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PROPAGATION OF A DOUBLE STREAM INSTABILITY IN A PLASMA*

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ABSTRACT

The problem of the propagation of a perturbation initially localized is considered in the case of an unstable plasma. We use a linearized theory (which puts a time limitation for the validity of the theory) and a one-dimensional geometry. One finds that the front of the instability propagates with a velocity v_0. Still the main part of the energy diffuses slowly, i.e., is localized in a zone the dimension of which increases as $\sqrt{t}$. This problem is connected with the problem of absolute versus convective instability.

I. INTRODUCTION

In view of the current interest in double stream instability phenomena, it seems interesting to extend a calculation developed in a preceding article(1) for stable plasma to an unstable one. The problem is the following: We initiate at time 0, a plane perturbation localized in space. [In (1) the initial perturbation was a δ function, but the main results here are not sensitive to the precise initial conditions.]

Our purpose here is to calculate how the electric field propagates, i.e., to obtain $E(x,t)$. Consequently we have to integrate the double Fourier Laplace transform both on frequency [this is the problem solved by Landau(2) to get $E(k,t)$] and on wavelength. New physical insight is brought about by the integration on k. As in the Landau problem exact calculations turned out to be very difficult, but asymptotic expressions can be obtained. The problem is one dimensional.

A difficulty arises in the non-stable case. Due to the linear character of the theory, our results cannot represent the behavior for very long times where obviously the non-linear interactions play an important role. Still if the initial perturbation is small enough we can find a time interval where the asymptotic results of the linear theory are correct.

The main results are the following: In a double stream type instability, we find that the main part of the energy diffuses, i.e., is localized on a distance which increases like $\sqrt{t}$. In addition the front of the instability propagates with a velocity $\pm v_0$. In other types of instability [more precisely when the pole of the dispersion has both a real and imaginary part for the frequency* ($s_k = \gamma_k + i\omega_k$)], this zone is carried with the velocity

*In the double stream case, $\omega_k = 0$ when $\gamma_k \neq 0$.

$v_G = d\omega_k/dk)k = k_0$ where k_0 is the wavenumber for which γ_k is maximum. This result helps to clear up the concept of convective versus non-convective instability.

We first review very briefly the results of the dispersion relation, then proceed to the k integration and treat in some detail the two beam case. The connection with the criteria of instability[(3),(4),(5)] is finally established.

II. DISPERSION RELATION AND ASYMPTOTIC BEHAVIOR OF E(k,t)

We suppose a velocity distribution as indicated in Fig. 1 (v must be understood as x component of the velocity, F(v) is supposed even and normalized). The asymptotic behavior of E(k,t) is given by the pole of the dispersion relation and is exp $s_k t$ with $s_k = \gamma_k + i\omega_k$ ($\gamma_k > 0$). With the distribution of Fig. 1, the unstable k go from 0 to a maximum value k_M. The variation is indicated in Fig. 2.

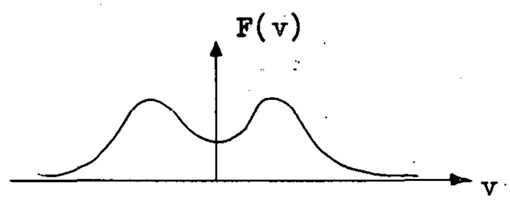

Fig. 1

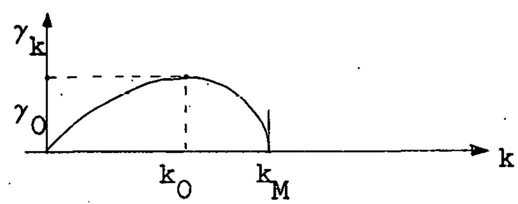

Fig. 2

In this interval $\omega_k = 0$, k_M is given by

$$k_M^2 = \omega_p^2 \int \frac{dF/dv}{v}\, dv .$$

For $k \to 0$ $\gamma_k \to 0$ with $\gamma_k \sim k\xi$ with ξ given by the equation

$$1 = \int_{-\infty}^{\infty} \frac{v\ dF/dv}{\xi^2 + v^2}\, dv.$$

III. INTEGRATION ON k

The problem is to calculate $E(x,t) = \int E(k,t)\exp{-ikx}\, dk$. We are interested in the asymptotic solutions $t \to +\infty$ so the only important wavelengths are the unstable ones and

$$E^{as}(x,t) = \int_0^{k_M} A(k)\exp s_k t \exp -ikx\, dk + C.C. \qquad (1)$$

C.C. (complex conjugate) takes care of the wavelength from $-k_M$ to 0. A(k) is a function only of the initial conditions.

To calculate the integral in Eq. (1) we notice that as $t \to \infty$ $\exp s_k t$ exhibits a sharper and sharper resonance around $k = k_0$ (value for which γ_k is maximum). Consequently only the most unstable modes and the neighboring modes will play a role in the asymptotic solution which is quite reasonable. We then develop around $k = k_0$

$$s_k = \gamma_0 - \frac{\alpha}{2}(k-k_0)^2 \quad \text{with } \alpha = -d^2\gamma_k/dk^2)\, k_0 \text{ obviously positive.}$$

We can write the integral in Eq. (1) taking out A_k

$$A_{k_0}\exp(\gamma_0 t - ik_0 x)\int_0^{k_M} \exp -\frac{\alpha t}{2}(k-k_0)^2 \exp -i(k-k_0)x\, d(k-k_0). \qquad (2)$$

For the asymptotic solutions we take the limit $-\infty$ $+\infty$. Still we must be careful if x goes to infinity with t. If we write $y = (k-k_0)\sqrt{\alpha t}$, the integral in Eq. (2) can be written

$$\frac{1}{\sqrt{\alpha t}} \int_{-k_0\sqrt{\alpha t}}^{(k_M-k_0)\sqrt{\alpha t}} \exp - \frac{y^2}{2} \exp -i \frac{x}{\sqrt{\alpha t}} y \, dy \ .$$

The profile $\exp-y^2/2$ can be considered good as long as $\alpha(k-k_0)^2 \ll \gamma_0$, i.e., for $y \ll \sqrt{\gamma_0 t}$. The limits of the integral are of the order of $\pm k_0\sqrt{\alpha t}$. As t goes to infinity, it is perfectly legitimate to replace the limits by $-\infty$ $+\infty$ provided the neglected zones $|y| > \sqrt{\gamma_0 t}$ and $|y| > k_0\sqrt{\alpha t}$ play no role. We have to be careful because of the presence in the integral of the factor $\cos xy/\sqrt{\alpha t}$. If the frequency $x/\sqrt{\alpha t}$ is too high the neglected zones can give an important contribution and the results we obtain are wrong. The condition is that the frequency must be much smaller than the zone of validity of the approximation which means $x\sqrt{\alpha t} \ll \sqrt{\gamma_0 t}$. So our results are valid for $x \ll \sqrt{\alpha\gamma_0}t$. (More exactly, x/t must go to zero. We will see the exact meaning of $x \ll \sqrt{\alpha\gamma_0}t$ in the next paragraph.) Under this condition ($\gamma_0 t \gg 1$ $x \ll \sqrt{\alpha\gamma_0}t$) one gets

$$E^{as}(x,t) = A \frac{\exp \gamma_0 t}{\sqrt{\alpha t}} \cos k_0 x \exp - \frac{x^2}{2\alpha t} \ . \qquad (3)$$

Equation (3) is interesting. It shows how the zone of turbulence is not conveyed away and that the zone in which one finds the main part of the electrostatic energy of the instability diffuses slowly; i.e., its dimensions increase like $\sqrt{t}$. (This is obviously a zone where our condition $x/t \to 0$ is satisfied.)

IV. GROWTH RATE IN A MOVING SYSTEM

Propagation of the Front of the Instability

In Eq. (3) $\exp\left[\gamma_0 t - x^2/2\alpha t\right]$ is equal to 1 if $x = \sqrt{2\alpha\gamma_0}\, t$. But we have shown that the expression is no more valid in this condition. Still this raises an interesting question: How can an observer moving with a velocity, v, see the perturbation? We suspect the existence of a critical velocity v_0. For $v < v_0$ the observer will see a growth rate γ, for $v > v_0$ a decrease of the perturbation. This v_0 can be considered as the propagation of the "shock front" of the instability[6].

The problem can be solved in the following way. In a system moving with a velocity, v, the growth rate for a given wavelength is

$$s'_k = s_k - ikv .$$

In the moving system, what is the nature of the instability: absolute (i.e., in every point the electric field grows without limit); or convective (i.e., one has to move in order to always see a growing electric field)? We have to calculate

$$\int_{-\infty}^{\infty} \exp s'_k t \; dk .$$

We follow Polovin[5] and replace the integration in the k plane by an integration in the s' plane. (See Fig. 3)

$$\int_{(c)} \frac{\exp s't}{\left[ds'/dk\right]} \, ds' .$$

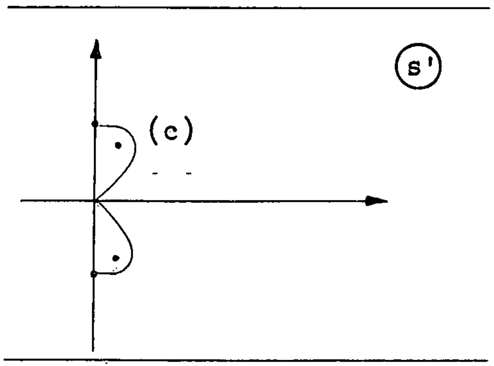

Fig. 3

The contour (c) in the (s') plane corresponds to the real axis on k plane. Now we can move (c) to the imaginary axis in (s') plane if there is no pole between (c) and this axis. If there is a pole, we have to consider the contribution of the poles. We suppose that for a value $k = k_1$, $ds'/dk = 0$. Around the point k_1 we can write $\frac{ds'}{dk} = A_1(k-k_1)$, and $s' = s_1' + \frac{A_1}{2}(k-k_1)^2$. We must calculate the integral

$$\int_{(c)} \frac{\exp s't}{\sqrt{2A_1}\sqrt{s'-s_1'}}\, ds'$$

and the asymptotic behavior will be $t^{-\frac{1}{2}} \exp s_1't$. If we except the factor $t^{-\frac{1}{2}}$ the growth will be given by $s_k - ikv$ where k is given by the solution of the equation

$$\frac{ds_k}{dk} = iv. \qquad (4)$$

Case of Double Stream Instability

In order to make clearer the above results, we are going to treat, in detail, the case of two streams of velocity -a and +a. ω_p is the plasma frequency corresponding to both streams

$$F(v) = \frac{1}{2}\left\{\delta(v+a) + \delta(v-a)\right\} .$$

We introduce the dimensionless unit $S = s\omega_p^{-1}$; $K = ka\omega_p^{-1}$; $V = v/a$. The pole is given by

$$S^2 = \frac{\sqrt{1+8K^2} - (1+2K^2)}{2} \tag{5}$$

Equation (4) is a 4 degree equation, but one can find an obvious root and consequently obtain a 3 degree equation which is easily solved. The result is

$$\alpha = \sqrt[3]{\frac{1-V}{1+V}} \qquad \beta = \sqrt[3]{\frac{1+V}{1-V}} \qquad z = \frac{\alpha+\beta}{2} + i\sqrt{3}\,\frac{\beta-\alpha}{2} .$$

$$K = K_1 + iK_2 = \sqrt{\frac{2z+z^2}{8}} \qquad S = \sqrt{\frac{2z-z^2}{8}} = S_1 + iS_2 .$$

The square root has to be taken such that the real value of S is positive and the imaginary value of K negative. The growth of the electric field in the system of velocity, V, is $\Gamma = S_1 + K_2 V$. Figure 4 shows the value of $\Gamma(V)$. Γ_0 corresponds to $V = 0$, $z = 1$, $S_1 = 1/\sqrt{8}$, $K_2 = 0$, and $\Gamma_0 = 1/\sqrt{8}$.

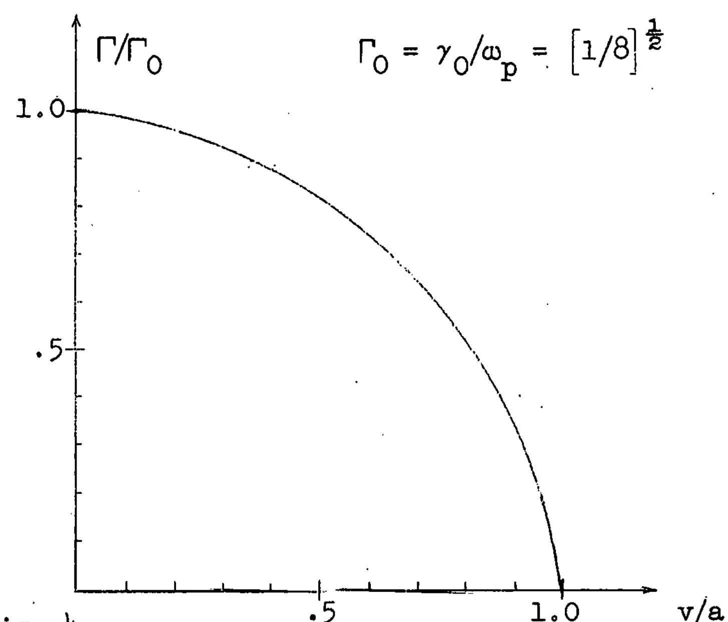

Fig. 4

We want to make two comments.

(1) If $v \to a$, $\Gamma \to 0$ which means that the velocity v_0 is in this case a. For $v > a$, the negative exponent we obtain has no meaning. In fact, we know that no signal propagates faster than the fastest velocity of the particles and if the perturbation is initially confined in the plane $x = 0$, we have $E \equiv 0$ for $|x| > at$. This is in agreement with an analysis of the problem by Sturrock[3]; he found that if two beams have velocity v_1 and v_2, the condition for an absolute instability is $v_1 v_2 < 0$. Here in the system moving with velocity v the two beams have the velocity $a-v$ and $-a-v$, and the condition for absolute instability, i.e., a growth, is $v^2-a^2 < 0$, i.e., $v < a$.

(2) If $v \to 0$, $\Gamma/\Gamma_0 \to 1$ and one can show that $\Gamma/\Gamma_0 \approx 1-(2v^2/3a^2)$. This is precisely the condition of validity of Eq. (3). If one puts $x = vt$ in Eq. (3) with $v \ll \sqrt{\alpha\gamma_0}$, one finds $\Gamma/\Gamma_0 = 1-v^2/2\alpha\gamma_0$ and $2\alpha\gamma_0$ is, in this case, $3a^2/2$. So for $v \ll \sqrt{\alpha\gamma_0}$, Eq. (3) gives the first term of the correction.

V. CASE WHERE s_k HAS AN IMAGINARY PART

When the behavior of $s_k = \gamma_k + i\omega_k$ is of the type indicated in Fig. 5, one can write

$$\omega_k = \omega_0 + (k-k_0)v_G + \frac{\beta}{2}(k-k_0)^2$$

$$v_G = \left.\frac{d\omega_k}{dk}\right)_{k_0} \qquad \beta = \left.\frac{d^2\omega_k}{dk^2}\right)_{k_0} .$$

We have to calculate

$$\int \left\{ \exp -i(x-v_G t)(k-k_0) \right\} \exp - \frac{(\alpha - i\beta)t}{2}(k-k_0)^2 d(k-k_0).$$

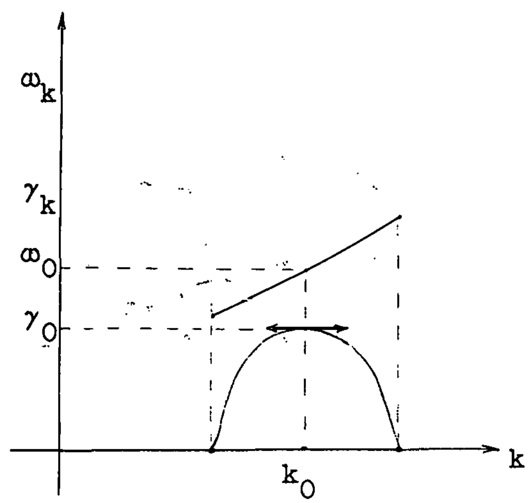

Fig. 5

Provided $x-v_G t$ does not go too quickly to infinity with t (more precisely $(x/t)-v_G$ must go to zero), the last integral can be written

$$\exp - \frac{(x-v_G t)^2}{2(\alpha - i\beta)t} = \exp - \frac{(x-v_G t)^2}{2(\alpha^2+\beta^2)t}(\alpha + i\beta) .$$

This means that in the system moving with velocity v_G the instability is absolute (or non-convective).

In the laboratory system, the instability can be absolute or convective. Still, as long as v_G is different from 0, the main part of the instability is always carried out with velocity v_G and diffuses around the position $v_G t$. [The diffusion constant is $\alpha + \beta^2/\alpha$.]

V. CONCLUSIONS

In a two stream type instability (when the pole has no imaginary part):

1.) The instability is absolute. The growth rate of the electric field in the laboratory system is γ_0 (maximum of γ_k).

2.) In a system moving with a velocity v, the growth rate is no longer γ_0, but smaller. For a certain velocity, v_0, the growth rate is zero. For systems moving with velocity $v > v_0$, the instability appears as a convective instability.

3.) The front of the instability moves with the velocity v_0. Still the main part of the electrostatic energy spreads much more slowly and more exactly diffuses.

When the pole s_k has an imaginary part, the instability is absolute in the system moving with velocity v_G, and in this system the preceding results are recovered.

ACKNOWLEDGEMENTS

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6. Weitzner, NYO-9601, p. 25-40 (unpublished). H. Weitzner has also derived the existence of a critical velocity v_c for the front of the perturbation. We have checked that his Eqs. (42a, b, c) can be written as $s_k - ikv_c = 0$ and $ds_k/dk = iv_c$ in agreement with our results.