previous section indicates that the higher terms of Eq. (20), ignored in the derivation of Eqs. (31) and (32), should be of minor importance for the dispersion law. The scale of the corrections to the dispersion law may be estimated by using the previous dispersion relation to evaluate λ_1 and λ_3 as functions of k. For more complete information, Eqs. (31) and (32) should be solved numerically after evaluating the Bessel functions by retaining dominant terms. This will be done here for m = 1 disturbances only. However, other m values give rise to similar analyses.

An analysis of m = 1 disturbances is given in order to clarify the nature of the corrections to the dispersion law. Attention is restricted to the $|hr_0| \ll 1$, $|\Omega^2| \ll 1$ regime. The dominant terms of the dispersion relation become for this case

$$A^2 h^2 r_0^2 = \Omega^2 - \frac{k r_0}{h^2 r_0^2}\,\Omega\left(\frac{\omega_B r_0}{u}\right) + \frac{k^2 r_0^2}{h^2 r_0^2} + \frac{\Omega^2}{2}\left(\frac{\omega_B r_0}{u}\right)^2, \qquad (36)$$

where $$A^2 = -\frac{1}{2}\ln\left|\frac{\gamma}{2} h r_0\right|.$$

Thus the dispersion law may be simplified to

$$A^2 h^2 r_0^2 = \Omega^2 - \frac{k r_0}{h^2 r_0^2}\left(\frac{\omega}{\omega_B}\right)\left(\frac{\omega_B r_0}{u}\right). \qquad (37)$$

Fairly complete evaluations of instability growth rates as functions of k may be made by solving Eq. (37) under the assumption that two terms dominate the equation and then using the solution to estimate the range of wavelengths in the regime for which the third term is negligible. The assumption that the equation is dominated by the previous macroscopic terms

$$A^2 h^2 r_0^2 = \Omega^2 \tag{38}$$

leads to the restrictions

$$kr_0 << \left(\frac{\omega_B r_0}{u}\right),$$

$$kr_0 << A^2 4\pi\sigma r_0 \left(\frac{\omega_B r_0}{u}\right)^2, \tag{39}$$

and

$$kr_0 >> \frac{1}{(4\pi\sigma r_0)}\left(\frac{\omega_B r_0}{u}\right)^2$$

for admissible wavelengths and yields the previously calculated growth rates in this range. These wavelengths correspond to the shortest wavelengths of the $|hr_0| << 1$ regime and yield the most rapid growth rates.

For wavelengths not meeting the conditions of Eq. (39), other terms dominate the dispersion relation. However, the only other solution which satisfies all requirements for consistency comes from the equation

$$A^2 h^2 r_0^2 = \frac{ikr_0}{(4\pi\sigma r_0)}. \tag{40}$$

This leads to the dispersion law

$$\left(\frac{\omega}{\omega_B}\right) = \frac{1}{A^2(4\pi\sigma r_0)^2}\left(\frac{u}{\omega_B r_0}\right) k r_0 \tag{41}$$

corresponding to a stable oscillation provided that the wavelength is sufficiently long. For the reasonable beam condition $\omega_B << 4\pi\sigma c A$, the condition for the validity of Eqs. (40) and (41) may be expressed very simply as

$$k r_0 << \frac{1}{(4\pi\sigma r_0)}\left(\frac{\omega_B r_0}{u}\right)^2 . \tag{42}$$

Otherwise, slightly more complex restrictions on kr_0 appear.

The above analysis may be generalized by removing the restriction $|\Omega^2| << 1$. In particular, for the opposite limit $|\Omega^2| >> 1$, the orbit integrals in the dominant terms of the dispersion relation become

$$\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \cos s = \frac{\Omega^2}{\Omega^2 - 1} \approx 1 + \frac{1}{\Omega^2}$$

and (43)

$$\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s = \frac{i\Omega}{\Omega^2 - 1} \approx \frac{i}{\Omega},$$

and the dispersion relation takes the form

$$1 = \frac{k^2 r_0^2}{h^2 r_0^2} - \frac{1}{\Omega^2} + \frac{1}{\Omega}\frac{k r_0}{h^2 r_0^2}\left(\frac{\omega_B r_0}{u}\right) + \frac{1}{2\Omega^2}\left(\frac{\omega_B r_0}{u}\right)^2, \tag{44}$$

where, as usual,

$$h^2 r_0^2 = i(4\pi\sigma r_0)\left(\frac{\omega_B r_0}{c}\right)\left(\frac{\omega}{\omega_B}\right) + k^2 r_0^2 - \left(\frac{\omega_B r_0}{c}\right)^2\left(\frac{\omega}{\omega_B}\right)^2 . \tag{45}$$

The last term of the dispersion relation again has little influence on the structure of solutions, and Eq. (44) may be approximated by

$$1 = \frac{k^2 r_0^2}{h^2 r_0^2} + \frac{1}{\Omega}\left(\frac{k r_0}{h^2 r_0^2}\right)\left(\frac{\omega_B r_0}{u}\right) - \frac{1}{\Omega^2}. \tag{46}$$

Consistent solutions to Eq. (46) may be found for a wider range of wavelengths than was the case for Eq. (37). However, all solutions are highly damped and hence are of little interest for an analysis of instabilities. In addition, the dispersion relation was derived under the assumption that the disturbances were growing waves. A prediction of only stable or damped roots may thus be taken seriously, but a determination of rates of damping should be made from a formalism that is derived from an initial value problem. For these reasons the solutions to Eq. (46) will not be given explicitly.

This analysis has led to a fairly detailed description of $m = 1$ disturbances in the $|hr_0| \ll 1$ regime. For wavelengths shorter than the betatron length, this regime yields no growing waves. At intermediate wavelengths a highly unstable disturbance occurs, and its behavior is governed primarily by macroscopic equations. However, at longer wavelengths additional terms come into play, stabilizing the mode.

Explicit expressions have been given here and in the analysis of Case I for the formal solutions and formal dispersion relations obtained in the last chapter. Thus when $|hr_o| << 1$ physical quantities may be calculated with high precision. However, the corrections to the macroscopic analysis are fairly small, and not all quantities are evaluated explicitly. Instead, sufficient detail is given to illustrate the convergence of the formal solutions and display their physical properties.

The corresponding analysis for first order corrections to the reduced integral equations may be carried out by similar techniques. The details of this process will not be given, since no qualitatively new features appear in the results.

Surface Dominated Perturbations

The asymptotic analysis of the previous section has established that when $|hr_o| << 1$ the dominant terms of the dispersion relation come from the first term of the iteration solution, and correspond physically to surface current driven disturbances. Since much of the tractability of this analysis is due to the dominance of surface terms, it is desirable to exploit this dominance more generally by finding all solutions for which the effects of surface currents control the development of perturbations. Of course, solutions for which $|hr_o| << 1$ will lead back to the principal results of the preceding section, but other solutions require a separate investigation. The full integral equations serve as a starting point for the discussion.

The regime is specified by using Eq. (22) to describe the perturbed electric field, leading to the truncated dispersion relation

$$1 = i\pi H_m^1(ihr_o)\left[J_m(ihr_o) - \int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_m(ihr_o \cos s)\right] + \frac{k^2}{h^2}$$

$$- \frac{k}{h}\left(\frac{\omega_B r_o}{u}\right)\frac{i\pi}{2} H_m^1(ihr_o)\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s\left[J_{m+1}(ihr_o \cos s) - J_{m-1}(ihr_o \cos s)\right], \quad (47)$$

as may be seen from the D_1 term of Eq. (31). Solutions to this equation provide an acceptable analysis of the dispersion relation provided that the remaining terms of the dispersion relation are found to be small when the solutions are used to evaluate them. When $|hr_o| << 1$ this condition is met, yielding, with minor modifications, the analysis of the preceding section. This will not be discussed further here.

To find other solutions requires the use of either intermediate or large values of $|hr_o|$ and leads immediately to the consideration of high frequency disturbances. Little information is gained by simply making a large $|hr_0|$ approximation in Eq. (47), however, since one cannot use asymptotic formulae for all values of s in carrying out the orbit integrals. Instead, the fact that large $|hr_0|$ implies large $|\Omega|$ suggests an approximate method for evaluating the orbit integrals accurately and simply, making it possible to extract the major new content of Eq. (47) quickly. The method is simply to express the integrals in the form of a power series in Ω^{-1} by use of integration by parts techniques and to obtain approximate expressions by truncating the series at the Ω^{-2} term. This yields for general p

$$\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} J_p(ihr_0 \cos s) = J_p(ihr_0) + \frac{ihr_0}{\Omega^2} J_p'(ihr_0)$$

and (48)

$$\int_{-\infty}^{0} ds\, i\Omega e^{i\Omega s} \sin s\, J_p(ihr_0 \cos s) = \frac{i}{\Omega} J_p(ihr_0),$$

where the prime indicates differentiation with respect to the argument of J_p. The dispersion relation then takes the form

$$1 - \left[-\left(\frac{ihr_0}{\Omega^2}\right) + \frac{i}{\Omega}\left(\frac{kr_0}{hr_0}\right)\left(\frac{\omega_B r_0}{u}\right)\right]\left[i\pi H_m^{1}(ihr_0) J_m'(ihr_0)\right] + \frac{k^2 r_0^2}{h^2 r_0^2}. \qquad (49)$$

The investigation of Eq. (49) may now be carried out through approximate evaluations of the Bessel functions.

When $|hr_o| << 1$ the equation becomes for $m \neq 0$

$$1 = \frac{k^2 r_o^2}{h^2 r_o^2} - \frac{1}{\Omega^2} + \frac{1}{\Omega} \frac{k r_o}{h^2 r_o^2} \left(\frac{a_B r_o}{u} \right) \qquad (50)$$

in agreement with Eq. (44). The additional term in Eq. (44) represents the dominant contribution of higher iterations in the formal series. As before, there are no growing disturbances for this regime.

The dispersion relation may also be simplified when $|hr_o| >> 1$ by means of the asymptotic formulae

$$J_m(ihr_o) = \left(\frac{1}{2\pi hr_o} \right)^{\frac{1}{2}} e^{im\frac{\pi}{2}} e^{hr_o}$$

and $\qquad (51)$

$$H_m^1(ihr_o) = \left(\frac{2}{\pi hr_o} \right)^{\frac{1}{2}} e^{-i(m+1)\frac{\pi}{2}} e^{-hr_o} \quad .$$

For this case Eq. (49) again reduces to Eq. (50) but now the equation is to be solved subject to the conditions

$$|\Omega| >> 1$$

and $\qquad (52)$

$$|hr_o| >> 1 .$$

Solutions are easily obtained for wavelengths such that $kr_o >> 1$, but correspond to highly damped disturbances. Thus, the unstable modes of the high frequency, short wavelength regime are not dominated by conditions at the surface of the beam.

The analysis of surface dominated perturbations has been straightforward and has increased the range of wavelengths for which explicit dispersion laws have been extracted from the formal dispersion relation. No further growing modes have been found, but a general method for evaluating high frequency orbit integrals has been derived in the course of the discussion. This method will be used to treat the wide class of extremely localized disturbances in the following sections.

Analysis of Localized High Frequency Disturbances

While the dispersion relation for the coupled integral equations - Eq. (117) of Chapter 4 - may be investigated directly in the high frequency regime by evaluating the orbit integrals to order Ω^{-r} and then continuing the analysis, this procedure is not an efficient way to extract the dominant terms of the dispersion relation for localized disturbances. One reason for this is that no clear-cut method is provided for determining in advance the relative sizes of certain terms. In addition, the whole procedure keeps very close track of the contributions of the surface terms, even though for localized disturbances they will play no role in the final expression for the dispersion law.

Instead, the orbit integrals appearing in the original expression of the perturbed distribution function f in terms of the perturbed electric field will be evaluated by the above asymptotic method. The perturbed currents are then derived by integration, and no difficulty as to the ordering of terms arises. For localized disturbances the surface current is discarded, and the derivation and analysis of the dispersion relation are then straightforward.

It is also convenient to work with a plane wave decomposition of the electric field and to carry out the analysis in rectangular coordinates. Thus the perturbed electric field takes the form

$$\underset{\sim}{E} = \underset{\sim}{F}\,\phi,$$

where (53)

$$F = \left(F_x, F_y, F_z\right)$$

and

$$\phi = e^{i(k_1 x + k_2 y + kz + \omega t)}$$

and the previously employed field variables $F^{\pm}(\ell)$ and $F_z(\ell)$ may be obtained by adding such plane waves, as in Eqs. (55) - (58) of Chapter 4. The perturbed distribution function then takes the form

$$f = \left(\frac{2en_0}{m\gamma_0}\right) h_0 g_0' \int_{-\infty}^{0} dt'\,\hat{\phi}\left[\left(\frac{\omega + k\hat{v}_z}{\omega}\right)\left(\hat{v}_x F_x + \hat{v}_y F_y\right) - \frac{(k_1\hat{v}_x + k_2\hat{v}_y)}{\omega}\hat{v}_z F_z\right], \qquad (54)$$

where the caret indicates that the designated variable is to be evaluated at time $\hat{t} = t + t'$ by means of the equilibrium beam orbits. Expansion of $\hat{\phi}$ yields in previously established notation

$$\tilde{\phi} = \bar{\phi}\, e^{i\bar{\Omega}s}\, e^{i(k_1\hat{x} + k_2\hat{y})}, \tag{55}$$

suggesting the asymptotic method to be used in evaluating the orbit integral in Eq. (54). Since the h_0 function will later impose the restriction $\bar{\Omega} = \Omega$, an expansion in powers of $\bar{\Omega}^{-1}$ is equivalent to an expression in Ω^{-1}. However, since the dynamic variables are expressed in terms of t' rather than s, it is also convenient to obtain the $\bar{\Omega}^{-1}$ expansion by means of the general formulae

$$\int_{-\infty}^{0} ds\, i\bar{\Omega}\, e^{i\bar{\Omega}s} f_1(t') = f_1(0) + \left(\frac{i}{\bar{\Omega}\omega_B}\right) f_1'(0) + \left(\frac{i}{\bar{\Omega}\omega_B}\right)^2 f_1''(0)$$

and (56)

$$\int_{-\infty}^{0} dt'\, e^{i\bar{\Omega}s} f_2(t') = -\left(\frac{i}{\bar{\Omega}\omega_B}\right) f_2(0) + \left(\frac{1}{\bar{\Omega}\omega_B}\right)^2 f_2'(0),$$

where

$$' = \frac{d}{dt'}.$$

which are correct to order $\bar{\Omega}^{-2}$. When use is made of the equilibrium beam orbits, this procedure yields the final result

$$f = \left(\frac{2en_0}{m\gamma_0 i\omega}\right) h_0 g_0' \phi \Bigg[\left(v_x F_x + v_y F_y\right) \left(1 - \frac{1}{\bar{\Omega}} \frac{(k_1 v_x + k_2 v_y)}{\omega_B} + \frac{1}{\bar{\Omega}^2} \frac{(k_1 v_x + k_2 v_y)^2}{\omega_B^2} + \frac{1}{\bar{\Omega}^2} + \frac{i}{\bar{\Omega}^2}(k_1 x + k_2 y) \right) + \tag{57}$$

$$+\left(xF_x+yF_y\right)\left(-\left(\frac{i\omega_B}{\Omega}\right)+\frac{2i}{\Omega^2}\left(k_1v_x+k_2v_y\right)\right)$$

$$\left.+v_zF_z\left(-\frac{1}{\Omega}\frac{(k_1v_x+k_2v_y)}{\omega_B}+\frac{1}{\Omega^2}\frac{(k_1v_x+k_2v_y)^2}{\omega_B^2}+\frac{i}{\Omega^2}(k_1x+k_2y)\right)\right]$$

The velocity integrations necessary to obtain the perturbed current from Eq. (57) are effected more readily by use of the previously defined coordinates v and α. Derivatives of delta functions are again removed by integration by parts procedures, and the z component of the current is specified by the expression

$$-\left(\frac{4\pi i\omega}{c}\right)j_z=\left(\frac{4\pi e^2n_0u}{m\gamma_0c^2}\right)\phi\int_0^{2\pi}d\alpha\left\{-\left[g_0[\;\;]\right]_{v=0}-\int_0^\infty dv\,g_0\frac{d}{dv}[\;\;]\right\}, \tag{58}$$

where

$$g_0=\frac{1}{\pi}\delta\left(v^2-\omega_B^2\mathcal{E}^2\right)$$

and [ ] denotes the expression in brackets in Eq. (57) but with v_z replaced by u and v_x and v_y expressed in terms of v and α. Direct evaluation yields the identity

$$\int_0^{2\pi}\frac{d\alpha}{\pi}[\;\;]=-\left(\frac{v^2}{\Omega\omega_B}\right)\left(k_1F_x+k_2F_y\right)-\left(\frac{2i\omega_B}{\Omega}\right)\left(xF_x+yF_y\right)$$

$$+\left[\left(\frac{v^2}{\Omega^2\omega_B^2}\right)\left(k_1^2+k_2^2\right)+\left(\frac{2i}{\Omega^2}\right)\left(k_1x+k_2y\right)\right]uF_z\,, \tag{59}$$

and substitution and rearrangement in Eq. (58) results in

$$-\left(\frac{4\pi i\omega}{c}\right)j_z = -\frac{2}{r_0}\delta(r-r_0)\left[\left(\frac{i}{\Omega^2}\right)(k_1x+k_2y)F_z-\left(\frac{i}{\Omega}\right)\left(\frac{\omega_B r_0}{u}\right)\left(\frac{xF_x+yF_y}{r_0}\right)\right]\phi$$

$$-2\left[\frac{(k_1^2+k_2^2)}{\Omega^2}F_z-\left(\frac{1}{\Omega}\right)\left(\frac{\omega_B r_0}{u}\right)\left(\frac{k_1F_x+k_2F_y}{r_0}\right)\right]\phi. \qquad (60)$$

For the study of highly localized disturbances, this expression may be simplified further. Such disturbances are formed by a superposition of plane waves involving wave numbers in the range $\sqrt{k_1^2+k_2^2}\, r_0 \gtrsim 1$ in a fashion which limits the entire disturbance to a region $r << r_0$. For these conditions the surface current given by Eq. (60) is of no interest and may be discarded from the start. The relevant current then becomes

$$-\left(\frac{4\pi i\omega}{c}\right)j_z = -2\left[\frac{(k_1^2+k_2^2)}{\Omega^2}F_z-\left(\frac{1}{\Omega}\right)\left(\frac{\omega_B r_0}{u}\right)\left(\frac{k_1F_x+k_2F_y}{r_0}\right)\right]\phi. \qquad (61)$$

Similarly, the x component of the current is specified by

$$-\left(\frac{4\pi i\omega}{c}\right)j_x = -\left(\frac{4\pi e^2 n_0}{m\gamma_0 c^2}\right)\phi\int_0^{2\pi}d\alpha\cos\alpha\int_0^{\infty}dv\,g_0\frac{d}{dv}\left(v[\;]\right), \qquad (62)$$

and the corresponding identity

$$\int_0^{2\pi} \frac{d\alpha}{\pi} \cos s \left[\; \right] \cdot \left[1 + \frac{1}{\Omega^2} + \left(\frac{i}{\Omega^2}\right)(k_1 x + k_2 y) \right] v F_x + \left(\frac{2iv}{\Omega^2}\right) k_1 (x F_x + y F_y)$$

$$+ \left(\frac{v^3}{\Omega^2 \omega_B^2}\right)\left[\left(\frac{3}{4} k_1^2 + \frac{1}{4} k_2^2\right) F_x + \frac{1}{2} k_1 k_2 F_y\right] \qquad (63)$$

$$- \left(\frac{k_1 v}{\Omega \omega_B}\right) u F_z$$

leads to

$$-\left(\frac{4\pi i \omega}{c}\right) j_x = -\left(\frac{4\pi e^2 n_0}{m \gamma_0 c^2}\right) \phi \left[\begin{array}{l} \left(1 + \frac{1}{\Omega^2}\right) F_x + \left(\frac{\epsilon^2}{\Omega^2}\right)\left[\left(\frac{3}{2} k_1^2 + \frac{1}{2} k_2^2\right) F_x + k_1 k_2 F_y\right] - \left(\frac{k_1 u}{\Omega \omega_B}\right) F_z \\ + \left(\frac{i}{\Omega^2}\right)\left[\left(3 k_1 x + k_2 y\right) F_x + 2 k_1 y F_y\right] \end{array} \right] . \qquad (64)$$

Again, further simplifications result from the restrictions

$$\sqrt{k_1^2 + k_2^2}\; r_0 \gtrsim 1$$

$$|\Omega^2| >> 1 \qquad (65)$$

and

$$r << r_0 ,$$

which apply to highly localized disturbances. To good approximation Eq. (64) may be replaced by the expression

$$-\left(\frac{4\pi i \omega}{c}\right) j_x = -\left(\frac{4\pi e^2 n_0}{m \gamma_0 c^2}\right) \phi \left[\left(1 + \frac{1}{\Omega^2}\right) F_x + \left(\frac{\epsilon^2}{\Omega^2}\right)\left[\left(\frac{3}{2} k_1^2 + \frac{1}{2} k_2^2\right) F_x + k_1 k_2 F_y\right] - \left(\frac{k_1 u}{\Omega \omega_B}\right) F_z\right] \qquad (66)$$

in which the entire dependence on position is contained in the factor ϕ . A similar argument then yields for j_y

$$-\left(\frac{4\pi i\omega}{c}\right)j_y = -\left(\frac{4\pi e^2 n_0}{m\gamma_0 c^2}\right)\phi\left[\left(1+\frac{1}{\Omega^2}\right)F_y + \left(\frac{r_0^2}{\Omega^2}\right)\left[\left(\frac{1}{2}k_1^2+\frac{3}{2}k_2^2\right)F_y + k_1k_2F_x\right] - \left(\frac{k_2 u}{\Omega\omega_B}\right)F_z\right]. \quad (67)$$

It is seen from Eqs. (61), (66), and (67) that for highly localized disturbances, the factor ϕ governs the entire spatial dependence of the perturbed beam current. In fact, the beam response to a perturbing electromagnetic field may be characterized by a frequency and wavelength dependent tensor conductivity. This, together with the frequency dependent scalar conductivity law

$$-\left(\frac{4\pi i\omega}{c}\right)\underset{\sim}{j}_p = -\left(\frac{4\pi i\omega\sigma}{c}\right)\phi\underset{\sim}{F} \quad (68)$$

for the plasma current, where σ is given in Chapter 2, indicates that the dispersion law for these disturbances may be found by a simple Fourier analysis of Maxwell's equations.

Each Fourier component may therefore be analyzed separately, and, without loss of generality, for each component a coordinate system may be adopted for which $k_2 = 0$. In this system the beam current becomes

$$-\left(\frac{4\pi i\omega}{c}\right)j_x = -\left(\frac{4\pi e^2 n_0}{m\gamma_0 c^2}\right)\phi\left[\left(1+\left[1+\frac{3}{2}k_1^2 r_0^2\right]\frac{1}{\Omega^2}\right)F_x - \left(\frac{k_1 u}{\Omega\omega_B}\right)F_z\right],$$

$$-\left(\frac{4\pi i\omega}{c}\right)j_y = -\left(\frac{4\pi e^2 n_0}{m\gamma_0 c^2}\right)\phi\left[\left(1+\left[1+\frac{1}{2}k_1^2 r_0^2\right]\frac{1}{\Omega^2}\right)F_y\right], \tag{69}$$

and

$$-\left(\frac{4\pi i\omega}{c}\right)j_z = -2\phi\left[\frac{k_1^2}{\Omega^2}F_z - \frac{1}{\Omega}\left(\frac{\omega_B r_0}{u}\right)\left(\frac{k_1 F_x}{r_0}\right)\right],$$

while the relevant Maxwell's equations may be written in the form

$$\left(\frac{i\omega}{c}\right)\nabla\times \underset{\sim}{B} + \frac{\omega^2}{c^2}\underset{\sim}{F} - \left(\frac{4\pi i\omega}{c}\right)\underset{\sim}{j}_p - \left(\frac{4\pi i\omega}{c}\right)\underset{\sim}{j} = 0$$

and (70)

$$\left(\frac{i\omega}{c}\right)\nabla\times \underset{\sim}{B} = \left(k_1 k F_z - k^2 F_x\,,\ -\left(k_1^2 + k^2\right)F_y\,,\ k k_1 F_x - k_1^2 F_z\right)\phi .$$

Substitution of Eqs. (68) and (69) into Eq. (70) then yields a set of three linear homogeneous equations for the components of $\underset{\sim}{F}$, and the solvability condition for the system is given by the determinant

$$\begin{Vmatrix} h^2 - k_1^2 + \frac{\omega_{b\perp}^2}{c^2} + \left(\frac{1}{\Omega^2}\right)\left(1+\frac{3}{2}k_1^2 r_0^2\right)\left(\frac{\omega_{b\perp}^2}{c^2}\right) & 0 & -k_1 k - \left(\frac{k_1 u}{\Omega\omega_B}\right)\frac{\omega_{b\perp}^2}{c^2} \\ 0 & h^2 + \frac{\omega_{b\perp}^2}{c^2} + \left(\frac{1}{\Omega^2}\right)\left(1+\frac{1}{2}k_1^2 r_0^2\right)\left(\frac{\omega_{b\perp}^2}{c^2}\right) & 0 \\ -k_1 k - \left(\frac{k_1 u}{\Omega\omega_B}\right)\frac{\omega_{b\perp}^2}{c^2} & 0 & h^2 - k^2 + \left(\frac{k_1 u}{\Omega\omega_B}\right)^2\frac{\omega_{b\perp}^2}{c^2} \end{Vmatrix} = 0, \tag{71}$$

where h^2 is now defined by

$$h^2 = k_1^2 + k^2 + \left(\frac{4\pi i\omega\sigma}{c}\right) - \frac{\omega^2}{c^2},$$

and

$$\omega_{b\perp}^2 = \frac{4\pi e^2 n_0}{m\gamma_0} \tag{72}$$

is the transverse plasma frequency for beam particles.

The simplest roots of Eq. (71) are obtained from the yy element of the determinant and satisfy the equation

$$h^2 + \frac{\omega_{b\perp}^2}{c^2} + \frac{1}{\Omega^2}\left(1 + \frac{1}{2}k_1^2 r_0^2\right)\left(\frac{\omega_{b\perp}^2}{c^2}\right) = 0. \tag{73}$$

They correspond to transverse electromagnetic disturbances and are stable. This may be seen from a perturbation solution of Eq. (73) using

$$h^2 + \frac{\omega_{b\perp}^2}{c^2} = 0 \tag{74}$$

as a first approximation. This is just the propagation equation for a transverse wave in a two component plasma, and inspection indicates its stability. As expected, higher order terms maintain this stability.

The remaining roots of Eq. (71) are determined from

$$\left[h^2 - k_1^2 + \frac{\omega_{b\perp}^2}{c^2} + \frac{1}{\Omega^2}\left(1 + \frac{3}{2}k_1^2 r_0^2\right)\frac{\omega_{b\perp}^2}{c^2}\right]\left[h^2 - k^2 + \frac{1}{\Omega^2}\left(\frac{k_1 u}{\omega_B}\right)^2 \frac{\omega_{b\perp}^2}{c^2}\right]$$

$$= k_1^2 k^2 + \left(\frac{2k_1 k}{\Omega}\right)\left(\frac{k_1 u}{\omega_B}\right)\frac{\omega_{b\perp}^2}{c^2} + \frac{1}{\Omega^2}\left(\frac{k_1 u}{\omega_B}\right)^2\left(\frac{\omega_{b\perp}^2}{c^2}\right)^2. \tag{75}$$

However, the individual terms of Eq. (71) are accurate only to order Ω^{-2}. Thus Eq. (75) must be expanded in powers of Ω^{-1}, and terms of higher order than Ω^{-2} must be discarded as spurious. Retention of all admissible terms yields after rearrangement

$$\left[\begin{array}{c}\left(\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)\left(\left[k_1^2+k^2\right]+\left[\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right]+\left[1+\left(\frac{1}{\Omega^2}\right)\left(\frac{k_1 u}{\omega_B}\right)^2\right]\frac{\omega_{bL}^2}{c^2}\right)\\ +\left(\frac{1}{\Omega^2}\right)\left(\frac{k_1^2\omega_{bL}^2}{c^2}\right)\left(\frac{\omega}{\omega_B}\right)^2+\left(\frac{1}{\Omega^2}\right)\left(1+\frac{3}{2}k_1^2 r_0^2\right)\frac{\omega_{bL}^2}{c^2}\left(k_1^2+\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)\end{array}\right]=0 \qquad (76)$$

which serves as the basic dispersion relation for unstable localized disturbances. The content of this equation is more readily explored by noting that the first and third terms of the equation have roughly similar structure, but the third term is much smaller. The dominant part of ω is therefore calculated from the simpler equation

$$\left(\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)\left(\left[k_1^2+k^2\right]+\left[\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right]+\frac{\omega_{bL}^2}{c^2}\right)+\left(\frac{4\pi i\omega\sigma}{c}\right)\left(\frac{1}{\Omega^2}\right)\left(\frac{k_1 u}{\omega_B}\right)^2\frac{\omega_{bL}^2}{c^2}=0 \qquad (77)$$

which follows from Eq. (76) by neglecting the third term and letting u equal c. Approximate solutions are found from the dominant terms of Eq. (77) and their range of validity is established from the correction terms of Eqs. (76) and (77).

Only those solutions of Eq. (77) which satisfy the constraints of Eq. (65) are of interest, and, again, the rates of growth of disturbances or the absence of growing solutions are reliably predicted by this analysis, but rates of damping require further study. When $|4\pi i\omega\sigma/c| << |\omega^2/c^2|$, Eq. (77) is solvable only for wavelengths satisfying the inequality

$$k^2c^2 + k_1^2c^2 >> \omega_p^2, \tag{78}$$

where ω_p is the background plasma frequency, and gives rise to a stable oscillation but no unstable modes. Similarly, when $|\omega^2/c^2| << |4\pi i\omega\sigma/c|$, the dispersion relation is solvable only under the conditions

$$\omega \simeq i\left(\frac{a^2 + a_1^2}{\frac{\omega_p^2}{\omega_{b\perp}^2} + a^2 + a_1^2}\right)\nu,$$

$$\left(\frac{\omega_B r_o}{u}\right) << k r_o, \tag{79}$$

and
$$\left(\frac{\omega_B r_o}{u}\right)|4\pi\sigma r_o| << k_1^2 r_o^2 + k^2 r_o^2 << |4\pi\sigma r_o|^2,$$

where ν is the collision frequency, corresponding to a highly damped wave.

Unstable modes are most readily found by an asymptotic analysis based on the condition $(4\pi i\omega\sigma/c) \approx \omega^2/c^2$. For this it is useful to adopt dimensionless variables, changing the scaling frequency from ω_B to $\omega_{b\perp}$. The variables

$$z = \frac{\omega}{\omega_{b\perp}},$$

$$N = \frac{4\pi\sigma c}{\omega_{b\perp}},$$

$$a = \frac{kc}{\omega_{b\perp}}, \tag{80}$$

$$a_1 = \frac{k_1 c}{\omega_{b\perp}},$$

and $\quad \epsilon = iNz - z^2,$

are suitable, and multiplication of Eq. (77) by $\left(c/\omega_{b\perp}\right)^4$ leads directly to the expression

$$\epsilon^2 + \left(a_1^2 + a^2 + 1 + \frac{a_1^2}{(z+a)^2}\right)\epsilon + \frac{a_1^2 z^2}{(z+a)^2} = 0 \tag{81}$$

for the dispersion relation. The restrictions

$$|z + a| >> 1$$

and

$$a_1^2 + a^2 \gtrsim \frac{c^2}{\omega_{b\perp}^2 r_0^2} \tag{82}$$

then characterize the regime of high frequency, localized disturbances and provide conditions that acceptable solutions of Eq. (81) must meet. The range of frequencies for which $(4\pi i\omega\sigma/c) \approx \omega^2/c^2$ is specified by the condition $|\epsilon| << 1$, which suggests that the term of Eq. (81) quadratic in ϵ be discarded, and application of the restrictions of Eq. (82) yields for these frequencies the approximate dispersion relation

$$\left(a_1^2 + a^2\right)\epsilon + a_1^2 \frac{z^2}{(z+a)^2} = 0, \tag{83}$$

or, on rearrangement,

$$1 = \frac{iNz}{z^2} + \left(\frac{a_1^2}{a_1^2 + a^2}\right)\frac{1}{(z+a)^2}. \tag{84}$$

Use of the definitions of Eq. (80) and of the value of σ determined in Chapter 2 leads to the expression of N as a comparatively simple rational function of z, whose coefficients depend on the particular plasma and beam conditions under study. Since for any given disturbance a and a_1 are known numbers, Eq. (84) may be written as a polynomial equation for z whose coefficients depend on the plasma and beam parameters and on the mode of disturbance. Solutions may be obtained numerically to any desired order of accuracy, but it then requires a lengthy parameter study to establish the behavior of the unstable disturbances. For this much detail it would be best to return directly to Eq. (81) and solve a slightly more complicated equation for z. In addition, for very high frequency instabilities, it would be desirable to take into account the effects of charge neutralization by modifying the derivation of σ to include the fact that the plasma electron density is $\bar{n} - n_o$ instead of $\bar{n}$. This would involve only minor modification of the final expression for σ.

However, a full numerical analysis of the dispersion relation is not needed to establish the major characteristics of the unstable disturbances in this regime. It suffices to assume that the conductivity is due to electron inertia and electron collisions, so that

$$N = \left(\frac{\omega_p^2}{\omega_{b\perp}^2}\right)\frac{1}{\mu + iz}, \qquad (85)$$

where

$$\mu = \frac{\nu}{\omega_{b\perp}}.$$

ν is the collision frequency and ω_p is the plasma frequency of the background plasma, and to search for solutions in the limiting cases of inertial and collisional dominance of the conductivity. Since the plasma is much denser than the beam, $\omega_p^2 >> \omega_{b\perp}^2$, which further simplifies the search for solutions.

When $|z| >> \mu$ or, more precisely, when $|\mathrm{Im}\, z| >> \mu$, the collisional frequency plays no significant role in governing disturbances, and the dispersion relation is well approximated by the equation

$$1 = \left(\frac{\omega_p^2}{\omega_{b\perp}^2}\right)\frac{1}{z^2} + \left(\frac{a_1^2}{a_1^2 + a^2}\right)\frac{1}{(z+a)^2}. \qquad (86)$$

The disturbances are thus governed by an equation identical in form with the dispersion relation for one-dimensional electrostatic modes of the two beam configuration, even though in this case the wave vector is not parallel to the z axis. This type of dispersion relation has been extensively studied [51-53] and gives rise to a continuum of unstable disturbances, including some with very rapid growth rates. The more rapidly growing solutions are most readily exhibited by means of the auxiliary variables

$$b = \frac{\omega_{b\perp}}{\omega_p} a,$$

$$w = \frac{\omega_{b\perp}}{\omega_p} z, \qquad (87)$$

and $$G^2 = \frac{\omega_{b\perp}^2}{\omega_p^2} \frac{a_1^2}{a_1^2 + a^2}$$

for which Eq. (86) takes the form

$$1 = \frac{1}{w^2} + \frac{G^2}{(\omega + b)^2}, \qquad (88)$$

where $$G^2 << 1.$$

Solutions are sought under the conditions $b \simeq 1$ and $w \approx -b$, and the most unstable mode is found to be given by

$$\omega = -\omega_p + e^{-\frac{i\pi}{3}} \left[\frac{1}{2} \left(\frac{k_1^2}{k_1^2 + k^2} \right) \omega_{b\perp}^2 \omega_p \right]^{\frac{1}{3}} \qquad (89)$$

corresponding to

$$k = \frac{\omega_p}{u}. \qquad (90)$$

The rate of growth is fairly insensitive to k values in this neighborhood, and growth remains large for a wide range of k, provided that

$$\frac{\omega_{b\perp}}{\omega_p} << \frac{k_1^2}{k_1^2 + k^2}$$

and (91)

$$\left(\frac{\omega_p}{\omega_{b\perp}}\right)\left(\frac{\omega_B^2 r_o^2}{u^2}\right) << k_1^2 r_o^2 + k^2 r_o^2 .$$

For somewhat larger wavelengths, the mode remains unstable, but its structure is altered by the other terms of the dispersion relation.

When the collision rate satisfies the inequalities

$$\left(\omega_{b\perp}^2 \omega_p\right)^{\frac{1}{3}} << \nu << \omega_p , \tag{92}$$

the conductivity is still controlled primarily by inertia, but the dispersion relation becomes

$$1 = \frac{1}{w}\left(\frac{1}{w - i[\nu/\omega_p]}\right) + \frac{G^2}{(w+b)^2} . \tag{93}$$

A substitution of the form $w = -1 + \rho_1$, $b = 1 + \rho_2$ leads to the equation

$$\frac{G^2}{(\rho_1 + \rho_2)^2} = i \frac{\nu}{\omega_p} \tag{94}$$

with solution

$$k = (1 + \rho_2)\left(\frac{\omega_p}{u}\right)$$

and (95)

$$\omega = -(1 + \rho_2)\,\omega_p + e^{-i\frac{\pi}{4}}\left(\frac{k_1^2}{k_1^2 + k^2}\right)^{\frac{1}{2}}\left(\frac{\omega_p}{\nu}\right)^{\frac{1}{2}}\omega_{b\perp} ,$$

yielding a slower but still large growth rate. When

$$\mathcal{G}_r << 1,$$

$$\left(\frac{\nu}{\omega_p}\right) << \frac{k_1^2}{k_1^2 + k^2}, \tag{96}$$

and

$$\left(\frac{\nu}{\omega_{b\perp}}\right)\left(\frac{\omega_p}{\omega_{b\perp}}\right)\frac{\omega_B^2 r_o^2}{u^2} << k_1^2 r_o^2 + k^2 r_o^2,$$

the unstable mode is well described by Eq. (95), but again the mode persists in modified form for somewhat larger wavelengths.

When the conductivity is collision dominated, N is real, and the solutions of

$$1 = \frac{iN}{z} + \left(\frac{a_1^2}{a_1^2 + a^2}\right)\frac{1}{(z+a)^2}, \tag{97}$$

are much altered in form. Highly damped solutions of Eq. (97) satisfy the restrictions necessary for consistency, but the growing solution, described approximately by

$$z \approx -a + e^{-i\frac{\pi}{4}}\left(\frac{a_1^2}{a_1^2 + a^2}\right)^{\frac{1}{2}}\left(\frac{a}{N}\right)^{\frac{1}{2}}, \tag{98}$$

fails to satisfy the high frequency condition $|\Omega| >> 1$ which is needed to derive the dispersion relation and is thus not admissible.

In summary, the dispersion relation Eq. (76) governing high frequency, well localized disturbances has been investigated analytically by examination of approximate solutions to the equation. Numerous damped solutions exist, but essentially only one unstable mode is permitted. This disturbance is essentially of electrostatic type for very short wavelengths, while for longer wavelengths, its structure is modified by other terms in the dispersion relation.

This analysis may be compared with the work of Bludman, Watson, and Rosenbluth,[53] which gives a non self-consistent analysis of the beam problem, ignoring the curvature of the beam particle orbits. When the longitudinal mass of the beam particles is taken to be infinite, their dispersion relation corresponds to Eq. (88) or Eq. (93), in contrast to Eq. (76). This has two effects. The additional terms of Eq. (76) cause some modification to the structure of the unstable modes. In addition, the frequency restrictions leading to Eq. (76), as well as the more complex structure of the equation, make a valid solution much more difficult to achieve. Thus, the more slowly growing disturbances predicted by their equation do not form valid approximate solutions to Eq. (76).

The treatment of high frequency, highly localized disturbances is generalized in the next section to include self-consistently the effects due to the finiteness of the beam particle longitudinal mass. Aside from an increased algebraic complexity, the discussion is quite similar to the present section.

The Effect of Finite Longitudinal Mass on Localized, High Frequency Disturbances

For highly relativistic beams, the infinite longitudinal mass approximation adopted throughout this work provides a good description of the orbits of beam particles. However, the approximation does ignore a small energy interchange that takes place between the longitudinal and transverse particle motions, and for the high frequency, highly localized disturbances studied in the previous section, this interchange could give rise to resonance effects missed in that analysis. Such effects are studied here by retaining first corrections to the perturbed distribution function and the unperturbed orbits due to the finiteness of the longitudinal mass, re-evaluating the perturbed current and rederiving the dispersion relation. A similar high frequency approximation is adopted for the evaluation of orbit integrals.

The perturbed distribution function f now takes the form $$f = f_1 + f_2 ,$$

where

$$f_1 = \left(\frac{2en_0h_0\rho_0'}{m\gamma_0 i\omega}\right)\int_{-\infty}^{0} i\,dt'\hat{\phi}\left[(\omega+k\hat{v}_z)(\hat{v}_x F_x + \hat{v}_y F_y) - (k_1\hat{v}_x + k_2\hat{v}_y)\hat{v}_z F_z\right] \tag{99}$$

and

$$f_2 = \left(\frac{en_0\rho_0 h_0'}{m\gamma_0^{\,3} i\omega}\right)\int_{-\infty}^{0} i\,dt'\hat{\phi}\left[(\omega+k_1\hat{v}_x+k_2\hat{v}_y)F_z - k(\hat{v}_x F_x + \hat{v}_y F_y)\right].$$

However, the two orbit integrals of Eq. (99) do not have the same significance. The finiteness of the longitudinal mass is to be taken into account by evaluating the perturbed currents to first order in $\omega_{b\parallel}^{2}$, where

$$\omega_{b\parallel}^{2} = \frac{4\pi e^2 n_0}{m\gamma_0^{\,3}} \tag{100}$$

corresponds to a longitudinal plasma frequency for the beam particles. Since f_2 is essentially proportional to $\omega_{b\parallel}^{2}$, the orbit integrals in its expression may be evaluated as in the previous section, assuming that the longitudinal mass is infinite, yielding

$$f_2 = \left(\frac{en_0\rho_0 h_0'}{m\gamma_0^{\,3} i\omega}\right)\phi\left\{\begin{array}{l}\left[\dfrac{1}{\bar{\Omega}\omega_B} - \dfrac{(k_1v_x+k_2v_y)}{\bar{\Omega}^2\omega_B^{\,2}}\right]\left[(\omega+k_1v_x+k_2v_y)F_z - k(v_xF_x+v_yF_y)\right] \\ -\left(\dfrac{i}{\bar{\Omega}^2}\right)\left[(k_1x+k_2y)F_z - k(xF_x+yF_y)\right]\end{array}\right\}. \tag{101}$$

In contrast, the expression for f_1, while formally identical to Eq. (54), must be evaluated along modified orbits in order to exhibit the effects of finite longitudinal mass. The orbits may be obtained exactly for a sharp-edged beam, and an adequate description is provided by the equations of motion

$$\dot{v}_x = -\omega_B^2 x\left(1 + \frac{3\omega_{b\parallel}^2 r^2}{8c^2}\right),$$

$$\dot{v}_y = -\omega_B^2 y\left(1 + \frac{3\omega_{b\parallel}^2 r^2}{8c^2}\right), \tag{102}$$

and

$$\dot{v}_z = u\,\frac{\omega_{b\parallel}^2}{2c^2}\left(x v_x + y v_y\right),$$

corresponding to

$$v_z = u\left(1 + \frac{1}{4}\,\frac{\omega_{b\parallel}^2 r^2}{c^2}\right)$$

and

$$v_x^2 + v_y^2 = \omega_0^2 \mathcal{E}^2, \tag{103}$$

where

$$\omega_0^2 = \omega_B^2\left(1 + \frac{3}{16}\,\frac{\omega_{b\parallel}^2}{c^2}\left[r_0^2 + r^2\right]\right).$$

Thus, there is a small coupling of longitudinal and transverse particle energy and a small shift in the oscillation frequency - both position dependent effects. Otherwise, the same high frequency approximation for the orbit integrals may be carried through, yielding

$$
f_1 = \left(\frac{ze n_0 h_0 g_0}{m r_0 i\omega}\right)\phi \left[\begin{aligned}
&\left[1 - \frac{(k_1 v_x + k_2 v_y)}{\Omega\omega_B} + \frac{(k_1 v_x + k_2 v_y)^2}{\Omega^2\omega_B^2} + i\frac{(k_1 x + k_2 y)}{\Omega^2} + \frac{1}{\Omega^2}\right](v_x F_x + v_y F_y) \\
&+ \left[-i\frac{\omega_B}{\Omega} + 2i\frac{(k_1 v_x + k_2 v_y)}{\Omega^2}\right](x F_x + y F_y) \\
&+ \left[\frac{(k_1 v_x + k_2 v_y)^2}{\Omega^2\omega_B^2} - \frac{(k_1 v_x + k_2 v_y)}{\Omega\omega_B} + i\frac{(k_1 x + k_2 y)}{\Omega^2}\right] u F_z \\
&+ \left(\frac{\omega_{b\parallel}^2 r^2}{4c^2}\right) \left\{ \begin{aligned}
&\left[\frac{ku(k_1 v_x + k_2 v_y)}{\Omega^2\omega_B^2} + \frac{3}{2} i \frac{(k_1 x + k_2 y)}{\Omega^2}\right](v_x F_x + v_y F_y) \\
&+ \left[\frac{iku}{\Omega^2} - \frac{3}{2} i \frac{\omega_B}{\Omega} + 3i\frac{(k_1 v_x + k_2 v_y)}{\Omega^2}\right](x F_x + y F_y) \\
&+ \frac{3}{2}\left(\frac{1}{\Omega^2}\right)\left[\left(\frac{3x^2 + y^2}{r^2}\right) v_x F_x + \frac{2xy}{r^2}(v_y F_x + v_x F_y) + \left(\frac{x^2 + 3y^2}{r^2}\right) v_y F_y\right] \\
&+ \left[\begin{aligned} &\frac{(k_1 v_x + k_2 v_y)^2}{\Omega^2\omega_B^2} - \frac{(k_1 v_x + k_2 v_y)}{\Omega\omega_B} + \frac{ku(k_1 v_x + k_2 v_y)}{\Omega^2\omega_B^2} \\ &+ \frac{5}{2} i \frac{(k_1 x + k_2 y)}{\Omega^2} - 2i\frac{(k_1 v_x + k_2 v_y)}{\Omega^2\omega_B^2 r^2}(x v_x + y v_y) \end{aligned} \right] u F_z
\end{aligned} \right\}
\end{aligned} \right] . \quad (104)
$$

The integrations necessary to obtain the perturbed current may be carried out by similar techniques, and the resulting expressions are evaluated up to terms of order Ω^{-2} and $\omega_{b\parallel}^2$. For highly localized disturbances it is again possible to choose $k_2 = 0$, discard surface currents and evaluate volume currents to zeroth order in the parameter (r^2/r_0^2). A lengthy calculation then yields

$$-\left(\frac{4\pi i\omega}{c}\right)j_x = -\left(\frac{\omega_{b\perp}^2}{c^2}\right)\phi\left[\left(1+\frac{1}{\Omega^2}+\frac{3}{2}\frac{k_1^2 r_0^2}{\Omega^2}\right)F_x-\left(\frac{k_1 u}{\Omega\omega_B}\right)F_z\right]$$

$$-\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\phi\left[\left(\frac{k^2}{\Omega^2}+\frac{9}{8}\left(\frac{\omega_B r_0}{u}\right)^2\frac{k_1^2}{\Omega^2}\right)F_x-\frac{k_1 k}{\Omega^2}F_z\right],$$

$$-\left(\frac{4\pi i\omega}{c}\right)j_y = -\left(\frac{\omega_{b\perp}^2}{c^2}\right)\phi\left[\left(1+\frac{1}{\Omega^2}+\frac{1}{2}\frac{k_1^2 r_0^2}{\Omega^2}\right)F_y\right] \qquad (105)$$

$$-\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\phi\left[\left(\frac{k^2}{\Omega^2}+\frac{3}{8}\left(\frac{\omega_B r_0}{u}\right)^2\frac{k_1^2}{\Omega^2}\right)F_y\right],$$

and

$$-\left(\frac{4\pi i\omega}{c}\right)j_z = -2\phi\left[\frac{k_1^2}{\Omega^2}F_z-\left(\frac{\omega_B k_1}{\Omega u}\right)F_x\right]$$

$$-\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\phi\left[\left(2\frac{\omega^2}{\Omega^2\omega_B^2 r_0^2}-\frac{k_1^2}{\Omega^2}\right)F_z+\frac{k_1 k}{\Omega^2}F_x\right].$$

To this approximation the entire spatial dependence of the perturbed current is again contained in the factor ϕ, so that substitution of Eq. (105) into Eq. (70) yields the dispersion relation directly. As before, the unstable modes are obtained from the x and z components of Eq. (70), and the relevant dispersion relation is obtained from the determinant

$$\begin{vmatrix} h^2-k_1^2+\frac{\omega_{b\perp}^2}{c^2}\left(1+\frac{1}{\Omega^2}+\frac{3}{2}\frac{k_1^2 r_0^2}{\Omega^2}\right)+\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\left(\frac{k^2}{\Omega^2}+\frac{9}{8}\left(\frac{\omega_B r_0}{u}\right)^2\frac{k_1^2}{\Omega^2}\right) & -k_1 k-\left(\frac{\omega_{b\perp}^2}{c^2}\right)\frac{k_1 u}{\Omega\omega_B}-\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\frac{k_1 k}{\Omega^2} \\ -k_1 k-\frac{\omega_{b\perp}^2}{c^2}\left(\frac{k_1 u}{\Omega\omega_B}\right)+\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\frac{k_1 k}{\Omega^2} & h^2-k^2+\frac{\omega_{b\perp}^2}{c^2}\left(\frac{k_1 u}{\Omega\omega_B}\right)^2+\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\left(2\frac{\omega^2}{\Omega^2\omega_B^2 r_0^2}-\frac{k_1^2}{\Omega^2}\right) \end{vmatrix}=0. \tag{106}$$

However, to achieve consistency with the previous analysis, Eq. (106) must be expanded and evaluated up to terms of order Ω^{-2} and $\omega_{b\parallel}^2$. This yields after simplification

$$\Bigg[\left(\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)\left\{\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}+k^2+k_1^2+\left(\frac{\omega_{b\perp}^2}{c^2}\right)\left(1+\frac{k_1^2u^2}{\Omega^2\omega_B^2}\right)+\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\left(\frac{k^2}{\Omega^2}-\frac{k_1^2}{\Omega^2}\left[1-\frac{9}{8}\left(\frac{\omega_B r_0}{u}\right)^2\right]\right)\right\}$$
$$+\frac{1}{\Omega^2}\left(\frac{\omega^2}{\omega_B^2}\right)\left[\frac{\omega_{b\perp}^2}{c^2}k_1^2+\left(\frac{\omega_{b\parallel}^2}{c^2}\right)\left(\frac{\omega_{b\perp}^2}{c^2}+k^2+\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)\right]$$
$$+\frac{1}{\Omega^2}\left(\frac{\omega_{b\perp}^2}{c^2}\right)\left[\left(1+\frac{3}{2}k_1^2r_0^2\right)\left(k_1^2+\frac{4\pi i\omega\sigma}{c}-\frac{\omega^2}{c^2}\right)+\left(\frac{\omega_{b\parallel}^2 r_0^2}{2c^2}\right)\left(\frac{9}{16}k_1^2r_0^2-1\right)k_1^2\right]\Bigg]=0. \tag{107}$$

No new unstable modes are predicted by this equation, but the instabilities found in the previous section are somewhat modified by the additional terms of Eq. (107). The dimensionless variables of Eq. (80) are again introduced, and the unstable modes are well determined from the approximate dispersion relation

$$\left(iNz - z^2\right)\left(a_1^2 + a^2 + \beta \frac{z^2}{(z+a)^2}\right) + \left(a_1^2 + \beta a^2\right)\frac{z^2}{(z+a)^2} = 0, \tag{108}$$

where

$$\beta = \frac{\omega_{b\parallel}^2}{\omega_{b\perp}^2} = \nu^{-2}.$$

For the case $|\mathrm{Im}\ z| \gg \mu$, Eq. (108) yields the solution

$$a = \frac{\omega_p}{\omega_{b\perp}}$$

and

$$\omega = -\omega_p + e^{-\frac{i\pi}{3}}\left[\frac{1}{2}\left(\frac{k_1^2+\beta k^2}{k_1^2+k^2}\right)\omega_{b\perp}^2\omega_p\right]^{1/3} + e^{-\frac{2\pi i}{3}}\left(\frac{2}{3}\right)^{1/3}\left(a_1^2+a^2\right)^{-2/3}\left(a_1^2+\beta a^2\right)^{-1/3}\left(\frac{\omega_p}{\omega_{b\perp}}\right)^{5/3}\left(\frac{\omega_{b\parallel}}{\omega_{b\perp}}\right)^2 \tag{109}$$

which shows that the maximum instability growth rate is increased by the additional terms in the dispersion law. Similar results may be obtained for neighboring wavelengths, both for this case and for regimes satisfying Eq. (92). The analysis shows that the previously obtained growth rates are somewhat modified but that no change appears in the character of the instabilities.

Summary of the Chapter

The formal solutions obtained in the previous chapter are examined in detail in the low frequency, long wavelength and the high frequency, short wavelength limits.

In the first case the iterations in the formal solution are carried out, and their convergence is shown. The dispersion relation is also obtained and is analyzed in detail for the case $m = 1$. The analysis is carried through for both the reduced integral equations and the full set of coupled integral equations. Corrections to the macroscopic dispersion relation appear, and, in particular, long wavelengths are stabilized.

For the high frequency, short wavelength disturbances, it is convenient to use asymptotic methods in the evaluation of the perturbed current, obtaining, for highly localized disturbances, expressions which lead directly to an algebraic dispersion relation. The fields and currents are resolved into three dimensional Fourier components, and the stability of each component is examined separately. The dispersion relation yields, in addition to stable and damped oscillations, unstable modes which are similar in structure to the two beam electrostatic instability. The analysis is repeated under the assumption that the longitudinal beam mass is finite but large, and corrections to the dispersion relation are obtained. The finiteness of the longitudinal mass affects the growth rates of instabilities but not their general character.

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APPENDIX I

In this section an explicit evaluation is given of the set of Green's functions $g_n(r,r_o)$ and their Hankel transforms

$$G_n(\ell,r_o)=\int_0^\infty r\,dr\,J_n(\ell r)g_n(r,r_o) \tag{A-1}$$

for use in the text of the paper. The Green's functions satisfy the equation

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-h^2\right]g_n(r,r_o)=\frac{1}{r_o}\delta(r-r_o), \tag{A-2}$$

the symmetry condition

$$g_n(r,r_o)=g_n(r_o,r), \tag{A-3}$$

and the boundary conditions

$$\lim_{r\to 0}|g_n(r,r_o)|<\infty \tag{A-4}$$

and

$$\lim_{r\to\infty} g_n(r,r_o)=0 \tag{A-5}$$

for $n=0,1,2,\ \ldots$ and Re $h>0$. A straightforward construction of $g_n(r,r_o)$ is given and $G_n(\ell,r_o)$ is then obtained by inspection.

The functions $g_n(r,r_o)$ may be obtained from solutions of Bessel's equation

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-h^2\right]y_n(r)=0 \tag{A-6}$$

by imposing two extra conditions on $g_n(r,r_o)$. These are

$$\lim_{r\to r_o^+} g_n(r,r_o)-\lim_{r\to r_o^-} g_n(r,r_o)=0 \tag{A-7}$$

and

$$\lim_{r\to r_0^+} \frac{d}{dr} g_n(r,r_0) - \lim_{r\to r_0^-} \frac{d}{dr} g_n(r,r_0) = \frac{1}{r_0}. \tag{A-8}$$

The only way to satisfy both Eq. (A-6) and Eq. (A-4) is to choose $y_n = C_1 J_n(ihr)$. Similarly, a simultaneous solution of Eqs. (A-6) and (A-5) must be given by $y_n = C_2 H_n^1(ihr)$. This suggests that $g_n(r,r_0)$ should be taken to be of the form

$$\begin{aligned} g_n(r,r_0) &= D H_n^1(ihr_0) J_n(ihr) \text{ for } r<r_0 \\ &= D J_n(ihr_0) H_n^1(ihr) \text{ for } r>r_0 \end{aligned} \tag{A-9}$$

where D is an undetermined constant. Such a choice satisfies Eqs. (A-3), (A-4), (A-5), and (A-7) identically and satisfies Eq. (A-2) for $r \neq r_0$. When D is chosen so that Eq. (A-8) is satisfied, the required Green's functions will have been constructed.

At this point it is useful to recall the relation

$$J_n(x) \frac{d}{dx} Y_n(x) - Y_n(x) \frac{d}{dx} J_n(x) = \frac{2}{\pi x}, \tag{A-10}$$

which is given on p. 76 of G. N. Watson's book, Bessel Functions. For present purposes this relation may be restated as

$$J_n(ihr_0) \frac{d}{dr} H_n^1(ihr)\Big|_{r=r_0} - H_n^1(ihr_0) \frac{d}{dr} J_n(ihr)\Big|_{r=r_0} = \frac{2i}{\pi r_0}. \tag{A-11}$$

From this it follows at once that the correct choice of D is $D = -(i\pi/2)$ and that $g_n(r,r_0)$ is given for all $n = 0,1,2,\ldots$ by

$$\begin{aligned} g_n(r,r_0) &= -(i\pi/2) H_n^1(ihr_0) J_n(ihr) \text{ for } r<r_0 \\ &= -(i\pi/2) J_n(ihr_0) H_n^1(ihr) \text{ for } r>r_0. \end{aligned} \tag{A-12}$$

A direct calculation of $G_n(\ell, r_0)$ from Eq. (A-12) and definition (A-1) could now be given, since the integrals involved are all known. However, it is simpler to make use of the relation

$$\int_0^\infty \ell d\ell J_n(\ell r)\left[\frac{J_n(\ell r_0)}{\ell^2 + h^2}\right] = I_n(hr)K_n(hr_0) = (i\pi/2)H_n^1(ihr_0)J_n(ihr) \text{ for } r < r_0$$
$$= I_n(hr_0)K_n(hr) = (i\pi/2)J_n(ihr_0)H_n^1(ihr) \text{ for } r > r_0, \quad \text{(A-13)}$$

which is given in the Bateman Manuscript Project book, Tables of Integral Transforms (TIT), Vol. 2, p. 49. When Eqs. (A-1) and (A-12) are kept in mind, the immediate result of applying a Hankel transform to both sides of Eq. (A-13) is

$$G_n(\ell, r_0) = -\left[\frac{J_n(\ell r_0)}{\ell^2 + h^2}\right]. \quad \text{(A-14)}$$

The results of this section are summarized by Eqs. (A-12) and (A-14). These equations furnish sufficient information about the Green's functions for the requirements of the paper.

APPENDIX II

The perturbed electric field of Case I, Chapter 3, will be derived here for large N_o. For convenience, the discussion is restricted to disturbances for which $|h^2 r_o^2| \ll 1$. The general form of the field is prescribed by Eqs. (24) and (29), but the constants must be determined from the boundary conditions (8) and (17). This yields $f_z(r)$ and $\lambda(r)$ directly, while $f_r(r)$ and $f_\theta(r)$ are easily obtained from $\lambda(r)$. A complete description of the field will be given for the region $r \leq r_o$ only, since this is sufficient for the analysis of the dispersion law.

The derivation is facilitated by the definitions

$$L^{\pm} = J_1(i\delta^{\pm} r_o), \quad M^{\pm} = \frac{d}{dr} J_1(i\delta^{\pm} r)\Big|_{r=r_o}$$
$$R^{\pm} = H'_{p^{\pm}}(ihr_o), \text{ and } S^{\pm} = \frac{d}{dr} H'_{p^{\pm}}(ihr)\Big|_{r=r_o}. \tag{A-15}$$

These may be used to express the boundary conditions as

$$\begin{aligned}
&(C^+R^+ - A^+L^+) + (C^-R^- - A^-L^-) = 0 \\
&b^+(C^+R^+ - A^+L^+) + b^-(C^-R^- - A^-L^-) = 0 \\
&b^+(C^+S^+ - A^+M^+) + b^-(C^-S^- - A^-L^-) = 0 \\
&(C^+S^+ - A^+M^+) + (C^-S^- - A^-L^-) = \left(\frac{4\pi i\omega a}{c} j_o\right),
\end{aligned} \tag{A-16}$$

and a simple reduction gives

$$(b^- - b^+)(C^+S^+ - A^+M^+) = b^-\left(\frac{4\pi i\omega a}{c} j_0\right)$$

$$(C^+R^+ - A^+L^+) = 0$$

and (A-17)

$$(b^+ - b^-)(C^-S^- - A^-M^-) = b^+\left(\frac{4\pi i\omega a}{c} j_0\right)$$

$$(C^-R^- - A^-L^-) = 0.$$

These equations may be solved by inspection yielding

$$A^+ = \left(\frac{b^-}{b^- - b^+}\right)\frac{R^+}{(S^+L^+ - R^+M^+)}\left(\frac{4\pi i\omega a}{c} j_0\right) \tag{A-18}$$

and

$$A^- = \left(\frac{b^+}{b^+ - b^-}\right)\frac{R^-}{(S^-L^- - R^-M^-)}\left(\frac{4\pi i\omega a}{c} j_0\right). \tag{A-19}$$

The expression for A^- is readily evaluated without further approximations. The leading term is obtained directly from Eq. (30) and becomes

$$\left(\frac{b^+}{b^+ - b^-}\right) = \left(1 - \frac{h^\gamma}{N_0^\gamma}\right). \tag{A-20}$$

Small argument expansions may be used to evaluate the other quantities, since $|hr_0| << 1$ and $|\delta^- r_0| << |hr_0|$. An excellent approximation to L^- and M^- is provided by

$$L^- = \left(\frac{i\delta^- r_0}{2}\right)$$

and (A-21)

$$M^{-} = \left(\frac{i\delta^{-}}{2}\right).$$

The evaluation of R^{-} and S^{-} is more intricate, and is based on the approximations

$$H^{1}_{1+\Delta p}(ix) \simeq H^{1}_{1}(ix) + \Delta p\left[\frac{d}{dp}H^{1}_{p}(ix)\right]_{p=1},$$

$$H^{1}_{p}(ix) \simeq -\frac{i}{\pi}(p-1)!\left(\frac{ix}{2}\right)^{-p}, \qquad \text{(A-22)}$$

and

$$\left[\frac{d}{dp}H^{1}_{p}(ix)\right]_{p=1} \simeq \left(\frac{2}{\pi x}\right)\ln\left|\frac{\gamma}{2}x\right|,$$

which are valid when $|\Delta p| << 1$ and $|x| << 1$. Substitution gives

$$H^{1}_{p^{-}}(ihr) = -\left(\frac{2}{\pi hr}\right)\left(1 + \frac{1}{2}h^{2}r^{2}\ln\left|\frac{\gamma}{2}hr\right| + \frac{1}{2}h^{2}r_{0}^{2}\ln\left|\frac{\gamma}{2}hr\right|\right), \qquad \text{(A-23)}$$

and a direct evaluation yields

$$R^{-} = -\left(\frac{2}{\pi h r_{0}}\right)\left(1 + h^{2}r_{0}^{2}\ln\left|\frac{\gamma}{2}hr_{0}\right|\right),$$

$$S^{-} = \left(\frac{2}{\pi h r_{0}^{2}}\right), \qquad \text{(A-24)}$$

$$(S^{-}L^{-} - R^{-}M^{-}) = \left(\frac{i2\delta^{-}}{\pi h r_{0}}\right)\left(1 + \frac{1}{2}h^{2}r_{0}^{2}\ln\left|\frac{\gamma}{2}hr_{0}\right|\right),$$

and

$$\frac{R^-}{(S^-L^- - R^-M^-)} = \left(\frac{i}{\delta^-}\right)\left(1 + \frac{1}{2}h^2r_o^2 \ln\left|\frac{\gamma}{2}hr_o\right|\right).$$

Combining terms yields the expression

$$A^- = \left(\frac{4\pi i \omega a}{c} j_o\right)\left(1 - \frac{h^2}{N_o^2}\right)\left(1 + \frac{1}{2}h^2r_o^2 \ln\left|\frac{\gamma}{2}hr_o\right|\right)\left(\frac{i}{\delta^-}\right), \qquad (A-25)$$

which may be rewritten as

$$A^- = -\left(\frac{2\pi i \omega a r_o}{c} j_o\right)\left(1 - \frac{h^2}{N_o^2}\right)\left(1 + \frac{1}{2}h^2r_o^2 \ln\left|\frac{\gamma}{2}hr_o\right|\right)\frac{1}{J_1(i\delta^- r_o)}. \qquad (A-26)$$

The evaluation of A^+ is more difficult, and the calculations will be given for two limiting cases only.

Case A: $N_o r_o << 1$

For this regime p^+ becomes

$$p^+ = 1 + \frac{1}{2}N_o^2 r_o^2 + \frac{1}{2}h^2 r_o^2, \qquad (A-27)$$

$\delta^+ r_o << 1$, and small argument expansions may be used to evaluate all quantities defined by Eq. (A-15). The derivation is similar to the derivation of Eq. (A-25) and yields

$$H'_{p^+}(ihr) = -\left(\frac{2}{\pi hr}\right)\left(1 + \frac{1}{2}\left[h^2r^2 - h^2r_o^2 - N_o^2r_o^2\right]\ln\left|\frac{\gamma}{2}hr\right|\right),$$

$$R^+ = -\left(\frac{2}{\pi hr_o}\right)\left(1 - \frac{1}{2}N_o^2r_o^2 \ln\left|\frac{\gamma}{2}hr_o\right|\right),$$

$$S^+ = \left(\frac{2}{\pi h r_o^2}\right)\left(1 - \left[h^2 r_o^2 + \tfrac{1}{2} N_o^2 r_o^2\right] \ln\left|\tfrac{\gamma}{2} h r_o\right|\right), \tag{A-28}$$

$$(S^+L^+ - R^+M^+) = \left(\frac{2 i \delta^+}{\pi h r_o}\right)\left(1 - \tfrac{1}{2}\left[h^2 r_o^2 + N_o^2 r_o^2\right] \ln\left|\tfrac{\gamma}{2} h r_o\right|\right),$$

and

$$\left(\frac{b^-}{b^- - b^+}\right) = \left(\frac{h^2}{N_o^2}\right)\left(1 - 3\frac{h^2}{N_o^2}\right).$$

Substitution then yields

$$A^+ = \left(\frac{4\pi i \omega a}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(1 - 3\frac{h^2}{N_o^2}\right)\left(1 + \tfrac{1}{2} h^2 r_o^2 \ln\left|\tfrac{\gamma}{2} h r_o\right|\right)\frac{i}{\delta^+} \tag{A-29}$$

or

$$A^+ = -\left(\frac{2\pi i \omega a r_o}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(1 - 3\frac{h^2}{N_o^2}\right)\left(1 + \tfrac{1}{2} h^2 r_o^2 \ln\left|\tfrac{\gamma}{2} h r_o\right|\right)\frac{1}{J_1(i\delta^+ r)}. \tag{A-30}$$

Equations (A-26) and (A-30) may be used to obtain the field variables in the form of an expansion in powers of the parameter (h^2/N_o^2). Direct substitution yields for $r \leq r_o$

$$f_z(r) = -\left(\frac{2\pi i \omega a}{c} j_o\right)\left(1 + \tfrac{1}{2} h^2 r_o^2 \ln\left|\tfrac{\gamma}{2} h r_o\right|\right) r + O\left(\frac{h^4}{N_o^4}\right)$$

and (A-31)

$$\lambda(r) = 0 + O\left(\frac{h^6}{N_o^6}\right).$$

Thus to good approximation the other components of the electric field are given by

$$f_r(r) = f_\theta(r) = 0, \tag{A-32}$$

and the new term in the dispersion relation becomes

$$\left(\frac{\pi e n_o}{a}\right)\int_0^{r_o} dr\, r\left(f_r(r) - i f_\theta(r)\right) = 0. \tag{A-33}$$

Since $f_z(r)$ is unchanged from Chapter 2, the dispersion law is unaltered.

Case B: $N_o r_o \gg 1$

In this case the conditions $\delta^+ r_o \gg 1$ and $p^+ \gg 1$ require that different techniques be used for the determination of A^+. Good approximate values for L^+ and M^+ are obtained from the asymptotic expansion

$$J_1(i\delta^+ r) = \frac{i e^{\delta^+ r}}{\sqrt{2\pi N_o r}}\left[1 - \frac{3}{8 N_o r} + O\left(\frac{1}{N_o^2 r_o^2}\right)\right]. \tag{A-32}$$

The result is

$$L^+ = \frac{i e^{\delta^+ r_o}}{\sqrt{2\pi N_o r_o}}\left(1 - \frac{3}{8 N_o r_o}\right)$$

and (A-33)

$$M^+ = \left(\frac{1}{r_o}\right)\frac{i N_o r_o e^{i\delta^+ r_o}}{\sqrt{2\pi N_o r_o}}\left(1 - \frac{7}{8 N_o r_o}\right).$$

Similarly, the expansions

$$p^+ = N_o r_o\left(1 + \frac{1}{2 N_o^2 r_o^2} + \frac{1}{2}\frac{h^2}{N_o^2}\right)$$

and (A-34)

$$H'_{p^+}(ihr) = -\left[\frac{i}{\sin p^+\pi(-p^+)!}\right]\left(\frac{ihr}{2}\right)^{-p^+}\left[1 - \frac{h^2r^2}{4N_or_o} + O\left(\frac{1}{N_o^2r_o^2}\right)\right]$$

are used to evaluate R^+ and S^+ and yield

$$R^+ = -\left[\frac{i}{\sin p^+\pi(-p^+)!}\right]\left(\frac{ihr_o}{2}\right)^{-p^+}\left[1 - \frac{h^2r_o^2}{4N_or_o}\right]$$

and (A-35)

$$S^+ = \left(\frac{1}{r_o}\right)\left[\frac{iN_or_o}{\sin p^+\pi(-p^+)!}\right]\left(\frac{ihr_o}{2}\right)^{-p^+}\left[1 - \frac{h^2r_o^2}{4N_or_o}\right].$$

Combining terms yields the denominator

$$\left(S^+L^+ - R^+M^+\right) = -2N_oR^+\frac{i\,e^{\delta^+r_o}}{\sqrt{2\pi N_or_o}}\left(1 - \frac{\sigma}{8N_or_o}\right),$$ (A-36)

and A^+ becomes

$$A^+ = -\left(\frac{2\pi i\omega a r_o}{c}\dot{\gamma}_o\right)\left(\frac{1}{N_or_o}\right)\left(\frac{h^2}{N_o^2}\right)\frac{1}{J_1(i\delta^+r_o)} + O\left(\frac{1}{N_o^4r_o^4}\right).$$ (A-37)

Direct substitution gives the field variables

$$f_z(r) = -\left(\frac{2\pi i\omega a r_o}{c}\dot{\gamma}_o\right)\left(1 - \frac{h^2}{N_o^2}\right)\left(1 + \frac{1}{2}h^2r_o^2\ln\left|\frac{\gamma}{2}hr_o\right|\right)\frac{J_1(i\delta^-r)}{J_1(i\delta^-r_o)}$$

$$-\left(\frac{2\pi i\omega a r_o}{c}\dot{\gamma}_o\right)\left(\frac{h^2}{N_o^2}\right)\left(\frac{1}{N_or_o}\right)\frac{J_1(i\delta^+r)}{J_1(i\delta^+r_o)}$$

and (A-38)

$$\lambda(r) = N_0\left(\frac{2\pi i\omega a r_0}{c} j_0\right)\left(\frac{h^2}{N_0^2}\right)\left(1 + \frac{1}{2}h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right)\frac{J_1(i\delta^- r)}{J_1(i\delta^- r_0)}$$

$$- N_0\left(\frac{2\pi i\omega a r_0}{c} j_0\right)\left(\frac{h^2}{N_0^2}\right)\left(\frac{1}{N_0 r_0}\right)\frac{J_1(i\delta^+ r)}{J_1(i\delta^+ r_0)} .$$

The component $f_\theta(r)$ is most readily determined from the equation

$$f_\theta = -\frac{i}{h^2}\frac{d}{dr}\lambda \qquad \text{(A-39)}$$

and is given by

$$f_\theta(r) = -\frac{i}{N_0}\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1 + \frac{1}{2}h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right)$$

$$+ \frac{i}{N_0}\left(\frac{2\pi i\omega a}{c} j_0\right)\left(\frac{1}{N_0 r_0}\right)\frac{r_0}{J_1(i\delta^+ r_0)}\frac{d}{dr}J_1(i\delta^+ r) . \qquad \text{(A-40)}$$

Similarly, the component $f_r(r)$ is determined from

$$f_r = -\left(N_0^2 r + \frac{1}{r}\right)\frac{1}{h^2}\lambda - N_0 r f_z \qquad \text{(A-41)}$$

and is given by

$$f_r(r) = -\frac{1}{N_0}\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1 + \frac{1}{2}h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right)$$

$$+\left(\frac{2\pi i\omega a}{c} j_0\right)\frac{r J_1(i\delta^+ r)}{J_1(i\delta^+ r_0)} + \frac{1}{N_0}\left(\frac{2\pi i\omega a}{c} j_0\right)\left(\frac{1}{N_0 r_0}\right)\frac{r_0 J_1(i\delta^+ r)}{r J_1(i\delta^+ r_0)} . \qquad \text{(A-42)}$$

Equations (A-38), (A-40), and (A-42) may be used to evaluate the dispersion law (5). To good approximation

$$f_z(r_0) = -\left(\frac{2\pi i \omega a r_0}{c} j_0\right)\left(1 + \frac{1}{2} h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right), \tag{A-43}$$

so that the magnetic term of Eq. (5) is unchanged from Chapter 2. The additional term

$$\begin{aligned} &\left(\frac{\pi e n_0}{a}\right)\int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right) \\ &= \left(\frac{\pi e n_0}{a N_0}\right) r_0 f_z(r_0) \\ &\quad + \left(\frac{\pi e n_0}{a J_1(i\delta^+ r_0)}\right)\left(\frac{2\pi i \omega a}{c} j_0\right)\int_0^{r_0} dr\, r^2 J_1(i\delta^+ r) \\ &\quad + \frac{1}{N_0^2}\left(\frac{\pi e n_0}{a J_1(i\delta^+ r_0)}\right)\left(\frac{2\pi i \omega a}{c} j_0\right)\int_0^{r_0} dr \frac{d}{dr}\left[r J_1(i\delta^+ r)\right] \end{aligned} \tag{A-44}$$

may be evaluated by using the identity

$$r^2 J_1(i\delta^+ r) = -\left(\frac{i}{N_0}\right)\frac{d}{dr} r^2 J_2(i\delta r) \tag{A-45}$$

to obtain the approximate value

$$\begin{aligned} &\left(\frac{\pi e n_0}{a}\right)\int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right) \\ &= \left(\frac{\pi r_0 e n_0}{a N_0}\right)\left[f_z(r_0) + \left(\frac{2\pi i \omega a r_0}{c} j_0\right)\right]. \end{aligned} \tag{A-46}$$

Although this additional term is non-zero, its value may be shown to be negligibly small in comparison with the magnetic term.

APPENDIX III

The perturbed electric fields of Case II, Chapter 3, will be obtained here for the regime $N_o^2 \gg |h^2|$ and $|h^2 r_o^2| \ll 1$. The fields will be given in detail for the region $r \leq r_o$, and the evaluation is quite similar to the treatment in Appendix II. General formulas will be derived for the coefficients $A^{\pm}$; the fields will then be evaluated in two limiting cases.

It is convenient to introduce the definitions

$$V^{\pm} = H'_{n\pm}(ihr_o) \text{ and } W^{\pm} = \frac{d}{dr} H'_{n\pm}(ihr)\Big|_{r=r_o} . \qquad \text{(A-47)}$$

By use of Eqs. (A-15) and (A-47) the boundary conditions may be written as

$$\begin{aligned} (C^+V^+ + C^-V^-) - (A^+L^+ + A^-L^-) &= 0 \\ h(C^+V^+ - C^-V^-) - (b^+A^+L^+ + b^-A^-L^-) &= 0 \\ h(C^+W^+ - C^-W^-) - (b^+A^+M^+ + b^-A^-M^-) &= 0 \end{aligned} \qquad \text{(A-48)}$$

and

$$(C^+W^+ + C^-W^-) - (A^+M^+ + A^-M^-) = \left(\frac{4\pi i\omega a}{c} j_o\right).$$

Equation (A-48) may be solved for $A^{\pm}$ in three steps. First, the equation is rewritten as the two pairs of equations

$$2hC^+V^+ - \left[(h+b^+)A^+L^+ + (h+b^-)A^-L^-\right] = 0$$

and

$$2hC^+W^+ - \left[(h+b^+)A^+M^+ + (h+b^-)A^-M^-\right] = h\left(\frac{4\pi i\omega}{a} j_o\right),$$

and

$$2hC^-V^- - \left[(h-b^+)A^+L^+ + (h-b^-)A^-L^-\right] = 0 \qquad \text{(A-49)}$$

and

$$2hC^-W^- - \left[(h-b^+)A^+M^+ + (h-b^-)A^-M^-\right] = h\left(\frac{4\pi i\omega}{a} j_o\right),$$

which are then reduced to the single pair of equations

$$(h+b^+)(W^+L^+ - V^+M^+)A^+ + (h+b^-)(W^+L^- - V^+M^-)A^- = hV^+\left(\frac{4\pi i\omega a}{c} j_o\right)$$

and (A-50)

$$(h-b^+)(W^-L^+ - V^-M^+)A^+ + (h-b^-)(W^-L^- - V^-M^-)A^- = hV^-\left(\frac{4\pi i\omega a}{c} j_o\right).$$

Finally, $A^{\pm}$ is determined from Eq. (A-50) to be

$$A^+ = \left(\frac{4\pi i\omega a}{c} j_o\right) \frac{h\left[(h-b^-)V^+(W^-L^- - V^-M^-) - (h+b^-)V^-(W^+L^- - V^+M^-)\right]}{\left[\begin{array}{l}(h+b^+)(h-b^-)(W^+L^+ - V^+M^+)(W^-L^- - V^-M^-) \\ -(h+b^-)(h-b^+)(W^-L^+ - V^-M^+)(W^+L^- - V^+M^-)\end{array}\right]}$$

and (A-51)

$$A^- = \left(\frac{4\pi i\omega a}{c} j_o\right) \frac{h\left[(h+b^+)V^-(W^+L^+ - V^+M^+) - (h-b^+)V^+(W^-L^+ - V^-M^+)\right]}{\left[\begin{array}{l}(h+b^+)(h-b^-)(W^+L^+ - V^+M^+)(W^-L^- - V^-M^-) \\ -(h+b^-)(h-b^+)(W^-L^+ - V^-M^+)(W^+L^- - V^+M^-)\end{array}\right]}.$$

For simplicity the explicit evaluation of $A^{\pm}$ will be given for two limiting cases only.

Case A: $N_o r_o << 1$

For this case small argument expansions may be used to evaluate all Bessel functions, and the expansion

$$n^{\pm} = 1 \pm \frac{1}{2} N_o r_o h r_o \qquad \text{(A-52)}$$

insures that (A-22) may be used to evaluate the Hankel functions. The result is

$$L^{\pm} = \frac{i\delta^{\pm} r_o}{2},$$

$$M^{\pm} = \frac{i\delta^{\pm}}{2},$$

$$H'_{n^{\pm}}(ihr) = -\left(\frac{2}{\pi h r}\right)\left(1 + \frac{1}{2}\left[h^2 r^2 \mp N_o r_o h r_o\right] \ln\left|\frac{\gamma}{2} h r\right|\right), \qquad \text{(A-53)}$$

$$V^{\pm} = -\left(\frac{2}{\pi h r_o}\right)\left(1 + \frac{1}{2}\left[h^2 r_o^2 \mp N_o r_o h r_o\right] \ln\left|\frac{\gamma}{2} h r_o\right|\right),$$

and

$$W^{\pm} = \left(\frac{2}{\pi h r_o^2}\right)\left(1 - \frac{1}{2}\left[h^2 r_o^2 \pm N_o r_o h r_o\right] \ln\left|\frac{\gamma}{2} h r_o\right|\right).$$

The needed combinations of these functions are easily obtained and are given by

$$(W^+ L^+ - V^+ M^+) = \left(\frac{2 i \delta^+}{\pi h r_o}\right)\left(1 - \frac{1}{2} N_o r_o h r_o \ln\left|\frac{\gamma}{2} h r_o\right|\right),$$

$$(W^+ L^- - V^+ M^-) = \left(\frac{2 i \delta^-}{\pi h r_o}\right)\left(1 - \frac{1}{2} N_o r_o h r_o \ln\left|\frac{\gamma}{2} h r_o\right|\right),$$

$$(W^-L^+ - V^-M^+) = \left(\frac{2i\delta^+}{\pi h r_o}\right)\left(1 + \frac{1}{2} N_o r_o h r_o \ln\left|\frac{\gamma}{2} h r_o\right|\right),$$

$$(W^-L^- - V^-M^-) = \left(\frac{2i\delta^-}{\pi h r_o}\right)\left(1 + \frac{1}{2} N_o r_o h r_o \ln\left|\frac{\gamma}{2} h r_o\right|\right), \quad \text{(A-54)}$$

and

$$\Delta = -\left(\frac{8\delta^+\delta^-}{\pi^2 h^2 r_o^2}\right) h\left(b^+ - b^-\right)$$

where Δ is defined as the denominator appearing in Eq. (A-51). The quantities $A^{\pm}$ are then readily evaluated as

$$A^+ = \left(\frac{4\pi i\omega a}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(1 - \frac{9h^2}{4N_o^2}\right)\left(1 + \frac{1}{2} h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)\left(\frac{i}{\delta^+}\right)$$

and (A-55)

$$A^- = \left(\frac{4\pi i\omega a}{c} j_o\right)\left(1 - \frac{h^2}{N_o^2}\right)\left(1 + \frac{1}{2} h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)\left(\frac{i}{\delta^-}\right).$$

When $r \leq r_o$ substitution gives for the field variables

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_o\right)\left(1 + \frac{1}{2} h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right) r + O\left(\frac{h^4}{N_o^4}\right)$$

and (A-56)

$$\lambda(r) = 0 + O\left(\frac{h^6}{N_o^6}\right).$$

Thus the fields and the dispersion law are unchanged from Chapter 2.

Case B: $N_o r_o \gg 1$ and $|N_o r_o h r_o| \ll 1$

Equations (A-53) and (A-54) of Case A may be used here, except that all quantities involving L^+ and M^+ must be re-evaluated. The asymptotic series for the relevant Bessel function is used, and a fairly good approximation is given by

$$L^+ = \frac{i\, e^{i\delta^+ r_o}}{\sqrt{2\pi N_o r_o}},$$

$$M^+ = N_o L^+,$$

$$(W^- L^+ - V^- M^+) = N_o L^+ \left(\frac{2}{\pi h r_o}\right)\left(1 + \frac{1}{2}\left[h^2 r_o^2 + N_o r_o h r_o\right] \ln\left|\frac{\gamma}{2} h r_o\right|\right),$$

$$(W^+ L^+ - V^+ M^+) = N_o L^+ \left(\frac{2}{\pi h r_o}\right)\left(1 + \frac{1}{2}\left[h^2 r_o^2 - N_o r_o h r_o\right] \ln\left|\frac{\gamma}{2} h r_o\right|\right), \qquad \text{(A-57)}$$

and

$$\Delta = \left(\frac{8 i \delta^-}{\pi^2 h^2 r_o^2}\right) N_o L^+ h (b^+ - b^-)\left(1 + \frac{1}{2} h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right).$$

Equation (A-51) may now be evaluated as

$$A^+ = -\left(\frac{4\pi i \omega a r_o}{c} j_o\right)\left(\frac{1}{N_o r_o}\right)\left(\frac{h^2}{N_o^2}\right)\frac{1}{J_1(i\delta^+ r_o)}$$

and (A-58)

$$A^- = \left(\frac{4\pi i \omega a}{c} j_o\right)\left(1 - \frac{h^2}{N_o^2}\right)\left(1 + \frac{1}{2} h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)\left(\frac{i}{\delta^-}\right).$$

Thus the only difference between Eq. (A-58) and the $A^{\pm}$ of Case B, Appendix II, is in A^+, which is twice as large here.

This means that very small modifications in Eqs. (A-38)-(A-46) give the field variables

$$f_z(r) = -\left(\frac{2\pi i\omega a r_o}{c} j_o\right)\left(1-\frac{h^2}{N_o^2}\right)\left(1+\frac{1}{2}h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)\frac{J_1(i\delta^- r)}{J_1(i\delta^- r_o)}$$

$$-\left(\frac{4\pi i\omega a r_o}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(\frac{1}{N_o r_o}\right)\frac{J_1(i\delta^+ r)}{J_1(i\delta^+ r_o)}$$

and (A-59)

$$\lambda(r) = N_o\left(\frac{2\pi i\omega a r_o}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(1+\frac{1}{2}\ln\left|\frac{\gamma}{2} h r_o\right|\right)\frac{J_1(i\delta^- r)}{J_1(i\delta^- r_o)}$$

$$-N_o\left(\frac{4\pi i\omega a r_o}{c} j_o\right)\left(\frac{h^2}{N_o^2}\right)\left(\frac{1}{N_o r_o}\right)\frac{J_1(i\delta^+ r)}{J_1(i\delta^+ r_o)},$$

the transverse fields

$$f_r(r) = -\frac{1}{N_o}\left(\frac{2\pi i\omega a}{c} j_o\right)\left(1+\frac{1}{2}h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)$$

$$+\left(\frac{4\pi i\omega a}{c} j_o\right) r\,\frac{J_1(i\delta^+ r)}{J_1(i\delta^+ r_o)}$$

$$+\left(\frac{1}{N_o r_o}\right)^2\left(\frac{4\pi i\omega a}{c} j_o\right)\frac{r_o^2\, J_1(i\delta^+ r)}{r\; J_1(i\delta^+ r_o)}$$

and (A-60)

$$f_\theta(r) = -\frac{i}{N_o}\left(\frac{2\pi i\omega a}{c} j_o\right)\left(1+\frac{1}{2}h^2 r_o^2 \ln\left|\frac{\gamma}{2} h r_o\right|\right)$$

$$+\, i\left(\frac{1}{N_o r_o}\right)^2\left(\frac{4\pi i\omega a}{c} j_o\right)\frac{r_o^2}{J_1(i\delta^+ r_o)}\frac{d}{dr}J_1(i\delta^+ r),$$

and the electrostatic term of the dispersion law

$$\left(\frac{\pi e n_0}{a}\right)\int_0^{r_0} dr\, r\left(f_r(r) - i f_\theta(r)\right)$$

$$= \left(\frac{\pi r_0 e n_0}{a N_0}\right)\left[f_z(r_0) + \left(\frac{4\pi i \omega a r_0}{c}\, j_0\right)\right]. \qquad \text{(A-61)}$$

Equations (A-59) and (A-61) contribute corrections to the dispersion law of Chapter 2, but these corrections are negligibly small. The present treatment indicates that while Hall currents affect the values of the perturbed fields, they do not greatly affect the dispersion law for the disturbances.

The case $|N_o r_o h r_o| \gg 1$ is also of interest, but it involves lengthy analysis and will not be considered here.

APPENDIX IV

The mathematical problem involved in evaluating $f_z(r)$ from Eqs. (46)-(49) of Chapter 3 is to determine the constants A, $\hat{A}$, δ, and C from four equations which are linear in A, $\hat{A}$, and C but non-linear in δ. This is done here. The first step is to express A and $\hat{A}$ in terms of δ. Substitution of Eq. (46) into Eq. (48) gives

$$-h^2 r_0^2\left(\frac{\pi i\omega a}{2c} j_0\right)\left(1+N_0^2 r_0^2\right)\ln\left[\delta^2\left(1+\frac{1}{N_0^2 r_0^2}\right)\right]+\frac{\hat{A}}{r_0^2}$$

$$= -h^2 r_0^2\left(\frac{\pi i\omega a}{2c} j_0\right)\left(\frac{1+N_0^2 r_0^2}{N_0^4 r_0^4}\right)\ln\left(1+N_0^2 r_0^2\right)+A$$

and (A-62)

$$-h^2 r_0^2\left(\frac{\pi i\omega a}{2c} j_0\right)\left[\left(1+N_0^2 r_0^2\right)\ln\left[\delta^2\left(1+\frac{1}{N_0^2 r_0^2}\right)\right]+2\right]-\frac{\hat{A}}{r_0^2}$$

$$= -h^2 r_0^2\left(\frac{\pi i\omega a}{2c} j_0\right)\left[\frac{(N_0^2 r_0^2-1)}{N_0^4 r_0^4}\ln\left(1+N_0^2 r_0^2\right)+\frac{2}{N_0^2 r_0^2}\right]+A,$$

which gives for A and $\hat{A}$

$$A = -h^2 r_0^2\left(\frac{\pi i\omega a}{2c} j_0\right)\left[\ln\left[\delta^2\left(1+\frac{1}{N_0^2 r_0^2}\right)\right]+\left(1-\frac{1}{N_0^2 r_0^2}\right)-\frac{1}{N_0^2 r_0^2}\ln\left(1+N_0^2 r_0^2\right)\right]$$

and (A-63)

$$\hat{A} = -h^2 r_0^2 \left(\frac{\pi i \omega a}{2c} j_0\right) \begin{bmatrix} -N_0^2 r_0^2 \ln\left[\delta^2 \left(1 + \frac{1}{N_0^2 r_0^2}\right)\right] + \left(1 - \frac{1}{N_0^2 r_0^2}\right) \\ + \frac{1}{N_0^4 r_0^4} \ln\left(1 + N_0^2 r_0^2\right) \end{bmatrix} .$$

When $N_o r_o \ll 1$ the fields and the dispersion law are unchanged from Chapter 2. For this reason only the case $N_o r_o \gg 1$ is treated in detail below. Other conditions on parameters will be imposed in the course of the discussion in order to obtain a solution which depends weakly on r_p. The constant A is well approximated in this range by

$$A = -\left(\frac{h^2 r_0^2}{2}\right)\left(\frac{2\pi i \omega a}{c} j_0\right) \ln\left[\left(\frac{\delta}{N_0 r_0}\right)\sqrt{1 + N_0^2 r_0^2}\right], \tag{A-64}$$

but the constant δ must be determined from Eq. (49) before $f_z(r)$ can be evaluated. The conditions

$$r_p \gg r_0$$

and (A-65)

$$|h r_p| \ll 1$$

have already been imposed on r_p. Thus Eq. (49) may be evaluated as

$$-C\left(\frac{2}{\pi h r_p}\right)\left(1+\frac{1}{2}h^2 r_p^2 \ln\left|\frac{\gamma}{2} h r_p\right|\right)$$

$$= \frac{\hat{A}}{r_p} - \left(\frac{2\pi i\omega a}{c} j_0\right)\frac{r_0^2}{r_p}$$

$$- \frac{1}{r_p}h^2 r_0^2\left(\frac{\pi i \omega a}{2c} j_0\right)\left(r_p^2+N_0^2 r_0^4\right)\ln\left[\delta^2\left(1+\frac{r_p^2}{N_0^2 r_0^4}\right)\right]$$

and (A-66)

$$-C\left(\frac{2}{\pi h r_p}\right)\left(1-\frac{1}{2}h^2 r_p^2 \ln\left|\frac{\gamma}{2} h r_p\right|\right)$$

$$= -\frac{\hat{A}}{r_p} - \left(\frac{2\pi i\omega a}{c} j_0\right)\frac{r_0^2}{r_p}$$

$$- \frac{1}{r_p}h^2 r_0^2\left(\frac{\pi i \omega a}{2c} j_0\right)\left[\left(r_p^2-N_0^2 r_0^4\right)\ln\left[\delta^2\left(1+\frac{r_p^2}{N_0^2 r_0^4}\right)\right]+2r_p^2\right].$$

The constants C and δ given by Eq. (A-66) depend weakly on r_p only when

$$r_p^2 \gg N_0 r_0^2. \tag{A-67}$$

For this case approximate values of C and δ may be found by equating separately the terms of Eq. (A-66) which involve log r_p and the terms which are independent of this expression. This procedure gives the equations

$$-\left(\frac{2}{\pi h}\mathcal{C}\right)\frac{1}{2}h^2 r_p^2 \ln\left|\frac{\gamma}{2} h r_p\right|$$

$$= -h^2 r_0^2\left(\frac{\pi i \omega a}{2c} j_0\right)\left(r_p^2 + N_0^2 r_0^4\right)\ln\left[\delta^2\left(\frac{r_p^2}{N_0^4 r_0^2}+1\right)\right]$$

and (A-68)

$$-\left(\frac{2}{\pi h}\mathcal{C}\right) = -r_0^2\left(\frac{2\pi i\omega a}{c} j_0\right) + \hat{A},$$

which have the approximate solution

$$\mathcal{C} = \left(\frac{\pi h r_0^2}{2}\right)\left(\frac{2\pi i\omega a}{c} j_0\right)$$

and (A-69)

$$\delta = N_0 r_0 \left|\frac{\gamma}{2} h r_0\right| .$$

It is perhaps worth noticing that Eqs. (A-65) and (A-67) together imply the restriction

$$\left|N_0 r_0 h r_0\right| << 1, \qquad \text{(A-70)}$$

which must be satisfied if the approximation procedure is to yield acceptable results.

The field $f_z(r)$ is now readily determined to be

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c} j_0\right)\left(1+\frac{1}{2}h^2 r_0^2 \ln\left[\left|\frac{\gamma}{2}h r_0\right|\sqrt{1+N_0^2 r_0^2}\right]\right) r$$

$$-\frac{1}{N_0^2 r_0^2}\left(\frac{2\pi i\omega a}{c} j_0\right)\frac{1}{4}h^2 r_0^2\left(\frac{1+N_0^2 r^2}{N_0^2 r}\right)\ln\left(1+r^2 N_0^2\right), \qquad \text{(A-71)}$$

and is thus well approximated by the equation

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c}\, j_0\right)\left(1 + \frac{1}{2}h^2 r_0^2 \ln\left[\left|\frac{\gamma}{2} h r_0\right|\sqrt{1+N_0^2 r_0^2}\,\right]\right). \tag{A-72}$$

Comparison with the field

$$f_z(r) = -\left(\frac{2\pi i\omega a}{c}\, j_0\right)\left(1 + \frac{1}{2}h^2 r_0^2 \ln\left|\frac{\gamma}{2} h r_0\right|\right) \tag{A-73}$$

of Chapter 2 shows that the main effect of Hall currents is to increase the effective beam radius in the argument of the logarithm.