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A FIRST- AND SECOND-ORDER MATRIX THEORY FOR THE DESIGN OF BEAM TRANSPORT SYSTEMS AND CHARGED PARTICLE SPECTROMETERS

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I. INTRODUCTION

Since the invention of the alternating gradient principle and the subsequent design of the Brookhaven and CERN proton-synchrotrons based on this principle, there has been a rapid evolution of the mathematical and physical techniques applicable to charged particle optics. In this report a matrix algebra formalism will be used to develop the essential principles governing the design of charged particle beam transport systems, with a particular emphasis on the design of high energy magnetic spectrometers. A notation introduced by John Streib[1] has been found to be useful in conveying the essential physical principles dictating the design of such beam transport systems. In particular to first order, the momentum dispersion,the momentum resolution, the particle path length, and the necessary and sufficient conditions for zero dispersion, achromaticity and isochronicity may all be expressed as simple integrals of particular first-order trajectories (matrix elements) characterizing a system.

This formulation provides direct physical insight into the design of beam transport systems and charged particle spectrometers. An intuitive grasp of the mechanism of second-order aberrations also results from this formalism; for example, the effect of magnetic symmetry on the minimization or elimination of second-order aberrations is immediately apparent.

The equations of motion will be derived and then the matrix formalism introduced, developed and evolved into useful theorems. It is hoped that the information supplied will provide the reader with the necessary tools whereby he can design any beam transport system or spectrometer suited to his particular needs.

The theory has been developed to second order in a Taylor expansion about a central trajectory, characterizing the system. This seems to be adequate for most high energy physics applications. For studying details beyond second order we have found computer ray tracing programs to be the best technique for verification of matrix calculations, and as a means for further refinement of the optics if needed.

In the design of actual systems for high energy beam transport applications it has proved convenient to express the results via a multipole expansion about a central trajectory. In this expansion, the constant term proportional to the field strength at the central trajectory is the dipole term. The term proportional to the first derivative of the field (with respect to the transverse dimensions) about the central trajectory is a quadrupole term and the second derivative with respect to the transverse dimensions is a sextupole term, etc.

A considerable design simplification results at high energies if the dipole, quadrupole and sextupole functions are physically separated such that cross product terms among them do not appear, and if the fringing field effects are small compared to the contributions of the multipole elements comprising the system. At the risk of over-simplification, the basic function of the multipole elements may be identified in the following way: The purpose of the dipole element(s) is to bend the central trajectory of the system and disperse the beam — that is, it is the means of providing the first-order momentum dispersion for the system. The quadrupole element(s) generate the first-order imaging. The sextupole terms couple with the second-order aberrations; and a sextupole element introduced into the system is a mechanism for minimizing or eliminating a particular second-order aberration that may have been generated by dipole or quadrupole elements.

Quadrupole elements may be introduced in any one of three characteristic forms: (1) via an actual physical quadrupole consisting of four poles such that a first field derivative exists in the field expansion about the central trajectory; (2) via a rotated input or output face of a bending magnet; and (3) via a transverse field gradient in the dipole elements of the system. Clearly any one of these three fundamental mechanisms may be used as a means of achieving first-order imaging in a system. Of course dipole elements will tend to image in the radial bending plane independent of whether a transverse field derivative does or does not exist in the system, but imaging perpendicular to the plane of bend is not possible without the introduction of a first-field derivative.

In addition to their fundamental purpose, dipoles and quadrupoles will also introduce higher-order aberrations. If these aberrations are second order, they may be eliminated or at least modified by the introduction of sextupole elements at appropriate locations.

In regions of zero dispersion, a sextupole will couple with and modify only geometric aberrations. However, in a region where momentum dispersion is present, sextupoles will also couple with and modify chromatic aberrations.

Similar to the quadrupole, a sextupole element may be generated in one of several ways. First by incorporating an actual sextupole – that is, a six-pole magnet — into the system. However, any mechanism which introduces a second derivative of the field with respect to the transverse dimensions is in effect introducing a sextupole component. Thus a second-order curved surface on the entrance or exit face of a bending magnet or a second-order transverse curvature on the pole surfaces of a bending magnet is also a sextupole component.

As a first illustration of systems possessing dipole, quadrupole and sextupole elements, consider the $n = 1/2$ double-focusing spectrometer which is widely used for low and medium energy physics applications. Clearly there is a dipole element resulting from the presence of a magnetic field component along the central trajectory of the spectrometer. A distributed quadrupole element exists as a consequence of the $n = 1/2$ field gradient. In this particular case, since the transverse imaging forces are proportional to $\sqrt{n}$ and the radial imaging forces are proportional to $\sqrt{1 - n}$, the restoring forces are equal in both planes, hence the reason for the "double focusing" properties. In addition to the first derivative of the field $n = -(r_o/B_o)(\partial B/\partial r)$, there are usually second- and higher-order transverse field derivatives present. The second derivative of the field $\beta = \frac{1}{2}(r_o^2/B_o)(\partial^2 B/\partial r^2)$ introduces a distributed sextupole along the entire length of the spectrometer. Thus to second-order a typical $n = 1/2$ spectrometer consists of a single dipole with a distributed quadrupole and sextupole superimposed along the entire length of the dipole element. Higher-order multipoles may also be present, but will be ignored in this discussion.

In the preceding example the dipole, quadrupole and sextupole functions are integrated in the same magnet. However, in many high energy physics applications it is often more economical to use separate magnetic elements for each of the multipole functions. As additional examples, consider the SLAC spectrometers. These instruments provide examples of solutions which combine the multipole functions into a single magnet as well as solutions using separate multipole elements. Three spectrometers have been designed: one for a maximum energy of 1.6 GeV/c to study large backward angle scattering processes; a second for 8 GeV/c to study intermediate forward angle production processes, and finally a 20-GeV/c spectrometer for small forward angle production. All of these instruments are to be used in conjunction with primary electron and gamma-ray energies in the range of 10-20 GeV/c.

The 1.6-GeV/c instrument is a single magnet, bending the central trajectory a total of 90°, thus constituting the dipole contribution to the optics of the system. Two "quadrupole" elements are present in the magnet;i.e., the input and output pole faces of the magnet are rotated so as to provide transverse focusing and the 90° bend provides radial focusing via the $\sqrt{1 - n}$ factor characteristic of any dipole magnet. The net optical result is point-to-point imaging in the plane of bend and parallel-to-point imaging in the plane transverse to the plane of bend. The solid angle and resolution requirements of the 1.6-GeV/c spectrometer are such that three sextupole components are needed to acheive the required performance. In this application the sextupoles are generated by machining an appropriate transverse second-order curvature on the magnet pole face at three different locations along the 90° bend of the system. In summary, the 1.6-GeV/c spectrometer consists of one dipole, bending a total of 90°, two quadrupole elements and a sextupole triplet with the quadrupole and sextupole strengths chosen to provide the first- and second-order properties demanded of the system.

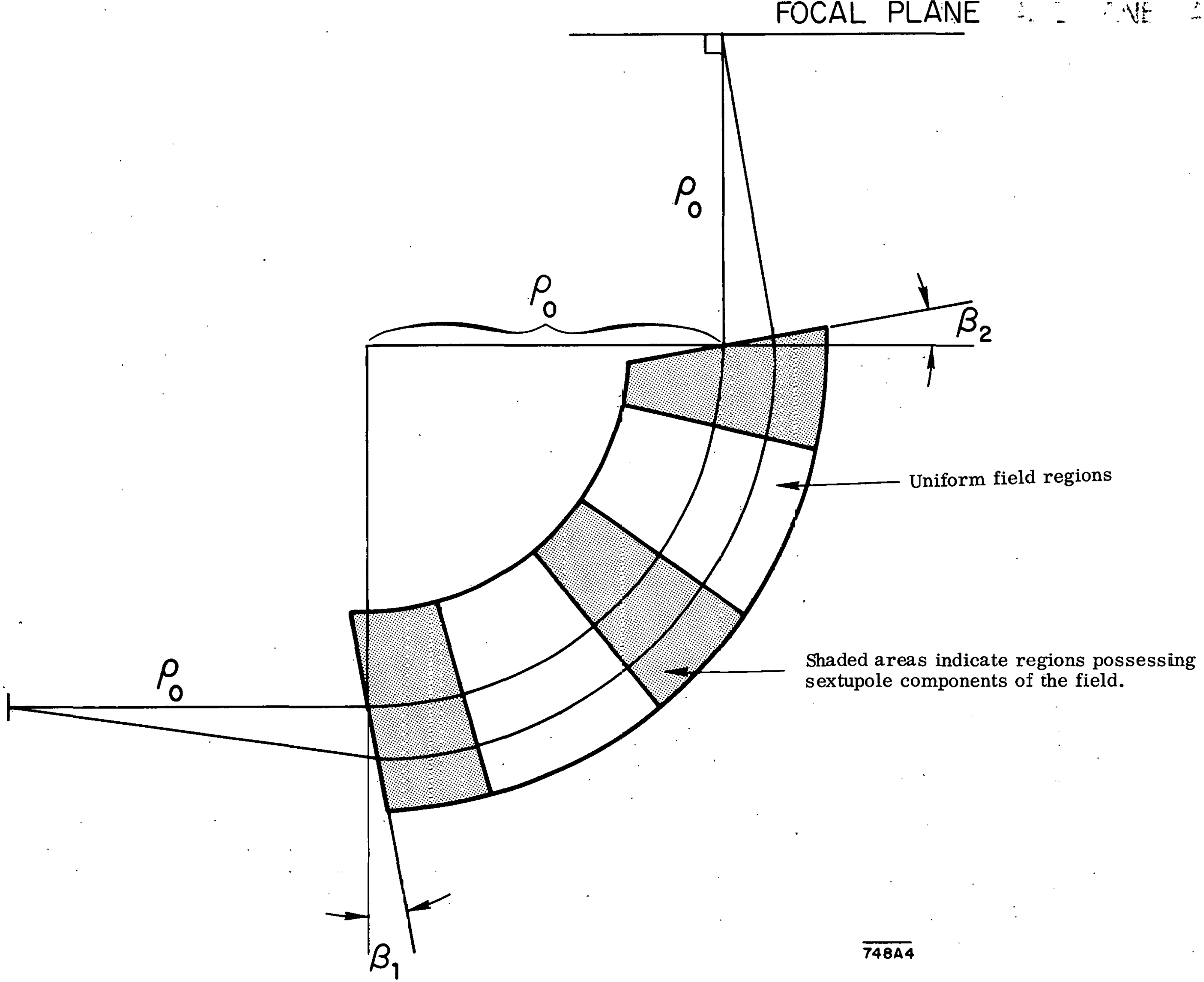

FIG. 1--1.6 GeV/c SPECTROMETER

Optically, the 8-GeV/c spectrometer is relatively simple. It consists of two dipoles, each bending 15° making a total of a 30° bend, and three quadrupoles (two preceding and one following the dipole elements) to provide point-to-point imaging in the plane of bend and parallel-to-point imaging in the plane transverse to the bending plane. The solid angle and resolution requirements of the instrument are sufficiently modest that no sextupole components are needed. The penalty paid for not adding sextupole components is that the focal plane angle with respect to the optic axis at the end of the system is a relatively small angle (13.7°). With the addition of one sextupole element near the end of the system, the focal plane could have been rotated to a much larger angle. However, the 13.7° angle was acceptable for the focal plane counter array and it was ultimately decided to omit the additional sextupole element.

The 20-GeV/c spectrometer is a more complex design. The increased momentum requires an $\int B \cdot dl$ twice that of the 8-GeV/c spectrometer. The final instrument is composed of four dipole elements (bending magnets), two bending in one sense and the other two bending in the opposite sense, so the beam emanating from the instrument is parallel to the incident primary particles. The first-order imaging is achieved via four quadrupoles. The chromatic aberrations generated by the quadrupoles in this system are more serious than in the 8-GeV/c case because of an intermediate image required at the midpoint of the system. As a result, the focal plane angle with respect to the central trajectory would have been in the range of 2-4 degrees. As a consequence, sextupoles were introduced in order to rotate the focal plane to a more satisfactory angle for the counter array. A final compromise placed the focal plane angle at 45° with respect to

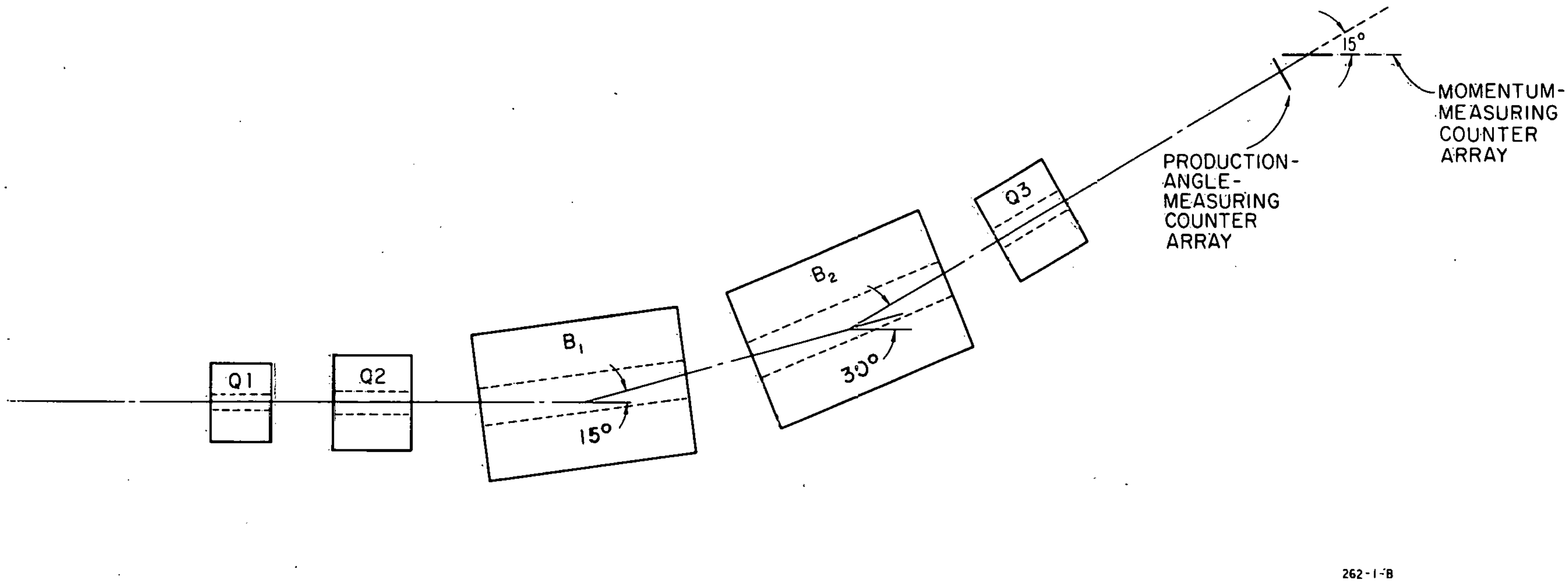

FIG. 2--MAGNET ARRANGEMENT, 3-GeV/c SPECTROMETER

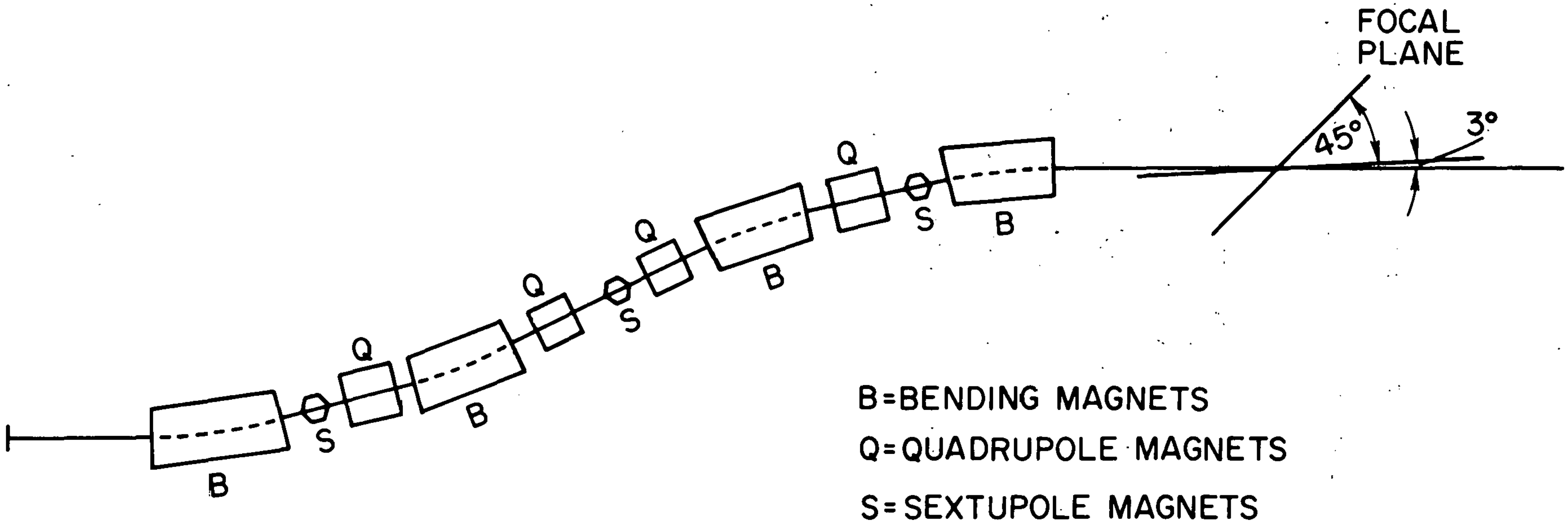

FIG. 3--20 GeV/c SPECTROMETER

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the optic axis of the system via the introduction of three sextupoles. Thus the 20-GeV/c spectrometer consists of four dipoles, with an intermediate cross-over following the first two dipoles; a quadrupole triplet to achieve first-order imaging and finally a sextupole triplet to compensate for the chromatic aberrations introduced by the quadrupoles. Optically, the 20-GeV/c spectrometer is very similar to the 1.6-GeV/c spectrometer and yet physically it is radically different because of the method of introducing the various multipole components.

Having provided some representative examples of spectrometer design, we now wish to introduce and develop the theoretical tools for creating other designs.

II. A GENERAL FIRST- AND SECOND-ORDER THEORY OF BEAM TRANSPORT OPTICS

The fundamental objective is to study the trajectories described by charged particles in a static magnetic field. To maintain the desired generality only one major restriction will be imposed on the field configuration: Relative to a plane that will be designated as the magnetic midplane, the magnetic scalar potential φ shall be an odd function in the transverse coordinate y (the direction perpendicular to the midplane), i.e., $\varphi(x,y,t) = -\varphi(x,-y,t)$. This restriction greatly simplifies the calculations; and from experience in designing beam transport systems, it appears that for most applications there is little, if any, advantage to be gained from a more complicated field pattern. The trajectories will be described by means of a Taylor's expansion about a particular trajectory (which lies entirely within the magnetic midplane) designated henceforth as the central trajectory. Referring to Fig. 4, the coordinate t is the arc length measured along the central trajectory; and x,y, and t form a right-handed curvilinear coordinate system. The results will be valid for describing trajectories lying close to and making small angles with the central trajectory.

The basic steps in formulating the solution to the problem are as follows:

1) A general vector differential equation is derived describing the trajectory of a charged particle in an arbitrary static magnetic field which possesses "midplane symmetry."

2) A Taylor's series solution about the central trajectory is then assumed; this is substituted into the general differential equation and terms to second-order in the initial conditions are retained.

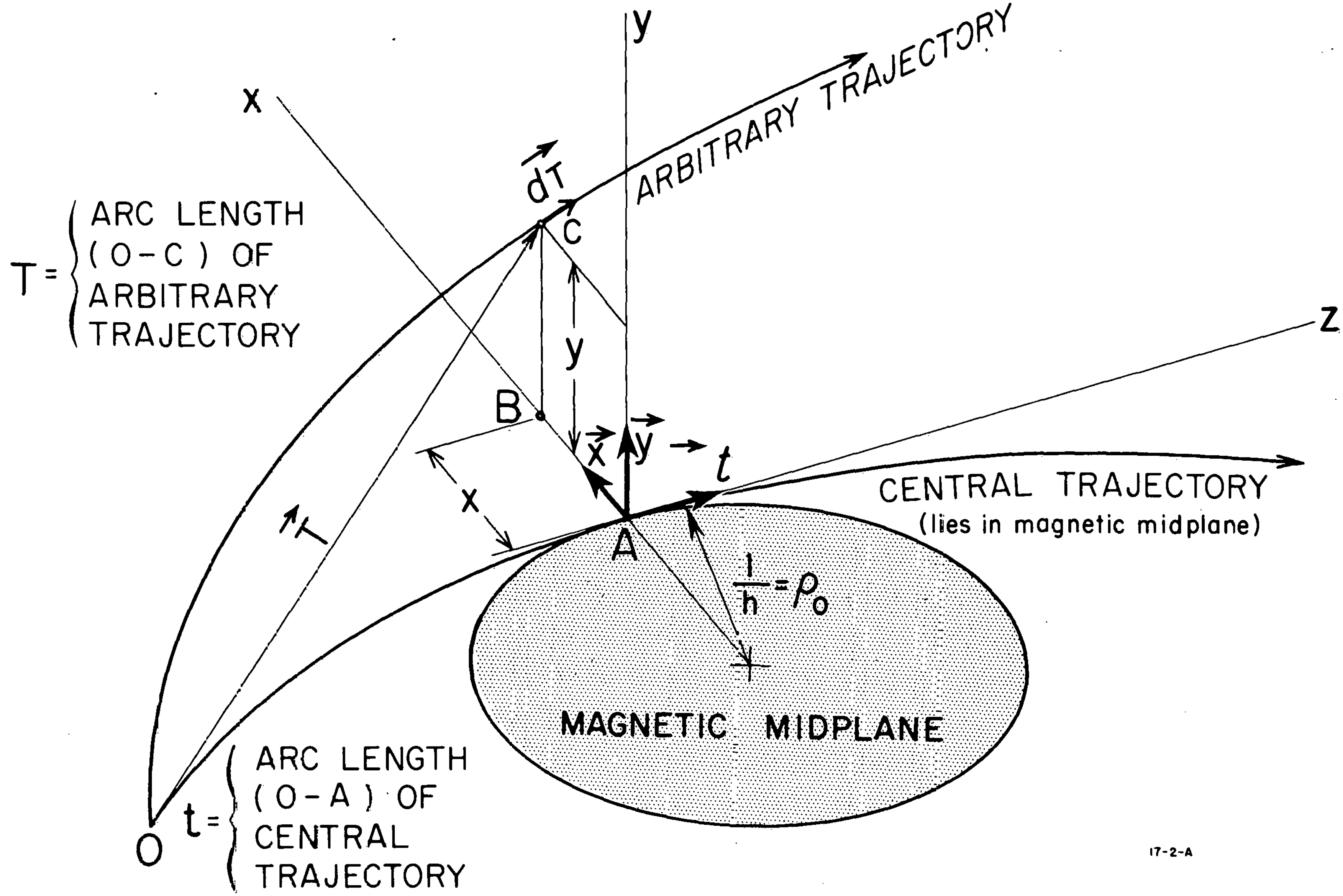


FIG. 4--CURVILINEAR COORDINATE SYSTEM USED IN DERIVATION OF EQUATIONS OF MOTION

3) The first-order coefficients of the Taylor's expansion (for mono-energetic rays) satisfy homogeneous second-order differential equations characteristic of simple harmonic oscillator theory; and the first-order dispersion and the second-order coefficients of the Taylor's series satisfy second-order differential equations having "driving terms."

4) The first-order dispersion term and the second-order coefficients are then evaluated via a Green's function integral containing the driving function of the particular coefficient being evaluated and the characteristic solutions of the homogeneous equations.

In other words, the basic mathematical solution for beam transport optics is similar to the theory of forced vibrations or to the theory of the classical harmonic oscillator with driving terms.

It is useful to express the second-order results in terms of the first-order coefficients of the Taylor's expansion. These first-order coefficients have a one-to-one correspondence with the following five characteristic first-order trajectories (matrix elements) of the system (identified by their initial conditions at $t = 0$); where prime denotes the derivative with respect to t:

1) The unit sine-like function $s_x(t)$ in the plane of bend (the magnetic midplane) where $s_x(0) = 0$; $s'_x(0) = 1$

2) The unit cosine-like function $c_x(t)$ in the plane of bend where $c_x(0) = 1$; $c'_x(0) = 0$

3) The dispersion function $d_x(t)$ in the plane of bend where $d_x(0) = 0$; $d'_x(0) = 0$

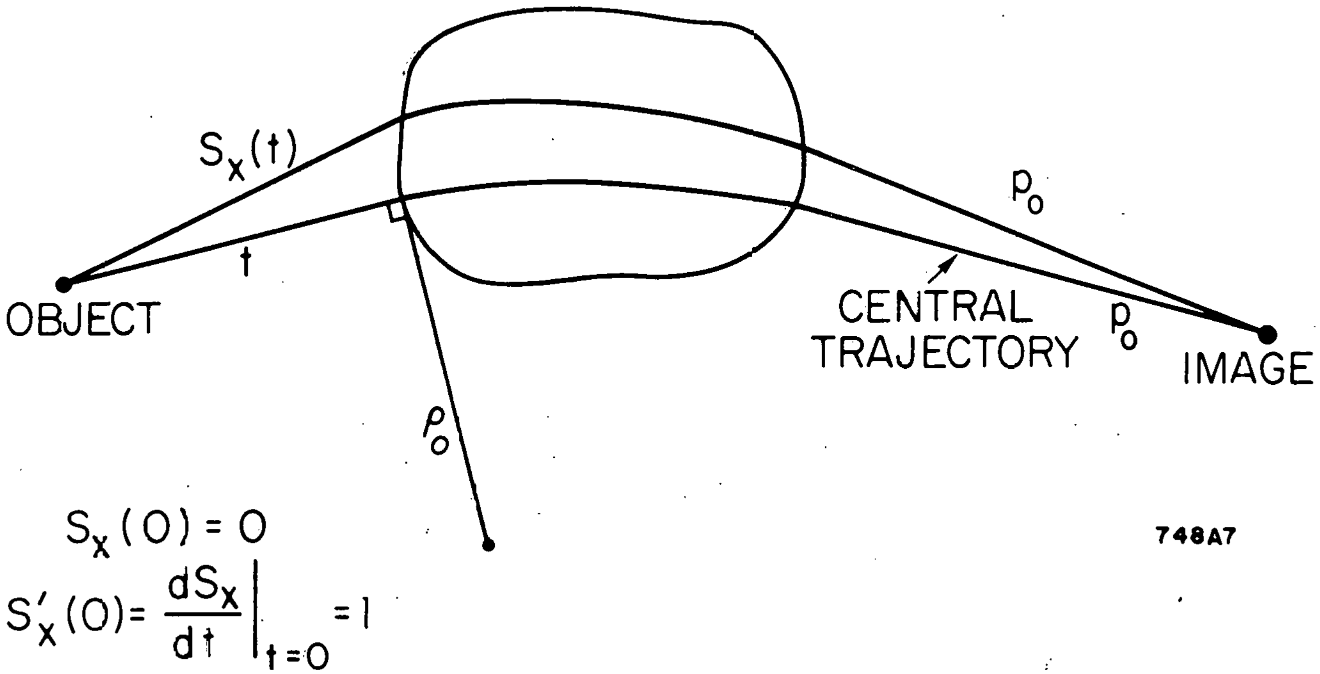
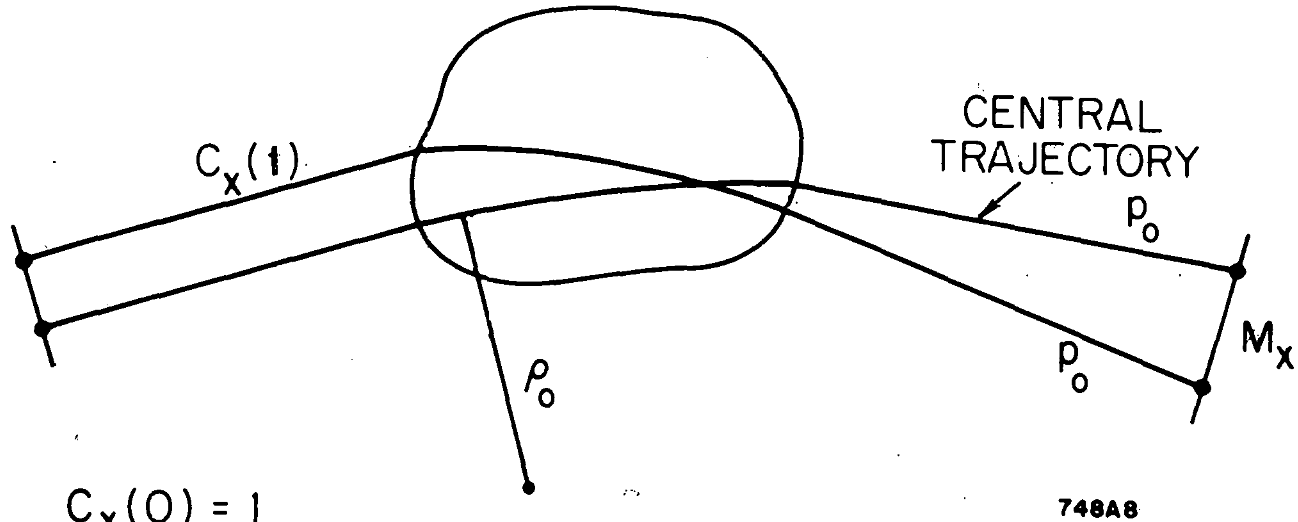


FIG. 5--SINE-LIKE FUNCTION $s_x(t)$ IN MAGNETIC MIDPLANE



$$C_x(0) = 1$$

$$C_x'(0) = \left.\frac{dC_x}{dt}\right|_{t=0} = 0$$

FIG. 6--COSINE-LIKE FUNCTION $c_x(t)$ IN MAGNETIC MIDPLANE

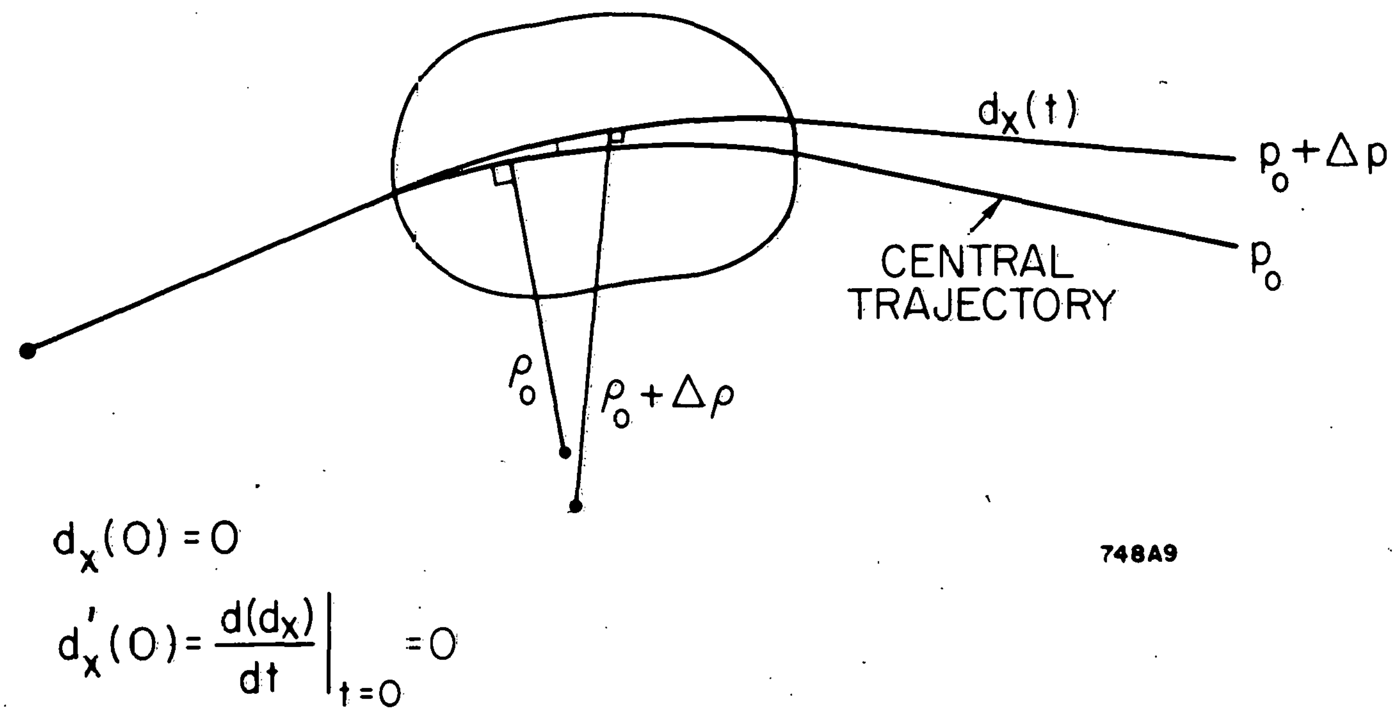


FIG. 7--DISPERSION FUNCTION $d_x(t)$ IN MAGNETIC MIDPLANE

4) The unit sine-like function $s_y(t)$ in the non-bend plane where $s_y(0) = 0$; $s'_y(0) = 1$

5) The unit cosine-like function $c_y(t)$ in the non-bend plane where $c_y(0) = 1$; $c'_y(0) = 0$

When the transverse position of an arbitrary trajectory at position t is written as a first-order Taylor's expansion in terms of its initial conditions, the above five quantities are just the coefficients appearing in the expansion for the transverse coordinates x and y as follows:

$$x(t) = c_x(t)\, x_o + s_x(t)\, x'_o + d_x(t) \left(\frac{\Delta p}{p_o}\right)$$

and

$$y(t) = c_y(t)\, y_o + s_y(t)\, y'_o$$

where x_o and y_o are the initial transverse coordinates and x'_o and y'_o are the initial angles (in the paraxial approximation) the arbitrary ray makes with respect to the central trajectory. $\frac{\Delta p}{p_o}$ is the fractional momentum deviation of the ray from the central trajectory. The prime (') denotes total derivative, along the trajectory, with respect to <u>t</u>.

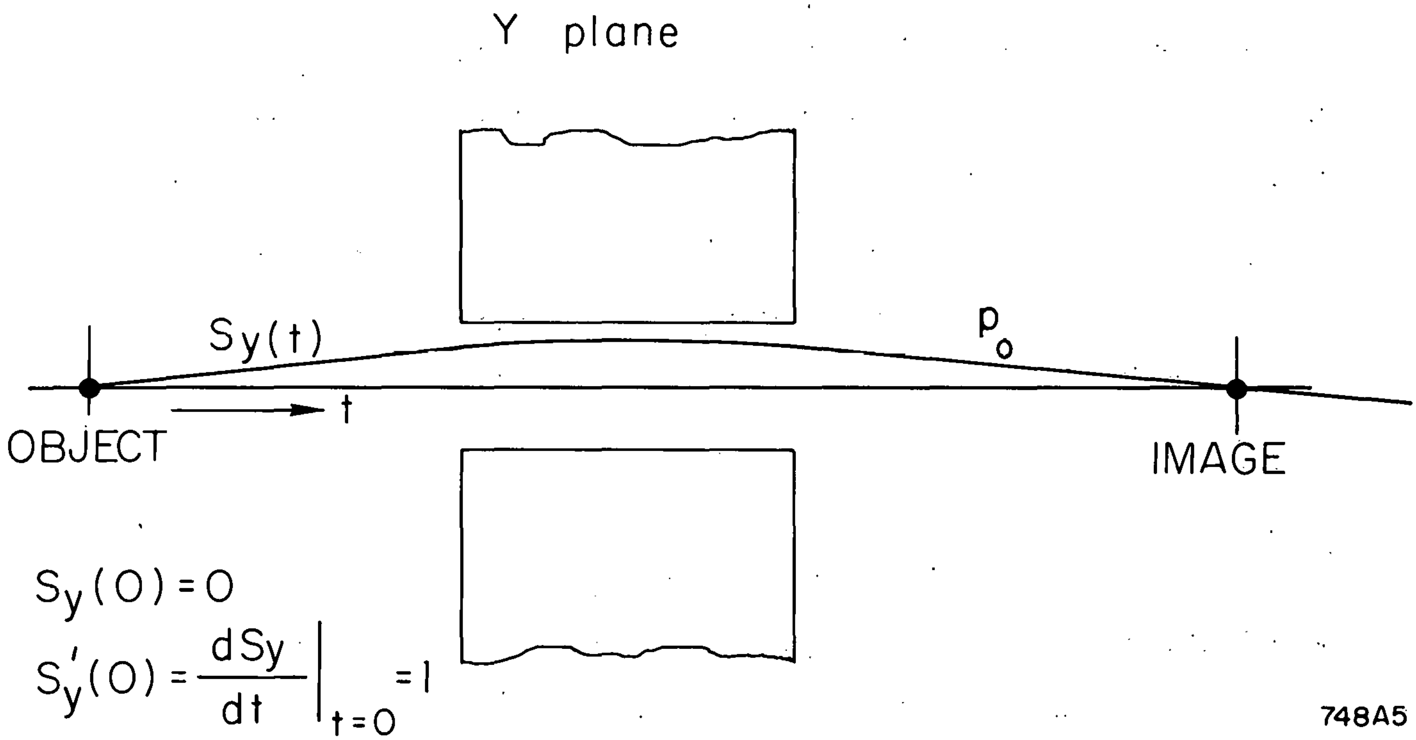


FIG. 8--SINE-LIKE FUNCTION $s_y(t)$ IN NON-BEND (y) PLANE

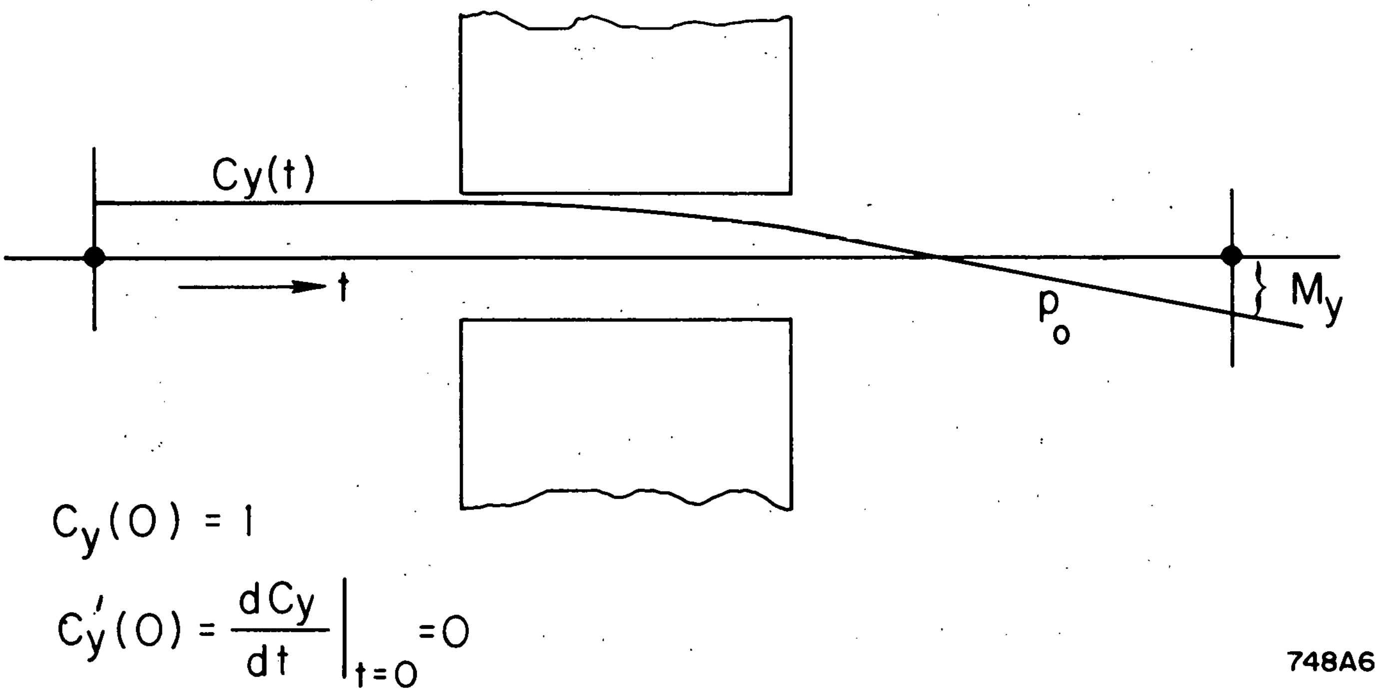


FIG. 9--COSINE-LIKE FUNCTION $c_y(t)$ IN THE NON-BEND (y) PLANE

THE VECTOR DIFFERENTIAL EQUATION OF MOTION

We begin with the usual vector relativistic equation of motion for a charged particle in a static magnetic field equating the time rate of change of the momentum to the Lorentz force:

$$\dot{\underset{\sim}{P}} = e(\underset{\sim}{V} \times \underset{\sim}{B})$$

and immediately transform this equation to one in which time has been eliminated as a variable and we are left only with spatial coordinates. The curvilinear coordinate system used is shown in Fig. 4 . Note that the variable t is not time but is the arc distance measured along the central trajectory. With a little algebra, the equation of motion is readily transformed to the vector forms shown below.

Let e be the charge of the particle, V its speed, P its momentum magnitude, $\underset{\sim}{T}$ its position vector, and T the distance traversed. The unit tangent vector of the trajectory is $d\underset{\sim}{T}/dT$. Thus, the velocity and momentum of the particle are, respectively, $(d\underset{\sim}{T}/dT)V$ and $(d\underset{\sim}{T}/dT)P$. The vector equation of motion then becomes:

$$V \frac{d}{dT}\left(\frac{d\underset{\sim}{T}}{dT} P\right) = eV \left(\frac{d\underset{\sim}{T}}{dT} \times \underset{\sim}{B}\right)$$

or

$$P \frac{d^2\underset{\sim}{T}}{dT^2} + \frac{d\underset{\sim}{T}}{dT}\left(\frac{dP}{dT}\right) = e \left(\frac{d\underset{\sim}{T}}{dT} \times \underset{\sim}{B}\right) ,$$

where $\underset{\sim}{B}$ is the magnetic induction. Then, since the derivative of a unit vector is perpendicular to the unit vector, $(d^2\underset{\sim}{T}/dT^2)$ is perpendicular to $d\underset{\sim}{T}/dT$. It follows that $dP/dT = 0$; that is, P is a constant of the motion as expected from the fact that the magnetic force is always perpendicular to the velocity in a static magnetic field. The final result is:

$$\frac{d^2 \underset{\sim}{T}}{dT^2} = \frac{e}{P} \left(\frac{d\underset{\sim}{T}}{dT} \times \underset{\sim}{B} \right) . \qquad (1)$$

THE COORDINATE SYSTEM

The general right-handed curvilinear coordinate system (x, y, t) used is illustrated in Fig. 4. A point O on the central trajectory is designated the origin. The direction of motion of particles on the central trajectory is designated the positive direction of the coordinate t . A point A on the central trajectory is specified by the arc length t measured along that curve from the origin O to point A. The two sides of the magnetic symmetry-plane are designated the positive and negative sides by the sign of the coordinate y . To specify an arbitrary point B which lies in the symmetry-plane, we construct a line segment from that point to the central trajectory (which also lies in the symmetry-plane) intersecting the latter perpendicularly at A ; the point A provides one coordinate t ; the second coordinate x is the length of the line segment BA, combined with a sign (+) or (-) according as an observer, on the positive side of the symmetry-plane, facing in the positive direction of the central trajectory, finds the point on the left or right side. In other words, x, y and t form a right-handed curvilinear coordinate system. To specify a point C which lies off the symmetry-plane, we construct a line segment from the point to the plane, intersecting the latter perpendicularly at B ; then

B provides the two coordinates, t and x ; the third coordinate y is the length of the line segment CB.

We now define three mutually perpendicular unit vectors $(\hat{x}, \hat{y}, \hat{t})$. $\hat{t}$ is tangent to the central trajectory and directed in the positive t-direction at the point A corresponding to the coordinate t ; $\hat{x}$ is perpendicular to the principal trajectory at the same point, parallel to the symmetry-plane, and directed in the positive x-direction. $\hat{y}$ is perpendicular to the symmetry-plane, and directed away from that plane on its positive side. The unit vectors $(\hat{x}, \hat{y}, \hat{t})$ constitute a right-handed system and satisfy the relations

$$\begin{aligned} \hat{x} &= \hat{y} \times \hat{t} \\ \hat{y} &= \hat{t} \times \hat{x} \\ \hat{t} &= \hat{x} \times \hat{y} \end{aligned} \qquad (2)$$

The coordinate t is the primary independent variable, and we shall use the prime to indicate the operation d/dt. The unit vectors depend only on the coordinate t, and from differential vector calculus, we may write

$$\begin{aligned} \hat{x}' &= h\hat{t} \\ \hat{y}' &= 0 \\ \hat{t}' &= -h\hat{x} \end{aligned} \qquad (3)$$

where $h(t) = \frac{1}{\rho_0}$ is the curvature of the central trajectory at point A defined as positive as shown in Fig. 4.

The equation of motion may now be rewritten in terms of the curvilinear coordinates defined above. To facilitate this, it is convenient to express $d\underset{\sim}{T}/dT$ and $d^2\underset{\sim}{T}/dT^2$ in the following forms:

$$\frac{d\underset{\sim}{T}}{dT} = \left(\frac{d\underset{\sim}{T}}{dt}\right) \Big/ \left(\frac{dT}{dt}\right) = \frac{\underset{\sim}{T}'}{T'}$$

$$\frac{d^2\underset{\sim}{T}}{dT^2} = \frac{1}{T'} \frac{d}{dt} \left(\frac{\underset{\sim}{T}'}{T'}\right)$$

or

$$(T')^2 \frac{d^2\underset{\sim}{T}}{dT^2} = \underset{\sim}{T}'' - \frac{1}{2} \frac{\underset{\sim}{T}'}{(T')^2} \frac{d}{dt} (T')^2$$

The equation of motion now takes the form

$$\underset{\sim}{T}'' - \frac{1}{2} \frac{\underset{\sim}{T}'}{(T')^2} \frac{d}{dt} (T')^2 = \frac{e}{P} T' (\underset{\sim}{T}' \times \underset{\sim}{B}) \quad (4)$$

In this coordinate system, the differential line element is given by:

$$d\underset{\sim}{T} = \hat{x}dx + \hat{y}dy + (1+hx)\hat{t}dt$$

and

$$(dT)^2 = d\underset{\sim}{T} \cdot d\underset{\sim}{T} = dx^2 + dy^2 + (1+hx)^2 \, dt^2$$

By differentiation of these equations with respect to t, it follows that:

$$T'^2 = x'^2 + y'^2 + (1+hx)^2$$

$$\frac{1}{2} \frac{d}{dt} (T')^2 = x'x'' + y'y'' + (1+hx)(hx'+h'x)$$

$$\underset{\sim}{T}' = \hat{x}x' + \hat{y}y' + (1+hx)\,\hat{t}$$

and

$$\underset{\sim}{T}'' = \hat{x}x'' + \hat{x}'x' + \hat{y}y'' + \hat{y}'y' + (1+hx)\hat{t}' + \hat{t}(hx'+h'x)$$

Use of the differential vector relations of Eq. (3), reduces the expression for $\underset{\sim}{T}''$ to

$$\underset{\sim}{T}'' = \hat{x}[x'' - h(1+hx)] + \hat{y}y'' + \hat{t}[2hx' + h'x]$$

The vector equation of motion may now be separated into its component parts; the result is:

$$
\begin{aligned}
&\hat{x}\left\{[x'' - h(1+hx)] - \frac{x'}{(T')^2}[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\} \\
&\quad + \hat{y}\left\{y'' - \frac{y'}{(T')^2}[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\} \\
&\quad + \hat{t}\left\{(2hx'+h'x) - \frac{(1+hx)}{(T')^2}[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\}
\end{aligned} \tag{5}
$$

$$
= \frac{e}{P} T'(\underset{\sim}{T'} \times \underset{\sim}{B}) = \frac{e}{P} T' \left\{\hat{x}[y'B_t - (1+hx)B_y] + \hat{y}[(1+hx)B_x - x'B_t] + \hat{t}[x'B_y - y'B_x]\right\} .
$$

Note that in this form, no approximations have been made; the equation of motion is still valid to all orders in the variables x and y and their derivatives.

If now we retain only terms through second order in x and y and their derivatives and note that $(T')^2 = 1+hx + \ldots$, the x and y components of the equation of motion become

$$
\begin{aligned}
x'' - h(1+hx) - x'(hx'+h'x) &= \frac{e}{P} T'[y'B_t - (1+hx)B_y] \\
y'' - y'(hx'+h'x) &= \frac{e}{P} T'[(1+hx)B_x - x'B_t]
\end{aligned} \tag{6}
$$

The equation of motion of the central orbit is readily obtained by setting x and y and their derivatives equal to zero. We thus obtain:

$$
h = \frac{e}{P_o} B_y(o,o,t) \quad \text{or} \quad B\rho_o = \frac{P_o}{e} \tag{7}
$$

This result will be useful for simplifying the final equations of motion. P_o is the momentum of a particle on the central trajectory. Note that this equation establishes the sign convention between h, e, and B_y .

EXPANDED FORM OF A MAGNETIC FIELD HAVING MEDIAN PLANE SYMMETRY

We now evolve the field components of a static magnetic field possessing median or midplane symmetry. We define median plane symmetry as follows: Relative to the plane containing the central trajectory, the magnetic scalar potential φ is an odd function in y ; i.e., $\varphi(x,y,t) = -\varphi(x,-y,t)$. Stated in terms of the magnetic field components B_x, B_y and B_t, this is equivalent to saying that:

$$B_x(x,y,t) = -B_x(x,-y,t)$$

$$B_y(x,y,t) = B_y(x,-y,t)$$

and

$$B_t(x,y,t) = -B_t(x,-y,t)$$

It follows immediately that on the midplane $B_x = B_t = 0$ and only B_y remains non-zero; in other words, on the midplane $\underset{\sim}{B}$ is always normal to the plane. As such, any trajectory initially lying in the midplane will remain in the midplane throughout the system.

The expanded form of a magnetic field with median plane symmetry has been worked out by many people; however, a convenient and comprehensible reference is not always available. L. C. Teng[2] has provided us with such a reference which is reproduced essentially in its original form in the following paragraphs.

For the magnetic field in vacuum, the field may be expressed in terms of a scalar potential φ by $\underset{\sim}{B} = \underset{\sim}{\nabla}\varphi$.* The scalar potential will be expanded in the curvilinear coordinates about the central trajectory lying in the median plane $y = 0$. The curvilinear coordinates have been

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*For convenience, we omit the minus sign since we are restricting the problem to static magnetic fields.

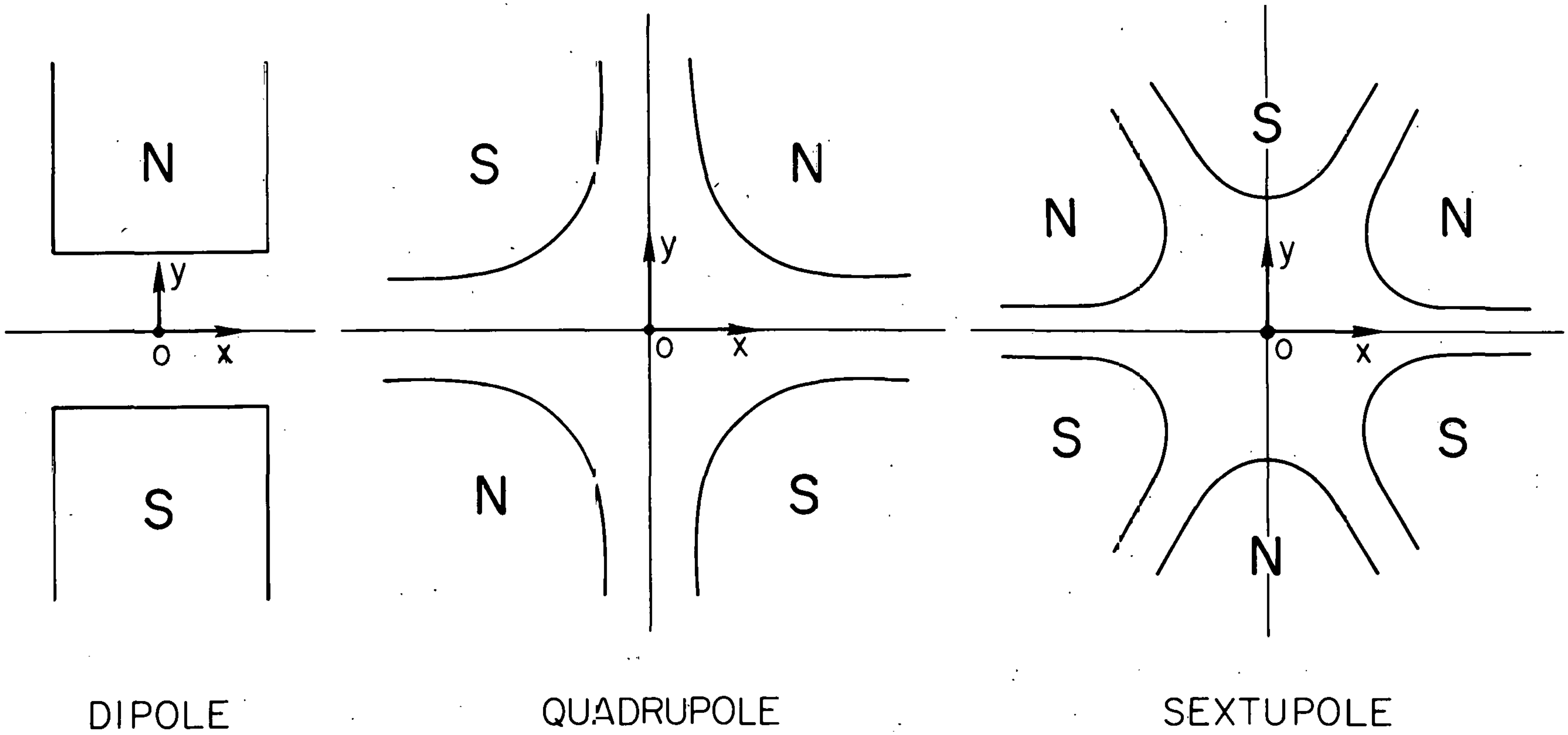

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FIG. 10--ILLUSTRATION OF THE MAGNETIC MIDPLANE (X AXIS) FOR DIPOLE, QUADRUPOLE AND SEXTUPOLE ELEMENTS. THE MAGNETIC POLARITIES INDICATE MULTIPOLE ELEMENTS THAT ARE POSITIVE IN RESPECT TO EACH OTHER.

defined in Fig. 1 where x is the outward normal distance in the median plane away from the central trajectory, y is the perpendicular distance from the median plane, t is the distance along the central trajectory, and h = h(t) is the curvature of the central trajectory. As stated previously, these coordinates (x, y, and t) form a right-handed orthogonal curvilinear coordinate system.

Since the existence of the median plane requires that ϕ be an odd functio of y, i.e., $\phi(x,y,t) = -\phi(x,-y,t)$; the most general expanded form of ϕ may be expressed as follows:

$$\begin{aligned}\varphi(x,y,t) &= \left(A_{10} + A_{11}x + A_{12}\frac{x^2}{2!} + A_{13}\frac{x^3}{3!} + \dots\right) y \\ &\quad + \left(A_{30} + A_{31}x + A_{32}\frac{x^2}{2!} + \dots\right)\frac{y^3}{3!} + \dots \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!}\end{aligned} \tag{8}$$

where the coefficients $A_{2m+1,n}$ are functions of t.

In this coordinate system, the differential line element dT is given by

$$dT^2 = dx^2 + dy^2 + (1+hx)^2\,(dt)^2 \tag{9}$$

and the Laplace equation has the form

$$\nabla^2\varphi = \frac{1}{(1+hx)}\frac{\partial}{\partial x}\left[(1+hx)\frac{\partial\varphi}{\partial x}\right] + \frac{\partial^2\varphi}{\partial y^2} + \frac{1}{(1+hx)}\frac{\partial}{\partial t}\left[\frac{1}{(1+hx)}\frac{\partial\varphi}{\partial t}\right] = 0 \tag{10}$$

Substitution of (8) into (10) gives the following recursion formula for the coefficients:

$$-A_{2m+3,n} = A''_{2m+1,n} + nhA''_{2m+1,n-1} - nh'A'_{2m+1,n-1} + A_{2m+1,n+2}$$

$$+ (3n+1)hA_{2m+1,n+1} + n(3n-1)h^2A_{2m+1,n} + n(n-1)^2h^3A_{2m+1,n-1} \quad (11)$$

$$+ 3nhA_{2m+3,n-1} + 3n(n-1)h^2A_{2m+3,n-2} + n(n-1)(n-2)h^3A_{2m+3,n-3}$$

where prime means $\frac{d}{dt}$, and where it is understood that all coefficients A with one or more negative subscripts are zero. This recursion formula expresses all the coefficients in terms of the midplane field $B_y(x,o,t)$: where

$$A_{1,n} = \left(\frac{\partial^n B_y}{\partial x^n}\right)_{\substack{x=o\\y=o}} = \text{functions of } t. \quad (12)$$

Since φ is an odd function of y, on the median plane we have $B_x = B_t = 0$. The normal (in x direction) derivatives of B_y on the reference curve defines B_y over the entire median plane, hence the magnetic field $\vec{B}$ over the whole space. The components of the field are expressed in terms of φ explicitly by $\underset{\sim}{B} = \underset{\sim}{\nabla}\varphi$ or

$$B_x = \frac{\partial\varphi}{\partial x} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} A_{2m+1,n+1} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!}$$

$$B_y = \frac{\partial\varphi}{\partial y} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} A_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m}}{(2m)!} \quad (13)$$

$$B_t = \frac{1}{(1+hx)} \frac{\partial\varphi}{\partial t} = \frac{1}{(1+hx)} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} A'_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!}$$

where B_t is not expressed in a pure power expansion form. This form can be obtained straightforwardly by expanding $\frac{1}{1+hx}$ in a power series of hx and multiplying out the two series; however, there does not seem to be any advantage gained over the form given in Eq. (13).

The coefficients up to the 6th degree terms in x and y are given explicitly below from Eq. (11)

$$A_{30} = - A''_{10} - A_{12} - hA_{11}$$

$$A_{31} = - A''_{11} + 2hA''_{10} + h'A'_{10} - A_{13} - hA_{12} + h^2A_{11}$$

$$A_{32} = - A''_{12} + 4hA''_{11} + 2h'A'_{11} - 6h^2A''_{10} - 6hh'A'_{10} - A_{14} - hA_{13} + 2h^2A_{12} - 2h^3A_{11}$$

$$A_{33} = - A''_{13} + 6hA''_{12} + 3h'A'_{12} - 18h^2A''_{11} - 18hh'A'_{11} + 24h^3A''_{10} + 36h^2 h'A'_{10}$$
$$- A_{15} - hA_{14} + 3h^2A_{13} - 6h^3A_{12} + 6h^4A_{11} \qquad (14)$$

$$A_{50} = A''''_{10} + 2A''_{12} - 2hA''_{11} + h''A_{11} + 4h^2A''_{10} + 5hh'A'_{10} + A_{14} + 2hA_{13}$$
$$- h^2A_{12} + h^3A_{11}$$

$$A_{51} = A''''_{11} - 4hA''''_{10} - 6h'A'''_{10} - 4h''A''_{10} - h'''A'_{10} + 2A''_{13} - 6hA''_{12} - 2h'A'_{12}$$
$$+ h''A_{12} + 10h^2A''_{11} + 7hh'A'_{11} - 4hh''A_{11} - 3h'^2A_{11} - 16h^3A''_{10}$$
$$- 29h^2h'A'_{10} + A_{15} + 2hA_{14} - 3h^2A_{13} + 3h^3A_{12} - 3h^4A_{11} \qquad (15)$$

In the special case when the field has cylindrical symmetry about $\hat{y}$, we can choose a circle with radius $\rho_o = \frac{1}{h}$ = a constant for the reference curve. The coefficients $A_{2m+1,n}$ in Eq. (8) and the curvature h of the reference curve are then all independent of t. Eqs. (14) and (15) are greatly simplified by putting all terms with primed quantities equal to zero.

FIELD EXPANSION TO SECOND-ORDER ONLY

If the field expansion is terminated with the second order terms, the results may be considerably simplified. For this case, the scalar potential φ and the field $\underset{\sim}{B} = \underset{\sim}{\nabla} \varphi$ become:

$$\varphi(x,y,t) = (A_{10} + A_{11}x + \frac{1}{2!} A_{12}x^2 + \ldots)\, y + (A_{30} + \ldots)\frac{y^3}{3!} + \ldots$$

$$A_{1n} = \left.\frac{\partial^n B_y}{\partial x^n}\right|_{\substack{x=0\\y=0}} = \text{functions of t only}$$

and

$$A_{30} = -\,[A''_{10} + hA_{11} + A_{12}]$$

where prime means the total derivative with respect to t. Then $\underset{\sim}{B} = \underset{\sim}{\nabla} \varphi$ from which

$$B_x(x,y,t) = \frac{\partial\varphi}{\partial x} = A_{11}y + A_{12}xy + \ldots$$

$$B_y(x,y,t) = \frac{\partial\varphi}{\partial y} = A_{10} + A_{11}x + \frac{1}{2!} A_{12}x^2 + \frac{1}{2!} A_{30}y^2 + \ldots \qquad (16)$$

$$B_t(x,y,t) = \frac{1}{(1+hx)}\frac{\partial\varphi}{\partial t} = \frac{1}{(1+hx)}[A'_{10}y + A'_{11}xy + \ldots]$$

By inspection it is evident that B_x, B_y and B_t are all expressed in terms of A_{10}, A_{11} and A_{12} and their derivatives with respect to t.

Consider then B_y on the midplane only

$$B_y(x,o,t) = A_{1o} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \dots$$

$$\underline{\text{dipole}} \quad \underline{\text{quadrupole}} \qquad \underline{\text{sextupole}} \quad \underline{\text{etc}} \tag{17}$$

$$= B_y\Big|_{\substack{x=o\\y=o}} + \frac{\partial B_y}{\partial x}\Big|_{\substack{x=o\\y=o}} x + \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}\Big|_{\substack{x=o\\y=o}} x^2 + \dots$$

The successive derivatives identify the terms as being dipole, quadrupole, sextupole, octupole, etc., in the expansion of the field. To eliminate the necessity of continually writing these derivatives, it is useful to express the midplane field in terms of dimensionless quantities $n(t)$, $\beta(t)$, etc., or

$$B_y(x,o,t) = B_y(o,o,t)\,[1 - nhx + \beta h^2x^2 + \gamma h^3x^3 + \dots] \tag{18}$$

where as before $h(t) = 1/\rho_o$ and n, β and γ are functions of t. Direct comparison of Eqs. (17) and (18) yields

$$n = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=o\\y=o}} \quad \text{and} \quad \beta = \left[\frac{1}{2!h^2B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=o\\y=o}} \tag{19}$$

We now make use of Eq. (7), the equation of motion of the central trajectory;

$$B_y(o,o,t) = \frac{hP_o}{e} .$$

Combining Eqs. (7) and (19), the coefficients of the field expansions become

$$
\begin{aligned}
A_{10} &= B_y(o,o,t) = h\left(\frac{P_o}{c}\right) \\
A_{11} &= \left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=o\\y=o}} = -nh^2\left(\frac{P_o}{e}\right) \\
\frac{1}{2!}A_{12} &= \frac{1}{2!}\left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=o\\y=o}} = \beta h^3\left(\frac{P_o}{e}\right) \\
A_{30} &= -[h'' - nh^3 + 2\beta h^3]\left(\frac{P_o}{e}\right) \\
A'_{10} &= h'\left(\frac{P_o}{e}\right) \\
A'_{11} &= -[2nhh' + n'h^2]\left(\frac{P_o}{e}\right) \quad .
\end{aligned}
\tag{20}
$$

To second order the expansions for the magnetic field components may now be expressed in the form:

$$
\begin{aligned}
B_x(x,y,t) &= \frac{P_o}{e}\,[-nh^2y + 2\beta h^3xy + \dots] \\
B_y(x,y,t) &= \frac{P_o}{e}\,[h - nh^2x + \beta h^3x^2 - \tfrac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2 + \dots] \\
B_t(x,y,t) &= \frac{P_o}{e}\,[h'y - (n'h^2 + 2nhh' + hh')xy + \dots]
\end{aligned}
\tag{21}
$$

where P_o is the momentum of the central trajectory.

IDENTIFICATION OF n AND β WITH PURE QUADRUPOLE AND SEXTUPOLE FIELDS

The scalar potential of a pure quadrupole field in cylindrical and in rectangular coordinates is given by:

$$\varphi = \frac{B_o r^2}{2a} \sin 2\alpha = \frac{B_o xy}{a} \qquad (22a)$$

where B_o is the field at the pole, a is the radius of the quadrupole aperture and r and α are the cylindrical coordinates, such that $x = r \cos \alpha$ and $y = r \sin \alpha$. From $\underset{\sim}{B} = \underset{\sim}{\nabla}\varphi$, it follows that

$$B_x = \frac{B_o y}{a} \quad \text{and} \quad B_y = \frac{B_o x}{a} \qquad (22b)$$

Using the second of Eqs. (20) and (22a-b)

$$\left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=o\\y=o}} = \frac{B_o}{a} = -nh^2 \left(\frac{P_o}{e}\right)$$

we define a quantity k_q^2 as follows:

$$k_q^2 = -\,nh^2 = \left(\frac{B_o}{a}\right)\left(\frac{e}{P_o}\right) = \left(\frac{B_o}{a}\right)\left(\frac{1}{B\rho}\right) . \qquad (23)$$

Similarly for a pure sextupole field,

$$\varphi = \frac{B_o r^3}{3a^2} \sin 3\alpha = \frac{B_o}{3a^2}\left[3x^2 y - y^3\right]$$

$$B_x = \frac{\partial\varphi}{\partial x} = \frac{2B_o xy}{a^2} \quad \text{and} \quad B_y = \frac{B_o}{a^2}(x^2 - y^2) \qquad (24)$$

where B_o is the field at the pole and a is the radius of the sextupole aperture.

Using the third of Eqs. (20) and (24)

$$\frac{1}{2!}\left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=o\\y=o}} = \frac{B_o}{a^2} = \beta h^3\left(\frac{P_o}{e}\right)$$

we define a quantity k_s^2 as follows:

$$k_s^2 = \beta h^3 = \left(\frac{B_o}{a^2}\right)\left(\frac{e}{P_o}\right) = \left(\frac{B_o}{a^2}\right)\left(\frac{1}{B\rho}\right) \qquad (25)$$

These definitions, Eqs. (23) and (25), are useful in the derivation of the equations of motion and the matrix elements for pure quadrupole and sextupole fields.

THE EQUATIONS OF MOTION IN THEIR FINAL FORM TO SECOND ORDER

Having derived Eq. (21), we are now in a position to substitute into the general second-order equations of motion, Eq. (6). Combining Eq. (6) (the equation of motion) with the expanded field components of Eq. (21), we find for x

$$x''-h(1+hx)-x'(hx'+h'x) = \frac{P_o}{P}\,T'\left\{(1+hx)\left[-h+nh^2x - \beta h^3x^2+ \frac{1}{2}(h''-nh^3+2\beta h^3)y^2\right] + h'yy'+\ldots\right\}$$

and for y

$$y''- y'(hx'+h'x) = \frac{P_o}{P}\,T'\left\{-x'h'y - (1+hx)\left[nh^2y - 2\beta h^3xy\right] + \ldots\right\}$$

Note that we have eliminated the charge of the particle e in the equations of motion. This has resulted from the use of the equation of motion of the central trajectory.

Inserting a second-order expansion for $T' = (x'^2 + y'^2 + (1+hx)^2)^{\frac{1}{2}}$ and letting

$$\frac{P_o}{P} = \frac{P_o}{P_o(1 + \delta)} \cong 1 - \delta + \delta^2 + \dots, \qquad (26)$$

we finally express the differential equations for x and y to second-order as follows:

$$x''+(1-n)h^2x = h\delta + (2n-1-\beta)h^3x^2 + h'xx' + \frac{1}{2}hx'^2 + (2-n)h^2x\delta$$
$$+ \frac{1}{2}(h''-nh^3+2\beta h^3)y^2 + h'yy' - \frac{1}{2}hy'^2 - h\delta^2$$
$$+ \text{higher-order terms} \qquad (27)$$

$$y''+nh^2y = 2(\beta-n)h^3xy + h'xy' - h'x'y + hx'y' + nh^2y\delta$$
$$+ \text{higher-order terms} \qquad (28)$$

From Eqs. (27) and (28) the familiar equations of motion for the first-order terms may be extracted:

$$x'' + (1-n)h^2x = h\delta \quad \text{and} \quad y'' + nh^2y = 0 \qquad (29)$$

Substituting $k_q^2 = - nh^2$ from Eq. (23) into Eqs. (27) and (28) and taking the limit $h \to 0$, $h' \to 0$ and $h'' \to 0$, we find the second-order equations of motion for a pure quadrupole field:

$$x'' + k_q^2x = k_q^2x\delta$$

$$y'' - k_q^2y = -k_q^2y\delta$$

where

$$k_q^2 = \left(\frac{B_o}{a}\right)\left(\frac{e}{P_o}\right) = \left(\frac{B_o}{a}\right)\left(\frac{1}{B\rho_o}\right) \tag{30}$$

Similarly, to find the second-order equations of motion for a pure sextupole field, we make use of Eq. (25) $\beta h^3 = k_s^2$ and, again, take the limit $h \to 0$, $h' \to 0$ and $h'' \to 0$. The results are:

$$x'' + k_s^2(x^2 - y^2) = 0$$

$$y'' - 2k_s^2 xy = 0$$

where

$$k_s^2 = \beta h^3 = \left(\frac{B_o}{a^2}\right)\left(\frac{e}{P_o}\right) = \left(\frac{B_o}{a^2}\right)\left(\frac{1}{B\rho_o}\right) \tag{31}$$

THE DESCRIPTION OF THE TRAJECTORIES. THE COEFFICIENTS OF THE TAYLOR'S EXPANSION.

The deviation of an arbitrary trajectory from the central trajectory is described by expressing x and y as functions of t. The expressions will also contain x_o, y_o, x_o', y_o' and δ, where the subscript o indicates that the quantity is evaluated at $t = 0$; these five boundary values will have the value zero for the central trajectory itself. The procedure for expressing x and y as a five-fold Taylor expansion will be considered in a general way using these boundary values, and detailed formulas will be developed for the calculations of the coefficients through the quadratic terms. The expansions are written:

$$\begin{aligned} x &= \Sigma \left(x \middle| x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi}\right) x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi} \\ y &= \Sigma \left(y \middle| x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi}\right) x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi} \end{aligned} \tag{32}$$

Here, the parentheses are symbols for the Taylor coefficients; the first part of the symbol identifies the coordinate represented by the expansion, and the second indicates the term in question. These coefficients are functions of t to be determined. The symbol Σ indicates summation over zero and all positive integer values of the exponents κ, λ, μ, ν, χ ; however, the detailed calculations will involve only the terms up to the second power. The constant term is zero, and the terms that would indicate a coupling between the coordinates x and y are also zero; this results from the midplane symmetry. Thus we have

$$(x|1) = (y|1) = 0$$

and

$$(x|y_o) = (y|x_o) = 0 \quad ; \qquad (33)$$

$$(x|y'_o) = (y|x'_o) = 0$$

Here, the first line is a consequence of choosing the central trajectory as the reference axis, while the second and third lines follow directly from considerations of symmetry, or more formally, from the formulas at the end of this section.

As mentioned in the introduction, it is convenient to introduce the following abbreviations for the first-order Taylor coefficients:

$$(x|x_o) = c_x(t) \qquad (x|x'_o) = s_x(t) \qquad (x|\delta) = d(t)$$

$$(y|y_o) = c_y(t) \qquad (y|y'_o) = s_y(t) \qquad (34)$$

Retaining terms to second-order and using Eqs. (33) and (34), the Taylor's expansions of Eq. (32) reduce to the following terms:

$$
\begin{aligned}
x = {} & \overbrace{(x|x_o)}^{c_x}x_o + \overbrace{(x|x_o')}^{s_x}x_o' + \overbrace{(x|\delta)}^{d_x}\delta \\
& + (x|x_o^2)x_o^2 + (x|x_o x_o')x_o x_o' + (x|x_o\delta)x_o\delta \\
& + (x|x_o'^2)x_o'^2 + (x|x_o'\delta)x_o'\delta + (x|\delta^2)\delta^2 \\
& + (x|y_o^2)y_o^2 + (x|y_o y_o')y_o y_o' + (x|y_o'^2)y_o'^2
\end{aligned}
$$

and

$$
\begin{aligned}
y = {} & \overbrace{(y|y_o)}^{c_y}y_o + \overbrace{(y|y_o')}^{s_y}y_o' \\
& + (y|x_o y_o)x_o y_o + (y|x_o y_o')x_o y_o' + (y|x_o' y_o)x_o' y_o \\
& + (y|x_o' y_o')x_o' y_o' + (y|y_o\delta)y_o\delta + (y|y_o'\delta)y_o'\delta
\end{aligned}
\tag{35}
$$

Substituting these expansions into Eqs. (27) and (28), we derive a differential equation for each of the first- and second-order coefficients contained in the Taylor's expansions for x and y. When this is done, a systematic pattern evolves: namely,

$$
\begin{aligned}
c_x'' + k_x^2 c_x &= 0 & \qquad c_y'' + k_y^2 c_y &= 0 \\
s_x'' + k_x^2 s_x &= 0 & \qquad s_y'' + k_y^2 s_y &= 0 \\
q_x'' + k_x^2 q_x &= f_x & \qquad q_y'' + k_y^2 q_y &= f_y
\end{aligned}
\tag{36}
$$

where $k_x^2 = (1-n)h^2$ and $k_y^2 = nh^2$ for the x and y motions, respectively. The first two of these equations represent the equations of motion for the first-order monoenergetic terms s_x, c_x, s_y and c_y. That there are two solutions, one for c and one for s, is a manifestation of the fact that the differential equation is second-order; hence, the two solutions differ only by the initial conditions of the characteristic s and c functions. The third differential equation for q is a type form which represents the solution for the first-order dispersion d_x and for any one of the coefficients of the second-order aberrations in the system where the driving term f has a characteristic form for each of these coefficients. The driving function f for each aberration is obtained from the substitution of the Taylor's expansions of Eq. (35) into the general differential Eqs. (27) and (28).

The coefficients satisfy the boundary conditions:

$$\begin{aligned} c(0) &= 1 \quad & c'(0) &= 0 \\ s(0) &= 0 \quad & s'(0) &= 1 \\ d(0) &= 0 \quad & d'(0) &= 0 \\ q(0) &= 0 \quad & q'(0) &= 0 \end{aligned} \tag{37}$$

The driving term f is a polynomial, peculiar to the particular q, whose terms are the coefficients of order less than that of q, and their derivatives. The coefficients in these polynomials are themselves polynomials in h, h', ..., with coefficients that are linear functions of n, β, ... For example, for $q = (x|x_0^2)$, we have

$$f = (2n-1-\beta)\, h^3 c_x^2 + h' c_x c_x' + \frac{1}{2} h c_x'^2 \quad ; \tag{38}$$

In Table I, page 43, are listed the f functions for the remaining linear coefficient, the momentum dispersion d(t) and all of the non-zero quadratic coefficients, shown in Eq. (35), which represent the second-order aberrations of a system.

The coefficients c and s (with identical subscripts) satisfy the same differential equation which has the form of the homogeneous equation of an harmonic oscillator. Here, the stiffness k^2 is a function of t and may be of either sign. In view of their boundary conditions, it is natural to consider c and s as the analogs of the two fundamental solutions of a simple harmonic oscillator, namely $\cos \omega t$ and $(\sin \omega t)/\omega$. The function q is the response of the hypothetical oscillator when, starting at equilibrium and at rest, it is subjected to a driving force f.

The stiffness parameters k_x^2 and k_y^2 represent the converging powers of the field for the two respective coordinates. It is possible for either to be negative, in which case, it actually represents a diverging effect. Addition of k_x^2 and k_y^2 yields

$$k_x^2 + k_y^2 = h^2 \tag{39}$$

For a specific magnitude of h, k_x^2 and k_y^2 may be varied by adjusting n, but the total converging power is unchanged; any increase in one converging power is at the expense of the other. The total converging power is positive; this fact admits the possibility of double-focusing.

A special case of interest is provided by the uniform field; here h = constant and n = 0; then $k_x^2 = h^2$ and $k_y^2 = 0$. Thus, there is a converging effect for x resulting in the familiar semicircular focusing, which is accompanied by no convergence or divergence of y.

Another important special case is given by $n = 1/2$; here, $k_x^2 = k_y^2 = h^2/2$. Thus, both coordinates experience an identical positive convergence, and $c_x = c_y$ and $s_x = s_y$; that is, in the linear approximation, the two coordinates behave identically, and if the trajectory continues through a sufficiently extended field, a double focus is produced.

The method of solution of the equations for c and s will not be discussed here, since they are standard differential equations. The most suitable approach to the problem must be determined in each case. In many cases it will be a satisfactory approximation to consider h and n, and therefore k^2 also, as piecewise-uniform. Then, c and s are represented in each interval of uniformity by a sinusoidal function, a hyperbolic function, or a linear function of t or simply a constant. Using Eq. (36), it follows for either the x or y motions that:

$$\frac{d}{dt}(cs' - c's) = 0$$

Upon integrating and using the initial conditions on c and s in Eq. (37), we find

$$cs' - c's = 1. \tag{40}$$

This expression is just the determinant of the first-order transport matrix representing either the x or y equations of motion. It can be demonstrated that the fact that the determinant is equal to one is equivalent to Liouville's theorem which states that phase areas are conserved throughout the system in either the x or y plane motions.

The first-order dispersion and each of the coefficients of the second-order aberrations (represented by the symbol q) are evaluated using the Green's function integral

$$q = \int_{0}^{t} f(\tau)G(t,\tau)\, d\tau \qquad (41)$$

where

$$G(t,\tau) = s(t)c(\tau) - s(\tau)c(t) \qquad (42)$$

or

$$q = s(t)\int_{0}^{t} f(\tau)c(\tau)\, d\tau - c(t)\int_{0}^{t} f(\tau)s(\tau)\, d\tau\ . \qquad (43)$$

To verify the correctness of this result, we differentiate Eq. (43) and make use of Eq. (40) and the first two of Eq.'s 36 to establish an identity with the last of Eq.'s 36. Thus:

and

$$q' = s'(t)\int_{0}^{t} f(\tau)c(\tau)\, d\tau - c'(t)\int_{0}^{t} f(\tau)s(\tau)\, d\tau \qquad (44a)$$

$$q'' = f + s''(t)\int_{0}^{t} f(\tau)c(\tau)\, d\tau - c''(t)\int^{t} f(\tau)s(\tau)\, d\tau$$

$$\equiv f - k^2 q \qquad (44b)$$

This along with the obvious results $q(o) = 0$ [Eq. (43)] and $q'(o) = 0$ [Eq. (44a)] shows that Eq. (43) is the desired solution of the differential equation for q.

The driving terms tabulated in Table I, combined with Eqs. (43) and (44), complete the solution of the general second-order theory. It now remains to find explicit solutions for specific systems or elements of systems. This will be done in later sections of this report.

It will be seen from Table I that several coefficients are absent, including the linear terms that would represent a coupling between x and y. The absence of these terms is a direct consequence of the initial assumption of midplane symmetry. If midplane symmetry is destroyed any or all of these missing terms may appear in the solution.

TABLE I

The Driving Terms for the Coefficients

Listed in the first column are the coefficients in the expressions for the coordinates x and y; they are indicated by means of the notation introduced in Eq. (32); in addition, the abbreviations given in Eq. (34) are used. For general considerations, q has been used to represent any one of these coefficients. Listed in the second column are the corresponding driving functions f, which are related to the coefficients as shown by Eq. (36), This list includes all those functions f for the linear and quadratic coefficients which do not vanish identically.

q	f			
$d_x = (x \mid \delta)$	h			
$(x \mid x_o^2)$		$+ (2n - 1 - \beta)h^3 c_x^2$	$+ h'c_x c'_x$	$+\frac{1}{2} hc'^2_x$
$(x \mid x_o x'_o)$		$+ 2(2n - 1 - \beta)h^3 c_x s_x$	$+ h'(c_x s'_x + c'_x s_x)$	$+ hc'_x s'_x$
$(x \mid \delta x_o)$	$(2 - n)h^2 c_x$	$+ 2(2n - 1 - \beta)h^3 c_x d$	$+ h'(c_x d' + c'_x d)$	$+ hc'_x d'$
$(x \mid x'^2_o)$		$+ (2n - 1 - \beta)h^3 s_x^2$	$+ h's_x s'_x$	$+\frac{1}{2} hs'^2_x$
$(x \mid \delta x'_o)$	$(2 - n)h^2 s_x$	$+ 2(2n - 1 - \beta)h^3 s_x d$	$+ h'(s_x d' + s'_x d)$	$+ hs'_x d'$
$(x \mid \delta^2)$	$-h + (2 - n)h^2 d$	$+ (2n - 1 - \beta)h^3 d^2$	$+ h'dd'$	$+\frac{1}{2} hd'^2$
$(x \mid y_o^2)$		$+\frac{1}{2}(h'' - nh^3 + 2\beta h^3)c_y^2$	$+ h'c_y c'_y$	$-\frac{1}{2} hc'^2_y$
$(x \mid y_o y'_o)$		$+ (h'' - nh^3 + 2\beta h^3)c_y s_y$	$+ h'(c_y s'_y + c'_y s_y)$	$- hc'_y s'_y$
$(x \mid y'^2_o)$		$+\frac{1}{2}(h'' - nh^3 + 2\beta h^3)s_y^2$	$+ h's_y s'_y$	$-\frac{1}{2} hs'^2_y$

TABLE I - Continued

q	f
$(y \mid x_o y_o)$	$2(\beta - n)h^3 c_x c_y + h'(c_x c'_y - c'_x c_y) + hc'_x c'_y$
$(y \mid x_o y'_o)$	$2(\beta - n)h^3 c_x s_y + h'(c_x s'_y - c'_x s_y) + hc'_x s'_y$
$(y \mid x'_o y_o)$	$2(\beta - n)h^3 s_x c_y + h'(s_x c'_y - s'_x c_y) + hs'_x c'_y$
$(y \mid x'_o y'_o)$	$2(\beta - n)h^3 s_x s_y + h'(s_x s'_y - s'_x s_y) + hs'_x s'_y$
$(y \mid \delta y_o)$	$nh^2 c_y + 2(\beta - n)h^3 c_y d - h'(c_y d' - c'_y d) + hc'_y d'$
$(y \mid \delta y'_o)$	$nh^2 s_y + 2(\beta - n)h^3 s_y d - h'(s_y d' - s'_y d) + hs'_y d'$

TRANSFORMATION FROM CURVILINEAR COORDINATES TO A RECTANGULAR COORDINATE SYSTEM AND "TRANSPORT" NOTATION

All results so far have been expressed in terms of the general curvilinear coordinate system (x,y,t). It is useful to transform these results to the local rectangular coordinate system (x,y,z), shown in Fig. 4, to facilitate matching boundary conditions between the various components comprising a beam transport system. This is accomplished by introducing the coordinates θ and φ defined as the x and y slopes in the local rectangular system:

$$\theta = \frac{dx}{dz} = \frac{x'}{z'} = \frac{x'}{1 + hx}$$
$$\varphi = \frac{dy}{dz} = \frac{y'}{z'} = \frac{y'}{1 + hx} \qquad (45)$$

where, as before, prime means the derivative with respect to t.

Note that $\theta = \tan\theta$ and $\varphi = \tan\varphi$, are correct to second order, so that in the present discussion θ and φ may be considered as angles relative to the local z-axis.

Using these definitions and those of Eqs. (34) and (35), it is now possible to express the Taylor's expansions for x, θ, y and φ in terms of the rectangular coordinate system. For the sake of completeness and to clearly define the notation used, the complete Taylor's expansions for x, θ, y, and φ at the end of a system as a function of the initial variables are given below:

$$\begin{aligned} x = {} & \overbrace{(x|x_0)}^{c_x}x_0 + \overbrace{(x|\theta_0)}^{s_x}\theta_0 + \overbrace{(x|\delta)}^{d_x}\delta \\ & + (x|x_0^2)x_0^2 + (x|x_0\theta_0)x_0\theta_0 + (x|x_0\delta)x_0\delta \\ & + (x|\theta_0^2)\theta_0^2 + (x|\theta_0\delta)\theta_0\delta + (x|\delta^2)\delta^2 \\ & + (x|y_0^2)y_0^2 + (x|y_0\varphi_0)y_0\varphi_0 + (x|\varphi_0^2)\varphi_0^2 \end{aligned}$$

$$\theta = \overbrace{(\theta|x_o)}^{c'_x}x_o + \overbrace{(\theta|\theta_o)}^{s'_x}\theta_o + \overbrace{(\theta|\delta)}^{d'_x}\delta$$
$$+ (\theta|x_o^2)x_o^2 + (\theta|x_o\theta_o)x_o\theta_o + (\theta|x_o\delta)x_o\delta$$
$$+ (\theta|\theta_o^2)\theta_o^2 + (\theta|\theta_o\delta)\theta_o\delta + (\theta|\delta^2)\delta^2$$
$$+ (\theta|y_o^2)y_o^2 + (\theta|y_o\varphi_o)y_o\varphi_o + (\theta|\varphi_o^2)\varphi_o^2$$

$$y = \overbrace{(y|y_o)}^{c_y}y_o + \overbrace{(y|\varphi_o)}^{s_y}\varphi_o$$
$$+ (y|x_oy_o)x_oy_o + (y|x_o\varphi_o)x_o\varphi_o + (y|\theta_oy_o)\theta_oy_o$$
$$+ (y|\theta_o\varphi_o)\theta_o\varphi_o + (y|y_o\delta)y_o\delta + (y|\varphi_o\delta)\varphi_o\delta$$

$$\varphi = \overbrace{(\varphi|y_o)}^{c'_y}y_o + \overbrace{(\varphi|\varphi_o)}^{s'_y}\varphi_o$$
$$+ (\varphi|x_oy_o)x_oy_o + (\varphi|x_o\varphi_o)x_o\varphi_o + (\varphi|\theta_oy_o)\theta_oy_o$$
$$+ (\varphi|\theta_o\varphi_o)\theta_o\varphi_o + (\varphi|y_o\delta)y_o\delta + (\varphi|\varphi_o\delta)\varphi_o\delta \qquad (46)$$

Using the definitions of Eq. (45), the coefficients appearing in Eq. (46) may be easily related to those appearing in Eq. (35). At the same time, we will introduce the abbreviated notation used in the Stanford TRANSPORT Program[3] where the subscript 1 means x; 2 means θ, 3 means y; 4 means Φ, and 6 means δ. The subscript 5 is the path length difference ℓ between an arbitrary ray and the central trajectory. The symbol R_{ij} will be used to signify a first-order matrix element and the symbol T_{ijk} will signify a second-order matrix element. Thus, we may write Eq. (46) in the general form

$$x_i = \sum_{j=1}^{6} R_{ij} x_j(0) + \sum_{j=1}^{6} \sum_{k=j}^{6} T_{ijk}\, x_j(0) x_k(0) \tag{47}$$

where

$$x_1 = x\ ,\quad x_2 = \theta\ ,\quad x_3 = y\ ,\quad x_4 = \varphi\ ,\quad x_5 = \ell \ \text{ and } \ x_6 = \delta$$

denotes the subscript notation.

Using Eq. (45) defining θ and φ, we find the following identities among the various matrix element definitions:

For the Taylor's expansions for x we have;

$$R_{11} = (x|x_o) = c_x$$
$$R_{12} = (x|\theta_o) = (x|x'_o) = s_x$$
$$R_{16} = (x|\delta) = d_x$$

$$T_{111} = (x|x_o^2)$$
$$T_{112} = (x|x_o\theta_o) = (x|x_o x'_o) + h(0)s_x$$
$$T_{116} = (x|x_o\delta)$$
$$T_{122} = (x|\theta_o^2) = (x|x'^2_o)$$
$$T_{126} = (x|\theta_o\delta) = (x|x'_o\delta)$$
$$T_{166} = (x|\delta^2)$$
$$T_{133} = (x|y_o^2)$$
$$T_{134} = (x|y_o\varphi_o) = (x|y_o y'_o)$$
$$T_{144} = (x|\varphi_o^2) = (x|y'^2_o) \tag{48}$$

For the θ terms we have:

$$R_{21} = (\theta|x_o) = (x'|x_o) = \frac{d}{dt}(x|x_o) = c'_x$$

$$R_{22} = (\theta|\theta_o) = (x'|x'_o) = s'_x$$

$$R_{26} = (\theta|\delta) = (x'|\delta) = d'_x$$

$$T_{211} = (\theta|x_o^2) = (x'|x_o^2) - h(t)c_x c_x'$$

$$T_{212} = (\theta|x_o\theta_o) = (x'|x_o x_o') + h(0)s_x' - h(t)[c_x s_x' + c_x' s_x]$$

$$T_{216} = (\theta|x_o\delta) = (x'|x_o\delta) - h(t)[c_x d_x' + c_x' d_x]$$

$$T_{222} = (\theta|\theta_o^2) = (x'|x_o'^2) - h(t)s_x s_x'$$

$$T_{226} = (\theta|\theta_o\delta) = (x'|x_o'\delta) - h(t)[s_x d_x' + s_x' d_x]$$

$$T_{266} = (\theta|\delta^2) = (x'|\delta^2) - h(t)d_x d_x'$$

$$T_{233} = (\theta|y_o^2) = (x'|y_o^2)$$

$$T_{234} = (\theta|y_o\varphi_o) = (x'|y_o y_o')$$

$$T_{244} = (\theta|\varphi_o^2) = (x'|y_o'^2) \qquad (49)$$

For the y terms in the Taylor's expansion:

$$R_{33} = (y|y_o) = c_y$$

$$R_{34} = (y|\varphi_o) = (y|y_o') = s_y$$

$$T_{313} = (y|x_o y_o)$$

$$T_{314} = (y|x_o\varphi_o) = (y|x_o y_o') + h(0)s_y$$

$$T_{323} = (y|\theta_o y_o) = (y|x_o' y_o)$$

$$T_{324} = (y|\theta_o\varphi_o) = (y|x_o' y_o')$$

$$T_{336} = (y|y_o\delta)$$

$$T_{346} = (y|\varphi_o\delta) = (y|y_o'\delta)$$

$$(50)$$

and finally for the φ terms we have:

$$R_{43} = (\varphi|y_o) = (y'|y_o) = \frac{d}{dt}\ (y|y_o) = c_x'$$

$$R_{44} = (\varphi|\varphi_o) = (y'|y_o') = s_y'$$

$$T_{413} = (\varphi|x_0 y_0) = (y'|x_0 y_0) - h(t)c_x c'_y$$
$$T_{414} = (\varphi|x_0 \varphi_0) = (y'|x_0 y'_0) + h(0)s'_y - h(t)c_x s'_y$$
$$T_{423} = (\varphi|\theta_0 y_0) = (y'|x'_0 y_0) - h(t)s_x c'_y$$
$$T_{424} = (\varphi|\theta_0 \varphi_0) = (y'|x'_0 y'_0) - h(t)s_x s'_y$$
$$T_{436} = (\varphi|y_0 \delta) = (y'|y_0 \delta) - h(t)c'_y d_x$$
$$T_{446} = (\varphi|\varphi_0 \delta) = (y'|y'_0 \delta) - h(t)s'_y d_x \qquad (51)$$

All of the above terms are understood to be evaluated at the terminal point of the system except for the quantity $h(0)$ which is to be evaluated at the beginning of the system. In practice, $h(0)$ will usually be equal to $h(t)$; but to retain generality in the formalism we show them as being different here.

All non-listed matrix elements are identically equal to zero.

FIRST- AND SECOND-ORDER MATRIX FORMALISM OF BEAM TRANSPORT OPTICS

The solution of first-order beam transport problems using matrix algebra has been extensively documented.[4,5,6] However, it does not seem to be generally known that matrix methods may be used to solve second- and higher-order beam transport problems. A general proof of the validity of extending matrix algebra to include second-order terms has been given by Brown, Belbeoch, and Bounin;[7] the results of which are summarized below in the notation of this report and in "TRANSPORT" notation.

Consider again Eq. (47). From Ref. 3, the matrix formalism may be logically extended to include second-order terms by extending the definition of the column matrices x_i and x_j in the first-order matrix algebra to include the second-order terms as shown in Tables II, III, IV and V.

In addition it is necessary to calculate and include the coefficients shown in the lower right-hand portion of the square matrix such that the set of simultaneous equations represented by Tables II through V are valid. Note that the second-order equations, represented by the lower right-hand portion of the matrix, are derived in a straightforward manner from the first-order equations, represented by the upper left-hand portion of the matrix. For example, consider the matrix in Table II; we see from row 1 that

$$x = c_x x_o + s_x \theta_o + d_x \delta + \text{second-order terms.}$$

Hence, row 4 is derived directly by squaring the above equation as follows:

$$x^2 = \left(c_x x_o + s_x \theta_o + d_x \delta\right)^2$$

$$= c_x^2 x_o^2 + 2c_x s_x x_o \theta_o + 2c_x d_x x_o \delta$$

$$+ s_x^2 \theta_o^2 + 2s_x d_x \theta_o \delta + d_x^2 \delta^2$$

The remaining rows are derived in a similar manner.

If now $x_1 = R_1 x_o$ represents the complete first- and second-order transformation from 0 to 1 in a beam transport system and $x_2 = R_2 x_1$ is the transformation from 1 to 2, then the first- and second-order transformation from 0 to 2 is simply $x_2 = R_2 x_1 = R_2 R_1 x_o$; where R_1 and R_2 are matrices fabricated as shown in Tables II and III in our notation or as shown in Tables IV and V in "TRANSPORT" notation.

TABLE II

x		c_x	s_x	d_x	$(x\|x_0^2)$	$(x\|x_0\theta_0)$	$(x\|x_0\delta)$	$(x\|\theta_0^2)$	$(x\|\theta_0\delta)$	$(x\|\delta^2)$	$(x\|y_0^2)$	$(x\|y_0\varphi_0)$	$(x\|\phi_0^2)$	x_0
θ		c'_x	s'_x	d'_x	$(\theta\|x_0^2)$	$(\theta\|x_0\theta_0)$	etc.							θ_0
δ		0	0	1	0	0	0	0	0	0	0	0	0	δ
x^2					c_x^2	$2s_xc_x$	$2c_xd_x$	s_x^2	$2s_xd_x$	d_x^2	0	0	0	x_0^2
$x\theta$					$c_xc'_x$	$c_xs'_x + c'_xs_x$	etc.							$x_0\theta_0$
$x\delta$														$x_0\delta$
θ^2	=		0											θ_0^2
$\theta\delta$														$\theta_0\delta$
δ^2														δ^2
y^2														y_0^2
$y\varphi$														$y_0\varphi_0$
φ^2														φ_0^2

Formulation of the second-order matrix for the bend (x)-plane.

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TABLE III

Vector		Matrix								Vector
y		c_y	s_y	$(y\|x_o y_o)$	$(y\|x_o \varphi_o)$	$(y\|\theta_o y_o)$	$(y\|\theta_o \varphi_o)$	$(y\|y_o \delta)$	$(y\|\varphi_o \delta)$	y_o
φ		c'_y	s'_y	$(\varphi\|x_o y_o)$	$(\varphi\|x_o \varphi_o)$	$(\varphi\|\theta_o y_o)$	$(\varphi\|\theta_o \varphi_o)$	$(\varphi\|y_o \delta)$	$(\varphi\|\varphi_o \delta)$	φ_o
xy				$c_x c_y$	$c_x s_y$	$s_x c_y$	$s_x s_y$	$d_x c_y$	$d_x s_y$	$x_o y_o$
$x\varphi$				$c_x c'_y$	$c_x s'_y$	$s_x c'_y$	etc.			$x_o \varphi_o$
θy	=	0								$\theta_o y_o$
$\theta\varphi$										$\theta_o \varphi_o$
$y\delta$										$y_o \delta$
$\varphi\delta$										$\varphi_o \delta$

Formulation of the second-order matrix for the non-bend-(y) plane.

TABLE IV

x		R_{11}	R_{12}	R_{16}	T_{111}	T_{112}	T_{116}	T_{122}	T_{126}	T_{166}	T_{133}	T_{134}	T_{144}		x_0
θ		R_{21}	R_{22}	R_{26}	T_{211}	T_{212}	T_{216}	T_{222}	T_{226}	T_{266}	T_{233}	T_{234}	T_{244}		θ_0
δ		0	0	1	0	0	0	0	0	0	0	0	0		δ
x^2					R_{11}^2	$2R_{11}R_{12}$	$2R_{11}R_{16}$	R_{12}^2	$2R_{12}R_{16}$	R_{16}^2	0	0	0		x_0^2
$x\theta$					$R_{11}R_{21}$	$R_{11}R_{22}$ $+R_{12}R_{21}$	$R_{11}R_{26}$ $+R_{16}R_{21}$	etc.			0	0	0		$x_0\theta_0$
$x\delta$															$x_0\delta$
θ^2	=		0												θ_0^2
$\theta\delta$															$\theta_0\delta$
δ^2															δ^2
y^2															y_0^2
$y\varphi$															$y_0\varphi_0$
φ^2															φ_0^2

Formulation of second-order matrix in the bend (x)-plane using TRANSPORT notation.

TABLE V

Output										Input
y	=	R_{33}	R_{34}	T_{313}	T_{314}	T_{323}	T_{324}	T_{336}	T_{346}	y_o
φ		R_{43}	R_{44}	T_{413}	T_{414}	T_{423}	T_{424}	T_{436}	T_{446}	φ_o
xy		0		$R_{11}R_{33}$	$R_{11}R_{34}$	$R_{12}R_{33}$	$R_{12}R_{34}$	$R_{16}R_{33}$	$R_{16}R_{34}$	$x_o y_o$
$x\varphi$				$R_{11}R_{43}$	$R_{11}R_{44}$	$R_{12}R_{43}$	etc.			$x_o \varphi_o$
θy										$\theta_o y_o$
$\theta\varphi$										$\theta_o \varphi_o$
$y\delta$										$y_o \delta$
$\varphi\delta$										$\varphi_o \delta$

Formulation of second-order matrix in non-bend (y)-plane using TRANSPORT notation.

III. REDUCTION OF THE GENERAL FIRST- AND SECOND-ORDER THEORY TO THE CASE OF THE IDEAL MAGNET

Part II of this report was devoted to the derivation of the general second-order differential equations of motion of charged particles in a static magnetic field. In Part II no restrictions were placed on the variation of the field along the central orbit, i.e., h,n, and β were assumed to be functions of t. As such, the final results were left in either a differential equation form or expressed in terms of an integral containing the driving function $f(t)$, and a Green's function $G(t,\tau)$ derived from the first-order solutions of the homogeneous equations. We now limit the generality of the problem by assuming h, n, and β to be constants over the interval of integration. With this restriction, the solutions to the homogeneous differential equation [Eq.(36)of Section II] are the following simple trigonometric functions:

$$c_x(t) = \cos k_x t \qquad s_x(t) = \frac{1}{k_x} \sin k_x t$$

$$c_y(t) = \cos k_y t \qquad s_y(t) = \frac{1}{k_y} \sin k_y t \qquad (52)$$

where now

$$k_x^2 = (1-n)h^2 \,, \quad k_y^2 = nh^2 \quad \text{and} \quad h = \frac{1}{\rho_o}$$

become constants of the motion. ρ_o is the radius of curvature of the central trajectory.

The solution of the inhomogeneous differential equations [the third of Eqs. (36)] for the remaining matrix elements is solved as indicated in Part II, using the Green's functions integral Eq. (41) and the driving functions listed in Table I. With the restrictions that k_x and k_y are constants, the Green's functions reduce to the following simple trigonometric forms:

$$G_x(t,\tau) = \frac{1}{k_x} \sin k_x(t-\tau)$$

and

$$G_y(t,\tau) = \frac{1}{k_y} \sin k_y(t-\tau) \tag{53}$$

The resulting matrix elements are tabulated below in terms of the key integrals listed in Table VI, the five characteristic first-order matrix elements s_x, c_x, d_x, c_y, and s_y and the constants h, n, and β.

The constants n and β are defined by the midplane field expansion [Eq.(18) of section II]:

$$B_y(x,o,t) = B_y(o,o,t) \left[1 - nhx + \beta h^2 x^2 + \gamma h^3 x^3 + \ldots\right] \tag{18}$$

or from Eq.(19) of section II:

$$n = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=o\\y=o}} \quad \text{and} \quad \beta = \left[\frac{1}{2!h^2 B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=o\\y=o}} \tag{19}$$

TABULATION OF THE FIRST- AND SECOND-ORDER MATRIX ELEMENTS FOR AN IDEAL MAGNET IN TERMS OF THE KEY INTEGRALS LISTED IN TABLE VI

$$R_{11} = (x|x_o) = c_x(t) = \cos k_x t$$

$$R_{12} = (x|\theta_o) = s_x(t) = \frac{1}{k_x}\sin k_x t$$

$$R_{16} = (x|\delta) = d_x(t) = \frac{h}{k_x^2}\left[1-c_x(t)\right]$$

Definitions:

$$k_x^2 = (1-n)h^2 \qquad h = \frac{1}{\rho_o}$$

$$k_y^2 = nh^2$$

$$
\begin{aligned}
T_{111} &= (x|x_o^2) &&= (2n-1-\beta)h^3 I_{111} + \tfrac{1}{2} k_x^4 h\, I_{122}\\
T_{112} &= (x|x_o\theta_o) &&= h\, s_x(t) + 2(2n-1-\beta)h^3 I_{112} - k_x^2 h\, I_{112}\\
T_{116} &= (x|x_o\delta) &&= (2-n)h^2 I_{11} + 2(2n-1-\beta)h^3 I_{116} - k_x^2 h^2 I_{122}\\
T_{122} &= (x|\theta_o^2) &&= (2n-1-\beta)h^3 I_{122} + \tfrac{1}{2} h\, I_{111}\\
T_{126} &= (x|\theta_o\delta) &&= (2-n)h^2 I_{12} + 2(2n-1-\beta)h^3 I_{126} + h^2 I_{112}\\
T_{166} &= (x|\delta^2) &&= -h\, I_{10} + (2-n)h^2 I_{16} + (2n-1-\beta)h^3 I_{166} + \tfrac{1}{2} h^3 I_{122}\\
T_{133} &= (x|y_o^2) &&= \beta h^3 I_{133} - \tfrac{1}{2} k_y^2 h\, I_{10}\\
T_{134} &= (x|y_o\varphi_o) &&= 2\beta h^3 I_{134}\\
T_{144} &= (x|\varphi_o^2) &&= \beta h^3 I_{144} - \tfrac{1}{2} h\, I_{10}
\end{aligned}
$$

$$R_{21} = (\theta|x_o) = c_x'(t) = -k_x^2\, s_x(t)$$

$$R_{22} = (\theta|\theta_o) = s_x'(t) = c_x(t)$$

$$R_{26} = (\theta|\delta) = d_x'(t) = h\, s_x(t)$$

$$T_{211} = (\theta|x_o^2) = (2n-1-\beta)h^3\, I_{211} + \frac{1}{2} k_x^4\, h\, I_{222} - h\, c_x(t) c_x'(t)$$

$$T_{212} = (\theta|x_o\theta_o) = h\, s_x'(t) \quad 2(2n-1-\beta)h^3\, I_{212} - k_x^2\, h\, I_{212} - h\left[c_x(t)s_x'(t) + c_x'(t)s_x(t)\right]$$

$$T_{216} = (\theta|x_o\delta) = (2-n)h^2\, I_{21} + 2(2n-1-\beta)h^3\, I_{216} - k_x^2\, h^2\, I_{222} - h\left[c_x(t)d_x'(t) + c_x'(t)d_x(t)\right]$$

$$T_{222} = (\theta|\theta_o^2) = (2n-1-\beta)h^3\, I_{222} + \frac{1}{2}\, h\, I_{211} - h\, s_x(t)s_x'(t)$$

$$T_{226} = (\theta|\theta_o\delta) = (2-n)h^2\, I_{22} + 2(2n-1-\beta)h^3\, I_{226} + h^2\, I_{212} - h\left[s_x(t)d_x'(t) + s_x'(t)d_x(t)\right]$$

$$T_{266} = (\theta|\delta^2) = -h\, I_{20} + (2-n)h^2\, I_{26} + (2n-1-\beta)h^3\, I_{266} + \frac{1}{2}\, h^3\, I_{222} - h\, d_x(t)d_x'(t)$$

$$T_{233} = (\theta|y_o^2) = \beta\, h^3\, I_{233} - \frac{1}{2} k_y^2\, h\, I_{20}$$

$$T_{234} = (\theta|y_o\varphi_o) = 2\beta\, h^3\, I_{234}$$

$$T_{244} = (\theta|\varphi_o^2) = \beta\, h^3\, I_{244} - \frac{1}{2}\, h\, I_{20}$$

$$R_{33} = (y|y_o) = c_y(t) = \cos k_y t$$

$$R_{34} = (y|\varphi_o) = s_y(t) = \frac{1}{k_y} \sin k_y t$$

$$\begin{aligned}
T_{313} &= (y|x_o y_o) = & &+ 2(\beta-n)h^3\, I_{313} + k_x^2\, k_y^2\, h\; I_{324} \\
T_{314} &= (y|x_o \varphi_o) = h\, s_y(t) & &+ 2(\beta-n)h^3\, I_{314} - k_x^2\; h\; I_{323} \\
T_{323} &= (y|\theta_o y_o) = & &+ 2(\beta-n)h^3\, I_{323} - k_y^2\, h\; I_{314} \\
T_{324} &= (y|\theta_o \varphi_o) = & &+ 2(\beta-n)h^3\, I_{324} + h\; I_{313} \\
T_{336} &= (y|y_o \delta) = k_y^2\, I_{33} & &+ 2(\beta-n)h^3\, I_{336} - k_y^2\, h^2\, I_{324} \\
T_{346} &= (y|\varphi_o \delta) = k_y^2\, I_{34} & &+ 2(\beta-n)h^3\, I_{346} + h^2\, I_{323}
\end{aligned}$$

$$R_{43} = (\varphi|y_o) = c_y'(t) = -k_y^2\, s_y(t)$$

$$R_{44} = (\varphi|\varphi_o) = s_y'(t) = c_y(t)$$

$$T_{413} = (\varphi|x_o y_o) = 2(\beta-n)h^3\, I_{413} + k_x^2\, k_y^2\, h\, I_{424} - h\, c_x(t) c_y'(t)$$

$$T_{414} = (\varphi|x_o \varphi_o) = h\, s_y'(t) + 2(\beta-n)h^3\, I_{414} - k_x^2\, h\, I_{423} - h\, c_x(t) s_y'(t)$$

$$T_{423} = (\varphi|\theta_o y_o) = 2(\beta-n)h^3\, I_{423} - k_y^2\, h\, I_{414} - h\, s_x(t) c_y'(t)$$

$$T_{424} = (\varphi|\theta_o \varphi_o) = 2(\beta-n)h^3\, I_{424} + h\, I_{413} - h\, s_x(t) s_y'(t)$$

$$T_{436} = (\varphi|y_o \delta) = k_y^2\, I_{43} + 2(\beta-n)h^3\, I_{436} - k_y^2\, h^2\, I_{424} - h\, d_x(t) c_y'(t)$$

$$T_{446} = (\varphi|\varphi_o \delta) = k_y^2\, I_{44} + 2(\beta-n)h^3\, I_{446} + h^2\, I_{423} - h\, d_x(t) s_y'(t) \qquad (54)$$

TABLE VI

Tabulation of Key Integrals Required for the Numerical Evaluation of the Second-Order Aberrations of Ideal Magnets

The results are expressed in terms of the five characteristic first order matrix elements $s_x(t)$, $c_x(t)$, $d_x(t)$, $c_y(t)$ and $s_y(t)$ and the quantities h and n (assumed to be constant for the ideal magnet over the interval of integration $\tau = o$ to $\tau = t$). The path length of the central trajectory is t. From the solutions of the differential equations (Eq. 29 of Section II), the first order matrix elements for the ideal magnet are:

$$c_x(t) = \cos k_x t \qquad s_x(t) = \frac{1}{k_x} \sin k_x t \qquad d_x(t) = \frac{h}{k_x^2}\left(1 - c_x(t)\right)$$

$$c_y(t) = \cos k_y t \qquad s_y(t) = \frac{1}{k_y} \sin k_y t$$

where $\quad k_x^2 = (1-n)\, h^2 \qquad k_y^2 = nh^2$ and $h = \frac{1}{\rho_o}$

ρ_o is the radius of curvature of the central trajectory.

$$I_{10} = \int_0^t G_x(t,\tau)\,d\tau = \left(\frac{d_x(t)}{h}\right)$$

$$I_{11} = \int_0^t c_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{2}\,t s_x(t)$$

$$I_{12} = \int_0^t s_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{2k_x^2}\left(s_x(t) - tc_x(t)\right)$$

$$I_{16} = \int_0^t d_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left(I_{10} - I_{11}\right) = \frac{h}{k_x^2}\left(\frac{d_x(t)}{h} - \frac{t}{2}\,s_x(t)\right)$$

$$I_{111} = \int_0^t c_x^2(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{3}\left(s_x^2(t) + \frac{d_x(t)}{h}\right)$$

$$I_{112} = \int_0^t c_x(\tau)\,s_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{3}\,s_x(t)\left(\frac{d_x(t)}{h}\right)$$

$$I_{116} = \int_0^t c_x(\tau)\,d_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left(I_{11} - I_{111}\right) = \frac{h}{k_x^2}\left[\frac{t}{2}\,s_x(t) - \frac{1}{3}\left(s_x^2(t) + \frac{d_x(t)}{h}\right)\right]$$

$$I_{122} = \int_0^t s_x^2(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{k_x^2}\left(I_{10} - I_{111}\right) = \frac{1}{3k_x^2}\left(2\,\frac{d_x(t)}{h} - s_x^2(t)\right)$$

$$I_{126} = \int_0^t s_x(\tau)\,d_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left(I_{12} - I_{112}\right) = \frac{h}{k_x^2}\left[\frac{1}{2k_x^2}\left(s_x(t) - tc_x(t)\right) - \frac{1}{3}\,s_x(t)\left(\frac{d_x(t)}{h}\right)\right] = \frac{h}{6k_x^4}\left[s_x(t) + 2s_x(t)\,c_x(t) - 3tc_x(t)\right]$$

$$I_{166} = \int_0^t d_x^2(\tau) G_x(t,\tau)\, d\tau = \frac{h^2}{k_x^4} \left(I_{10} - 2I_{11} + I_{111} \right) = \frac{h^2}{k_x^4} \left[\frac{4}{3} \left(\frac{d_x(t)}{h} \right) + \frac{1}{3}\, s_x^2(t) - t s_x(t) \right]$$

$$I_{133} = \int_0^t c_y^2(\tau)\, G_x(t,\tau)\, d\tau = \left(\frac{d_x(t)}{h} \right) - \left(\frac{k_y^2}{k_x^2 - 4k_y^2} \right) \left(s_y^2\,(t) - 2\, \frac{d_x(t)}{h} \right) = I_{10} - k_y^2 I_{144} = I_{10} - k_y^2 I_{144}$$

$$I_{134} = \int_0^t c_y(\tau)\, s_y(\tau) G_x(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2} \left(s_y(t) c_y(t) - s_x(t) \right)$$

$$I_{144} = \int_0^t s_y^2\,(\tau) G_x(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2} \left(s_y^2(t) - 2\, \frac{d_x(t)}{h} \right)$$

$$I_{20} = I'_{10} = \frac{d}{dt} \int_0^t G_x(t,\tau)\, d\tau = s_x(t)$$

$$I_{21} = I'_{11} = \frac{d}{dt} \int_0^t c_x(\tau) G_x(t,\tau)\, d\tau = \frac{1}{2} \left(s_x(t) + t c_x(t) \right)$$

$$I_{22} = I'_{12} = \frac{d}{dt} \int_0^t s_x(\tau) G_x(t,\tau)\, d\tau = \frac{1}{2}\, t s_x(t) = I_{11}$$

$$I_{26} = I'_{16} = \frac{d}{dt} \int_0^t d_x(\tau) G_x(t,\tau)\, d\tau = \frac{h}{2k_x^2} \left(s_x(t) - t c_x(t) \right)$$

$$I_{211} = I'_{111} = \frac{d}{dt}\int_0^t c_x^2(\tau)\,G_x(t,\tau)\,d\tau = \frac{s_x(t)}{3}\left(1 + 2c_x(t)\right)$$

$$I_{212} = I'_{112} = \frac{d}{dt}\int_0^t c_x(\tau)\,s_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{3}\left[2s_x^2(t) - \frac{d_x(t)}{h}\right]$$

$$I_{216} = I'_{116} = \frac{d}{dt}\int_0^t c_x(\tau)\,d_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left[\frac{tc_x(t)}{2} + \frac{s_x(t)}{6} - \frac{2s_x(t)c_x(t)}{3}\right]$$

$$I_{222} = I'_{122} = \frac{d}{dt}\int_0^t s_x^2(\tau)\,G_x(t,\tau)\,d\tau = \frac{2}{3}\,s_x(t)\left(\frac{d_x(t)}{h}\right)$$

$$I_{226} = I'_{126} = \frac{d}{dt}\int_0^t s_x(\tau)\,d_x(\tau)\,G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left[\frac{1}{2}\,ts_x(t) - \frac{2}{3}\,s_x^2(t) + \frac{1}{3}\left(\frac{d_x(t)}{h}\right)\right]$$

$$I_{266} = I'_{166} = \frac{d}{dt}\int_0^t d_x^2(\tau)\,G_x(t,\tau)\,d\tau = \frac{h^2}{k_x^4}\left[\frac{1}{3}\,s_x(t) + \frac{2}{3}\,s_x(t)\,c_x(t) - tc_x(t)\right]$$

$$I_{233} = I'_{133} = \frac{d}{dt}\int_0^t c_y^2(\tau)\,G_x(t,\tau)\,d\tau = s_x(t) - \frac{2k_y^2}{k_x^2 - 4k_y^2}\left(s_y(t)c_y(t) - s_x(t)\right)$$

$$I_{234} = I'_{134} = \frac{d}{dt}\int_0^t c_y(\tau)\,s_y(\tau)\,G_x(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[2c_y^2(t) - 1 - c_x(t)\right] = \frac{1}{k_x^2 - 4k_y^2}\left[k_x^2\left(\frac{d_x(t)}{h}\right) - 2k_y^2\,s_y^2(t)\right]$$

$$I_{244} = I'_{144} = \int_0^t s_y^2(\tau) G_x(t,\tau) d\tau = \frac{2}{k_x^2 - 4k_y^2} \left[s_y(t) c_y(t) - s_x(t) \right]$$

$$I_{30} = \int_0^t G_y(t,\tau) d\tau = \frac{1 - c_y(t)}{k_y^2}$$

$$I_{33} = \int_0^t c_y(\tau) G_y(t,\tau) d\tau = \frac{1}{2} t s_y(t)$$

$$I_{34} = \int_0^t s_y(\tau) G_y(t,\tau) d\tau = \frac{1}{2k_y^2} \left(s_y(t) - t c_y(t) \right)$$

$$I_{313} = \int_0^t c_x(\tau) c_y(\tau) G_y(t,\tau) d\tau = \frac{1}{k_x^2 - 4k_y^2} \left[c_y(t) \left(1 - c_x(t) \right) - 2k_y^2 s_x(t) s_y(t) \right] = \frac{1}{k_x^2 - 4k_y^2} \left[k_x^2 c_y(t) \left(\frac{d_x(t)}{h} \right) - 2k_y^2 s_x(t) s_y(t) \right]$$

$$I_{314} = \int_0^t c_x(\tau) s_y(\tau) G_y(t,\tau) d\tau = \frac{1}{k_x^2 - 4k_y^2} \left[2 s_x(t) c_y(t) - s_y(t) \left(1 + c_x(t) \right) \right]$$

$$I_{323} = \int_0^t s_x(\tau) c_y(\tau) G_y(t,\tau) d\tau = \frac{1}{k_x^2 - 4k_y^2} \left[2 \left(\frac{k_y^2}{k_x^2} \right) s_y(t) \left(1 + c_x(t) \right) - s_x(t) c_y(t) \right] + \frac{s_y(t)}{k_x^2}$$

$$I_{324} = \int_0^t s_x(\tau) s_y(\tau) G_y(t,\tau) d\tau = \frac{1}{k_x^2 - 4k_y^2} \left[\frac{2 c_y(t) \left(1 - c_x(t) \right)}{k_x^2} - s_x(t) s_y(t) \right] = \frac{1}{k_x^2 - 4k_y^2} \left[2 c_y(t) \left(\frac{d_x(t)}{h} \right) - s_x(t) s_y(t) \right]$$

$$I_{336} = \int_0^t c_y(\tau)\, d_x(\tau)\, G_y(t,\tau)\, d\tau = \frac{h}{k_x^2}\left(I_{33} - I_{313}\right) = \frac{h}{k_x^2}\left[\frac{t}{2}\, s_y(t) - \frac{1}{k_x^2 - 4k_y^2}\left(c_y(t)\left(1 - c_x(t)\right) - 2k_y^2 s_x(t)\, s_y(t)\right)\right]$$

$$I_{346} = \int_0^t s_y(\tau)\, d_x(\tau)\, G_y(t,\tau)\, d\tau = \frac{h}{k_x^2}\left(I_{34} - I_{314}\right) = \frac{h}{k_x^2}\left[\frac{1}{2k_y^2}\left(s_y(t) - t c_y(t)\right) - \frac{1}{k_x^2 - 4k_y^2}\left[2s_x(t)\, c_y(t) - s_y(t)\left(1 + c_x(t)\right)\right]\right]$$

$$I_{40} = I'_{30} = \frac{d}{dt}\int_0^t G_y(t,\tau)\, d\tau = s_y(t)$$

$$I_{43} = I'_{33} = \frac{d}{dt}\int_0^t c_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{2}\left[s_y(t) + t c_y(t)\right]$$

$$I_{44} = I'_{34} = \frac{d}{dt}\int_0^t s_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{2}\, t s_y(t) = I_{33}$$

$$I_{413} = I'_{313} = \frac{d}{dt}\int_0^t c_x(\tau)\, c_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[\left(k_x^2 - 2k_y^2\right) s_x(t)\, c_y(t) - k_y^2\, s_y(t)\left(1 + c_x(t)\right)\right]$$

$$I_{414} = I'_{314} = \frac{d}{dt}\int_0^t c_x(\tau)\, s_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[\left(k_x^2 - 2k_y^2\right) s_x(t)\, s_y(t) - c_y(t)\left(1 - c_x(t)\right)\right]$$

$$I_{423} = I'_{323} = \frac{d}{dt}\int_0^t s_x(\tau)\, c_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[2\left(\frac{k_y^2}{k_x^2}\right) c_y(t)\left(1 + c_x(t)\right) - c_x(t)\, c_y(t) - k_y^2\, s_x(t)\, s_y(t)\right] + \frac{c_y(t)}{k_x^2}$$

$$I_{424} = I'_{324} = \frac{d}{dt}\int_0^t s_x(\tau)\, s_y(\tau)\, G_y(t,\tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[c_y(t)\, s_x(t) - c_x(t)\, s_y(t) - 2k_y^2\, s_y(t)\left(\frac{d_x(t)}{h}\right)\right]$$

$$I_{436} = I'_{336} = \frac{d}{dt}\int_0^t c_y(\tau)\, d_x(\tau)\, G_y(t,\tau)\, d\tau = \frac{h}{k_x^2}\left[\frac{t}{2}\, c_y(t) + \frac{s_y(t)}{2} + \frac{1}{k_x^2 - 4k_y^2}\left[k_y^2 s_y(t)\left(1 + c_x(t)\right) - (k_x^2 - 2k_y^2)\, s_x(t)\, c_y(t)\right]\right]$$

$$I_{446} = I'_{346} = \frac{d}{dt}\int_0^t s_y(\tau)\, d_x(\tau)\, G_y(t,\tau)\, d\tau = \frac{h}{k_x^2}\left[\frac{t s_y(t)}{2} - \frac{1}{k_x^2 - 4k_y^2}\left[\left(k_x^2 - 2k_y^2\right) s_x(t)\, s_y(t) - c_y(t)\left(1 - c_x(t)\right)\right]\right]$$

Matrix Elements for a Pure Quadrupole Field

For a pure quadrupole, the matrix elements are derived from those of the general case by letting $\beta = 0$, $k_x^2 = k_q^2$ and $k_y^2 = - k_q^2$, where

$$k_q^2 = - nh^2 = \left(\frac{B_o}{a}\right)\left(\frac{1}{B\rho}\right)$$

and then taking the limit $h \to 0$. The results are:

$$R_{11} = \cos k_q t$$

$$R_{12} = \frac{1}{k_q} \sin k_q t$$

$$T_{116} = \frac{1}{2} k_q t \sin k_q t$$

$$T_{126} = \frac{1}{2 k_q} \sin k_q t - \frac{t}{2} \cos k_q t$$

$$R_{21} = -k_q \sin k_q t$$

$$R_{22} = \cos k_q t$$

$$T_{216} = \frac{k_q}{2} \left[k_q t \cos k_q t + \sin k_q t\right]$$

$$T_{226} = \frac{1}{2} k_q t \sin k_q t$$

$$R_{33} = \cosh k_q t$$

$$R_{34} = \frac{1}{k_q} \sinh k_q t$$

$$T_{336} = - \frac{1}{2} k_q t \sinh k_q t$$

$$T_{346} = \frac{1}{2} \left[\frac{1}{k_q} \sinh k_q t - t \cosh k_q t\right]$$

$$R_{43} = k_q \sinh k_q t$$

$$R_{44} = \cosh k_q t$$

$$T_{436} = -\frac{k_q}{2}\left[k_q t \cosh k_q t + \sinh k_q t\right]$$

$$T_{446} = -\frac{1}{2} k_q t \sinh k_q t \qquad (55)$$

all non-listed matrix elements are identically zero.

Matrix Elements for a Pure Sextupole Field

For a pure sextupole, the matrix elements are derived from those of the general case by letting

$$\beta h^3 = k_s^2 = \left(\frac{B_o}{a^2}\right)\left(\frac{1}{B\rho}\right)$$

and then taking the limit $h \longrightarrow 0$. The results are:

$$R_{11} = 1$$

$$R_{12} = t$$

$$T_{111} = -\frac{1}{2} k_s^2 t^2$$

$$T_{112} = -\frac{1}{3} k_s^2 t^3$$

$$T_{122} = -\frac{1}{12} k_s^2 t^4$$

$$T_{133} = \frac{1}{2} k_s^2 t^2$$

$$T_{134} = \frac{1}{3} k_s^2 t^3$$

$$T_{144} = \frac{1}{12} k_s^2 t^4$$

$$R_{21} = 0$$

$$R_{22} = 1$$

$$T_{211} = - k_s^2 \, t$$

$$T_{212} = - k_s^2 \, t^2$$

$$T_{222} = - \frac{1}{3} k_s^2 \, t^3$$

$$T_{233} = k_s^2 \, t$$

$$T_{234} = k_s^2 \, t^2$$

$$T_{244} = \frac{1}{3} k_s^2 \, t^3$$

$$R_{33} = 1$$

$$R_{34} = t$$

$$T_{313} = k_s^2 \, t^2$$

$$T_{314} = \frac{1}{3} k_s^2 \, t^3$$

$$T_{323} = \frac{1}{3} k_s^2 \, t^3$$

$$T_{324} = \frac{1}{6} k_s^2 \, t^4$$

$$R_{43} = 0$$

$$R_{44} = 1$$

$$T_{413} = 2 \, k_s^2 \, t$$

$$T_{414} = k_s^2 \, t^2$$

$$T_{423} = k_s^2 \, t^2$$

$$T_{424} = \frac{2}{3} k_s^2 \, t^3 \qquad (56)$$

All non-listed matrix elements are identically zero.

First- and Second-Order Matrix Elements for a Curved, Inclined Magnetic Field Boundary. Fringing Field Effects.

Matrix elements for the fringing fields of bending magnets have been derived using an impulse approximation.[7,8] These computations combined with a correction term[9] to the R_{43} element (to correct for the finite extent of actual fringing fields) have produced results which are in substantial agreement with precise ray tracing calculations and with experimental measurements made on actual magnets.

We introduce four new variables (illustrated in Fig. 11); the angle of inclination β_1 of the entrance face of a bending magnet, the radius of curvature R_1 of the entrance face, the angle of inclination β_2 of the exit face, and the radius of curvature R_2 of the exit face. The sign convention of β_1 and β_2 is considered positive for positive focusing in the transverse (y) direction. The sign convention for R_1 and R_2 is positive if the field boundary is convex outward; (a positive R represents a negative sextupole component of strength $k_s^2 L = -\left(\frac{h}{2R}\right) \sec^3\beta$). The sign conventions adopted here are in agreement with Penner,[4] and Brown, Belbeoch, and Bounin.[7]

The results of these calculations yield the following matrix elements for the fringing fields of the entrance face of a bending magnet:

$$R_{11} = 1$$

$$R_{12} = 0$$

$$T_{111} = -\frac{h}{2} \tan^2\beta_1$$

$$T_{133} = \frac{h}{2} \sec^2\beta_1$$

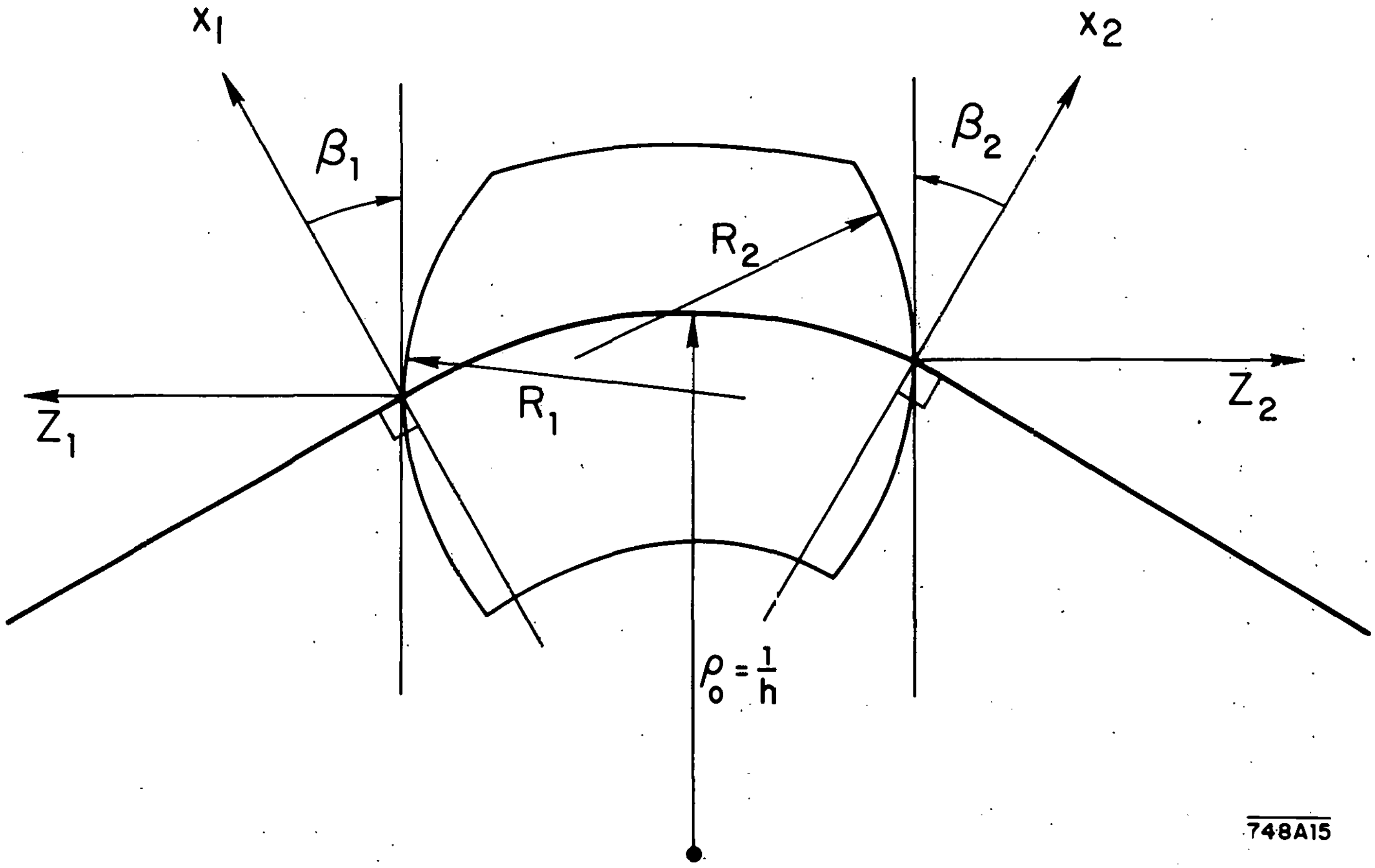


FIG. 11--FIELD BOUNDARIES FOR BENDING MAGNETS

The TRANSPORT sign conventions for x, β, R and h are all positive as shown in the figure. The positive y direction is out of the paper. Positive β's imply transverse focusing. Positive R's (convex curvatures) represent negative sextupole components of strength $S = \left(-\frac{h}{2R}\right)\sec^3\beta$. (See page 71.)

$$R_{21} = -\frac{1}{f_x} = h \tan \beta_1$$

$$R_{22} = 1$$

$$T_{211} = \frac{h}{2R_1} \sec^3 \beta_1 - nh^2 \tan \beta_1$$

$$T_{212} = h \tan^2 \beta_1$$

$$T_{216} = - h \tan \beta_1$$

$$T_{233} = h^2 \left(n + \frac{1}{2} + \tan^2 \beta_1\right) \tan \beta_1 - \frac{h}{2R_1} \sec^3 \beta_1$$

$$T_{234} = - h \tan^2 \beta_1$$

$$R_{33} = 1$$

$$R_{34} = 0$$

$$T_{313} = h \tan^2 \beta_1$$

$$R_{43} = -\frac{1}{f_y} = - h \tan (\beta_1 - \psi_1)$$

$$R_{44} = 1$$

$$T_{413} = -\frac{h}{R_1} \sec^3 \beta_1 + 2h^2 n \tan \beta_1$$

$$T_{414} = - h \tan^2 \beta_1$$

$$T_{423} = - h \sec^2 \beta_1$$

$$T_{436} = h \tan \beta_1 - h\psi_1 \sec^2(\beta_1 - \psi_1) \qquad (57)$$

All nonlisted matrix elements are equal to zero. The quantity ψ_1 is the correction to the transverse focal length when the finite extent of the fringing fields are included.[9]

$$\psi_1 = K_1 hg\ (\sec \beta_1)(1 + \sin^2 \beta_1) + \text{higher order terms in } (hg)$$

where g = the distance between the poles of the magnet at the central orbit (i.e., the magnet gap) and

$$K_1 = \int_{-\infty}^{+\infty} \frac{B_y(z)\left[B_o - B_y(z)\right]}{gB_o^2}\ dz$$

$B_y(z)$ is the magnitude of the fringing field on the magnetic mid-plane at a position z. z is the perpendicular distance measured from the entrance face of the magnet to the point in question. See Fig. 11. B_o is the asymptotic value of $B_y(z)$ well inside the magnet entrance. Typical values of K_1 for actual magnets may range from 0.3 to 1.0 depending upon the detailed shape of the magnet profile and the location of the energizing coils.

The matrix elements for the fringing fields of the exit face of a bending magnet are:

$R_{11} = 1$

$R_{12} = 0$

$T_{111} = \frac{h}{2} \tan^2 \beta_2$

$T_{133} = - \frac{h}{2} \sec^2 \beta_2$

$R_{21} = - \frac{1}{f_x} = h \tan \beta_2$

$R_{22} = 1$

$T_{211} = \frac{h}{2R_2} \sec^3 \beta_2 - h^2(n + \frac{1}{2} \tan^2 \beta_2) \tan \beta_2$

$T_{212} = - h \tan^2 \beta_2$

$$T_{216} = -h \tan \beta_2$$

$$T_{233} = h^2\left(n - \frac{1}{2} \tan^2 \beta_2\right) \tan \beta_2 - \frac{h}{2R_2} \sec^3 \beta_2$$

$$T_{234} = h \tan^2 \beta_2$$

$$R_{33} = 1$$

$$R_{34} = 0$$

$$T_{313} = -h \tan^2 \beta_2$$

$$R_{43} = -\frac{1}{f_y} = -h \tan(\beta_2 - \psi_2)$$

$$R_{44} = 1$$

$$T_{413} = -\frac{h}{R_2} \sec^3 \beta_2 + h^2(2n + \sec^2 \beta_2) \tan \beta_2$$

$$T_{414} = h \tan^2 \beta_2$$

$$T_{423} = h \sec^2 \beta_2$$

$$T_{436} = h \tan \beta_2 - h\psi_2 \sec^2 (\beta_2 - \psi_2) \tag{58}$$

All nonlisted matrix elements are zero.

$$\psi_2 = K_1 hg \sec \beta_2(1 + \sin^2 \beta_2) + \text{higher order terms in } (hg)$$

and K_1 is evaluated for the exit fringing field.

Matrix Elements for a Drift Distance

For a drift distance of length L, the matrix elements are simply as follows:

$$R_{11} = R_{22} = R_{33} = R_{44} = R_{55} = R_{66} = 1$$

$$R_{12} = R_{34} = L$$

All remaining first- and second-order matrix elements are zero.

IV. SOME USEFUL FIRST-ORDER OPTICAL RESULTS DERIVED FROM THE GENERAL THEORY OF SECTION II.[10,11]

We have shown in Section II, Eq. (47), that beam transport optics may be reduced to a process of matrix multiplication. To first-order, this is represented by the matrix equation

$$x_i(t) = \sum_{j=1}^{6} R_{ij} x_j(o) \qquad (59)$$

where

$$x_1 = x \quad x_2 = \theta \quad x_3 = y \quad x_4 = \varphi \quad x_5 = \ell \quad \text{and} \quad x_6 = \delta$$

We have also proved that the determinant $|R| = 1$ results from the basic equation of motion and is a manifestation of Liouville's theorem of conservation of phase space volume.

The six simultaneous linear equations represented by Eq.(59) may be expanded in matrix form as follows:

$$\begin{bmatrix} x(t) \\ \theta(t) \\ y(t) \\ \varphi(t) \\ \ell(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \\ y_o \\ \varphi_o \\ \ell_o \\ \delta_o \end{bmatrix} \qquad (60)$$

where the transformation is from an initial position $\tau = o$ to a final position $\tau = t$.

The zero elements $R_{13} = R_{14} = R_{23} = R_{24} = R_{31} = R_{32} = R_{41} = R_{42} = R_{36} = R_{46} = 0$ in the R matrix are a direct consequence of midplane symmetry. If midplane symmetry is destroyed, these elements will in general become non-zero. The zero elements in column five occur because the variables x, θ, y, φ, and δ are independent of the path length difference ℓ. The zero's in row six result from the fact that we have restricted the problem to static magnetic fields, i.e., the scalar momentum is a constant of the motion.

We have already attached a physical significance to the non-zero matrix elements in the first four rows in terms of their identification with characteristic first-order trajectories. We now wish to relate the elements appearing in column six with those in row five and calculate both sets in terms of simple integrals of the characteristic first-order elements $c_x(t) = R_{11}$ and $s_x(t) = R_{12}$. In order to do this, we make use of the Green's function integral, Eq. (43) of Section II, and of the expression for the differential path length in curvilinear coordinates

$$dT = \left[(dx)^2 + (dy)^2 + (1+hx)^2\,(dt)^2\right]^{1/2} \qquad (61)$$

used in the derivation of the equation of motion.

First-Order Dispersion

The spatial dispersion $d_x(t)$ of a system at position t is derived using the Green's function integral, Eq. (43), and the driving term $f = h(\tau)$ for the dispersion (see Table I). The result is

$$d_x(t) = R_{16} = s_x(t) \int_0^t c_x(\tau)\, h(\tau)d\tau - c_x(t) \int_0^t s_x(\tau)\, h(\tau)d\tau \qquad (62)$$

where τ is the variable of integration. Note that $h(\tau)d\tau = d\alpha$ is the differential angle of bend of the central trajectory at any point in the system. Thus first-order dispersion is generated only in regions where the central trajectory is deflected (i.e., in dipole elements.) The angular dispersion is obtained by direct differentiation of $d_x(t)$ with respect to t;

$$d_x'(t) = R_{26} = s_x'(t) \int_0^t c_x(\tau)\, h(\tau) d\tau - c_x'(t) \int_0^t s_x(\tau)\, h(\tau) d\tau \qquad (63)$$

where

$$c_x'(t) = R_{21} \text{ and } s_x'(t) = R_{22}$$

First-Order Path Length

The first-order path length difference is obtained by expanding Eq. (61) and retaining only the first-order term, i.e.,

$$\ell - \ell_o = (T - t) = \int_0^t x(\tau)\, h(\tau) d\tau + \text{higher order terms}$$

from which

$$\ell = x_o \int_0^t c_x(\tau)\, h(\tau) d\tau + \theta_o \int_0^t s_x(\tau)\, h(\tau) d\tau + \ell_o + \delta \int_0^t d_x(\tau)\, h(\tau) d\tau$$

$$= R_{51} x_o + R_{52} \theta_o + \ell_o + R_{56} \delta \qquad (64)$$

Inspection of Eqs.(62), (63), and (64) yields the following useful theorems:

Achromaticity: A system is defined as being achromatic if $d_x(t) = d_x'(t) = 0$. Therefore it follows from Eq's.(62) and (63) that the necessary and sufficient conditions for achromaticity are that

$$\int_0^t s_x(\tau)\, h(\tau) d\tau = \int_0^t c_x(\tau)\, h(\tau) d\tau = 0 \qquad (65)$$

By comparing Eq. (64) with Eq. (65), we note that if a system is achromatic, all particles of the same momentum will have equal (first-order) path lengths through the system.

Isochronicity: It is somewhat unfortunate that this word has been used in the literature to mean equal path lengths since equal path lengths only imply equal transit times for highly relativistic particles. Nevertheless, from Eq. (64), the necessary and sufficient conditions that the first-order path length of all particles (independent of their initial momenta) will be the same through a system are that

$$\int_0^t c_x(\tau)\, h(\tau)\, d\tau = \int_0^t s_x(\tau)\, h(\tau)\, d\tau = \int_0^t d_x(\tau)\, h(\tau)\, d\tau = 0 \qquad (66)$$

First-Order Imaging

First-order point-to-point imaging in the x plane occurs when x(t) is independent of the initial angle θ_0. This can only be so when

$$s_x(t) = R_{12} = 0. \qquad (67)$$

Similarly first-order point-to-point imaging occurs in the y plane when

$$s_y(t) = R_{34} = 0. \qquad (68)$$

First-order parallel-to-point imaging occurs in the x plane when x(t) is independent of the initial particle position x_0. This will occur only if

$$c_x(t) = R_{11} = 0 \qquad (69)$$

and correspondingly in the y plane, parallel-to-point imaging occurs when

$$c_y(t) = R_{33} = 0. \qquad (70)$$

Magnification

For point-to-point imaging in the x-plane, the magnification is given by

$$M_x = \left|\frac{x(t)}{x_o}\right| = \left|R_{11}\right| = \left|c_x(t)\right|$$

and in the y plane by $$M_y = \left|R_{33}\right| = \left|c_y(t)\right| \qquad (71)$$

First-Order Momentum Resolution

For point-to-point imaging the first-order momentum resolving power R_1 (not to be confused with the matrix R) is the ratio of the momentum dispersion to the image size: Thus

$$R_1 = \left|\frac{R_{16}}{R_{11}\ x_o}\right| = \left|\frac{d_x(t)}{c_x(t)\ x_o}\right|$$

For point-to-point imaging $\left(s_x(t) = 0\right)$ using Eq. (62), the dispersion at an image is

$$d_x(t) = -\ c_x(t) \int_o^t s_x(\tau)\ h(\tau)d\tau \qquad (72)$$

from which the first-order momentum resolving power R_1 becomes

$$R_1\ x_o = \left|\frac{d_x(t)}{c_x(t)}\right| = \left|\int_o^t s_x(\tau)\ h(\tau)\ d\tau\right| = \frac{1}{\theta_o}\int_o^t x d\alpha = \frac{(\ell - \ell_o)}{\theta_o} \qquad (73)$$

where x_o is the total source size.

Zero Dispersion

For point-to-point imaging, using Eq. (72), the necessary and sufficient condition for zero dispersion at an image is

$$\int_o^t s_x(\tau)\ h(\tau)d\tau = 0 \qquad (74)$$

For parallel to point imaging, (i.e., $c_x(t) = 0$), the condition for zero dispersion at the image is

$$\int_0^t c_x(\tau)\, h(\tau) d\tau = 0. \tag{75}$$

Focal Length

It can be readily demonstrated from simple lens theory[4] that the physical interpretations of R_{21} and R_{43} are:

$$c_x'(t) = R_{21} = -\frac{1}{f_x} \quad \text{and} \quad c_y'(t) = R_{43} = -\frac{1}{f_y} \tag{76}$$

where f_x and f_y are the system focal lengths in the x and y planes respectively between $\tau = 0$ and $\tau = t$.

Evaluation of the First-Order Matrix for Ideal Magnets

From the results of Section III, we conclude that for an ideal magnet the matrix elements of R are simple trigonometric or hyperbolic functions. The general result for an element of length L is

$$R = \begin{bmatrix} \cos k_xL & \frac{1}{k_x}\sin k_xL & 0 & 0 & 0 & \frac{h}{k_x^2}\left[1-\cos k_xL\right] \\ -k_x \sin k_xL & \cos k_xL & 0 & 0 & 0 & \left(\frac{h}{k_x}\right)\sin k_xL \\ 0 & 0 & \cos k_yL & \frac{1}{k_y}\sin k_yL & 0 & 0 \\ 0 & 0 & -k_y \sin k_yL & \cos k_yL & 0 & 0 \\ \frac{h}{k_x}\sin k_xL & \frac{h}{k_x^2}\left[1-\cos k_xL\right] & 0 & 0 & 1 & \frac{h^2}{k_x^3}\left[k_xL-\sin k_xL\right] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{77}$$

where for a dipole (bending) magnet, we have defined

$$k_x^2 = (1-n)\, h^2 \quad \text{and} \quad k_y^2 = nh^2 \ .$$

For a pure quadrupole, the R matrix is evaluated by letting

$$k_x^2 = k_q^2 \quad \text{and} \quad k_y^2 = - k_q^2$$

and taking the limiting case $h \rightarrow 0$, where

$$k_q^2 = - nh^2 = \left(\frac{B_o}{a}\right)\left(\frac{1}{B\rho}\right) \ .$$

Taking these limits, the R matrix for a quadrupole is:

$$R = \begin{bmatrix} \cos k_q L & \frac{1}{k_q}\sin k_q L & 0 & 0 & 0 & 0 \\ -k_q \sin k_q L & \cos k_q L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh k_q L & \frac{1}{k_q}\sinh k_q L & 0 & 0 \\ 0 & 0 & k_q \sinh k_q L & \cosh k_q L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{78}$$

Note that the trigonometric and hyperbolic functions will interchange if the sign of B_o is reversed.

The R Matrix Transformed to the Principal Planes

The positions Z of the principal planes of a magnetic element (measured from its ends towards the center of the element) may be derived from the following matrix equation:

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ R_{21} & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{43} & 1 & 0 & 0 \\ X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -Z_{2x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{2y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} R \begin{bmatrix} 1 & -Z_{1x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{1y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{79}$$

Solving this equation, we have

$$Z_{1x} = \frac{R_{22}-1}{R_{21}} \qquad Z_{2x} = \frac{R_{11}-1}{R_{21}}$$

$$Z_{1y} = \frac{R_{44}-1}{R_{43}} \qquad Z_{2y} = \frac{R_{33}-1}{R_{43}} \tag{80}$$

For the ideal magnet, the general result for the transformation matrix R_{pp} between the principal planes is

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k_x \sin k_x L & 1 & 0 & 0 & 0 & \frac{h}{k_x} \sin k_x L \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -k_y \sin k_y L & 1 & 0 & 0 \\ \frac{h}{k_x} \sin k_x L & 0 & 0 & 0 & 1 & \frac{h^2}{k_x^3}\left[k_x L - \sin k_x L\right] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{81}$$

and because of symmetry

$$Z_x = Z_{1x} = Z_{2x} = \frac{1}{k_x} \tan\left(\frac{k_x L}{2}\right)$$

and

$$Z_y = Z_{1y} = Z_{2y} = \frac{1}{k_y} \tan\left(\frac{k_y L}{2}\right) \tag{82}$$

Correspondingly for the ideal quadrupole, R_{pp} is derived by letting

$$k_x^2 = k_q^2 \quad \text{and} \quad k_y^2 = -k_q^2$$

and taking the limit $h \to 0$ for each of the matrix elements.

the result is:

$$
R_{pp} = \left[\begin{array}{c|c|c|c|c|c}
1 & 0 & 0 & 0 & 0 & 0 \\
-k_q \sin k_q L & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & k_q \sinh k_q L & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \qquad (83)
$$

where now

$$
Z_x = \frac{1}{k_q} \tan \frac{k_q L}{2}
$$

$$
Z_y = \frac{1}{k_q} \tanh \frac{k_q L}{2} \quad . \qquad (84)
$$

V. SOME GENERAL SECOND-ORDER THEOREMS DERIVED FROM THE GENERAL THEORY OF SECTION II

We have established in Section II that any second-order aberration coefficient q may be evaluated via the Green's function integral, Eq.(43), i.e.,

$$q(t) = s(t)\int_0^t f(\tau)c(\tau)d\tau - c(t)\int_0^t f(\tau)s(\tau)d\tau$$

A second-order aberration may therefore be determined as soon as a first-order solution for the system has been established since the polynomial expressions for the driving terms $f(\tau)$ have all been expressed as functions of the characteristic first-order matrix elements (Table I). Usually one is interested in knowing the value of the aberration at an image point of which there are two cases of interest; point-to-point imaging $s(t) = 0$ and parallel-to-point imaging $c(t) = 0$.

Thus for point-to-point imaging:

$$q = - c(t)\int_0^t f(\tau)s(\tau)d(\tau)$$

where $\tau = t$ is the location of an image and $|c(t)| = M$ is the first-order spatial magnification at the image, and for parallel-to-point imaging;

$$q = s(t)\int_0^t f(\tau)c(\tau)d(\tau)$$

where $\tau = t$ is the position of the image and $s(t)$ is the angular dispersion at the image.

If a system possesses first-order optical symmetries, then it can be immediately determined if a given second-order aberration is identically zero as a consequence of the first-order symmetry. We observe that for point-to-point imaging a second-order aberration coefficient q will be identically zero if the product of the corresponding driving term $f(\tau)$ and the first-order matrix element $s(\tau)$ form an odd function about the midpoint of the system.

As an example of this, consider the transformation between principal planes for the two symmetric achromatic systems illustrated in Fig.12 and Fig. 13 . We assume in both cases that the elements of the system have been chosen such as to transform an initial parallel beam of particles into a final parallel beam; i.e. , $R_{21} = -\frac{1}{f_x} = 0$ for midplane trajectories. We further assume parallel-to-point imaging at the midpoint of the system. With these assumptions, the first-order matrix transformation for midplane trajectories between principal planes is:

$$\begin{bmatrix} x(t) \\ x'(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x'(0) \\ \delta(0) \end{bmatrix}$$

Thus $c_x(t) = -1$, $s_x(t) = 0$, $c_x'(t) = 0$, $s_x'(t) = -1$, and of course $d_x(t) = d_x'(t) = 0$. About the midpoint of the system, the following symmetries exist for the characteristic first-order matrix elements and for the curvature $h(\tau) = \frac{1}{\rho_0}$ of the central trajectory; we classify them as being either odd or even functions about the midpoint of the system. The results are:

$c_x(\tau)$ = odd $\quad$ $s_x(\tau)$ = even $\quad$ $d_x(\tau)$ = even $\quad$ $h(\tau)$ = even

$c_x'(\tau)$ = even $\quad$ $s_x'(\tau)$ = odd $\quad$ $d_x'(\tau)$ = odd $\quad$ $h'(\tau)$ = odd

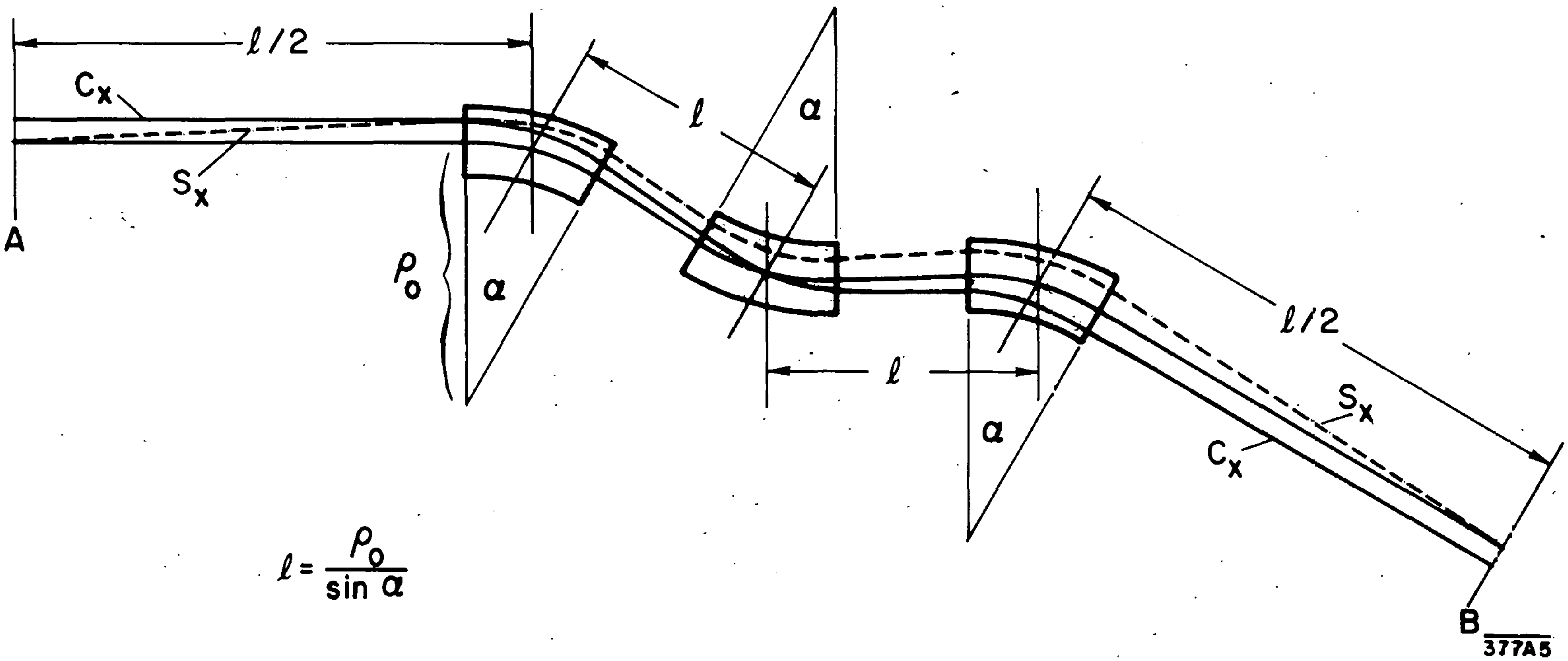


FIG. 12--THREE-BENDING MAGNET ACHROMATIC SYSTEM.
A AND B ARE LOCATIONS OF PRINCIPAL PLANES

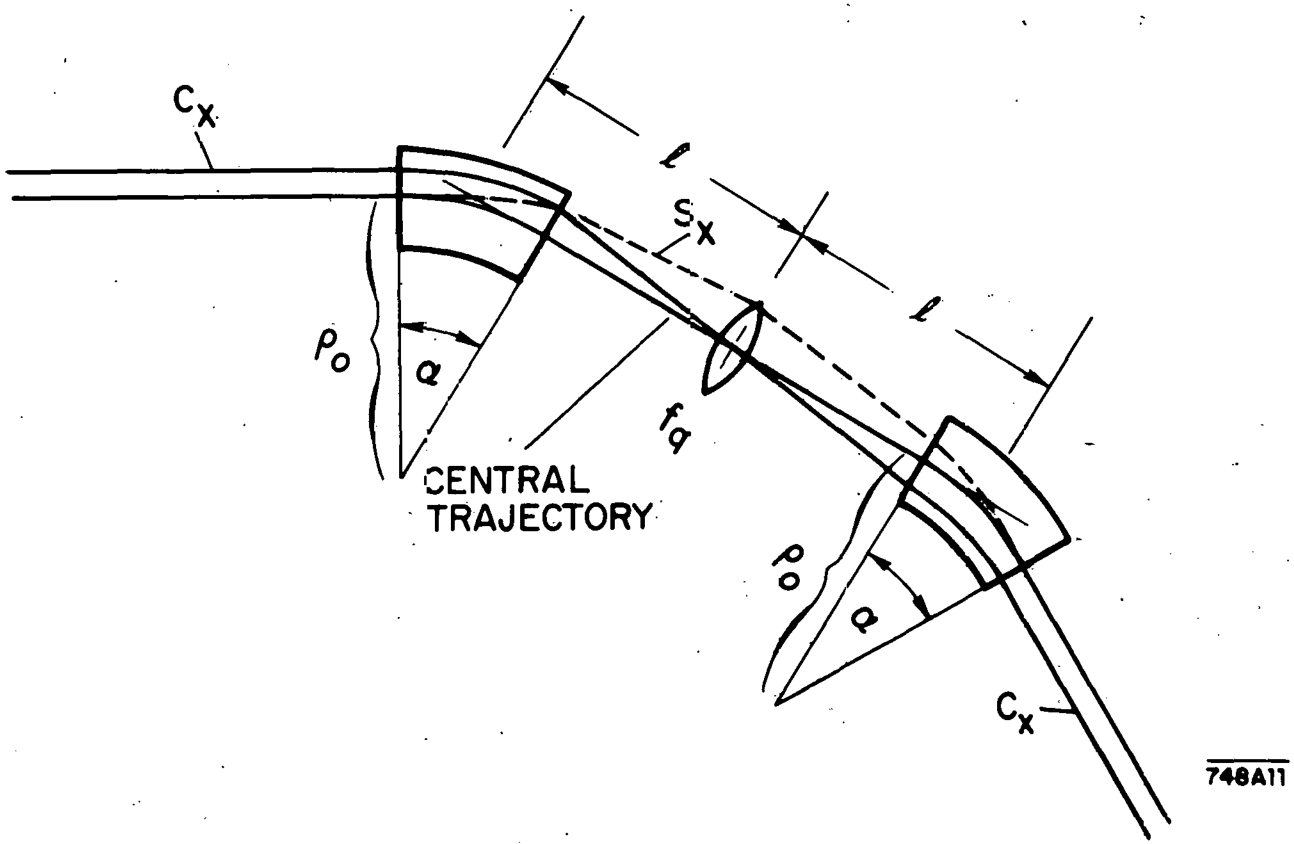


FIG. 13--ACHROMATIC SYSTEM WITH QUADRUPOLE AT CENTER TO ACHIEVE ACHROMATIC IMAGING. THE PRINCIPAL PLANES ARE LOCATED AT CENTERS OF THE BENDING MAGNETS

As a consequence of these symmetries, the following second-order coefficients are uniquely zero for the transformation between principal planes.

$$(x|x_0x_0') = (x|x_0\delta) = (x'|x_0^2) = (x'|x_0'^2) = (x'|x_0'\delta) = (x'|\delta^2) = 0$$

This result is valid independent of the details of the fringing fields of the magnets provided symmetry exists about the midpoint.

Second-Order Optical Symmetries in $n = 1/2$ Magnetic Systems

In magnetic-optical systems composed of $n = 1/2$ magnets having normal entry and exit of the central trajectory (i.e., nonrotated entrance and exit faces), several general mathematical relationships result from the $n = 1/2$ symmetry. Since $k_x^2 = (1-n)\,h^2$ and $k_y^2 = nh^2$, for $n = 1/2$ it follows that $c_x(\tau) = c_y(\tau)$ and $s_x(\tau) = s_y(\tau)$ at any position τ along the system; thus as is well known, an $n = 1/2$ system possesses first-order double focusing properties.

In addition to the above first-order results, at any point t in an $n = 1/2$ system, the sums of the following second-order aberration coefficients are constants independent of the distribution or magnitude of the sextupole components throughout the system:

$$
\begin{aligned}
(x|x_o'^2) + (x|y_o'^2) &= \text{a constant independent of } \beta h^3.\\
2(x|x_o^2) + (y|x_o y_o) &= \text{a constant independent of } \beta h^3.\\
(x|x_o x_o') + (y|x_o y_o') &= \text{a constant independent of } \beta h^3.\\
(x|x_o\delta) + (y|y_o\delta) &= \text{a constant independent of } \beta h^3.\\
2(x|x_o'^2) + (y|x_o'y_o') &= \text{a constant independent of } \beta h^3.\\
(x|x_o'\delta) + (y|y_o'\delta) &= \text{a constant independent of } \beta h^3.\\
(x|x_o^2) + (x|y_o^2) &= \text{a constant independent of } \beta h^3.\\
(x|x_o x_o') + (x|y_o y_o') &= \text{a constant independent of } \beta h^3. \qquad (85)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(x'|x_o'^2) + (x'|y_o'^2) &= \text{a constant independent of } \beta h^3.\\
2(x'|x_o^2) + (y'|x_o y_o) &= \text{a constant independent of } \beta h^3.
\end{aligned}
$$

$$(x'|x_0x_0') + (y'|x_0y_0') = \text{ a constant independent of } \beta h^3.$$
$$(x'|x_0\delta) + (y'|y_0\delta) = \text{ a constant independent of } \beta h^3.$$
$$2(x'|x_0'^2) + (y'|x_0'y_0') = \text{ a constant independent of } \beta h^3.$$
$$(x'|x_0'\delta) + (y'|y_0'\delta) = \text{ a constant independent of } \beta h^3.$$
$$(x'|x_0^2) + (x'|y_0^2) = \text{ a constant independent of } \beta h^3.$$
$$(x'|x_0x_0') + (x'|y_0y_0') = \text{ a constant independent of } \beta h^3. \qquad (86)$$

Of the above relations, the first is perhaps the most interesting in that it shows the impossibility of simultaneously eliminating both the $(x|x_0'^2)$ and $(x|y_0'^2)$ aberrations in an $n = 1/2$ system; i.e., either $(x|x_0'^2)$ or $(x|y_0'^2)$ may be eliminated by the appropriate choice of sextupole elements but not both.

VI. AN APPROXIMATE EVALUATION OF THE SECOND-ORDER ABERRATIONS FOR HIGH ENERGY PHYSICS

Quite often it is desirable to estimate the magnitude of various second-order aberrations in a proposed system to obtain insight into what constitutes an optimum solution to a given problem. A considerable simplification occurs in the formalism in the high-energy limit where ρ_o is much much greater than the transverse amplitudes of the first-order trajectories and where the dipole, quadrupole and sextupole functions are physically separated into individual elements. It is also assumed that fringing-field effects are small compared to the contributions of the various multipole elements.

Under these circumstances, the second-order chromatic aberrations are generated predominately in the quadrupole elements; the geometric aberrations are generated in the dipole elements (bending magnets); and, depending upon their location in the system, the sextupole elements couple with either the chromatic or geometric aberrations or both.

We have tabulated in Tables VII, VIII and IX the approximate formulae for the high-energy limit for three cases of interest; point-to-point imaging in the x (bend) plane, Table VII; point-to-point imaging in the y (nonbend) plane, Table VIII; and parallel-to-point imaging in the y plane, Table IX.

For the purpose of clearly illustrating the physical principles involved, we assume that the amplitudes of the characteristic first-order matrix elements c_x, s_x, d_x, c_y, and s_y are constant within any given quadrupole or sextupole element, and we define the strengths of the quadrupole and sextupole elements as follows:

$$\int_0^L k_q^2 \, d\tau = k_q^2 L_q \cong \frac{1}{f_q}$$

where L_q is the effective length of the quadrupole and where $1/f_q = k_q \sin k_q L$ is the reciprocal of the focal length of the q^{th} quadrupole; and for the j^{th} sextupole of length L_s, we define its strength as

$$\int_0^L k_s^2 \, d\tau = k_s^2 L_s = S_j \quad .$$

The results are tabulated in the tables in terms of integrals over the bending magnets and summations over the quadrupole and sextupole elements. Note that under these circumstances the quadrupole and sextupole contributions to the aberration coefficients are proportional to the amplitudes of the characteristic first-order trajectories within these elements, whereas the dipole contributions are proportional to the derivatives of the first-order trajectories within the dipole elements.

As an example of the above concepts, we shall calculate the angle ψ between the momentum focal plane and the central trajectory for some representative cases.

For point-to-point imaging, it may be readily verified that

$$\tan \psi = -\left(\frac{d_x(i)}{c_x(i)}\right) \frac{1}{(x_i|x_o'\delta)} = \frac{\int_0^i s_x d\alpha}{(x_i|x_o'\delta)} = \frac{R_1 x_o}{(x_i|x_o'\delta)} \tag{87}$$

where the subscript o refers to the object plane and the subscript i to the image plane.

Let us now consider some representative quadrupole configurations and assume that the bending magnets are placed in a region having a large amplitude of the unit sine-like function s_x (so as to optimize the first-order momentum resolving power R_1).

Case I

Consider the simple quadrupole configuration shown in Fig. 14 with the bending magnets located in the region between the quadrupoles and $s'_x \cong 0$ in this region. For these conditions, $f_1 = \ell_1$, $s_x = \ell_1$ at the quadrupoles, and $f_2 = \ell_3$. From Table VII, we have:

$$(x_i|x'_o\delta) \cong - c_x(i) \sum_q \frac{s_x^2}{f_q} = - c_x(i)\ell_1 \left(1 + \frac{\ell_1}{\ell_3}\right) = \ell_1(1 + M_x)$$

where we make use of the fact that $(\ell_3/\ell_1) = M_x = - c_x(i)$. M_x is the first-order magnification of the system.

Hence,

$$\tan \psi = \frac{\int_o^i s_x d\alpha}{(x_i|x'_o\delta)} \cong \frac{\alpha}{(1 + M_x)} \quad . \tag{88}$$

Case II

For a single quadrupole, Fig. 15, the result is similar

$$\tan \psi = \frac{K\alpha}{(1 + M_x)} \tag{89}$$

except for the factor $K < 1$ resulting from the fact that s_x cannot have the same amplitude in the bending magnets as it does in the quadrupole. Therefore

$$\int_o^i s_x d\alpha = K \ell_1 \alpha \quad .$$

$$\tan \psi = \frac{\int_0^i s_x d\alpha}{(x_i | x_0' \delta)} \cong \frac{\alpha}{(1 + M_x)}$$

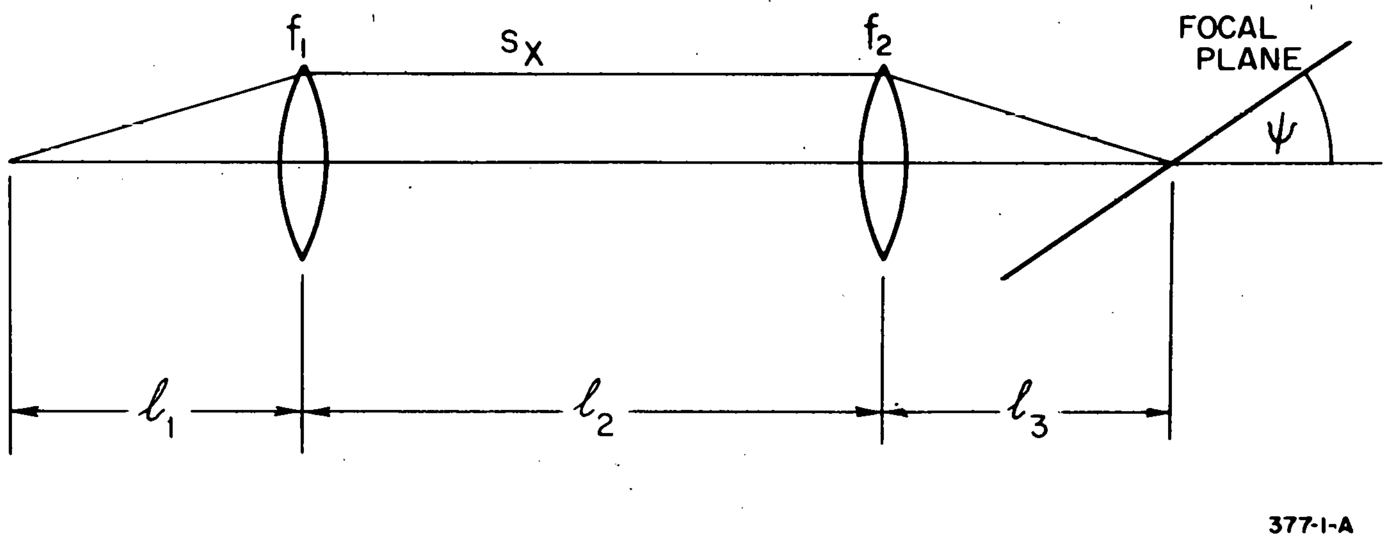


FIG. 14-- Focal plane tilt for symmetric quadrupole doublet spectrometer.

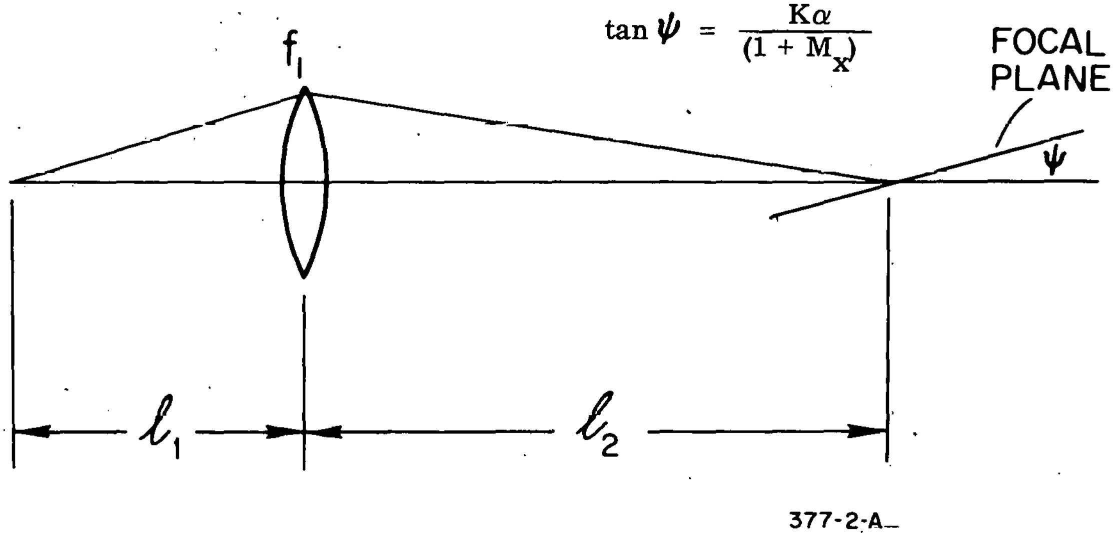


FIG. 15--Focal plane tilt for a quadrupole singlet spectrometer.

Case III

Now let us consider a symmetric four-quadrupole array, Fig. 16, such that we have an intermediate image. Then

$$(x_i|x_o'\delta) = -2c_x(i)\ell_1 \left[1 + (\ell_1/\ell_3)\right] = \text{twice that for Case I,}$$

because of symmetry, $c_x(i) = M_x = 1$. Thus, we conclude

$$\tan\psi = -\alpha/2 \left[1 + (\ell_1/\ell_3)\right] \qquad (90)$$

In other words, the intermediate image has introduced a factor of two in the denominator and has changed the sign of ψ.

$$\tan \psi = -\alpha/2 \left[1 + (\ell_1/\ell_3) \right]$$

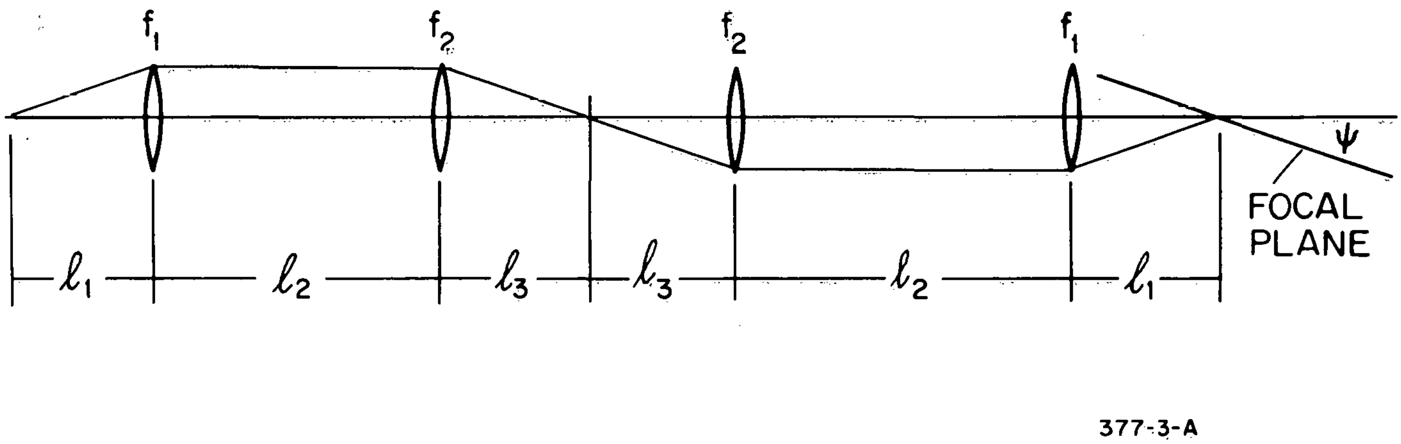


FIG. 16--Focal plane tilt for a symmetric array of 4 quadrupoles.

TABLE VII

Applying the Greens' function solution, Eq. (22), in the high-energy limit as defined above for point-to-point imaging in the x(bend) plane, the second-order matrix elements reduce to:

$$(x|x_0^2) \cong -\frac{1}{2} c_x(i) \int_0^i c_x'^2 s_x d\alpha + c_x(i) \sum_j S_j c_x^2 s_x$$

$$(x|x_0 x_0') \cong - c_x(i) \int_0^i c_x' s_x' s_x d\alpha + 2c_x(i) \sum_j S_j c_x s_x^2$$

$$(x|x_0\delta) \cong - c_x(i) \int_0^i c_x' d_x' s_x d\alpha + 2c_x(i) \sum_j S_j c_x s_x d_x - c_x(i) \sum_q \frac{c_x s_x}{f_q}$$

$$(x|x_0'^2) \cong -\frac{1}{2} c_x(i) \int_0^i s_x'^2 s_x d\alpha + c_x(i) \sum_j S_j s_x^3$$

$$(x|x_0'\delta) \cong - c_x(i) \int_0^i s_x' d_x' s_x d\alpha + 2c_x(i) \sum_j S_j s_x^2 d_x - c_x(i) \sum_q \frac{s_x^2}{f_q}$$

$$(x|\delta^2) \cong -\frac{c_x(i)}{2} \int_0^i (d_x')^2 s_x d\alpha + c_x(i) \sum_j S_j s_x d_x^2 - c_x(i) \sum_q \frac{s_x d_x}{f_q}$$

$$(x|y_0^2) \cong \frac{1}{2} c_x(i) \int_0^i c_y'^2 s_x d\alpha - c_x(i) \sum_j S_j c_y^2 s_x$$

$$(x|y_0 y_0') \cong c_x(i) \int_0^i c_y' s_y' s_x d\alpha - 2c_x(i) \sum_j S_j c_y s_y s_x$$

$$(x|y_0'^2) \cong \frac{1}{2} c_x(i) \int_0^i s_y'^2 s_x d\alpha - c_x(i) \sum_j S_j s_y^2 s_x$$

TABLE VIII

For point-to-point imaging in the y (non-bend) plane, Eq. (23), the high-energy limit yields:

$$(y|x_0 y_0) \cong - c_y(i) \int_0^i c'_x c'_y s_y \, d\alpha - 2c_y(i) \sum_j S_j c_x c_y s_y$$

$$(y|x_0 y'_0) \cong - c_y(i) \int_0^i c'_x s'_y s_y \, d\alpha - 2c_y(i) \sum_j S_j c_x s_y^2$$

$$(y|x'_0 y_0) \cong - c_y(i) \int_0^i s'_x c'_y s_y \, d\alpha - 2c_y(i) \sum_j S_j s_x c_y s_y$$

$$(y|x'_0 y'_0) \cong - c_y(i) \int_0^i s'_x s'_y s_y \, d\alpha - 2c_y(i) \sum_j S_j s_x s_y^2$$

$$(y|y_0 \delta) \cong - c_y(i) \int_0^i c'_y d'_x s_y \, d\alpha - 2c_y(i) \sum_j S_j c_y d_x s_y + c_y(i) \sum_q \frac{c_y s_y}{f_q}$$

$$(y|y'_0 \delta) \cong - c_y(i) \int_0^i s'_y d'_x s_y \, d\alpha - 2c_y(i) \sum_j S_j d_x s_y^2 + c_y(i) \sum_q \frac{s_y^2}{f_q}$$

TABLE IX

For parallel-(line)-to-point imaging in the y (non-bend) plane, Eq. (24), the high energy limit yields:

$$(y|x_o y_o) \cong s_y(i) \int_o^i c'_x c'_y c_y \, d\alpha + 2s_y(i) \sum_j S_j c_x c_y^2$$

$$(y|x_o y'_o) \cong s_y(i) \int_o^i c'_x s'_y c_y \, d\alpha + 2s_y(i) \sum_j S_j c_x s_y c_y$$

$$(y|x'_o y_o) \cong s_y(i) \int_o^i s'_x c'_y c_y \, d\alpha + 2s_y(i) \sum_j S_j s_x c_y^2$$

$$(y|x'_o y'_o) \cong s_y(i) \int_o^i s'_x s'_y c_y \, d\alpha + 2s_y(i) \sum_j S_j s_x s_y c_y$$

$$(y|y_o \delta) \cong + s_y(i) \int c'_y d' c_y \, d\alpha + 2s_y(i) \sum_j S_j c_y^2 d_x - s_y(i) \sum_q \frac{c_y^2}{f_q}$$

$$(y|y'_o \delta) \cong + s_y(i) \int s'_y d' c_y \, d\alpha + 2s_y(i) \sum_j S_j s_y c_y d_x - s_y(i) \sum_q \frac{s_y c_y}{f_q}$$

VII A FIRST-ORDER MATRIX PHASE ELLIPSE FORMALISM FOR BEAM TRANSPORT OPTICS

In accelerators and in external beam transport systems, the behavior of an individual particle is often of less concern than is the behavior of a bundle of particles " the BEAM" of which an individual particle is a member. An extension of the first-order matrix algebra of Eq. (59), page 77, provides a useful and convenient means for defining and transforming this "BEAM" through a beam transport system. We assume that the bundle of rays constituting the BEAM may be adequately represented in phase space by an ellipsoid whose coordinates are the six parameters

$$x_1 = x \quad x_2 = \theta \quad x_3 = y \quad x_4 = \phi \quad x_5 = \ell \quad \text{and} \quad x_6 = \delta$$

introduced previously in this report.

The validity and interpretation of this phase ellipse formalism must be ascertained for each system being designed. For charged particle beams in or emanating from accelerators, the assumption of representing the BEAM by an ellipsoid usually corresponds reasonably well with physical reality. For other applications, such as charged particle spectrometers, considerable caution must be exercised in the use of the phase ellipse formalism.

For the remainder of this discussion, we shall assume that (to first-order) it is valid to represent the actual distribution of a bundle of rays by an ellipsoid in 6-dimensional phase space, where the projection of the ellipsoid in any two dimensions (for example; x and θ) is an ellipse. To simplify the discussion, we shall proceed by first formulating the matrix equation of a two-dimensional ellipse, derive and discuss its properties and then generalize the result to an $n(n = 6)$ dimensional ellipsoid.

Matrix Equation for an Ellipse

Consider a two-dimensional real, positive definite, symmetric matrix,

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad ; \qquad (91)$$

and, its inverse

$$\sigma^{-1} = \frac{1}{|\sigma|} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \quad ;$$

where we define $|\sigma|$ as the determinant of σ.

Define the column matrix or "vector"

$$X = \begin{pmatrix} x \\ \theta \end{pmatrix} \quad ; \qquad (92)$$

and, its transpose

$$X^T = (x \quad \theta) \; .$$

Then,

$$X^T \sigma^{-1} X = 1 \qquad (93)$$

i.e.,

$$\sigma_{22} x^2 - 2\sigma_{12} x\theta + \sigma_{11} \theta^2 = |\sigma|$$

is the equation of an ellipse in x, θ space.

Transformation Properties of the Ellipse Under a Coordinate Rotation

Let us now study the transformation properties of σ under a coordinate rotation. Suppose we define the coordinates x_0, θ_0 as those corresponding to the directions of the major and minor axes of an ellipse. Then, the equation of this ellipse is

$$X_0^T \sigma_0^{-1} X_0 = 1. \tag{94}$$

where

$$X_0 = \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix} \quad ; \quad \sigma_0 = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \quad ;$$

and

$$\sigma_0^{-1} = \frac{1}{|\sigma_0|} \begin{pmatrix} \sigma_{22} & 0 \\ 0 & \sigma_{11} \end{pmatrix}$$

from which the equation of the ellipse in an expanded algebraic form is

$$\sigma_{22} x_0^2 + \sigma_{11} \theta_0^2 = |\sigma| = \sigma_{11} \sigma_{22}$$

or

$$\frac{x_0^2}{\sigma_{11}} + \frac{\theta_0^2}{\sigma_{22}} = 1 \tag{95}$$

From the type form Eq. (95), we conclude that the area of the ellipse is

$$A = \pi(\sigma_{11}\sigma_{22})^{\frac{1}{2}} = \pi|\sigma_0|^{\frac{1}{2}} \tag{96}$$

and

$$\begin{aligned} \sigma_{11} &= x_{max}^2 \\ \sigma_{22} &= \theta_{max}^2 \end{aligned} \tag{97}$$

Now, consider a rotation of the coordinate system from the x_0, θ_0 axes to the x_1, θ_1 axes by an angle α via the matrix equation

$$X_1 = MX_0$$

where

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} , \tag{98}$$

is the coordinate rotation matrix. For this matrix

$$M^T = M^{-1} \quad \text{and} \quad |M| = |M^T| = |M^{-1}| = 1$$

We may therefore rewrite Eq. (94) in the following forms;

$$X_0^T M^T {M^T}^{-1} \sigma_0^{-1} M^{-1} M X_0 = 1$$

or

$$(MX_0)^T (M\sigma_0 M^T)^{-1} (MX_0) = 1$$

or, finally

$$X_1^T \sigma_1^{-1} X_1 = 1 \tag{99}$$

where

$$\sigma_1 = M\sigma_0 M^T = M\sigma_0 M^{-1} \tag{100}$$

Since the determinant of the product of two matrices is equal to the product of the determinants, it follows that:

$$|\sigma_1| = |M\sigma_0 M^{-1}| = |M|\,|\sigma_0|\,|M^{-1}| = |\sigma_0| \quad .$$

The area of the ellipse is a constant under a coordinate rotation, therefore, we conclude that the invariant equation for the area of the ellipse is:

$$A = \pi|\sigma_0|^{\frac{1}{2}} = \pi|\sigma|^{\frac{1}{2}} \tag{101}$$

independent of the orientation of the coordinate system.

Phase Ellipse Transformations Through a Beam Transport System

We have shown in previous sections (e.g., Eq. (59), page 77) that the physics of beam transport optics may be reduced to a process of matrix multiplication.

$$X_2 = RX_1 \tag{102}$$

where the matrix R describes the action of the magnetic system on the particle coordinates. We have also proved that the differential equations of motion require that the determinant

$$|R| = 1 \quad .$$

We further note that for any matrix

$$|R| = |R^T| \quad .$$

If now we begin with an arbitrary phase space ellipse represented by the matrix

$$\sigma_1 = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

the inverse of which is

$$\sigma_1^{-1} = \frac{1}{|\sigma|} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

The equation of this ellipse in X_1 coordinate phase space is

$$X_1^T \sigma_1^{-1} X_1 \leqq 1 \tag{103}$$

The less-than sign is now added to include all the phase points (particles) inside the ellipse.

If now we rewrite this equation as follows:

$$X_1^T \; R^T \; R^{T^{-1}} \; \sigma_1^{-1} R^{-1} \; RX_1 \leqq 1 \quad ;$$

$$\left(RX_1\right)^T \left(R\sigma_1 R^T\right)^{-1} \; \left(RX_1\right) \leqq 1$$

or, finally

$$X_2^T \; \sigma_2^{-1} \; X_2 \leqq 1 \tag{104}$$

this is the equation of an ellipse in the X_2 coordinate system; where

$$\sigma_2 = R\sigma_1 \, R^T \tag{105}$$

is the transformation relating σ_2 to σ_1.

Using the property $|R| = |R^T| = 1$, we immediately conclude that $|\sigma_2| = |\sigma_1|$ hence, the phase area is preserved. We see then that the fact that $|R| = 1$ is equivalent to Liouville's theorem of phase space conservation.

Some General Properties of an Ellipse

Consider the general ellipse

$$X^T \; \sigma^{-1} \; X = 1$$

We have already verified that the area of the ellipse is

$$A = \pi \, |\sigma|^{\frac{1}{2}}$$

The maximum values of x and θ are simply

$$x_{max} = \pm\sqrt{\sigma_{11}} \quad \text{and} \quad \theta_{max} = \pm\sqrt{\sigma_{22}} \tag{106}$$

The x intercepts at $\theta = 0$ are

$$x_{int} = \pm\sqrt{\sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}} = \pm\sqrt{\sigma_{11}\,(1 - r_{12}^2)} \tag{107}$$

where we define

$$r_{12}^2 = \frac{\sigma_{12}^2}{\sigma_{11}\sigma_{22}} \tag{108}$$

and the θ intercepts at $x = 0$ are

$$\theta_{int} = \pm\sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} = \pm\sqrt{\sigma_{22}(1 - r_{12}^2)} \tag{109}$$

From which the area of an ellipse may be expressed in the following additional forms:

$$A = \pi|\sigma|^{\frac{1}{2}} = \pi x_{max}\,\theta_{int} = \pi x_{int}\,\theta_{max}. \tag{110}$$

The physical interpretation of the phase ellipse parameters are shown in Fig. 17 for the two dimensional case.

<u>Generalization to an n-Dimensional Ellipsoid</u>

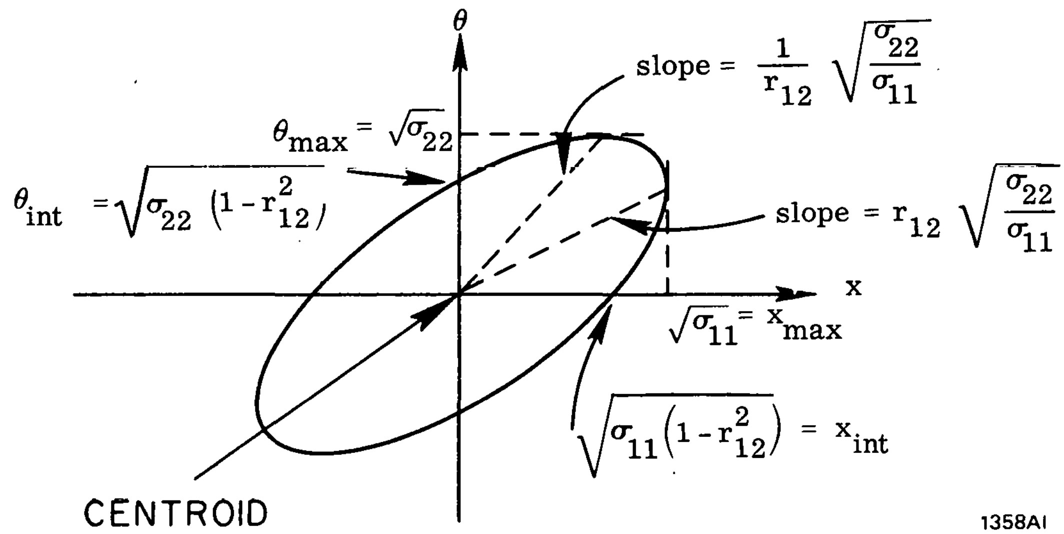
The two dimensional results may be generalized to n dimensions by expanding the column matrix X (Eq. (92)) to include all six of the phase space variables as follows:

$$X = \begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix},$$

and also expanding σ to a six by six symmetric array. Equation (93) then becomes the equation of a six dimensional ellipsoid whose volume is

$$\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}|\sigma|^{1/2}$$

The phase ellipse in any two dimensions (e.g., x and θ) is a projection of the general six dimensional ellipsoid.



A Two Dimensional BEAM Phase Ellipse

Fig. 17

The beam matrix carried in the computer for a TRANSPORT calculation has the following construction:

	x	θ	y	ϕ	ℓ	δ
x	σ_{11}					
θ	σ_{21}	σ_{22}				
y	σ_{31}	σ_{32}	σ_{33}			
ϕ	σ_{41}	σ_{42}	σ_{43}	σ_{44}		
ℓ	σ_{51}	σ_{52}	σ_{53}	σ_{54}	σ_{55}	
δ	σ_{61}	σ_{62}	σ_{63}	σ_{64}	σ_{65}	σ_{66}

The matrix is symmetric so that only a triangle of elements is needed.

In the printed output this matrix has a somewhat different format for ease of interpretation:

			x	θ	y	ϕ	ℓ
x	$\sqrt{\sigma_{11}}$	CM					
θ	$\sqrt{\sigma_{22}}$	MR	r_{21}				
y	$\sqrt{\sigma_{33}}$	CM	r_{31}	r_{32}			
ϕ	$\sqrt{\sigma_{44}}$	MR	r_{41}	r_{42}	r_{43}		
ℓ	$\sqrt{\sigma_{55}}$	CM	r_{51}	r_{52}	r_{53}	r_{54}	
δ	$\sqrt{\sigma_{66}}$	PC	r_{61}	r_{62}	r_{63}	r_{64}	r_{65}

The units are always printed with the matrix.

Physical Interpretation of the Phase Ellipse Formalism

For the six-dimensional BEAM matrix, the physical interpretation of the $\sqrt{\sigma_{ii}}$'s is as follows:

$\sqrt{\sigma_{11}} = \frac{x_{max}}{2}$ = The maximum (half)-width of the beam envelope in the x(bend) plane.

$\sqrt{\sigma_{22}} = \frac{\theta_{max}}{2}$ = The maximum (half)-angular divergence of the beam envelope in the x(bend) plane.

$\sqrt{\sigma_{33}} = \frac{y_{max}}{2}$ = The maximum (half)-height of the beam envelope.

$\sqrt{\sigma_{44}} = \frac{\phi_{max}}{2}$ = The maximum (half)-angular divergence of the beam envelope in the y(non-bend)-plane.

$\sqrt{\sigma_{55}} = \frac{\ell_{max}}{2} = \frac{1}{2}$ the longitudinal extent of the bunch of particles.

$\sqrt{\sigma_{66}} = \frac{\delta}{2}$ = The half-width $\frac{1}{2}\left(\frac{\Delta p}{p}\right)$ of the momentum interval being transmitted by the system.

The units appearing next to the $\sqrt{\sigma_{ii}}$'s in a TRANSPORT printout sheet are the units chosen for the initial x, θ, y, ϕ ℓ and $\delta = \frac{\Delta p}{p}$ coordinates at the beginning of the data set via the units Card entry.

The physical interpretation of the off-diagonal terms in the beam matrix are as shown in Fig. 17. The magnitude of these off-diagonal terms are a measure of the orientation of the ellipsoid. A case of particular interest in any given plane (e.g., x and θ) is when the off-diagonal terms are equal to zero (i.e., an erect ellipse). This corresponds to a so-called "waist" in the BEAM.

It is important to understand correctly the meaning of a waist: For an existing beam, it is the location of the minimum beam size in a given region of the system (i.e., there may be several waists in an entire beam transport system). It is not the minimum beam size that can be achieved at a particular physical location; nor does a waist necessarily coincide with the first-order image ($R_{12}=0$ or $R_{34}=0$) of a system. Only in the limit of zero phase space area do these three quantities occur at the same location. A useful criterion that determines the physical proximity of these quantities is the following: Suppose a system has been adjusted to provide the smallest spot size possible at a given fixed location, then if the size of the beam at the principal planes of an optical system is large compared to its size at the waist, at the first-order image, or at the minimum spot size, then the location of these three quantities will closely coincide; if, on the other hand, the size of the beam does not change substantially throughout the system, then the locations of a waist, the minimum size and the first-order image may (and usually will) differ substantially.*

If an arbitrary beam transport system is reduced to the most elementary first-order form of representing it as an initial drift distance, followed by a lens action between two principal planes, and a final drift distance; then we observe that there are only two basic phase ellipse transformations of interest.

(1) An arbitrary DRIFT distance and

(2) A LENS action

Each of these elementary cases are illustrated on Fig. 18 for both a parallogram as well as ellipse phase space transformations. Note that a DRIFT followed by a LENS action is not necessarily equal to a LENS action followed by a DRIFT; i.e., the matrices do not necessarily commute.

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*See the appendix of Ref. 3 for a more extensive discussion of this general subject.

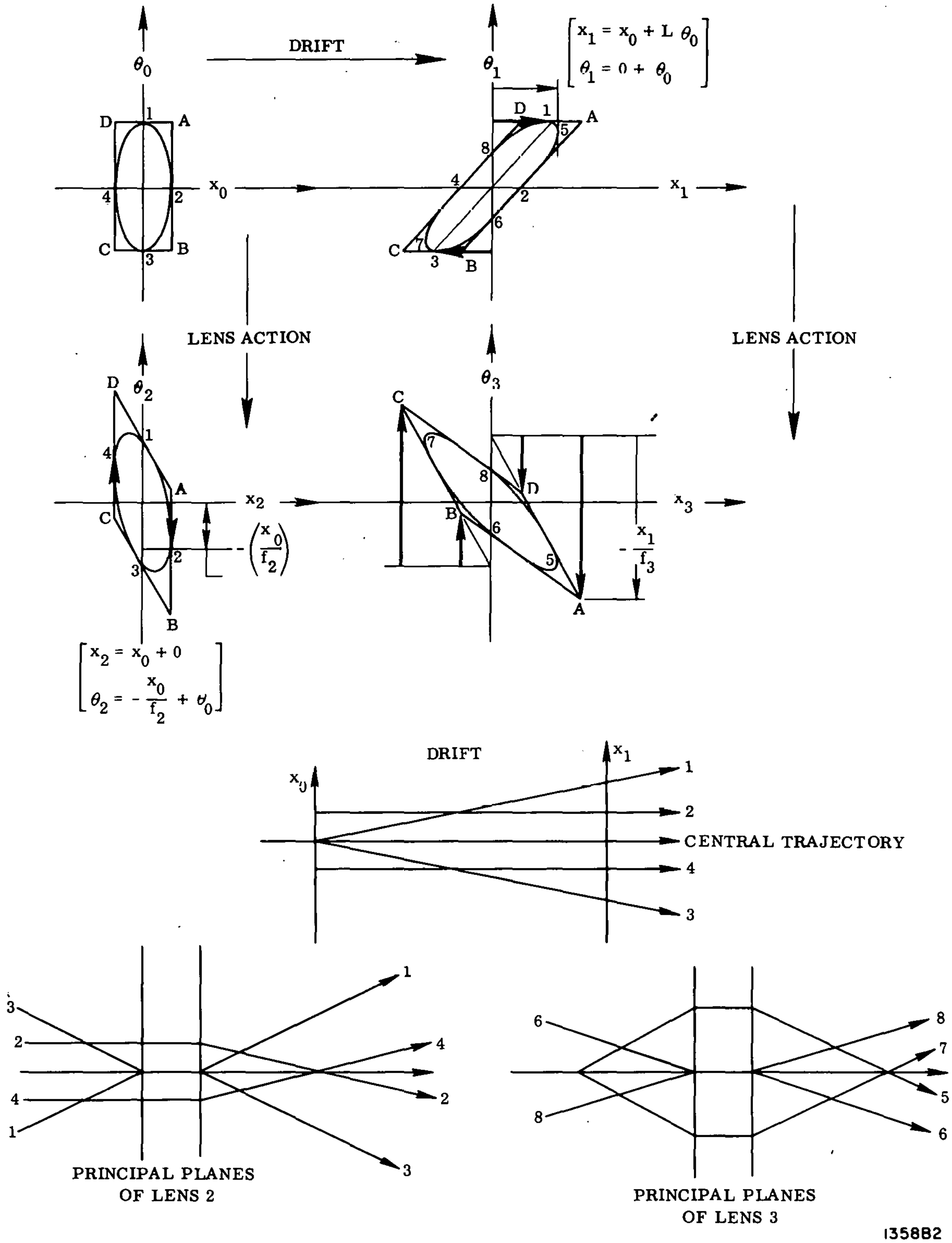
DRIFT
$x_1 = x_0 + L\,\theta_0$
$\theta_1 = 0 + \theta_0$
LENS ACTION
LENS ACTION
$x_2 = x_0 + 0$
$\theta_2 = -\frac{x_0}{f_2} + \theta_0$
$-\left(\frac{x_0}{f_2}\right)$
$-\frac{x_1}{f_3}$
DRIFT
CENTRAL TRAJECTORY
PRINCIPAL PLANES
OF LENS 2
PRINCIPAL PLANES
OF LENS 3
1358B2

Fig. 18

REFERENCES

1. J. F. Streib, "Design Considerations for Magnetic Spectrometers," HEPL Report No. 104, High-Energy Physics Lab., Stanford University, Stanford, California (1960).
2. L. C. Teng, "Expanded Form of Magnetic Field with Median Plane," ANL Internal Memorandum, dated December 15, 1962. ANL-LCT-28.
3. K. L. Brown, S. K. Howry, "TRANSPORT/360 A Computer Program for Designing Beam Transport Systems," Report No. SLAC 91, Stanford Linear Accelerator Center, Stanford University, Stanford, California.
4. S. Penner, "Calculations of Properties of Magnetic Deflection Systems," Rev. Sci. Instr. 32, No. 2, 150-160 (February 1961).
5. K. G. Steffen, High Energy Beam Optics, Interscience Monographs and Tests in Physics and Astronomy, Vol. 17 (John Wiley and Sons, 1965).
6. A. P. Banford, The Transport of Charged Particle Beams, (E. and F. N. Spon, Ltd., London, 1966).
7. K. L. Brown (with R. Belbeoch and P. Bounin), "First- and Second-Order Magnetic Optics Matrix Equations for the Midplane of Uniform-Field Wedge Magnets, Rev. Sci. Instr. 35, 481 (1964).
8. R. H. Helm, "First- and Second-Order Beam Optics of a Curved, Inclined Magnetic Field Boundary in the Impulse Approximation," SLAC Report No. 24, Stanford Linear Accelerator Center, Stanford, California(1963).
9. The expression for ψ_1 and ψ_2 was calculated in unpublished work of A. C. Hindmarsh and K. L. Brown at SLAC for insertion into the TRANSPORT Program (Summer 1965).

10. K. L. Brown, "Some Useful Theorems Based on a General First-Order Theory of Image Formation, "SLAC Internal Report, TN-64-18, Stanford Linear Accelerator Center, Stanford, California.

11. K. L. Brown, "A General First- and Second-Order Theory of Beam Transport Optics and its Application to the Design of High-Energy Particle Spectrometers," Proc. of the Fifth International Conference on High Energy Accelerators, Frascati, Italy(September 1965), pp 507-513; and "A General First- and Second-Order Theory of Beam Transport Optics," Proc. of the International Symposium on Magnet Technology, Stanford, California (1965), p. 141.

12. The contents of this Report (SLAC 75) are also published in essentially the same form in Advances in Particle Physics 1, 71 - 134 (1967).