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The Notion of Complexity

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ABSTRACT

The notion of the arithmetic complexity $|n|$ of an integer n is defined in terms of the minimum number of additions, multiplications, and exponentiations required to combine 1's to form n. The value of $|n|$ is calculated for $n < 2^{10}$. n is called complicated if $|n| > |n_1|$ for every $n_1 < n$. Of the first 19 complicated numbers, 14 are prime. A conjecture about a relation between complexity and entropy is proposed. Some computations are presented to support this conjecture.

I. INTRODUCTION

In this report we discuss notions of complexity in some algebraic structures. These notions are also applicable to more general combinatorial situations that perhaps lack any algebraic pattern in the classical sense. We concentrate on a few special cases for which we studied and calculated a special notion of complexity. Essentially, we examined a special notion of complexity for ordinary integers with a little excursion on such a notion for integers <u>modulo a prime</u>.

The notion of complexity, in our view, is separate, though associated with the idea of the amount of <u>information</u> or <u>entropy</u> of a system. We mention briefly a possible axiomatic approach to defining a real number called complexity for elements of a set or of a class on which certain <u>operations</u> are performed. These could be binary operations; our set could be a set of integers, and the operations could be addition, multiplication, and exponentiation, for example. It is this case that was examined on a computing machine and to which most of this report is devoted.

Another case would be a class of subsets of a given set, with allowed operations being the Boolean operations of <u>union</u> and <u>intersection</u> or <u>union</u> and <u>complementation</u>. One could add other operations, for example, the direct product of sets and also <u>projection</u>. This would correspond to allowing quantifiers in our theory. One can study a notion of complexity for vectors in a countable space or even in the continuum. An important study would be that of a <u>relative</u> complexity; that is to say, complexity of elements or "expressions" when the complexity of certain symbols is normalized to 1. In what has been sometimes called "speculation" on constants in physical theories, for example, the whole art seems to depend on the success of attempts to define some known important numbers, e.g., the dimensionless ratios

$$M_{proton}/M_{electron} = 1836.11...$$

and

$$e^2/hc = 137.1...$$

by use of only a few artificially introduced constants which should be as "simple" as possible. (cf. the attempts by Eddington[1] and some very recent ones by Good[2] and Wyler.[3])

Considered "genetically," a mathematical theory resembles a tree in that one obtains from a given number of symbols corresponding to "variables" and from a number of allowed operations, expressions that elongate by branching. The simplifications and abbreviations may then reduce the length of the expressions.

One could try to define complexity in a mathematical structure by postulating certain of its properties, somewhat like postulating properties of a measure.

Let the structure, S, consist of elements x, y, ... It may be finite or infinite. We have in the set S a number of, say, binary operations R_1, R_2, ... R_n. We want to assign a number $c(x) \geq 0$ to each element x of S and to each R_i ($i = 1 \ldots n$) so that the following properties should hold.

a. If $z = R_i(x,y)$, then $c(z) = c(R_i(x,y)) \leq c(x) + c(y) + c(R_i) \quad i = 1 \ldots n$.

b. For each element z, if $z = R_j(x,y)$, we should have for one case at least, $c(z) = c(x) + c(y) + c(R_j)$.

c. $z(x_0) = z(x_1) = \ldots z(x_n)$ for some preassigned elements $x_0 \ldots x_n$.

Needless to say, one can define analogous desiderata for the case in which the operations are more general than binary ones.

Obviously, in the case to which our exercise is devoted, these postulates are satisfied. Moreover, they define the complexity uniquely if, as must be the case in general, the complexity was normalized for some elements. (In our case, we assume the complexity of the integer 0 to be equal to 1. We hope to study this notion more thoroughly for the more general case and also to perform experiments to determine complexity functions for the case in which S is a class of sets.) Ultimately, one would wish to discuss the complexity of genetic codes and biological organisms quantitatively.

("Integer" always means a positive integer.)

II. ARITHMETIC COMPLEXITY OF INTEGERS

The arithmetic complexity $|n|$ of an integer n is defined as the fewest number of operators: +, x, xx (addition, multiplication, and exponentiation) which combine 1's to form n. Thus, $|1| = 0$; $|2| = 1$ since $2 = 1 + 1$; and $|5| = 4$ since $5 = (1 + 1)\text{xx}(1 + 1) + 1$ and not fewer than four operators with 1's will form five. Obviously, for a and b integers, $|a + b|$, $|ab|$, and $|a^b|$ are each not more than $|a| + |b| + 1$. For an infinity of integers n, the relation $|n + 1| = |n| + 1$ holds.

For the purpose of calculating the complexity of some integers, all correct formulas (up to some number of operators) involving +, x, xx, and the number 1 were enumerated using parenthesis-free notation on a computer. It required one hour of computer time to enumerate the integers with complexity ≤ 6. Ralph Cooper made the following observation. Each correct formula involving n (> 0) operators is the composition of two formulas, one formula with n_1 operators and one formula with n_2 operators such that $n = n_1 + n_2 + 1$. One generates the integers of complexity n by first generating tables of integers of complexity $< n$. One partitions $n - 1$ into $n_1 + n_2$ in all ways and combines the integers of complexity n_1 with the integers of complexity n_2 to produce integers of complexity not larger than n. This method is considerably more efficient than the previous method. Table I lists the complexity of all integers $< 2^{10}$.

From the above construction, one sees that an upper bound $\ell^1(k)$ to $\ell(k)$, the number of integers of complexity k, is given by the solution of

$$\ell^1(k + 1) = \sum_{j=0}^{k} \ell^1(j)\, \ell^1(k - j),$$

with $\ell^1(0) = 1$. The solution to this equation is given by

$$\ell^1(k) = \frac{1}{k+1}\binom{2k}{k} 2^{-k},$$

which implies that

$$\ell(k) \leq \frac{2^k}{k\sqrt{\pi k}} + O\left(2^k\, k^{-5/2}\right).$$

Two additional forms of complexity have been considered and calculated.

a. Complement complexity. To make complexity symmetric in 0's and 1's, we introduce a slightly different complexity, the complement complexity $\overline{K}(y|n)$. Define the complement operation C by $C(x|n) = 2^n - 1 - x$. $\overline{K}(y|n)$ is defined as the fewest operations of addition, multiplication, exponentiation, and complementation that combine 1's to form y. In the count of operations, the

TABLE I. COMPLEXITY OF INTEGERS $< 2^{10}$.

Complexity	Integer
0	1
1	2
2	3
3	4
4	5 6 8 9
5	7 10 16 27
6	11 12 17 18 25 28 32 36 64 81 256 512
7	13 14 15 19 20 24 26 29 33 37 49 54 65 82 100 125 128 216 243 257 513 729 1024
8	21 22 30 34 38 48 50 55 56 66 72 83 101 121 126 129 144 162 217 244 258 289 324 343 514 625 730 784 1000
9	23 31 35 39 40 45 51 52 57 58 67 73 74 75 84 96 98 102 108 122 127 130 145 163 164 169 192 196 200 218 225 245 250 259 290 325 344 361 400 432 486 515 576 626 676 731 768 785 841 1001
10	41 42 44 46 53 59 60 63 68 76 78 80 85 87 90 97 99 103 109 110 111 112 123 131 132 135 146 147 165 166 170 193 195 197 201 202 219 226 242 246 251 252 260 286 291 300 326 345 362 375 384 401 433 434 441 484 487 488 516 577 578 627 648 677 686 732 769 771 786 842 900 1002
11	43 47 61 62 69 70 77 79 86 88 89 91 104 113 114 116 124 133 134 136 140 148 150 153 160 167 168 171 180 189 194 198 203 204 220 224 227 247 249 253 254 261 262 264 265 270 292 301 303 320 327 328 338 346 363 376 378 385 387 392 402 405 435 436 442 450 485 489 490 500 517 518 520 521 529 579 580 628 649 650 651 678 687 688 722 733 770 772 774 787 800 843 864 867 901 961 972 1003
12	71 92 93 95 105 106 115 117 118 119 120 137 141 149 151 152 154 156 161 172 174 175 176 181 185 190 199 205 206 208 221 222 228 232 234 248 255 263 266 271 272 280 283 293 294 296 297 302 304 306 321 329 330 332 333 339 340 347 360 364 366 377 379 381 386 388 390 393 394 403 404 406 410 437 438 443 448 451 452 459 491 492 501 502 504 507 519 522 528 530 539 567 581 582 585 588 600 629 640 652 654 656 675 679 689 690 723 724 734 735 737 738 750 756 773 775 777 788 801 802 810 844 865 866 868 870 882 902 962 968 973 974 975 976 1004
13	94 107 138 142 155 157 158 159 173 177 178 182 186 187 191 207 209 223 229 231 233 235 240 267 268 273 274 275 281 284 295 298 305 307 308 309 322 331 334 336 337 341 342 348 349 351 352 365 367 369 370 380 382 389 391 395 396 407 408 411 415 416 425 439 440 444 449 453 454 455 460 464 476 493 494 495 498 503 505 506 508 510 523 524 531 537 540 544 548 568 574 583 584 586 589 591 592 593 594 601 602 603 605 606 612 630 631 633 634 641 645 653 655 657 664 680 691 692 700 702 704 720 725 726 736 739 745 747 751 752 753 757 776 778 780 783 789 790 792 793 803 804 808 811 820 845 869 871 872 873 875 883 884 891 896 903 909 963 969 970 977 978 980 999 1005 1006 1008 1009
14	139 143 179 183 184 188 210 212 230 236 237 238 241 269 276 279 282 285 286 299 310 312 315 316 319 323 335 350 353 356 359 368 371 372 383 397 398 399 409 412 417 420 426 445 456 461 462 465 468 472 475 477 480 496 499 509 511 525 526 527 532 536 538 541 542 545 549 550 560 561 566 569 575 587 590 595 604 607 608 609 610 613 632 635 637 642 646 658 660 665 666 672 681 682 684 685 693 694 701 703 705 707 715 721 727 728 740 741 746 748 754 755 758 759 761 762 765 779 781 791 794 795 805 806 809 812 815 816 821 825 830 832 833 846 847 849 850 874 876 880 885 886 892 897 904 910 918 924 925 928 936 960 964 971 979 981 982 984 985 1007 1010 1014 1016
15	211 213 214 239 277 278 287 311 313 314 317 318 354 355 357 373 374 413 414 418 421 423 424 427 429 446 447 457 458 463 466 469 470 473 478 481 483 497 533 534 543 546 551 555 562 570 596 597 599 611 614 615 616 618 621 624 636 638 643 644 647 659 661 662 663 667 668 670 673 674 683 695 696 698 706 708 714 716 742 743 744 749 760 763 764 766 782 796 798 807 813 814 817 822 824 826 829 831 834 836 837 840 848 851 854 855 857 858 877 878 879 881 887 888 889 893 898 905 906 908 911 912 913 919 920 926 927 929 931 935 937 945 950 952 957 965 983 986 987 988 990 996 1011 1012 1015 1017 1018 1020
16	215 358 419 422 428 430 467 471 474 479 482 535 547 552 556 557 558 559 563 564 565 571 572 573 598 617 619 620 622 639 669 671 697 699 709 711 712 713 717 718 767 797 799 818 819 823 827 828 835 838 852 853 856 859 861 890 894 899 907 914 915 916 917 921 922 930 932 938 944 946 951 953 954 958 966 967 989 991 992 993 997 998 1013 1019 1021 1022 1023
17	431 553 554 623 710 719 839 860 862 895 923 933 939 940 941 942 947 948 949 955 956 959 994 995

first three are given the value 1 and the last is given the value zero. Thus $\overline{K}(y|n) = \overline{K}(2^n - 1 - y|n)$. Table II gives the values of $\overline{K}(y|n)$ for $y < 2^{10}$ and $n = 10$.

b. Modulo a prime p complexity. In addition to the operations of +, x, and xx, the operation of mod_p is allowed and is defined by $mod_p(x) = x - p[x/p]$ where p is a fixed prime and [] denotes the greatest integer. Table III gives the modulo prime p = 137 complexity for integers < 137. Table IV gives the modulo prime p = 1009 complexity for integers < 1009.

III. COMPLICATED NUMBERS

One defines n to be a complicated number if $|n| > |n_1|$ for every $n_1 < n$. The complicated numbers $< 2^{10}$ are 1, 2, 3, 4, 5, 7, 11, 13, 21, 23, 41, 43, 71, 94, 139, 211, 215, 431, and 863. (Those underlined are also prime.) Obviously, there are an infinity of complicated numbers. We propose the following conjectures.

a. There exists K such that all complicated numbers $K_1 > K$ are prime.

b. Every sufficiently large integer n is the sum of $k < \log n$ complicated integers.

c. There exists c such that every sufficiently large n satisfies $|n| < c + \sqrt{\log n}$.

IV. COMPLEXITY AND ENTROPY

Kolmogorov[4,5] has introduced the notion of complexity of a finite string over a given alphabet. For simplicity, suppose the alphabet to be {0,1}. Let A be an algorithm that transforms finite binary sequences into binary sequences. By an algorithm is meant any of the various equivalent concepts used in logic. For a binary string x, one defines the complexity by

$$K_A(x) = \begin{cases} \min\limits_{A(p)=x} \ell(p) \\ \infty \quad \text{if no p exists such that } A(p) = x, \end{cases}$$

where $\ell(p)$ denotes the length of the binary string p. Analogously, one defines conditional complexity. Let A(p,x) be an algorithm defined from pairs of binary strings to binary strings. Put

$$K_A(y|x) = \begin{cases} \min\limits_{A(p,x)=y} \ell(p) \\ \infty \quad \text{if no p exists such that } A(p,x) = y. \end{cases}$$

$K_A(y|x)$ is called the conditional complexity of y with respect to x. Kolmogorov regards complexity as analogous to entropy. We make the following conjecture.

Conjecture. Let a discrete binary information source S in the sense of Shannon[6] be given with entropy $H = -p \log p - (1-p) \log (1-p)$ where probability (0) = p and probability (1) = 1 -p; $0 < p < 1$. Let $\{x_1, x_2, \ldots, x_{2^n}\}$ be the set of all binary strings of length n arranged in order of decreasing probability. Let k(n) be the least integer so that $\sum_{i=1}^{k(n)} \text{prob}(x_i) > r$ where $1/2 < r < 1$. Then asymptotically for large n,

$$H \approx \frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n). \tag{1}$$

(In Eq. (1), K_A should be normalized so that when p = 1/2,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n) = 1.)$$

In other words, the most likely sequences from A have complexity approximately equal to the entropy of S.

In order to test the conjecture expressed in Eq. (1), we replaced $K_A(x_i|n)$ by $\lambda\overline{K}(y|n)$, where λ is selected so that when p = 1/2,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \lambda\overline{K}(x_i|n) = 1.$$

Graphs of $H_1 = -p \log p - (1-p) \log (1-p)$ and

$$H_2 = \frac{1}{k(n)} \sum_{i=1}^{n} \lambda\overline{K}(x_i|n)$$

when n = 10 and r = .75 are shown in Fig. 1

TABLE II. COMPLEMENT COMPLEXITY OF INTEGERS $< 2^{10}$.

Complement Complexity	Integer																			
0	0	1	1022	1023																
1	2	1021																		
2	3	1020																		
3	4	1019																		
4	5	6	8	9	1014	1015	1017	1018												
5	7	10	16	27	996	1007	1013	1016												
6	11	12	15	17	18	25	26	28	32	36	64	81	256	511	512	767	942	959	987	991
	995	997	998	1005	1006	1008	1011	1012												
7	13	14	19	20	24	29	31	33	35	37	49	54	63	65	80	82	100	125	128	216
	243	255	257	294	510	513	729	766	768	780	807	895	898	923	941	943	958	960	969	974
	986	988	990	992	994	999	1003	1004	1009	1010										
8	21	22	23	30	34	38	48	50	52	53	55	56	62	66	72	79	83	99	101	121
	124	126	127	129	144	162	215	217	225	239	242	244	254	258	269	293	295	324	343	347
	398	509	514	625	676	680	699	728	730	734	765	769	779	781	784	798	806	808	861	879
	894	896	897	899	902	922	924	940	944	951	957	961	967	968	970	971	973	975	985	989
	993	1000	1001	1002																
9	39	40	45	47	51	57	58	61	67	70	71	73	74	75	78	84	96	98	102	108
	120	122	123	130	143	145	160	161	163	164	169	182	192	196	200	214	218	224	226	238
	240	241	245	250	253	259	288	290	292	296	323	325	342	344	346	348	361	397	399	400
	432	435	447	486	508	515	537	576	588	591	623	624	626	662	675	677	679	681	698	700
	727	731	733	735	764	770	773	778	782	783	785	797	799	805	809	823	827	831	841	854
	859	860	862	863	878	880	893	900	901	903	915	921	925	927	939	945	948	949	950	952
	953	956	962	965	966	972	976	978	983	984										
10	41	42	44	46	59	60	68	69	76	77	85	87	90	93	95	97	103	104	105	106
	107	109	110	111	112	119	131	132	135	141	142	146	147	158	159	165	166	168	170	181
	183	189	191	193	195	197	198	199	201	202	213	219	223	227	237	246	248	249	251	252
	260	287	291	297	300	322	326	329	337	341	345	349	360	362	375	384	396	401	430	431
	433	434	436	437	441	445	446	448	450	478	484	485	487	488	494	507	516	529	535	536
	538	539	545	573	575	577	578	582	586	587	589	590	592	593	622	627	639	648	661	663
	674	678	682	686	694	697	701	723	726	732	736	763	771	772	774	775	777	786	796	800
	804	810	821	822	824	825	826	828	830	832	834	840	842	853	855	857	858	864	865	876
	877	881	882	888	891	892	904	911	912	913	914	916	917	918	919	920	926	928	930	933
	936	938	946	947	954	955	963	964	977	979	981	982								
11	43	86	88	89	91	92	94	113	114	116	118	133	134	136	138	140	148	150	153	156
	157	167	171	180	184	186	188	190	194	203	204	208	212	220	222	228	229	234	236	247
	261	262	264	265	270	286	298	299	301	303	306	320	321	327	328	330	331	335	336	338
	339	340	350	359	363	364	372	373	374	376	377	378	381	383	385	387	392	395	402	405
	428	429	438	439	440	442	443	444	449	451	452	476	477	479	480	482	483	489	490	493
	495	500	502	503	504	505	506	517	518	519	520	521	523	528	530	533	534	540	541	543
	544	546	547	571	572	574	579	580	581	583	584	585	594	595	618	621	628	631	636	638
	640	642	645	646	647	649	650	651	659	660	664	673	683	684	685	687	688	692	693	695
	696	702	703	717	720	722	724	725	737	753	758	759	761	762	776	787	789	794	795	801
	803	811	815	819	820	829	833	835	837	839	843	852	856	866	867	870	873	875	883	885
	887	889	890	905	907	909	910	929	931	932	934	935	937	980						
12	115	117	137	139	149	151	152	154	155	172	174	175	176	179	185	187	205	206	207	209
	210	211	221	230	231	232	233	235	263	266	267	269	271	272	273	279	280	282	283	284
	285	302	304	305	307	309	315	316	318	319	332	333	334	351	358	365	366	367	369	371
	379	380	382	386	388	390	391	393	394	403	404	406	410	416	423	426	427	453	454	456
	459	474	475	481	491	492	496	498	499	501	522	524	525	527	531	532	542	548	549	564
	567	569	570	596	597	600	607	613	617	619	620	629	630	632	633	635	637	641	643	644
	652	654	656	657	658	665	672	689	690	691	704	705	707	708	714	716	718	719	721	738
	739	740	741	743	744	750	751	752	754	756	757	760	788	790	791	792	793	802	812	813
	814	816	817	818	836	838	844	847	848	849	851	868	869	871	872	874	884	886	906	908
13	173	177	178	268	274	275	276	277	278	281	308	310	312	314	317	352	353	354	355	356
	357	368	370	389	407	408	409	411	415	417	418	420	421	422	424	425	455	457	458	460
	463	464	465	468	472	473	497	526	550	551	555	558	559	560	563	565	566	568	598	599
	601	602	603	605	606	608	612	614	615	616	634	653	655	666	667	668	669	670	671	706
	709	711	713	715	742	745	746	747	748	749	755	845	846	850						
14	311	313	412	413	414	419	461	462	466	467	469	470	471	552	553	554	556	557	561	562
	604	609	610	611	710	712														

TABLE III. MODULO PRIME p = 137 COMPLEXITY OF INTEGERS < 137.

Complexity	Integer																			
0	1																			
1	2																			
2	3																			
3	4																			
4	5	6	8	9																
5	7	10	16	27																
6	11	12	17	18	25	28	32	36	64	81	101	119								
7	13	14	15	19	20	24	26	29	33	37	44	49	50	54	61	65	79	82	92	100
	102	106	120	122	125	128														
8	21	22	30	34	38	41	45	48	51	55	56	60	62	63	66	68	69	72	77	80
	83	88	93	99	103	107	109	117	118	121	123	126	129	130	132	133				
9	23	31	35	39	40	42	46	47	52	53	57	58	59	67	70	73	74	75	76	78
	84	87	89	94	96	98	104	108	110	111	112	113	115	124	127	131	134	136		
10	0	43	71	85	86	90	95	97	105	114	116	135								
11	91																			

V. COMPLEXITY OF N-TUPLES OF INTEGERS

Matijasevič[7] has proved the following theorem. There exists a fifth-degree polynomial $Q(y_1, \ldots, y_k; z)$ with integer coefficients such that any enumerable set m of natural numbers (for example, the set of prime numbers) coincides with the set of natural values of the polynomial $Q(y_1, \ldots, y_k; a_m)$ where a_m is a certain number effectively constructed for the set m. From the result, it follows that if one could discuss complexity of n-tuples of integers, then one could discuss the complexity of enumerable sets of natural numbers by equating such complexity to the complexity of the associated polynomial Q.

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TABLE IV. <u>MODULO PRIME</u> p = 1009 COMPLEXITY OF INTEGERS < 1009.

Complexity	Integer																			
0	1																			
1	2																			
2	3																			
3	4																			
4	5	6	8	9																
5	7	10	16	27																
6	11	12	17	18	25	28	32	36	64	81	256	512								
7	13	14	15	19	20	24	26	29	33	37	49	54	65	82	100	125	128	216	243	257
	507	513	548	729	960															
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15	856																			

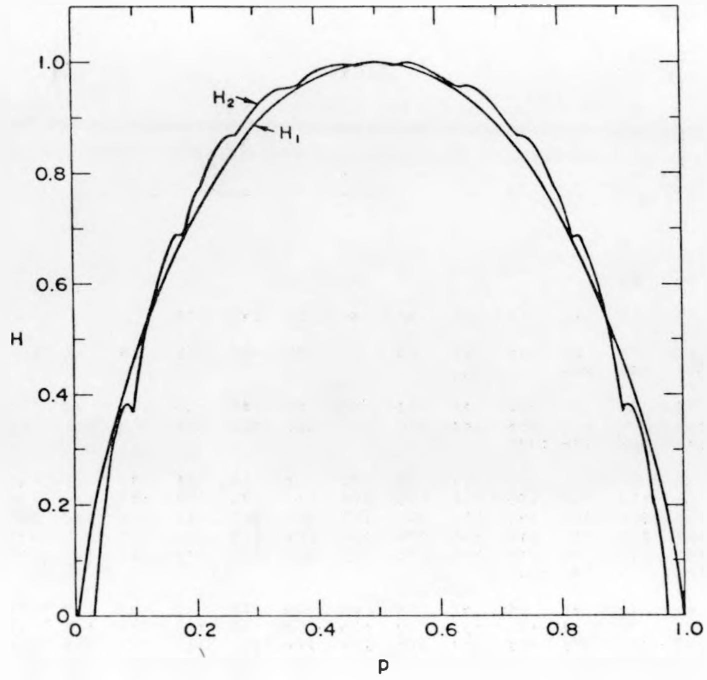

Fig. 1. Comparison of entropy $H_1 = -\sum p_i \log p_i$ and complement complexity H_2 as defined and discussed in text.

KT/bk: 304 (50)