CPT 128
ORO 3992-68
January 1972

THE EXISTENCE OF CROSSING-SYMMETRIC AND UNITARY AMPLITUDES IN S-MATRIX THEORY*†

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I. Introduction

As you know, in 1958 Mandelstam,(1) following some similar but not identical proposals by other people, suggested that a two-particle strong-interaction amplitude should be simultaneously analytic in the two independent variables upon which it depends, say s and t. He did more than this, in that he wrote down a double-spectral representation, which of course is now generally called the Mandelstam representation. He did this by combining partly heuristic, and partly rigor-

*Lectures given at the University of Texas at Austin.

†Supported in part by the U.S. Atomic Energy Commission under contract #AEC AT(40-1)3992.

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ous reasoning, as we shall see. The rigorous part was a combination of the elastic unitarity condition in one channel with a dispersion relation in the crossed channel, while the nonrigorous part was an Ansatz that respected crossing symmetry, without spoiling elastic unitarity in the elastic region. In this way, he was able to write down an expression for the double-discontinuity of the amplitude. What I propose to do first is to remind you how he did this, and then to point out a slight generalization of this method.

You will recall that in the years following 1958, people were very interested in finding solutions of the system of Mandelstam equations. This is not an easy undertaking, since the equations are nonlinear, thanks to the quadratic nature of the unitarity condition. Hideous approximations were made, divergent series were truncated merrily at their second terms; and the field was obfuscated by unnecessary discussions about bootstraps. The situation is in principle straightforward: we have a nonlinear, singular equation, and we want to know if there are solutions; and, if so, how they may be obtained. This is the question we shall consider in these lectures, and we shall find at least a partial answer.

To summarize our findings: we discover that the Mandelstam equations for the $\pi\pi$ system have many solutions, consistent with crossing symmetry and elastic unitarity. A subset of these solutions also satisfy the inelastic constraints above the inelastic threshold. There are two sources of non-uniqueness. One source is the freedom to insert inelastic contributions in the interior of the double-spectral function; and the other source is the freedom to include CDD poles[2] in a finite number of partial waves. We shall follow the treatment of a number of recently published papers,[3] as we explain the details of the proofs. For simplicity, I shall omit isospin, but this can be included without too much extra complication.[3]

II. Elastic Unitarity

Let p_1, p_2, p_3, p_4 be the four-momenta of the pions (Fig. 1). We will use the following two variables:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2. \qquad (2.1)$$

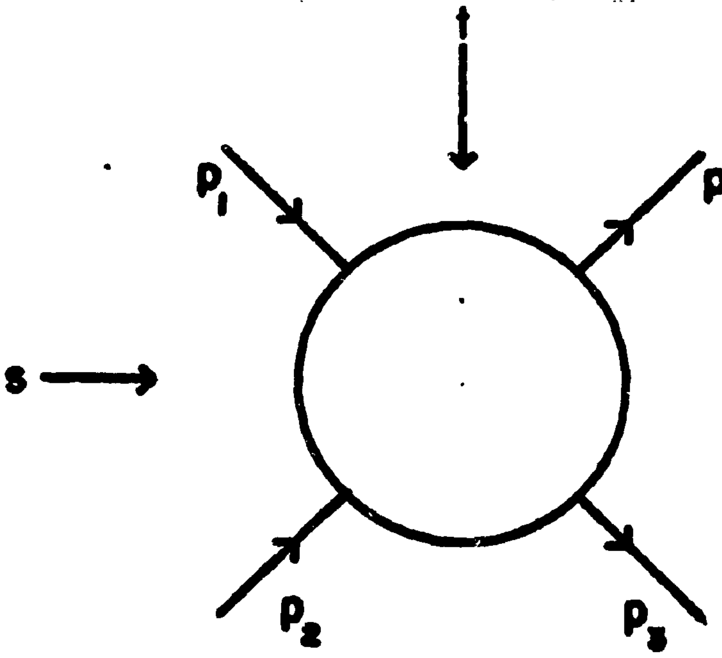

Figure 1

Definitions of Kinematic Variables

The two-pion scattering amplitude, A(s,t), is a function of these two variables. In the so-called elastic region, $4 \leq s \leq 16$, in which only two pions are allowed in the intermediate state (Fig. 2), the elastic unitarity condition is exact. It may be written

$$\text{Im } A(s,t) = \frac{q}{4\pi} \int d\Omega' \; A^*(s,t') \; A(s,t''), \qquad (2.2)$$

where

$$q = \left(\frac{s - 4}{s}\right)^{\frac{1}{2}} . \tag{2.3}$$

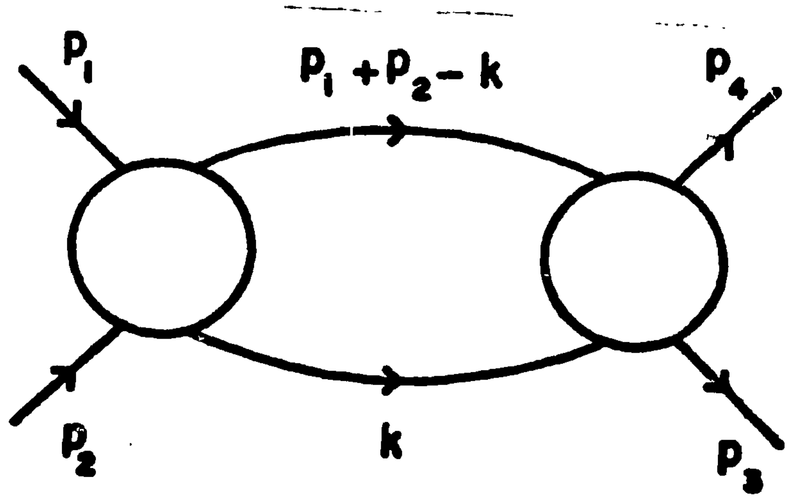

Figure 2

Elastic Unitarity Diagram

Consider the s-channel center-of-mass frame, defined by

$$\vec{p}_1 + \vec{p}_2 = 0. \tag{2.4}$$

With $p \equiv | \vec{p}_1 |$, one finds

$$s = 4(p^2 + 1)$$

$$t = -2p^2(1 - z), \tag{2.5}$$

where z is the cosine of the angle between the initial and final center-of-mass directions. Analogously,

$$t' = -2p^2(1 - z'),$$

$$t'' = -2p^2(1 - z''), \tag{2.6}$$

which implies, and is implied by

$$A_t(s,t) = A_u(s,u). \tag{2.10}$$

Substitute (2.8) into (2.2):

$$\begin{aligned}\mathrm{Im}\,A(s,t) = {} & \frac{q}{4\pi^3}\int d\Omega'\int\frac{dt_1}{t_1-t'}\,A_t^*(s,t_1)\int\frac{dt_2}{t_2-t''}\,A_t(s,t_2)\\ & + \frac{q}{4\pi^3}\int d\Omega'\int\frac{dt_1}{t_1-t'}\,A_t^*(s,t_1)\int\frac{du_2}{u_2-u''}\,A_u(s,4-s-u_2)\\ & + \frac{q}{4\pi^3}\int d\Omega'\int\frac{du_1}{u_1-u'}\,A_u^*(s,4-s-u_1)\int\frac{dt_2}{t_2-t''}\,A_t(s,t_2)\\ & + \frac{q}{4\pi^3}\int d\Omega'\int\frac{du_1}{u_1-u'}\,A_u^*(s,4-s-u_1)\int\frac{du_2}{u_2-u''}\,A_u(s,4-s-u_2)\ .\end{aligned} \tag{2.11}$$

Let us write

$$\mathrm{Im}\,A = \mathrm{Im}\,A_1 + \mathrm{Im}\,A_2 + \mathrm{Im}\,A_3 + \mathrm{Im}\,A_4, \tag{2.12}$$

corresponding to the four terms of (2.11), and consider $\mathrm{Im}\,A_1$ first. Change from t's to z's, using (2.5), (2.6) and, analogously,

$$\begin{aligned} t_1 &= -2p^2(1 - z_1)\\ t_2 &= -2p^2(1 - z_2). \end{aligned} \tag{2.13}$$

Then we obtain

$$\mathrm{Im}\,A_1(s,t) =$$

$$\frac{q}{4\pi^3}\int_{z_0}^{\infty} dz_1 A_t^*(s,t_1)\int_{z_0}^{\infty} dz_2 A_t(s,t_2)\int\frac{d\Omega'}{(z_1-z')(z_2-z'')}. \tag{2.14}$$

where t_1, t_2 are to be regarded as functions of z_1, z_2 (and s), and $z_0 = 1 + 2/p^2$. We will now evaluate the integral

$$I(z) = \int_{-1}^{+1} \frac{dz'}{z'-z_1} \int_0^{2\pi} \frac{d\phi'}{zz_1 + \cos\phi' [(1-z^2)(1-z_1^2)]^{\frac{1}{2}} - z_2}. \qquad (2.15)$$

The ϕ'-integral can be performed with the help of the formula

$$\int_0^{2\pi} \frac{d\phi'}{A + B\cos\phi'} = \frac{2\pi}{(A^2 - B^2)^{\frac{1}{2}}} . \qquad (2.16)$$

This gives

$$I(z) = 2\pi \int_{-1}^{1} \frac{dz'}{z'-z_1} \; k^{-\frac{1}{2}}(z,z',z_2), \qquad (2.17)$$

where k is defined by

$$k(z,z',z_2) = z^2 + z'^2 + z_2^2 - 2zz'z_2 - 1. \qquad (2.18)$$

This kernal has many interesting properties. It will often be useful to write it as

$$k(z,z',z_2) = [z' - z_+(z,z_2)][z' - z_-(z,z_2)], \qquad (2.19)$$

where $z_\pm$ are the two roots

$$z_\pm(z,z_2) = zz_2 \pm [(1 - z^2)(1 - z_2^2)]^{\frac{1}{2}}. \qquad (2.20)$$

The integral (2.17) is a little messy to do, so I will just state the answer:

$$I(z) = 2\pi k^{-\frac{1}{2}}(z,z_1,z_2) \log \frac{z - z_1z_2 + k^{\frac{1}{2}}(z,z_1,z_2)}{z - z_1z_2 - k^{\frac{1}{2}}(z,z_1,z_2)} . \qquad (2.21)$$

In working through this, remember that

$$z_1, z_2 \geq 1 + \frac{4}{s-4} > 1 \geq z \tag{2.22}$$

in the physical region of the s-channel.

For reasons that will become clear in a moment, I want to write a dispersion relation for $I(z)$. To do this, we must investigate the singularities and the asymptotic behavior of $I(z)$. At first sight, one might expect logarithmic branch-points when

$$z - z_1 z_2 \pm k^{\frac{1}{2}}(z, z_1, z_2) = 0, \tag{2.23}$$

but then

$$z^2 - 2zz_1z_2 + z_1^2 z_2^2 = z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1. \tag{2.24}$$

Since z concels out of this equation, it can never be satisfied (for $z_1 \neq 1$, $z_2 \neq 1$), and so (2.23) does not after all give branch-points. Since $k^{\frac{1}{2}}$ can be written in the form

$$k^{\frac{1}{2}}(z, z_1, z_2) = \{[z - z_+(z_1, z_2)][z - z_-(z_1, z_2)]\}^{\frac{1}{2}}, \tag{2.25}$$

we see that $z = z_\pm(z_1, z_2)$ will in general be branch-points. However, it turns out that, on the physical Riemann sheet, only $z = z_+(z_1, z_2)$ is in fact a branch-point. The physical sheet is specified uniquely by the requirement that $I(z)$ be real when $-1 \leq z \leq 1$. Since $z_1, z_2 > 1$, $z_+(z_1, z_2) > 1$. Moreover,

$$z_-(z_1, z_2) - 1 = \frac{(z_1 - z_2)^2}{z_+(z_1, z_2) - 1}, \tag{2.26}$$

and z, z', z'', are connected by the solid geometry relation,

$$z'' = zz' + \cos\phi'[(1 - z^2)(1 - z'^2)]^{\frac{1}{2}}, \quad (2.7)$$

where ϕ' is an azimuthal angle. The integration in (2.2) is over all intermediate directions,

$$d\Omega' = dz'\, d\phi',$$

$-1 \le z \le 1,\ 0 \le \phi' \le 2\pi.$

Following Mandelstam, we want to combine (2.2) with a fixed-s dispersion relation for A(s,t), namely

$$A(s,t) = \frac{1}{\pi}\int_4^\infty \frac{dt'\, A_t(s,t')}{t' - t} + \frac{1}{\pi}\int_4^\infty \frac{du'\, A_u(s,4 - s - u')}{u' - u}, \quad (2.8)$$

where $u = 4 - s - t$. Here A_t and A_u are the t- and u-channel absorptive parts, respectively. We will see their physical significance later.

The variables s, t, u will appear symmetrically in the Mandelstam representation. They represent, respectively, the square of the total energy when particles 1 and 2, 1 and 4, 1 and 3 are incoming, and the other two outgoing. Since we are considering pion scattering, in which all channels are identical, we will have, in particular

$$A(s,t) = A(s,u), \quad (2.9)$$

so that $z_-(z_1,z_2) \geq 1$. One finds, on continuing $I(z)$ from the physical region to z_-, and then encircling z_- once, that $k^{1/2}$ changes sign, but that the argument of the logarithm in (2.21) stays near zero. Hence $I(z)$ does not change, and so z_- is not a branch-point. However, when z_+ is reached, the logarithm changes by $2\pi i$ when z circles this point once. There is a cut from $z = z_+(z_1,z_2)$ to $z = \infty$, the discontinuity across which is

$$2\pi i[2\pi k^{-1/2}(z,z_1,z_2)]. \tag{2.27}$$

As far as the asymptotic behavior of $I(z)$ is concerned, $k^{1/2}(z,z_1,z_2) = z + 0(1)$ for large z, from which we see that

$$I(z) \underset{z\to\infty}{\sim} \frac{\log z}{z} \quad . \tag{2.28}$$

It follows from the above analysis of $I(z)$ that we may write it as the dispersion relation

$$I(z) = 4\pi \int_{z_+(z_1,z_2)}^{\infty} \frac{dz'}{z'-z} k^{-1/2}(z',z_1,z_2). \tag{2.29}$$

According to (2.14), we will have

$$\mathrm{Im}\, A_1(s,t) =$$

$$\frac{2}{\pi^2}\int_{z_0}^{\infty} dz_1 A_t^*(s,t_1)\int_{z_0}^{\infty} dz_2 A_t(s,t_2)\int_{z_+(z_1,z_2)}^{\infty}\frac{dz'}{z'-z} k^{-1/2}(z',z_1,z_2)$$

$$= \frac{q}{\pi^2}\int_{z_+(z_0,z_0)}^{\infty}\frac{dz'}{z'-z}\iint_{z_0 z_0}^{z'>z_+(z_1,z_2)} dz_1 dz_2 A_t^*(s,t_1)A_t(s,t_2)k^{-1/2}(z',z_1,z_2). \tag{2.30}$$

The expression (2.30) allows us to continue $\mathrm{Im}\, A_1(s,t)$ from $-1 \leq z \leq 1$ into the entire z plane. We should not call it $\mathrm{Im}\, A_1$ now, but rather A_{s1}, the first of the four contributions to the s-channel absorptive part. Eq. (2.12) becomes

$$A_s = A_{s1} + A_{s2} + A_{s3} + A_{s4}. \tag{2.31}$$

The t-discontinuity (or the z-discontinuity) of (2.30) may be called A_{st1}, and it may be written down by inspection from eq. (2.30):

$$A_{st1}(s,t) =$$

$$\frac{q}{\pi^2}\,\theta[z-z_+(z_0,z_0)]\iint_{z_0 z_0}^{z>z_+(z_1,z_2)} dz_1 dz_2 k^{-\frac{1}{2}}(z,z_1,z_2)A_t^*(s,t_1)A_t(s,t_2). \tag{2.32}$$

We need to analyze the limits in (2.32) in a little greater detail. Let us write (2.32) as a repeated integral $\int dz_1 \cdot \int dz_2$. The upper limit of the z_2 integral is given by $z = z_+(z_1,z_2)$, i.e., by $z_2 = z_-(z,z_1)$. The upper limit of the z_1 integral is the maximum value of z_1 for which $z_+(z_1,z_2) = z$, with $z_2 \geq z_0$. This point corresponds to $z_2 = z_0$ (see Fig. 3), and so the upper limit is $z_1 = z_+(z,z_0)$. The double-spectral function (2.32) vanishes when $z < z_+(z_0,z_0) \equiv 2z_0^2 - 1$.

Now let me sketch briefly what happens with the other three terms in (2.31). The term A_{s4}, corresponding to the

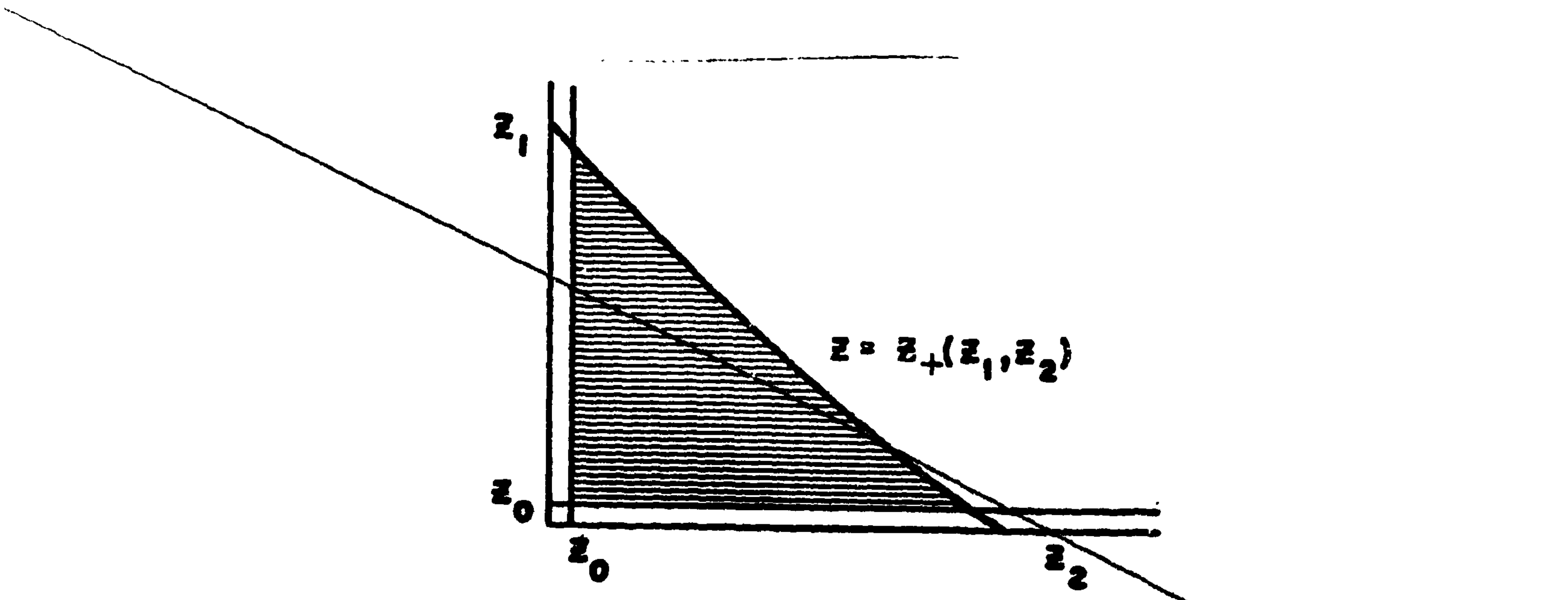

Figure 3

Boundary of $z_1 z_2$ Integration

last term in (2.11), turns out to be just equal to A_{s1}. On the other hand, A_{s2} gives a term like (2.30), but with z replaced by -z. The cut runs from $z = -[2z_0^2 - 1]$ to $-\infty$, which corresponds to positive u, so the discontinuity of A_{s2} is reasonably called A_{su2}. The term A_{s3} is exactly equal to A_{s2}. We will now collect these four terms together, and return to the variables t (and u) instead of z:

$$A_s(s,t) =$$

$$\frac{1}{\pi}\int_{\frac{16s}{s-4}}^{\infty} \frac{dt'}{t'-t}\, \rho^{e\ell}(s,t') + \frac{1}{\pi}\int_{\frac{16s}{s-4}}^{\infty} \frac{du'}{u'-u}\, \rho^{e\ell}(s,u'), \tag{2.33}$$

where

$$\rho^{e\ell}(s,t) =$$

$$\int_4^{g(s;t,4)} dt_1 \int_4^{g(s;t,t_1)} dt_2 K(s;t,t_1,t_2) A_t^*(s,t_1) A_t(s,t_2), \tag{2.34}$$

with

$$K(s;t,t_1,t_2) =$$

$$\frac{4}{\pi}\{s(s-4)[t^2+t_1{}^2+t_2{}^2-2tt_1-2tt_2-2t_1t_2-\frac{4tt_1t_2}{s-4}]\}^{-\frac{1}{2}} \tag{2.35}$$

and

$$g(s;t,t_1) =$$

$$t + t_1 + \frac{2tt_1}{s-4} - 2\{tt_1(1+\frac{t}{s-4})(1+\frac{t_1}{s-4})\}^{\frac{1}{2}}. \tag{2.36}$$

The equation (2.33) has been demonstrated only for $4 \leq s \leq 16$, the s-channel physical region. In this region, the t-discontinuity of $A_s(s,t)$, for positive t, is of course

$$A_{st}(s,t) = \theta[t - \frac{16s}{s-4}]\rho^{e\ell}(s,t), \tag{2.37}$$

The expression (2.34) is well-defined for all s and t, but the equality (2.37) only holds for $4 \leq s \leq 16$. We could have done the whole calculation by combining t-channel unitarity with a fixed t-dispersion relation, and then we could have obtained the s-discontinuity of A_t for positive s. We will assume that the s-discontinuity of the t-discontinuity of

$A(s,t)$ is the same as the t-discontinuity of the s-discontinuity, so we would then have the result

$$A_{st}(s,t) = \theta[s - \frac{16t}{t-4}]\rho^{e\ell}(t,s) \tag{2.38}$$

for $4 \leq t \leq 16$. Since the right-hand side of (2.37) vanishes for all $s \geq 4$ and $t \leq 16$, and the right-hand side of (2.38) vanishes for all $t \geq 4$ and $s \leq 16$, it follows that the function

$$v(s,t) = A_{st}(s,t) - \theta[t-\frac{16s}{s-4}]\rho^{e\ell}(s,t) - \theta[s-\frac{16t}{t-4}]\rho^{e\ell}(t,s) \tag{2.39}$$

must vanish for $4 \leq s \leq 16$ <u>and</u> for $4 \leq t \leq 16$. The Mandelstam assumption is that $v(s,t)$ is non-zero only for $s > 16$ and $t > 16$, and that there are no complex branch-points in $s \times t$. So for all s and t, eq. (2.33) is replaced by

$$A_s(s,t) = \frac{1}{\pi}\int^{\infty} \frac{dt'}{t'-t}\,\rho(s,t') + \frac{1}{\pi}\int^{\infty} \frac{du'}{u'-u}\,\rho(s,u'), \tag{2.40}$$

where

$$\rho(s,t) \equiv A_{st}(s,t) = \theta[t-\frac{16s}{s-4}]\rho^{e\ell}(s,t) + \theta[s-\frac{16t}{t-4}]\rho^{e\ell}(t,s) + v(s,t). \tag{2.41}$$

The support of $\rho(s,t)$ is shown in Fig. 4.

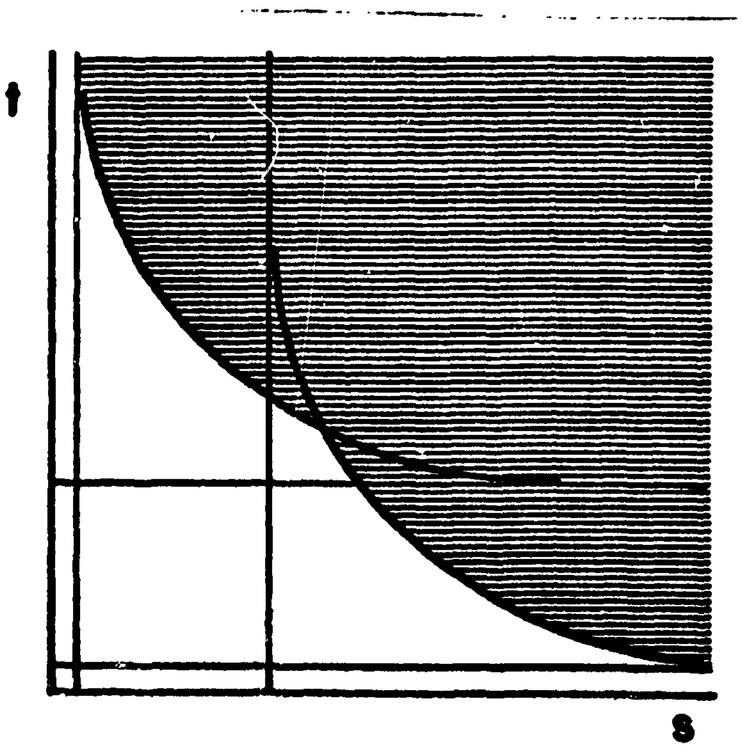

Figure 4

Support of $\rho(s,t)$

Lastly, I will derive the Mandelstam representation. A fixed-t dispersion relation for A(s,t), analogous to eq. (2.8), would be

$$A(s,t) =$$

$$\frac{1}{\pi}\int_4^{\infty}\frac{ds'}{s'-s}\,A_s(s',t) + \frac{1}{\pi}\int_4^{\infty}\frac{du'}{u'-u}\,A_u(4-t-u',t)\,. \qquad (2.42)$$

If we substitute

$$A_s(s',t) =$$

$$\frac{1}{\pi}\int^{\infty}\frac{dt'}{t'-t}\,\rho(s',t') + \frac{1}{\pi}\int^{\infty}\frac{du'}{u'-4+s'+t}\,\rho(s',u') \qquad (2.43)$$

and

$$A_u(4-t-u',t) =$$

$$\frac{1}{\pi}\int^{\infty} \frac{dt'}{t'-t}\,\rho(u',t') + \frac{1}{\pi}\int^{\infty} \frac{ds'}{s'-4+t+u'}\,\rho(u',s') \qquad (2.44)$$

into (2.42) we get

$$A(s,t) =$$

$$\frac{1}{\pi^2}\iint \frac{ds'dt'\rho(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\iint \frac{du'dt'\rho(u',t')}{(u'-u)(t'-t)}$$

$$+ \frac{1}{\pi^2}\iint ds'du' \frac{\rho(u',s')}{(s'-s)(u'+s'+t-4)} + \frac{\rho(u',s')}{(u'-u)(u'+s'+t-4)} \qquad (2.45)$$

where use has been made of the symmetry of $\rho(u',s')$. This reduces easily to

$$A(s,t) = f(s,t) + f(t,u) + f(u,s), \qquad (2.46)$$

where

$$f(s,t) = \frac{1}{\pi^2}\iint \frac{ds'dt'\rho(s',t')}{(s'-s)(t'-t)} \quad . \qquad (2.47)$$

III. Existence Proof

I am now going to show how to demonstrate the existence of solutions of the nonlinear system of equations that we have set up. Let us gather the key equations together. If we knew A_t, we could calculate ρ^{el} from

$$\rho^{el}(s,t) = \theta[t-\frac{16s}{s-4}]\iint dt_1 dt_2 K(s;t,t_1,t_2)A_t^*(s,t_1)A_t(s,t_2), \tag{3.1}$$

which is eq. (2.34), except that I have chosen to combine the θ-function into the definition of ρ^{el}. How do we know A_t? If we knew ρ^{el}, we would have, by combining eq. (2.41) with the t-channel absorptive part of eq. (2.46),

$$A_t(s,t) =$$
$$\frac{1}{\pi}\int^{\infty} ds[\frac{1}{s'-s} + \frac{1}{s'-u}][\rho^{el}(s',t)+\rho^{el}(t,s')+v(s',t)], \tag{3.2}$$

where v is to be a given function, and where the lower limit of integration is defined by the θ-function in eq. (3.1). The support of the double spectral function is sketched in Fig. 4.

Evidently the system (3.1)-(3.2) provides a nonlinear integral equation for ρ^{el}. Let us summarize it by

$$\rho^{el}(s,t) = T[\rho^{el}; s,t]. \tag{3.3}$$

I have suppressed the dependence of the operator, T, on the function, $v(s,t)$. It may be thought of as a parameter (but an infinite dimensional one!) Sometimes, I will suppress also the independent variables, s and t. There is nothing metaphysical about eq. (3.3): it merely summarizes eqs. (3.1)-(3.2).

I am actually going to apply the Contraction Mapping Principle[4] to eq. (3.3). This is a rigorous way of discussing when the iteration,

$$\rho_{n+1}^{e\ell} = T[\rho_n^{e\ell}], \tag{3.4}$$

converges to some limiting function,

$$\rho_n^{e\ell} \xrightarrow[n\to\infty]{} \rho_*^{e\ell}\ , \tag{3.5}$$

which satisfies eq. (3.3), i.e. for which

$$\rho_*^{e\ell} = T[\rho_*^{e\ell}]. \tag{3.6}$$

My discussion of the Contraction Mapping Theorem, or Banach-Cacciopoli Principle, will be, for the most part, general; but it may be helpful to consider in particular the equations (3.1)-(3.2) to concretize our ideas, and of course we are indeed specifically interested in these equations.

First of all, we have to define a space of functions in which we are going to work. The space we use depends on the nature of the equations, but there is no general way of finding a suitable space in which a given equation will contract: this initial step is a work of art. We want to be sure that, if $\rho_0^{e\ell}(s,t)$, the zeroth step in the iteration, has the properties which ensure that it belongs to the space, then $\rho_1^{e\ell}(s,t)$ also has these properties, and likewise $\rho_2^{e\ell}(s,t)$, and so on, so that the infinite sequence of iterates lies in the function space. The equation (3.2) contains a Cauchy singular integral, so we must restrict $\rho^{e\ell}$ to belong to a space that ensures the convergence of this singular integral. It would not be enough actually to require continuity, because, although the principal-value integral of a continuous function exists, it is not necessarily continuous. The equations would kick a function out of a space of merely continuous functions. However, the principal-value integral of a Hölder-continuous function is itself Hölder-continuous. A function, $f(x)$, is said to be Hölder-continuous on $0 \leq x \leq 1$, if

$$|f(x_1) - f(x_2)| \leq A|x_1 - x_2|^{\mu}, \qquad (3.7)$$

for any x_1, x_2 in $[0,1]$, where A and μ are constants, the latter being called the Hölder index. We will build Hölder-continuity into our space; and there are some other fine points that we will come to later.

Let us imagine that we have a suitable space. That is to say, suppose we have the specifications of a space of functions such that, if $\rho^{e\ell}(s,t)$ belongs to the space, then $\bar{\rho}^{e\ell}(s,t)$ also belongs to it, where

$$\bar{\rho}^{e\ell}(s,t) = T[\rho^{e\ell};\ s,t]. \tag{3.8}$$

It follows that if $\rho_0^{e\ell}$ belongs to the space, then the infinite sequence $\{\rho_n^{e\ell}\}$, $n = 0, 1, 2, 3, \ldots$ will also belong to the space.

In order to demonstrate the convergence of this infinite sequence, we have to show two things, both of which assume the existence of a suitable distance function, or metric. I will always be talking about a normed space, i.e. a space in which a suitable number, the norm, $||\rho^{e\ell}||$, is associated with each function, $\rho^{e\ell}(s,t)$. The distance between two functions, $\rho_a^{e\ell}$ and $\rho_b^{e\ell}$, is defined to be $||\rho_a^{e\ell} - \rho_b^{e\ell}||$. We have to show firstly that a closed set in the space is mapped into itself, and in practice this means showing that if

$$||\rho^{e\ell}|| \leq b \tag{3.9}$$

for some particular b, then

$$||\bar{\rho}^{e\ell}|| \leq b, \tag{3.10}$$

where $\bar{\rho}^{e\ell}$ is given by eq. (3.8). Evidently, this means that if $\rho_0^{e\ell}$ lies within the ball of radius b, then the whole sequence $\{\rho_n^{e\ell}\}$ is trapped within the ball. Secondly, we must show that, if $\rho_a^{e\ell}$ and $\rho_b^{e\ell}$ are any two functions lying in the ball, and

$$\bar{\rho}_a^{e\ell} = T[\rho_a^{e\ell}], \tag{3.11}$$

$$\bar{\rho}_b^{e\ell} = T[\rho_b^{e\ell}], \tag{3.12}$$

then

$$||\bar{\rho}_a^{e\ell} - \bar{\rho}_b^{e\ell}|| \le P||\rho_a^{e\ell} - \rho_b^{e\ell}|| , \tag{3.13}$$

where P is some number such that

$$0 < P < 1. \tag{3.14}$$

Clearly, if we set $\rho_a^{e\ell} = \rho_n^{e\ell}$, we shall have $\bar{\rho}_a^{e\ell} = \rho_{n+1}^{e\ell}$, and if we set $\rho_b^{e\ell} = \rho_{n+1}^{e\ell}$, we shall have $\bar{\rho}_b^{e\ell} = \rho_{n+2}^{e\ell}$, so that (3.13) will read

$$||\rho_{n+2}^{e\ell} - \rho_{n+1}^{e\ell}|| \le P||\rho_{n+1}^{e\ell} - \rho_n^{e\ell}|| . \tag{3.15}$$

In other words, successive iterates get closer and closer together (see Fig. 5).

We can show from eq. (3.15) that the sequence $\{\rho_n^{e\ell}\}$ necessarily converges, since, if m > n,

$$||\rho_m^{e\ell}-\rho_n^{e\ell}||\le||\rho_m^{e\ell}-\rho_{m-1}^{e\ell}||+||\rho_{m-1}^{e\ell}-\rho_{m-2}^{e\ell}||+\ldots+||\rho_{n+1}^{e\ell}-\rho_n^{e\ell}||$$

$$\le \{P^{m-1}+P^{m-2}+\ldots+P^n\}||\rho_1^{e\ell}-\rho_0^{e\ell}||$$

$$= P^n \frac{1-P^{m-n}}{1-P} ||\rho_1^{e\ell}-\rho_0^{e\ell}|| \le \frac{P^n}{1-P} ||\rho_1^{e\ell}-\rho_0^{e\ell}||. \tag{3.16}$$

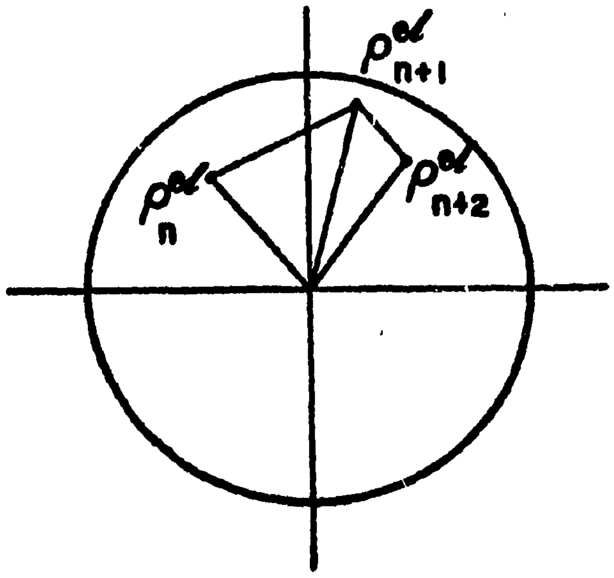

Figure 5

The General Principle of Contraction Mapping

Since $P<1$, the right-hand side of this inequality can be made as small as one likes, just by making n large enough. Hence $\{\rho^{e\ell}_n\}$ is a Cauchy sequence, in the sense of the norm: that is, the quantity $||\rho^{e\ell}_m - \rho^{e\ell}_n||$ can be made arbitrarily small, just by making m and n large enough. This means that $\{\rho^{e\ell}_n\}$ converges to some function, $\rho^{e\ell}_*$. Strictly speaking, $\rho^{e\ell}_*$ lies in the completion of our space; but we shall deal exclusively with a complete, normed, linear space (a Banach space), so we can ignore this point.

To say that $\rho^{e\ell}_n$ tends to $\rho^{e\ell}_*$ as $n \to \infty$, in the sense of the norm, means that, given any $\varepsilon > 0$, we can find an N such that

$$||\rho_m^{e\ell} - \rho_*^{e\ell}|| < \varepsilon \tag{3.17}$$

for all m > N. Therefore, from eq. (3.16),

$$\begin{aligned} ||\rho_n^{e\ell} - \rho_*^{e\ell}|| &\le ||\rho_n^{e\ell} - \rho_m^{e\ell}|| + ||\rho_m^{e\ell} - \rho_*^{e\ell}|| \\ &\le \frac{P^n}{1-P} ||\rho_1^{e\ell} - \rho_0^{e\ell}|| + \varepsilon \ . \end{aligned} \tag{3.18}$$

Since the left-hand side of this inequality, and also the final form of the right-hand side, do not depend on m, and we can make ε as small as we like, it is clear that we can drop ε, and write simply

$$||\rho_*^{e\ell} - \rho_n^{e\ell}|| \le \frac{P^n}{1-P} ||\rho_1^{e\ell} - \rho_0^{e\ell}|| \ . \tag{3.19}$$

This is a useful inequality, since it allows one to estimate the error that is involved in truncating the interation at the nth step.

We will now prove something that may seem obvious, but which actually needs to be proved, namely that $\rho_*^{e\ell}$, which has now been shown to exist, actually satisfies the equation. We have

$$\begin{aligned} ||\rho_*^{e\ell} - T(\rho_*^{e\ell})|| \le\ & ||\rho_*^{e\ell} - \rho_n^{e\ell}|| + ||\rho_n^{e\ell} - T(\rho_n^{e\ell})|| \\ & + ||T(\rho_n^{e\ell}) - T(\rho_*^{e\ell})|| \ . \end{aligned} \tag{3.20}$$

The first term on the right is bounded by (3.19), so if we are given any $\varepsilon > 0$, no matter how small, we can find an n so large that the term is less than ε. The second term in (3.20) is just

$$||\rho_n^{e\ell} - \rho_{n+1}^{e\ell}||, \tag{3.21}$$

because of eq. (3.4), and we can use (3.16), with m replaced by n + 1, to show that this too can be made smaller than ε, by making n large enough. The last term is not greater than

$$P||\rho_n^{e\ell} - \rho_*^{e\ell}|| , \tag{3.22}$$

according to the contraction condition, eqs. (3.11)-(3.13). So by eq. (3.19) again, we can make this smaller than ε. Finally, we have shown that, merely by choosing a suitable n, we can arrange that

$$||\rho_*^{e\ell} - T(\rho_*^{e\ell})|| < 3\varepsilon. \tag{3.23}$$

Since n does not appear here, and ε can be as small as one likes, the only possibility is that the left-hand side is zero. This means that eq. (3.6) is satisfied, since the only function with a zero norm is the null function itself.

This concludes the general discussion of the Contraction Mapping Theorem. Let us now look at the equations (3.1)-(3.2), with a view to applying the theorem to them. We have already indicated that Hölder-continuity will be built into the specification of the norm. We will be looking for a solution, $\rho^{e\ell}(s,t)$, in a space of functions, $f(s,t)$, that satisfy Hölder-continuity. Set $s = 4/x$ in inequality (3.7), to transform the interval (0, 1) into (4, ∞). We would like then to have

$$|f(s_1,t) - f(s_2,t)| \leq A\left|\frac{s_1-s_2}{s_1 s_2}\right|^{\mu}. \qquad (3.24)$$

However, in view of the occurrence of $\rho^{e\ell}(t,s')$ as well as $\rho^{e\ell}(s',t)$ in eq. (3.2), it is clear that we should require double Hölder-continuity, i.e.

$$|f(s_1,t_1) - f(s_2,t_2)| \leq A\left\{\left|\frac{s_1-s_2}{s_1 s_2 \bar{t}}\right|^{\mu} + \left|\frac{t_1-t_2}{t_1 t_2 \bar{s}}\right|^{\mu}\right\}, \qquad (3.24)$$

where $\bar{s} = \min(s_1,s_2)$, $\bar{t} = \min(t_1,t_2)$. If we also impose the restriction

$$f(s, \infty) = f(\infty, t), \qquad (3.26)$$

upon the functions that belong to our space, then eq. (3.25) implies

$$|f(s,t)| \leq A(st)^{-\mu}. \qquad (3.27)$$

Unfortunately, the bound (3.27) cannot be reproduced in general by eq. (3.1). In fact, if $\rho^{e\ell}$ and v in eq. (3.2). obey (3.25), we cannot show that $\rho^{e\ell}$ in eq. (3.1) satisfies a bound $t^{-\mu}$ for large t, as required by eq. (3.27), but only

$t^{-\mu}\log t$. Once again, the equations would expel functions from our space, and the Contraction Mapping Theorem would be inapplicable. However, we can show that, if

$$|A_t(s,t)| \leq Bt^{-\mu} (\log t)^{-1-\varepsilon} \tag{3.28}$$

$\varepsilon > 0$, $0 < \mu < \frac{1}{2}$, in eq. (3.1), then

$$|\rho^{e\ell}(s,t)| \leq Ct^{-\mu}(\log t)^{-1-\varepsilon}. \tag{3.29}$$

This prescribes the final form of our norm, namely

$$||f|| = \sup\left\{|f(s,t)|(st)^{\mu}(\log s\cdot\log t)^{1+\varepsilon} + \frac{|f(s_1,t_1) - f(s_2,t_2)|[\log \bar{s}\cdot\log \bar{t}]^{1+\varepsilon}}{\left|\frac{s_1-s_2}{s_1 s_2 \bar{t}}\right|^{\mu} + \left|\frac{t_1-t_2}{t_1 t_2 \bar{s}}\right|^{\mu}}\right\} \tag{3.30}$$

where the supremum is to be taken over s, t, s_1, s_2, t_1, t_2, in $(4, \infty)$. We see from eq. (3.30) that necessarily

$$|f(s,t)| \leq ||f||(st)^{-\mu}(\log s\cdot\log t)^{-1-\varepsilon}, \tag{3.31}$$

and

$$|f(s_1,t)-f(s_2,t)| \leq ||f||\left|\frac{s_1-s_2}{s_1 s_2}\right|^{\mu} t^{-\mu}(\log \bar{s}\cdot\log t)^{-1-\varepsilon}. \tag{3.32}$$

It can be shown that, if $\rho^{e\ell}$ and v in eq. (3.2) possess a norm (3.30), then $\rho^{e\ell}$ in eq. (3.1) also possesses a finite norm in this space. We say that the nonlinear operator, T, of eq. (3.3), maps the Banach space specified by the norm (3.30) into itself. The detailed proof of this involves a lot of algebra. I will not prove that, if $[\rho^{el}(s,t)+\rho^{el}(t,s)+v(s,t)]$ satisfies an inequality like (3.32), then $A_t(s,t)$, eq. (3.2) does so also. The result, that the Cauchy integral of a Hölder-continuous function is itself Hölder-continuous, is a standard one, and may be found in Muskhelishvili's book. We need a slight generalization, to carry along the extra logarithms, and the t-dependence, but this is not hard.

We will concentrate upon the physically more interesting logarithms. I will show that, if $|A_t(s,t)|$ satisfies a bound like (3.31), then $\rho^{el}(s,t)$, defined by eq. (3.1), will do so also. Suppose then that

$$|A_t(s,t)| \le C(st)^{-\mu}(\log s\cdot\log t)^{-1-\varepsilon}, \tag{3.33}$$

then

$$|\rho^{el}(s,t)| \le C^2 s^{-2\mu}(\log s)^{-2-2\varepsilon}\iint \frac{dz_1 dz_2 (t_1 t_2)^{-\mu}(\log t_1\cdot\log t_2)^{-1-\varepsilon}}{k^{\frac{1}{2}}(z,z_1,z_2)} \tag{3.34}$$

where I have changed back to the z-variable, eqs. (2.5), (2.13) and (2.18). Now the z_2-integral can be majorized by

$$\int_{z_0}^{z_-(z,z_1)} \frac{dz_2 t_2^{-\mu}(\log t_2)^{-1-\varepsilon}}{\{z_+(z,z_1)-z_-(z,z_1)\}^{\frac{1}{2}}\{z_-(z,z_1)-z_2\}^{\frac{1}{2}}} \;, \qquad (3.35)$$

where use has been made of the factorization (2.25), and of the fact that $z_2 \le z_-(z,z_1)$ in the integration. Since $t_2^{\frac{1}{2}-\mu}(\log t_2)^{-1-\varepsilon}$ can be certainly majorized by

$$C_1[\tfrac{s-4}{2}(z_--1)]^{\frac{1}{2}-\mu}[\log \tfrac{s-4}{2}(z_--1)]^{-1-\varepsilon}$$

for $1 \le z_2 \le z_-$, so long as $\mu < \frac{1}{2}$, and C_1 is some constant, we may majorize (3.35) by

$$2^{-\frac{1}{2}}[(z^2-1)(z_1^2-1)]^{-\frac{1}{2}}C_1[\tfrac{s-4}{2}(z_--1)]^{\frac{1}{2}-\mu}[\log \tfrac{s-4}{2}(z_--1)]^{-1-\varepsilon}$$

$$\cdot\int_{z_0}^{z_-} \frac{dz_2}{[(z_--z_2)(z_2-1)]^{\frac{1}{2}}} \left(\frac{2}{s-4}\right)^{\frac{1}{2}} . \qquad (3.36)$$

Now

$$\int_{z_0}^{z_-} dz_2 \left[(z_--z_2)(z_2-1)\right]^{-\frac{1}{2}}$$

$$\le \int_1^{z_-/2} dz_2\left[\frac{z_-}{2}(z_2-1)\right]^{-\frac{1}{2}} + \int_{z_-/2}^{z_-} dz_2 \left[(z_--z_2)\frac{z_-}{2}\right]^{-\frac{1}{2}}$$

$$= \left(\frac{2}{z_-}\right)^{\frac{1}{2}} 2\left[(z_2-1)^{\frac{1}{2}}\Big|_1^{z_-/2} - (z_--z_2)\Big|_{z_-/2}^{z_-}\right]$$

$$= 4 \; . \qquad (3.37)$$

So we find

$$|\rho^{e\ell}(s,t)| \leq C_2(st)^{-\mu}(\log s)^{-1-\varepsilon}$$

$$\cdot\int_{z_0}^{z/z_0} \frac{dz_1}{z_1-1}\,\{\log(z_1-1)\cdot\log(\frac{z}{z_1}-1)\}^{-1-\varepsilon}\,. \quad (3.38)$$

The z_1-integral here can be majorized in two pieces by

$$\int_{z_0}^{z^{1/2}} \frac{dz_1}{z_1-1}\,[\log(z_1-1)\cdot\log(z^{1/2}-1)]^{-1-\varepsilon}$$

$$+\int_{z^{1/2}}^{z/z_0} \frac{dz_1}{z_1-1}\,[\log(z^{1/2}-1)\cdot\log(\frac{z}{z_1}-1)]^{-1-\varepsilon}\,.$$

With the variable-change $z_1 \to z/z_1$ in the second piece here, we find that the sum of both pieces is not more than

$$2[\log(z^{1/2}-1)]^{-1-\varepsilon}\int_{z_0}^{z^{1/2}} \frac{dz_1}{z_1-1}\,[\log(z_1-1)]^{-1-\varepsilon}$$

$$= 2[\tfrac{1}{2}\log(z-2z^{1/2}+1)]^{-1-\varepsilon}\,\frac{[\log(z_1-1)]^{-\varepsilon}}{-\varepsilon}\Bigg|_{z_1=z_0}^{z^{1/2}}$$

$$\leq \frac{2^{2+\varepsilon}}{\varepsilon}\,[\log(z_0-1)]^{-\varepsilon}[\log(z-1)]^{-1-\varepsilon}. \quad (3.39)$$

I have written this out in some detail, so that you can see why we need $\varepsilon > 0$. If ε were negative, the dominant contribution to the z_1-integral would come from the upper limit, giving us an extra, unacceptable factor of $[\log(z^{1/2}-1)]^{|\varepsilon|}$.

Finally, one has the bound

$$|\rho^{e\ell}(s,t)| \leq C_3(st)^{-\mu}(\log s\cdot\log t)^{-1-\varepsilon}, \qquad (3.40)$$

where C_3 is a constant. Of course, one still has to show that $\rho^{e\ell}(s,t)$ is Hölder-continuous in s and t, in order to accommodate the second term in the norm (3.30). This involves a lot more algebra that I do not have time to describe: I can only refer to the original papers.[3]

The actual application of the Contraction Mapping Theorem to the equations (3.1)-(3.2) is now almost trivial. Since A_t depends linearly on $\rho^{e\ell}$ and v, [illegible]epends quadratically on A_t, it follows that, if $||v||$ is small enough, then the operator T of eq. (3.8) will map a sufficiently small ball of the space into itself, i.e. (3.10) follows from (3.9) for b small enough. Secondly, because of the quadratic structure of the equations, it is easy to see that we will obtain an inequality of the form (3.13), with P proportional to $2||\rho^{e\ell}|| + ||v||$. So by making b and $||v||$ small enough, we can ensure that P < 1, and so conclude the contraction proof.

IV. Subtractions

In this section, I want to explain how the treatment can be generalized to include subtractions. It is certain that we need subtractions, because the norm of the preceding section would only allow a total cross-section behaving like $s^{-1}(\log s)^{-2-\varepsilon}$ for large s, whereas a constant is perhaps the most likely asymptotic behaviour (although this is far from certain).

The unitarity equation, eq. (3.1), is still valid, but we now entertain the possibility that the infinite integral (3.2) might not converge without subtractions. From the Heine expansion,

$$\frac{1}{z'-z} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) Q_\ell(z'), \tag{4.1}$$

which converges if $|z + \sqrt{z^2-1}| < |z' + \sqrt{z'^2-1}|$, we infer that

$$\frac{1}{s'-s} - \frac{2}{t-4} \sum_{\ell=0}^{L} (2\ell+1) P_\ell(1+\frac{2s}{t-4}) Q_\ell(1+\frac{2s'}{t-4}), \tag{4.2}$$

and

$$\frac{1}{s'-u} - \frac{2}{t-4} \sum_{\ell=0}^{L} (2\ell+1)(-1)^\ell P_\ell(1+\frac{2s}{t-4}) Q_\ell(1+\frac{2s'}{t-4}), \tag{4.3}$$

both behave like s'^{-L-1} for large s', and both are orthogonal to the first $(L+1)$ partial waves in the t-channel. Hence we can replace eq. (3.2) by

$A_t(s,t) =$

$$\frac{1}{\pi}\int^{\infty} ds'\left[\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{4}{t-4}\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{L}(2\ell+1)P_\ell\left(1+\frac{2s}{t-4}\right)Q_\ell\left(1+\frac{2s'}{t-4}\right)\right]$$

$$\cdot\left[\rho^{e\ell}(s',t)+\rho^{e\ell}(t,s')+v(s',t)\right]$$

$$+\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{L}(2\ell+1)P_\ell\left(1+\frac{2s}{t-4}\right)\mathrm{Im}\,a_\ell(t), \qquad (4.4)$$

where the Im $a_\ell(t)$ are t-channel partial-wave absorptive parts.

We must envisage a suitably modified version of the norm (3.30), to allow $\rho^{e\ell}(s,t)$ to grow without limit as $t \to \infty$. Here, however, a difficulty arises. If $\rho^{e\ell}(s,t)$ behaves like t^N as $t \to \infty$ (apart from possible logarithms), then $\rho^{e\ell}(t,s)$ will behave like s^N as $s\to\infty$, and so $A_t(s,t)$ will be at least as bad as s^N as $s \to \infty$. The unitarity condition (3.1) contains A_t quadratically, and one finds, on careful analysis, that $\rho^{e\ell}(s,t)$ would behave like s^{2N-1} at least, which is worse than s^N if $N > 1$.

The trouble is rather artificial, however, since elastic unitarity is good only for $4 \leq s \leq 16$, so that we are free to modify (3.1) above $s = 16$, so long as it is left inviolate in the elastic region. We choose to replace (3.1) by

$$\rho^{e\ell}(s,t) =$$

$$h(s)\theta[t - \frac{16s}{s-4}]\iint dt_1 dt_2 K(s;t,t_1,t_2) A_t^*(s,t_1) A_t(s,t_2), \quad (4.5)$$

where $h(s)$ is a function that satisfies the following conditions:

(a) $h(s) = 1$, for $4 \leq s \leq 16$,

so that elastic unitarity is not affected in the elastic region.

(b) $|h(s) - h(s')| \leq C|s-s'|^{\mu}$, for $16 \leq s \leq \Lambda$,

where C and Λ are constants. In this way Hölder-continuity is not spoiled.

(c) $h(s) = 0$, for $s \geq \Lambda$, (4.6)

so that we can forget about the large-s behaviour of $\rho^{e\ell}(s,t)$. A suitable example of a function satisfying (a)-(c) is

$$h(s) = \left(\frac{\Lambda-s}{\Lambda-16}\right)^{\mu} \quad (4.7)$$

for $16 \leq s \leq \Lambda$, and (a) for $s \leq 16$, (c) for $s \geq \Lambda$.

A suitable generalized norm is

$$||f|| =$$

$$\sup\left\{\left[|f(s,t)| + \frac{|f(s_1,t_1)-f(s_2,t_2)|}{|s_1-s_2|^{\mu}+\left|\frac{t_1-t_2}{t_1 t_2}\right|^{\mu}}\right]\left(\frac{\bar{t}}{t_1 t_2}\right)^{L}\left(\log \bar{t}\right)^{1+\varepsilon}\right\}. \quad (4.8)$$

This guarantees the bounds

$$|f(s,t)| \le ||f||\, t^{L} (\log t)^{-1-\varepsilon} \tag{4.9}$$

and

$$|f(s_1,t) - f(s_2,t)| \le ||f||\ |s_1-s_2|^{\mu} t^{L} (\log t)^{-1-\varepsilon}. \tag{4.10}$$

Notice the complete lack of interest here in the behavior as $s \to \infty$: this is effectively mastered by the Hölder-continuous cut-off function, $h(s)$.

If one knew Im $a_\ell(t)$, $\ell=0, 2,\ldots L$, one could treat eqs. (4.4) and (4.5) as a mapping $\rho^{e\ell}(s,t) \to \rho^{e\ell}(s,t)$, and try to contract as before. This procedure works, in fact, if Im $a_\ell(t)$ is Hölder-continuous and not too big. Precisely, one needs Im $a_\ell(t)$ to belong to a space of functions of one variable, $f(t)$, for which the following norm exists:

$$||f|| = \sup\left\{\left[|f(t)| + \frac{|f(t_1)-f(t_2)|}{\left|\frac{t_1-t_2}{t_1t_2}\right|^{\mu}}\right] \left(\frac{\bar{t}}{t_1t_2}\right)^{L} \left(\log \bar{t}\right)^{1+\varepsilon}\right\}; \tag{4.11}$$

and one needs $||\text{Im } a_\ell||$, $\ell=0, 2,\ldots L$, to be sufficiently small.

I will sketch now the most straightforward way of determining the Im $a_\ell(t)$, $\ell=0,2,\ldots L$, consistent with elastic unitarity, and Mandelstam analyticity. Define the total amplitude as

$$A(s,t) = f(t,u) + f(u,s) + f(s,t), \tag{4.12}$$

where

$$f(t,u) = P(t,u) + \sum_{\ell=0}^{L} (2\ell+1)(s-4)^{\ell} P_{\ell}\left(1+\frac{2t}{s-4}\right) \frac{s^{L+1-\ell}}{\pi} \int_4^{\infty} \frac{ds'\rho_{\ell}(s')}{s'^{\,L+1-\ell}(s'-s)}$$

$$+ \frac{s^{L+1}}{\pi^2} \iint \frac{dt'du'\rho(t',u')}{(4-t'-u')^{L+1}(t'-t)(u'-u)} . \tag{4.13}$$

Here $P(t,u)$ is a symmetric, Lth order polynomial of t and u. The $\rho_{\ell}(s)$ are single-spectral-functions, and the $\rho(t,u)$ is the double-spectral-function. They are defined by

$$\rho(t,u) = \rho^{e\ell}(t,u) + \rho^{e\ell}(u,t) + v(t,u), \tag{4.14}$$

and by the requirement that the t-channel absorptive part, $A_t(s,t)$, as in eq. (4.4), is equal to

$$\sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{L} (2\ell+1)(t-4)^{\ell} P_{\ell}\left(1+\frac{2s}{t-4}\right)\rho_{\ell}(t) + \frac{u^{L+1}}{\pi}\int \frac{ds'\rho(s',t)}{(4-t-s')^{L+1}(s'-s)}$$

$$+ \frac{s^{L+1}}{\pi}\int \frac{du'\rho(u',t)}{(4-t-u')^{L+1}(u'-u)}. \tag{4.15}$$

Finally, the new value of Im a_{ℓ} is determined by

$$\text{Im } a_{\ell}(s) = q|A_{\ell}(s)|^2 + u_{\ell}(s), \tag{4.16}$$

for $\ell = 0, 2, \ldots, L$, where $A_{\ell}(s)$ is the partial-wave projection of eq. (4.12). Here $u_{\ell}(s)$ is an inelastic function that must vanish for $s \leq 16$, and that must have a finite norm of the type (4.11).

Now eqs. (4.4)-(4.16) constitute a set of equations for $\rho^{e\ell}(s,t)$ and Im $a_\ell(s)$, $\ell = 0, 2, \ldots L$.

We may summarize them as

$$\rho^{e\ell}(s,t) = T_1[\rho^{e\ell}; s,t] \qquad (4.17a)$$

$$\text{Im } a_\ell(s) = T_2[\text{Im } a; s,\ell]. \qquad (4.17b)$$

If we call the space defined by the norm (4.8) B; and if we define the space C by the norm

$$||\text{Im } a||_C = \sup_{\ell=0,2\ldots L} ||\text{Im } a_\ell||, \qquad (4.18)$$

where the norm on the right-hand side here is that of eq. (4.11); then we see that eq. (4.17a) maps $B \times C \to B$, while (4.17b) takes $B \times C \to C$. We can combine the two equations (4.17) into one equation, by inventing the quantity $\{\rho^{e\ell}(s,t), \text{Im } a_\ell(s)\}$, which belongs to $B \times C$, with the norm

$$||\{\rho^{e\ell}, \text{Im } a\}|| \equiv \max\{||\rho^{e\ell}||_B, ||\text{Im } a||_C\}. \qquad (4.19)$$

We write equations (4.17) as

$$\{\rho^{e\ell}(s,t), \text{Im } a_\ell(s)\} = T[\{\rho^{e\ell}, \text{Im } a\}; s,t,\ell]. \qquad (4.20)$$

The proof that eq. (4.20) defines a contraction mapping, if $||\{v,u\}||$ is small enough, now proceeds smoothly. The essential point, just as in the simpler proof of Section 3, is that T is quadratic in $\{\rho^{e\ell}, \mathrm{Im}\ a\}$, so that a sufficiently small ball in B x C will be mapped into itself. Moreover, the constant P in the contraction condition, eq. (3.13), is proportional to $||\{\rho^{e\ell}, \mathrm{Im}\ a\}||$, so it will be less than unity, if this norm is small enough.

V. The CDD Ambiguity

I come now to the discussion of the CDD ambiguity, which arises from the presence of Cauchy-singular integrals in the equations. I will limit myself, in the main, to a description of the new mappings, and an itemization of new difficulties. It should not be necessary by now to spell out all the details of the contraction proof.

The general idea is to replace the mapping T_2 of eq. (4.17b), for the partial-waves $\ell = 0, 2, \ldots, L$, by a different mapping, T_2', that is more general. The greater generality is possible by observing that it is enough to construct partial waves, $a_\ell(s)$, $\ell = 0, 2, \ldots, L$, that satisfy the following properties:

(a) $$\mathrm{Im}\, a_\ell(s) = \Delta_\ell(s) \equiv -\frac{2}{s-4} \int_4^{4-s} dt P_\ell\left(1+\frac{2t}{s-4}\right) \mathrm{Re} A_t(s,t), \tag{5.1}$$

for $s \leq 0$, in other words, the discontinuity on the left-hand cut agrees with that from the partial-wave projection of the Mandelstam representation (4.12)-(4.13).

(b) $$\mathrm{Im}\, a_\ell(s) = q|a_\ell(s)|^2 + u_\ell(s), \tag{5.2}$$

for $s \geq 4$, so unitarity is satisfied.

(c) The single-spectral-functions are determined, for $s \geq 4$, by

$$(s-4)^{\ell}\rho_{\ell}(s) =$$

$$\operatorname{Im} a_{\ell}(s) - \frac{1}{s-4}\int_{4-s}^{0} dt P_{\ell}\left(1+\frac{2t}{s-4}\right)$$

$$\cdot\left\{\frac{u^{L+1}}{\pi}\int \frac{dt'\rho(t',s)}{(4-s-t')^{L+1}(t'-t)} + (t\to u)\right\}, \quad (5.3)$$

This ensures that the discontinuities of the partial waves agree with the Mandelstam representation.

(d) $$\left(\frac{d}{ds}\right)^{n} a_{\ell}(s)\Big|_{s=2} =$$

$$C_{\ell}^{n} \equiv \left(\frac{d}{ds}\right)^{n} \frac{1}{s-4}\int_{4-s}^{0} dt P_{\ell}\left(1+\frac{2t}{s-4}\right) A(s,t)\Big|_{s=2}, \quad (5.4)$$

for $n = 0, 1, 2, \ldots L$; $\ell = 0, 2, \ldots, L$. This finally ensures that the real parts agree too.

The conditions (a)-(d) will be observed if $a_{\ell}(s)$ satisfies

$$a_{\ell}(s) =$$

$$\sum_{n=0}^{L} C_{\ell}^{n}(s-2)^{n} + \frac{(s-2)^{L+1}}{\pi}\int_{-\infty}^{0} \frac{ds'\Delta_{\ell}(s')}{(s'-2)^{L+1}(s'-s)}$$

$$+ \frac{(s-2)^{L+1}}{\pi}\int_{4}^{\infty} \frac{ds'\{q'|a_{\ell}(s')|^{2}+u_{\ell}(s')\}}{(s'-2)^{L+1}(s'-s)}. \quad (5.5)$$

If we simply use this equation as our new mapping $\{\rho^{e\ell}, a_{\ell}\} \to a'_{\ell}$ and make $a_{\ell}(s)$ Hölder-continuous, so that we can get past the singular integral, we shall reproduce the fixed-point of Section IV, although at the cost of more labor. It is to the possibility of finding alternative solutions of eq. (5.5), belonging to higher CDD classes, that we now turn.

To simplify the writing, we shall now take the simple case $L = 0$, so that we discuss separately only the S-wave. We make an N/D decomposition,

$$a_0(s) = N(s)/D(s), \tag{5.6}$$

and we define

$$B(s) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{ds'}{s'-s} \Delta_0(s'). \tag{5.7}$$

The N-function then satisfies the linear equation,

$$N(s) = B(s) + \sum_{n=1}^{M} d_n \frac{B(s)-B(s_n)}{s-s_n} + \frac{1}{\pi} \int_4^{\infty} ds' \frac{B(s')-B(s)}{s'-s} q'N(s'), \tag{5.8}$$

which may be shown to have a solution, if suitable restrictions are placed on B(s). Then D(s) is given by

$$D(s) = 1 + \sum_{n=1}^{M} \frac{d_n}{s-s_n} - \frac{1}{\pi} \int_4^{\infty} \frac{ds'}{s'-s} q'N(s'). \tag{5.9}$$

Where the poles here are the CDD poles. Finally, to make the mapping into one for Im $a_0(s)$, we set

$$\text{Im } a_0(s) = \frac{q[N(s)]^2}{|D(s)|^2} . \tag{5.10}$$

In the above equations, we have dropped the subtraction, and also assumed elastic unitarity. However, both of these deficiencies can be repaired, and one can show that a fixed point of the new mapping exists.[3]

In general, each CDD pole is associated with a resonance. This is because of Levinson's Theorem, which may be proved for the fixed point solution. It has the form

$$\delta(\infty) = n\pi, \tag{5.11}$$

where $\delta(s)$ is the phase-shift ($\delta(4) = 0$), and where n is the number of CDD poles. Evidently, the phase-shift must equal an odd multiple of $\pi/2$ at least n times, so there will certainly be n resonances. This is the main physical interest of CDD poles: they allow us to obtain resonances in our solutions.

The formula (5.11) is only correct if D(s) has no physical-sheet zeros, which would correspond in general to ghosts. It has been proved, in the case of weak CDD poles, that if the residues, d_n, of eq. (5.9), have the correct sign, then there are in fact no ghosts.[3]

VI. Positivity

So far, we have completely neglected the inelastic unitarity constraint, namely

$$\mathrm{Im}\, a_\ell(s) - q|a_\ell(s)|^2 \geq 0, \tag{6.1}$$

for $s \geq 16$, $\ell = 0, 2, 4, \ldots$

Let us consider the subtraction-free equations of Section III first. The partial waves may be written

$$a_\ell(s) = \frac{2}{\pi(s-4)} \int_4^\infty dt\ Q_\ell\left(1 + \frac{2t}{s-4}\right) A_t(s,t), \tag{6.2}$$

for $s \geq 4$, ℓ even. Now $Q_\ell(z)$ is real for real $z > 1$, so the imaginary part of $a_\ell(s)$ comes entirely from the imaginary part of $A_t(s,t)$, which may be obtained by inspection from eq. (3.3). We find

$$\mathrm{Im}\, a_\ell(s) =$$

$$\frac{2}{\pi(s-4)} \int dt\ Q_\ell\left(1 + \frac{2t}{s-4}\right) \left[\rho^{e\ell}(s,t) + \rho^{e\ell}(t,s) + v(s,t)\right], \tag{6.3}$$

where the integration extends over the support of the double-spectral-function. Consider the elastic piece of this integral, i.e. the part involving $\rho^{e\ell}(s,t)$. Insert the expression (3.1), and interchange orders of integration:

$$\frac{2}{\pi(s-4)} \iint dt_1 dt_2 A_t^*(s,t_1) A_t(s,t_2) \int dt Q_\ell\left(1+\frac{2t}{s-4}\right) K(s;t,t_1,t_2). \tag{6.4}$$

The integral over t here can be done explicitly. To see how this is done, let us return to the variable z. It may be proved, from the addition formula for Legendre functions, and the Heine expansion, that

$$4\pi \sum_{\ell=0}^{\infty} (2 \;+1) P_\ell(z) Q_\ell(z_1) Q_\ell(z_2) = \int \frac{d\Omega'}{(z'-z_1)(z''-z_2)} \quad , \tag{6.5}$$

where the right-hand side is precisely the integral I(z) of eq. (2.15), which we have already calculated. Accordingly, we must have

$$Q_\ell(z_1) Q_\ell(z_2) = 2 \int_{-1}^{1} dz\; P_\ell(z) \int_{z_+(z_1,z_2)}^{\infty} \frac{dz'}{z'-z} k^{-\frac{1}{2}}(z',z_1,z_2)$$

$$= 4 \int_{z_+(z_1,z_2)}^{\infty} dz' Q_\ell(z') k^{-\frac{1}{2}}(z',z_1,z_2) . \tag{6.6}$$

The right-hand side here is the required integral, to within an s-dependent factor. The term (6.4) becomes

$$\frac{4q}{\pi^2 (s-4)^2} \iint dt_1 dt_2 A_t^*(s,t_1) A_t(s,t_2) Q_\ell(1+\frac{2t_1}{s-4}) Q \;\; (1+\frac{2t_2}{s-4})$$

$$= q|a_\ell(s)|^2 . \tag{6.7}$$

Let us take the term (6.4) to the left-hand side of eq. (6.3), obtaining therefore

$$\mathrm{Im}\; a_\ell(s) - q|a_\ell(s)|^2 =$$

$$\frac{2}{\pi(s-4)} \int_{\frac{4s}{s-16}}^{\infty} dt Q_\ell(1+\frac{2t}{s-4}) [\rho^{e\ell}(t,s) + v(s,t)] . \tag{6.8}$$

We have to show that this integral is non-negative, for $s \geq 16$. The Legendre function itself is positive, so we shall show how to constrain the term $[\rho^{el}(t,s)+v(s,t)]$ to be positive also. We divide the integral into two pieces:

$$\int_{\frac{4s}{s-16}}^{\infty} dt\ldots \;=\int_{20}^{\infty} dt\ldots + \int_{\frac{4s}{s-4}}^{20} dt\ldots \quad . \tag{6.9}$$

On the domain $(20,\infty)$ we can require $v(s,t)$ to be such that

$$\int_{20}^{\infty} dt \; v(s,t) Q_{\ell}(1+\frac{2t}{s-4}) \geq 0. \tag{6.10}$$

Since $\rho^{el}(t,s)$ is quadratic in v, we can arrange for the positive contribution from $v(s,t)$ to dominate that from $\rho^{el}(t,s)$, for v small enough. However, for $t < 16$, $v(s,t)$ vanishes, and so we must arrange for $\rho^{el}(t,s)$ itself to be positive here. This is done by showing that the cone $\rho^{el}(s,t) \geq 0$, $4 \leq s \leq 20$, is mapped into itself (under the conditions of the contraction mapping). The proof of this is worked out by dividing the integrations over t_1 and t_2 in eq. (3.1) into parts below and above t_1, $t_2 = 20$. Above this point, the sign of $\mathrm{Re}A_t(s,t)$ is controlled by v, and can be made positive. Below this point, $A_t(s,t)$ is real (for $4 \leq s \leq 20$), and positive, since $\rho^{el}(t,s)$ is positive for $4 \leq t \leq 20$.

The above argument works also when there is one subtraction; but it breaks down with two or more subtractions. In fact, there is no known way of accommodating positivity with more than one subtraction. Let me explain why this is. Suppose that $\rho^{e\ell}(s,t)$ behaves like t^{α}, $\alpha > 0$, as $t \to \infty$ (except possibly for logarithms). Then $\rho^{e\ell}(t,s)$ behaves like s^{α} as $s \to \infty$. For fixed t, $Q_{\ell}(1+\frac{2t}{s-4}) \sim$ logs as $s \to \infty$. Hence, unless there is effective cancellation under the integral (6.8), we shall have

$$\operatorname{Im} a_{\ell}(s) - q|a_{\ell}(s)|^{2} \sim s^{\alpha-1} \log s, \tag{6.11}$$

which is a flagrant violation of the unitarity bound if $\alpha \geq 1$.

In principle, one could avoid this behavior if $[\rho^{e\ell}(t,s)+v(s,t)]$ were to oscillate infinitely, as a function of t at fixed s, in such a way that, although it is not bounded by $s^{\alpha-\varepsilon}$ as $s \to \infty$, t fixed, and so would need $[\alpha + 1]$ subtractions, nevertheless the integral (6.8) is bounded. A class of examples is furnished by the identity

$$\frac{1}{s-4} \int_{4-s}^{0} dt\, P_{\ell}(1+\frac{2t}{s-4})\, s^{\alpha(t)} =$$

$$\frac{1}{\pi(s-4)} \int_{t_0}^{\infty} dt\, Q_{\ell}(1+\frac{2t}{s-4}) s^{\operatorname{Re}\alpha(t)} \sin[\operatorname{Im} \alpha(t) \log s], \tag{6.12}$$

where we suppose $\alpha(t)$ to be a real analytic function of t, with a cut $t_0 \leq t < \infty$, and such that $\alpha(t) \leq 1$ for $t \leq 0$, and $\mathrm{Re}\,\alpha(t) \leq L$ for all t, and $\alpha(\infty) < 0$ (in all directions of the t-plane). Then we must take $\ell > L$ to ensure convergence of the right-hand side of (6.12). Although we can have $\mathrm{Re}\,\alpha(t)$ much greater than unity for some of the t-values in the integral (6.12), the oscillations of $\sin[\mathrm{Im}\,\alpha(t) \log s]$ succeed in reducing $s^{\max[\mathrm{Re}\,\alpha(t)]-1} \log s$ to a constant, asymptotically, as we can see from the left-hand side of (6.12).

The problem however is two-fold, and appears at present to be intractable: how does one set up a Banach space of functions with this subtle kind of oscillation, and how does one then show that the integrals (6.8) are non-negative?

Acknowledgments:

It is my pleasure to thank Professors E. C. G. Sudarshan and A. M. Gleeson for the hospitality extended to me at the Center for Particle Theory at Austin, Texas.

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