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Abstract

A method is described for obtaining an approximate solution to the equations describing a nucleon-meson cascade by using the angular dependence of the secondary particle production kernels as a perturbation. The usefulness of the method lies in the fact that in a slab geometry the equations which must be solved numerically are essentially the same as those which are used in the straight-ahead approximation and have been solved previously.

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I. Introduction

In a series of recent reports[1-4] the equations which describe a nucleon-meson cascade have been solved numerically in the straight-ahead approximation. In this report a method is described for treating the angular dependence of the secondary particle production kernels as a perturbation. The usefulness of the method lies in the fact that in a slab geometry (the only geometry considered in this report) the angles enter only parametrically in the first-order equations. For any specific values of the angles the equations which must be solved numerically are essentially the same as those which are used in the straight-ahead approximation.

In Section II the perturbation method is described and the first-order equations in the case of an infinite beam incident normally on a slab are given. In Section III the case of an infinite beam incident isotropically on a slab is discussed. In Section IV the case of a very narrow beam incident normally on a slab is considered and expressions for the lateral structure functions of the cascade are given. In Section V the equations governing the muon component of the cascade are given. In the Appendix the perturbation series in all orders is discussed. In particular, it is shown that for the case of an infinite beam incident either normally or isotropically on a slab the equation for the nth order flux is of the same form as the equation for the first-order flux. Thus, if a code exists which will solve the first-order equations, one can in principle by repeated use of this code obtain an exact solution.

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[1] R. G. Alsmiller, Jr., F. S. Alsmiller, and J. E. Murphy, Nucleon-Meson Cascade Calculations: Transverse Shielding for a 45-Gev Electron Accelerator (Parts I, II, and III), ORNL-3289 (1962), ORNL-3365 (1962), and ORNL-3412 (1963).

[2] R. G. Alsmiller, Jr. and J. E. Murphy, Nucleon-Meson Cascade Calculations: The Star Density Produced by a 24-Gev Proton Beam in Heavy Concrete, ORNL-3367 (1963).

[3] R. G. Alsmiller, Jr. and J. E. Murphy, Nucleon-Meson Cascade Calculations: Shielding Against an 800-Mev Proton Beam, ORNL-3406 (1963).

[4] R. G. Alsmiller, Jr. and J. E. Murphy, Space Vehicle Shielding Studies: Calculations of the Attenuation of a Model Solar Flare and Monoenergetic Proton Beams by Aluminum Slab Shields, ORNL-3317 (1963).

II. Infinite Beam Normally Incident on a Slab

The discussion which will be given below can easily be carried through for an arbitrary number of cascade components. However, in order to avoid unnecessary complexity, we shall restrict it to the consideration of neutrons, protons, charged pions, and muons.* Furthermore, since we shall assume that muons do not interact with nuclei, the muon equations are much simpler than those for the other components and will be treated in Section V.

Under these conditions the Boltzmann transport equations for the nucleon-meson cascade may be written

$$B_\alpha \, \Phi_\alpha(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \, Q_\beta(E') \, \Phi_\beta(\vec{R},E',\vec{\Omega}') \, d\Omega' \, dE', \qquad (2.1)$$

where

$$B_\alpha = \vec{\Omega}\cdot\nabla + Q_\alpha(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E} S_\alpha(E), \qquad (2.2)$$

α,β = subscripts which here and throughout this report take values N, P, π^+, and π^- corresponding to neutrons, protons, positive pions, and negative pions, respectively,

$\Phi_\alpha(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of particles of type α,

$\vec{R}$ = position vector,

E = kinetic energy,

$\vec{\Omega}$ = unit vector in the direction of the momentum,

Q_α = macroscopic cross section for the nonelastic collision of a particle of type α in the medium being considered,

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*The neutral pion decays very rapidly into two photons. These pions are not included here because we do not include photons.

$Q_{\alpha D}$ = probability per unit distance for the decay of the αth kind of particle,

S_α = energy loss per unit distance of a particle of type α in the medium being considered,

$F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})$ = the number of particles of type α per unit energy range per unit solid angle produced at energy E and direction $\vec{\Omega}$ when a particle of type β with energy E' and direction $\vec{\Omega}'$ undergoes a nonelastic collision,

E_0 = maximum kinetic energy of any particle considered.

To facilitate the application of boundary conditions, we use the fact that the primary flux, i.e., the flux of incident particles which have undergone neither nuclear collision nor decay, can be obtained analytically, and we separate the total flux into primary and secondary components. Let

$$B_\alpha \Phi_{i\alpha}(\vec{R},E,\vec{\Omega}) = 0, \tag{2.3}$$

$$B_\alpha \Phi_{s\alpha}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\, Q_\beta(E') \times \left[\Phi_{i\beta}(\vec{R},E',\vec{\Omega}') + \Phi_{s\beta}(\vec{R},E',\vec{\Omega}')\right] d\Omega'\, dE', \tag{2.4}$$

$$\Phi_\alpha(\vec{R},E,\vec{\Omega}) = \Phi_{i\alpha}(\vec{R},E,\vec{\Omega}) + \Phi_{s\alpha}(\vec{R},E,\vec{\Omega}), \tag{2.5}$$

where

$\Phi_{i\alpha}(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of primary particles of type α,

$\Phi_{s\alpha}(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of secondary particles of type α.

The integral terms containing the secondary fluxes in Eq. (2.4) are the terms which make the equation difficult to solve. For high-energy cascades the situation is simplified considerably by virtue of the fact that the secondary particles are preferentially emitted in the forward direction. At high energies $F_{\alpha\beta}$ may to a reasonable approximation be written

$$F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \simeq g_{\alpha\beta}(E',E)\,\frac{\delta(\vec{\Omega}'\cdot\vec{\Omega}-1)}{2\pi} + G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}), \tag{2.6}$$

where

$$g_{\alpha\beta}(E',E) = \int_{\Delta} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\,d\Omega, \tag{2.7}$$

Δ = a small region of solid angle centered about the unit vector $\vec{\Omega}'$ which covers the forward peak in $F_{\alpha\beta}$,

$G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})$ = a correction term which is to make Eq. (2.6) approximately correct.

It is clear that there exists a $G_{\alpha\beta}$ which will make Eq. (2.6) exact. However, since $F_{\alpha\beta}$ is not in general singular when $\vec{\Omega}'\cdot\vec{\Omega} = 1$, this exact $G_{\alpha\beta}$ would contain a singularity and would be inconvenient for further calculation.* In what follows $G_{\alpha\beta}$ is to be thought of as a function which is equal to $F_{\alpha\beta}$ outside of the region Δ and is negligibly small but continuous inside Δ.

In the straight-ahead approximation $G_{\alpha\beta}$ is taken to be zero. In this report we assume $G_{\alpha\beta}$ to be nonzero but sufficiently small that the terms containing $G_{\alpha\beta}$ may be treated as small perturbations.

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*See the discussion following Eq. (2.18) and see also the appendix.

Using Eq. (2.6), Eq. (2.4) may be written*

$$B_\alpha\, \Phi_{s\alpha}(\vec{R},E,\vec{\Omega}) \simeq \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E)\; Q_\beta(E')\; \Phi_{s\beta}(\vec{R},E',\vec{\Omega})\; dE'$$

$$+ \sum_\beta \int_E^{E_0} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\; Q_\beta(E')\; \Phi_{s\beta}(\vec{R},E',\vec{\Omega}')\; d\Omega'\; dE'$$

$$+ \sum_\beta \int_E^{E_0} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\; Q_\beta(E')\; \Phi_{i\beta}(\vec{R},E',\vec{\Omega}')\; d\Omega'\; dE'. \qquad (2.8)$$

Here, the angular distribution of the first-generation secondaries, i.e., the secondaries produced by primaries, is treated accurately and completely; the small-angle production of secondaries from secondaries is treated as being straight ahead, and the remaining wide-angle production of secondaries by secondaries is included in the term containing $G_{\alpha\beta}$.

Since the term containing $G_{\alpha\beta}$ is to be considered a small perturbation, we may introduce into this term a zeroth approximation $\Phi_{s\alpha}$. To obtain this we shall assume that $G_{\alpha\beta}$ may be put equal to zero everywhere and define the zeroth approximation Ψ_α through the equation**

$$B_\alpha\, \Psi_\alpha(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E)\; Q_\beta(E')\; \left[\Psi_\beta(\vec{R},E',\vec{\Omega}) + \Phi_{i\beta}(\vec{R},E,\vec{\Omega})\right] dE'. \qquad (2.9)$$

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*Here and throughout this report we used the general theorem

$$f(\vec{\Omega}) = \int_{\Omega'} f(\vec{\Omega}')\; \frac{\delta(\vec{\Omega}'.\vec{\Omega} - 1)}{2\pi}\; d\vec{\Omega}'.$$

**The use of $g_{\alpha\beta}$ in this equation is to a certain extent arbitrary. See the appendix.

We then write the equation for the first approximation to the angular flux, $\Phi_{s\alpha}^{(1)}$, as,

$$B_\alpha \, \Phi_{s\alpha}^{(1)}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_o} g_{\alpha\beta}(E',E) \; Q_\beta(E') \; \Phi_{s\beta}^{(1)}(\vec{R},E',\vec{\Omega}) \; dE' + s_\alpha^{(1)}(\vec{R},E,\vec{\Omega}), \tag{2.10}$$

where

$$s_\alpha^{(1)}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_o} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \; Q_\beta(E') \; \Psi_\beta(\vec{R},E',\vec{\Omega}') \; d\Omega' \; dE'$$

$$+ \sum_\beta \int_E^{E_o} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \; Q_\beta(E') \; \Phi_{i\beta}(\vec{R},E',\vec{\Omega}') \; d\Omega' \; dE' . \tag{2.11}$$

We now wish to apply this general perturbation theory to the specific case of an infinite beam incident on a slab. Taking the z axis to be normal to the slab, it is clear from symmetry considerations that the fluxes do not depend on x and y. Therefore, we have

$$B_\alpha = \omega \frac{\partial}{\partial z} + Q_\alpha(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E} S_\alpha , \tag{2.12}$$

where ω = cosine of the angle between the unit vector $\vec{\Omega}$ and the z axis, and Eq. (2.3) may be solved immediately to yield

$$\Phi_{i\alpha}(\vec{R},E,\vec{\Omega}) = \phi_{i\alpha}(z,E) \; \frac{\delta(\omega - 1)}{2\pi} , \tag{2.13}$$

$$\phi_{i\alpha}(z,E) = \phi_{i\alpha}(0,E_\alpha)\,\frac{S_\alpha(E_\alpha)}{S_\alpha(E)}\,e^{-\int_E^{E_\alpha}\left[\frac{Q_\alpha(E') + Q_{\alpha D}(E')}{S_\alpha(E')}\right]dE'} ,$$

$$\int_E^{E_\alpha(z,E)} \frac{dE'}{S_\alpha(E')} = z, \tag{2.14}$$

where $\phi_{i\alpha}(0,E)$ = arbitrary functions which must be specified as boundary conditions. Using Eq. (2.13) in Eq. (2.9), Ψ_α may be written

$$\Psi_\alpha(\vec{R},E,\vec{\Omega}) = \psi_\alpha(z,E)\,\frac{\delta(\omega - 1)}{2\pi} , \tag{2.15}$$

where

$$\left[\frac{\partial}{\partial z} + Q_\alpha(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E}\,S_\alpha(E)\right]\psi_\alpha(z,E)$$

$$= \sum_\beta \int_E^{E_o} g_{\alpha\beta}(E',E)\;Q_\beta(E')\left[\psi_\beta(z,E') + \phi_{i\beta}(z,E')\right]dE'. \tag{2.16}$$

Equation (2.16) is, of course, just the straight-ahead equation which has previously been solved numerically. Since $g_{\alpha\beta}(E',E)$ omits in each collision the secondaries produced at wide angles with respect to the direction of the initiating particle, ψ_α is to be regarded as an estimate of the angular flux integrated over a small angular region about the z axis.

Using Eqs. (2.13) and (2.15), the equation for the first approximation to the angular flux becomes

$$\left[\omega \frac{\partial}{\partial z} + Q_\alpha(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E} S_\alpha\right] \Phi_{s\alpha}^{(1)}(z,E,\omega)$$

$$= \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E)\; Q_\beta(E')\; \Phi_{s\beta}^{(1)}(z,E',\omega)\; dE' + s_\alpha^{(1)}(z,E,\omega), \qquad (2.17)$$

where

$$s_\alpha^{(1)}(z,E,\omega) = \sum_\beta \int_E^{E_0} G_{\alpha\beta}(E',E,\omega)\; Q_\beta(E')\; \psi_\beta(z,E')\; dE'$$

$$+ \sum_\beta \int_E^{E_0} F_{\alpha\beta}(E',E,\omega)\; Q_\beta(E')\; \phi_{i\beta}(z,E')\; dE'. \qquad (2.18)$$

It is to be noted that Eq. (2.17) is very similar to the equation which is used in the straight-ahead approximation. The whole point of the discussion is that the angles enter only parametrically in Eq. (2.17), so each value of ω may be treated separately using very nearly the same IBM code which was used previously.

Note also that if $G_{\alpha\beta}$ is defined in such a manner that Eq. (2.6) is satisfied exactly, then the source term, Eq. (2.18), contains a term which is proportional to $\delta(1 - \omega)$ and this means that $\Phi_{s\alpha}^{(1)}$ contains a part which is singular. Since such a singularity is unphysical, it seems preferable to define $G_{\alpha\beta}$ as being a nonsingular function which makes Eq. (2.6) only approximately correct. It must be understood that once the perturbation approximation is made, i.e., $\Phi_{s\alpha}$ is replaced by Ψ_α as in Eq. (2.10), there is no longer any very clear way of deciding what form of $G_{\alpha\beta}$ will lead to the best approximation for $\Phi_{s\alpha}$.*

---

*See the appendix.

It is still necessary to consider the boundary conditions which must be used in solving Eqs. (2.16) and Eqs. (2.17). Since the initial values of the fluxes have been incorporated in the primary solution, the boundary conditions on the secondary flux are that no particles enter the slab from the region outside of the slab, i.e.,

$$\begin{aligned} \Phi_{s\alpha}(0,E,\omega) &= 0, \qquad 0 < \omega \leq 1, \\ \Phi_{s\alpha}(\ell,E,\omega) &= 0, \qquad -1 \leq \omega < 0, \end{aligned} \tag{2.19}$$

where ℓ = thickness of the slab. These boundary conditions are, of course, to be applied to both the zeroth-order and the first-order flux.

In the case of the zeroth-order flux, ψ_α, the boundary conditions are satisfied by using

$$\psi_\alpha(0,E) = 0.$$

In the case of the first-order flux, these boundary conditions may be used directly in solving Eq. (2.17) and may be satisfied exactly. That is, for each value of ω the equation is solved using Eq. (2.19) as an initial value on $\Phi^{(1)}_{s\alpha}$. In the case when $\omega < 0$, it is necessary to make the substitution

$$z' = \ell - z \tag{2.20}$$

and solve the equation with z' going from zero to ℓ since it is only at $z' = 0$ that the initial values are known.

III. Infinite Beam Isotropically Incident on a Slab

In this section we apply the perturbation theory of the previous section to the case of an infinite beam isotropically incident on a slab.

Taking the z axis to be normal to the slab, it is again clear from symmetry considerations that the fluxes do not depend on x and y. For this case the primary flux equation may be solved to yield

$$\Phi_{i\alpha}(\vec{R},E,\vec{\Omega}) = \frac{1}{2\pi}\,\phi_{i\alpha}(R,E), \tag{3.1}$$

where

$$R = z/\omega, \tag{3.2}$$

ω has the same meaning as before and $\phi_{i\alpha}$ is the function defined by Eq. (2.14).

Using Eq. (3.1) in conjunction with Eq. (2.9), the zeroth approximation to the flux may be written

$$\Psi_{\alpha}(\vec{R},E,\vec{\Omega}) = \frac{1}{2\pi}\,\psi_{\alpha}(R,E), \tag{3.3}$$

where $\psi_{\alpha}(R,E)$ satisfies the equation

$$\left[\frac{\partial}{\partial R} + Q_{\alpha}(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E}\,S_{\alpha}\right]\psi_{\alpha}(R,E)$$

$$= \sum_{\beta}\int_{E}^{E_o} g_{\alpha\beta}(E',E)\;Q_{\beta}(E')\left[\psi_{\beta}(R,E') + \phi_{i\beta}(R,E')\right]dE'. \tag{3.4}$$

Equation (3.4) is, of course, exactly the same as Eq. (2.17).

Using Eqs. (3.1) and (3.3), the equation for the first approximation to the secondary flux may be written

$$\left[\omega\frac{\partial}{\partial z} + Q_{\alpha}(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E}\,S_{\alpha}(E)\right]\Phi_{s\alpha}^{(1)}(z,E,\omega)$$

$$= \sum_{\beta}\int_{E}^{E_o} g_{\alpha\beta}(E',E)\;Q_{\beta}(E')\;\Phi_{s\alpha}^{(1)}(z,E',\omega)\;dE' + s_{\alpha}^{(1)}(z,E,\omega), \tag{3.5}$$

where

$$
\begin{aligned}
s_{\alpha}^{(1)}&(z,E,\omega) \\
&= \sum_{\beta} \int_{E}^{E_o} \int_{o}^{\pi} \left\{ \int_{o}^{2\pi} G_{\alpha\beta}\left[E',E,\omega'\omega - \sqrt{1-\omega^2}\sqrt{1-\omega'^2}\cos(\phi'-\phi)\right] d(\phi'-\phi) \right\} \\
&\qquad\qquad \times\ Q_{\beta}(E')\, \frac{\psi_{\beta}\left(\frac{z}{\omega'}\,,\ E'\right)}{2\pi}\, d\omega'\, dE' \\
&+ \sum_{\beta} \int_{E}^{E_o} \int_{o}^{\pi} \left\{ \int_{o}^{2\pi} F_{\alpha\beta}\left[E',E,\omega'\omega - \sqrt{1-\omega^2}\sqrt{1-\omega'^2}\cos(\phi'-\phi)\right] d(\phi'-\phi) \right\} \\
&\qquad\qquad \times\ Q_{\beta}(E')\, \frac{\phi_{i\beta}\left(\frac{z}{\omega'}\,,\ E'\right)}{2\pi}\, d\omega'\, dE' . \qquad (3.6)
\end{aligned}
$$

Equation (3.5) is, of course, exactly the same as Eq. (2.17). The significant difference between the cases of an infinite beam incident normally and isotropically on a slab lies in form of the source terms, Eqs. (2.18) and (3.6). While in Eq. (2.18) the angle integrations could be carried out analytically, this is not the case in Eq. (3.6). Thus the computation required to obtain $\Phi_{s\alpha}^{(1)}$ with isotropic incidence is somewhat more lengthy than with normal incidence.

Since we are again considering a slab geometry, the boundary conditions on $\Phi_{s\alpha}$ are those given in Eq. (2.19). They may be satisfied in zeroth order by using

$$\psi_{\alpha}(0,E) = 0$$

and may be applied directly and exactly in first order, it being understood that when $\omega < 0$ the transformation given in Eq. (2.20) must be employed.

IV. Narrow Beam Normally Incident on a Slab

The case of a very narrow beam normally incident on a slab is both more interesting and more difficult than the cases treated previously in that we must consider the lateral spread as well as the longitudinal development of the cascade.

Let the z axis be normal to the slab and let the beam of particles be incident at the origin of coordinates. It is convenient to use cylindrical coordinates in position space and spherical coordinates in velocity space, so we let

r, z, ϕ_r = cylindrical coordinates of $\vec{R}$,

ω, ϕ_ω = spherical coordinates of the unit vector $\vec{\Omega}$,

where, as before, ω = cosine of the angle between the unit vector $\vec{\Omega}$ and the z axis.

Using this notation, Eq. (2.3) for the primaries may be solved to yield

$$\Phi_{i\alpha}(\vec{R},E,\vec{\Omega}) = \phi_{i\alpha}(z,E)\,\frac{\delta(\omega - 1)}{2\pi}\,\delta(r\sin\phi_r)\,\delta(r\cos\phi_r), \tag{4.1}$$

where $\phi_{i\alpha}$ is again given by Eq. (2.14).

Using Eq. (4.1) in Eq. (2.9), the zeroth-order flux, Ψ_α, may be written

$$\Psi_\alpha(\vec{R},E,\vec{\Omega}) = \psi_\alpha(z,E)\,\frac{\delta(\omega - 1)}{2\pi}\,\delta(r\sin\phi_r)\,\delta(r\cos\phi_r), \tag{4.2}$$

where $\psi_\alpha(z,E)$ satisfies Eq. (2.16). Since we are again considering a slab geometry, the boundary condition of $\psi_\alpha(z,E)$ is, from Eq. (2.19),

$$\psi_\alpha(0,E) = 0. \tag{4.3}$$

Introducing Eqs. (4.1) and (4.2) into Eq. (2.10), the equation for the first approximation to the angular flux may be written

$$B_\alpha \, \Phi_{s\alpha}^{(1)}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E) \; Q_\beta(E') \; \Phi_{s\beta}^{(1)}(\vec{R},E',\vec{\Omega}) \, dE' + s_\alpha^{(1)}(\vec{R},E,\vec{\Omega}), \tag{4.4}$$

where

$$B_\alpha = \omega \frac{\partial}{\partial z} + (1 - \omega^2)^{\frac{1}{2}} \cos\eta \frac{\partial}{\partial r} + (1 - \omega^2)^{\frac{1}{2}} \sin\eta \frac{\partial}{\partial \phi_r} + Q_\alpha(E) + Q_{\alpha D}(E) - \frac{\partial}{\partial E} S_\alpha(E), \tag{4.5}$$

$$\eta = \phi_\omega - \phi_r,$$

$$s_\alpha^{(1)}(\vec{R},E,\vec{\Omega}) = \delta(r \sin\phi_r) \; \delta(r \cos\phi_r) \left\{ \sum_\beta \int_E^{E_0} G_{\alpha\beta}(E',E,\omega) \; Q_\beta(E') \; \psi_\beta(z,E') \, dE' + \sum_\beta \int_E^{E_0} F_{\alpha\beta}(E',E,\omega) \; Q_\beta(E') \; \phi_{i\beta}(z,E') \, dE' \right\}. \tag{4.6}$$

Because of the three derivatives which occur in B_α, Eq. (4.4) is still not in a form suitable for numerical computation. To reduce the equation to a more suitable form, we introduce*

$$\delta(r \cos\phi_r) \; \delta(r \sin\phi_r) = \delta\left(\frac{r \sin\eta}{\cos\phi_\omega}\right) \delta(r \cos\phi_r) \tag{4.7}$$

---

*To prove this relation note that

$$\frac{r \sin\eta}{\cos\phi_\omega} = \frac{r}{\cos\phi_\omega} \left[\sin\phi_\omega \cos\phi_r - \cos\phi_\omega \sin\phi_r \right]$$

and that $\delta(r \cos\phi_r)$ requires that

$$r \cos\phi_r = 0.$$

and the ansatz

$$(1 - \omega^2)^{\frac{1}{2}} \, \Phi^{(1)}_{i\alpha}(\vec{R},E,\vec{\Omega}) = \pm \frac{\chi_\alpha(r,z,E,\omega)}{\cos\phi_\omega} \, \delta\left(\frac{r \sin\eta}{\cos\phi_\omega}\right) \Theta(\pm\, r \cos\phi_r), \qquad (4.8)$$

where

$$\begin{aligned} \Theta(x) &= 1 \quad \text{if} \quad x > 0, \\ &= 0 \quad \text{if} \quad x < 0, \end{aligned}$$

and the plus and minus sign is to be used to keep the flux positive and nonzero; i.e., the positive sign is used when $\cos\phi_r, \cos\phi_\omega$ is greater than zero and the negative sign is used when $\cos\phi_r, \cos\phi_\omega$ is less than zero.* Substituting the ansatz into the equation, we find

$$\begin{aligned} &\Big\{\omega \frac{\partial}{\partial z} \chi_\alpha(r,z,E,\omega) + (1 - \omega^2)^{\frac{1}{2}} \chi_\alpha(r,z,E,\omega) + \Big[Q_\alpha(E) + Q_{\alpha D}(E)\Big] \chi_\alpha(r,z,E,\omega) \\ &\quad - \frac{\partial}{\partial E}\Big[S_\alpha(E) \, \chi_\alpha(r,z,E,\omega)\Big] - \sum_\beta \int_E^{E_o} g_{\alpha\beta}(E',E) \, Q_\beta(E') \, \chi_\beta(r,z,E',\omega) \, dE'\Big\} \\ &\qquad\qquad \times \left[\pm \frac{\Theta(\pm r \cos\phi_r)}{\cos\phi_\omega}\right] \delta\left(\frac{r \sin\eta}{\cos\phi_\omega}\right) \\ &\quad + \Big\{\chi_\alpha(r,z,E,\omega) - \sum_\beta \int_E^{E_o} G_{\alpha\beta}(E',E,\omega) \, Q_\beta(E') \, \psi_\beta(z,E') \, dE' \\ &\quad - \sum_\beta \int_E^{E_o} F_{\alpha\beta}(E',E,\omega) \, Q_\beta(E') \, \phi_{i\beta}(z,E') \, dE'\Big\} \, (1 - \omega^2)^{\frac{1}{2}} \\ &\qquad\qquad \times \; \delta\left(\frac{r \sin\eta}{\cos\phi_\omega}\right) \delta(r \cos\phi_r) = 0. \qquad (4.9) \end{aligned}$$

---

*Note that $\delta(r \sin\eta/\cos\phi_\omega)$ requires $\phi_\omega = \phi_r$ everywhere except possibly at $r = 0$.

Equation (4.9) can have a solution only if the bracketed terms are separately zero, so we have*

$$\omega \frac{\partial}{\partial z} \chi_\alpha(r,z,E,\omega) + (1 - \omega^2)^{\frac{1}{2}} \frac{\partial}{\partial r} \chi_\alpha(r,z,E,\omega) + \Big[Q_\alpha(E) + Q_{\alpha D}(E) \Big] \chi_\alpha(r,z,E,\omega)$$
$$- \frac{\partial}{\partial E} \Big[S_\alpha(E)\ \chi_\alpha(r,z,E,\omega) \Big] = \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E)\ Q_\alpha(E)\ \chi_\alpha(r,z,E',\omega)\ dE', \tag{4.10}$$

$$\chi_\alpha(0,E,\omega) = \sum_\beta \int_E^{E_0} G_{\alpha\beta}(E',E,\omega)\ Q_\beta(E')\ \psi_\beta(z,E')\ dE'$$
$$+ \sum_\beta \int_E^{E_0} F_{\alpha\beta}(E',E,\omega)\ Q_\beta(E')\ \phi_{i\beta}(z,E')\ dE'. \tag{4.11}$$

Equation (4.10) is now a homogeneous equation subject to the boundary condition expressed in Eq. (4.11).

If we introduce the variables ρ and z_0 defined by

$$z - z_0 = \rho\omega,$$
$$r = \rho(1 - \omega^2)^{\frac{1}{2}}\ , \tag{4.12}$$

Eq. (4.10) and (4.11) may be written

$$\frac{\partial}{\partial \rho} \mathscr{Y}_\alpha(\rho,z_0,E,\omega) + \Big[Q_\alpha(E) + Q_{\alpha D}(E) \Big] \mathscr{Y}_\alpha(\rho,z_0,E,\omega) = \frac{\partial}{\partial E} \Big[S_\alpha(E)\, \mathscr{Y}_\alpha(\rho,z_0,E,\omega) \Big]$$
$$+ \sum_\beta \int_E^{E_0} g_{\alpha\beta}(E',E)\ Q_\beta(E')\, \mathscr{Y}_\alpha(\rho,z_0,E',\omega)\ dE', \tag{4.13}$$

---

*Since the coefficient of $\delta(r \sin\eta/\cos\phi_\omega)$ $\delta(r \cos\phi_r)$ contains the factor $(1 - \omega^2)^{\frac{1}{2}}$, the case $\omega = 1$ is included in the following discussion only in the sense that one may take the limit as ω approaches one.

$$\mathcal{Y}_\alpha(0,z_0,E,\omega) = \sum_\beta \int_E^{E_0} G_{\alpha\beta}(E',E',\omega)\, Q_\beta(E')\, \psi_\beta(z_0,E')\, dE'$$

$$+ \sum_\beta \int_E^{E_0} F_{\alpha\beta}(E',E,\omega)\, Q_\beta(E')\, \phi_\beta(z_0,E')\, dE', \qquad (4.14)$$

where

$$\mathcal{Y}_\alpha(\rho,z_0,E,\omega) = \chi_\alpha(r,z,E,\omega). \qquad (4.15)$$

Equation (4.13) is the usual straight-ahead equation subject to the boundary conditions expressed by Eq. (4.13). The quantities z_0 and ω occur as parameters in the equation, so each value of these variables may be treated separately and the flux, $\Phi_{s\alpha}^{(1)}$, obtained by repeatedly solving Eq. (4.13).

In terms of χ_α we must have

$$\chi_\alpha(r,0,E,\omega) = 0, \quad 0 < \omega \leq 1,$$
$$\chi_\alpha(r,\ell,E,\omega) = 0, \quad -1 \leq \omega < 0,$$

and using the transformation given in Eqs. (4.12) and (4.13) we must have

$$\mathcal{Y}_\alpha(\rho,z_0,E,\omega) = 0, \quad \ell > z_0 > 0. \qquad (4.16)$$

Thus for z_0 between 0 and ℓ the calculation is unrestricted and for all other values of z_0 the flux $\mathcal{Y}_\alpha$ is zero.

Assuming that $\Phi_\alpha^{(1)}$ is an adequate approximation to the particle flux, one may, of course, calculate a variety of quantities which are of interest. Of particular interest in the present case will be the lateral structure function of the cascade.

We shall define the lateral structure function of primary particles of type α, $L_{i\alpha}$, as*

$$L_{i\alpha}(r,z,E) = \int_0^{2\pi} \int_{\Omega} r\ \phi_{i\alpha}(\vec{R},E,\vec{\Omega})\ d\Omega\ d\phi_r \tag{4.17}$$

and the lateral structure function of secondary particles of type α (in first approximation), $L^{(1)}_{s\alpha}$, as*

$$L^{(1)}_{s\alpha}(r,z,E) = \int_0^{2\pi} \int_{\Omega} r\ \Phi^{(1)}_{s\alpha}(\vec{R},E,\vec{\Omega})\ d\Omega\ d\phi_r . \tag{4.18}$$

Using Eqs. (4.1) and (4.8) we have

$$L_{i\alpha}(r,z,E) = \phi_{i\alpha}(z,E)\ \delta(r), \tag{4.19}$$

$$L^{(1)}_{s\alpha}(r,z,E) = 2\pi\, \Theta(r) \int_{\arctan \frac{r}{z}}^{\arctan \frac{r}{z-\ell}} \chi_\alpha(r,z,E,\cos\theta)\ d\theta . \tag{4.20}$$

The limits in Eq. (4.20) come from the transformation given in Eq. (4.12). Since $\mathcal{Y}_\alpha$ has a nonzero value only when z_0 is between 0 and ℓ, the flux χ_α will have a value only when

$$0 \leq z - r\ \mathrm{ctn}\theta \leq \ell . \tag{4.21}$$

Strictly speaking, because of the Θ function in Eq. (4.20) we must take the limit as r approaches zero to obtain $L^{(1)}_{s\alpha}(0,z,E)$. However, from Eq. (4.11) it follows that

---

*Note that in defining $L_{i\alpha}$ and $L^{(1)}_{s\alpha}$ we have included an r factor to avoid a singularity in the functions and have carried out the integration over $d\phi_r$.

$$L_{s\alpha}^{(1)}(0,z,E) = 2\pi \int_0^{\pi} \left\{ \sum_{\beta} \int_E^{E_0} G_{\alpha\beta}(E',E,\cos\theta)\; Q_\beta(E')\; \psi_\beta(z,E')\; dE' \right.$$

$$\left. + \sum_{\beta} \int_E^{E_0} F_{\alpha\beta}(E',E,\cos\theta)\; Q_\beta(E')\; \phi_{i\beta}(z,E')\; dE' \right\} d\theta, \qquad (4.22)$$

and thus the value of the lateral structure function on the axis of the cascade may be obtained from a knowledge of ψ_β and $\phi_{i\beta}$.

V. Muon Components of the Cascade

The muons could easily have been included in the previous discussion; however, they constitute such a special case that they are best treated separately.

The very special nature of the muon equations arises from the fact that we may neglect the muon interaction with nuclei. Once a muon is formed -- by pion decay -- it has no further effect on the cascade.

The transport equations for the primary and secondary muon fluxes may be written

$$B_\mu\; \Phi_{i\mu\pm}(\vec{R},E,\vec{\Omega}) = 0, \qquad (5.1)$$

$$B_\mu\; \Phi_{s\mu\pm}(\vec{R},E,\vec{\Omega}) = s_{\mu\pm}(\vec{R},E,\vec{\Omega}), \qquad (5.2)$$

where

$$B_\mu = \vec{\Omega}\cdot\nabla + Q_{\mu D}(E) - \frac{\partial}{\partial E}\, S_\mu(E), \qquad (5.3)$$

$$s_{\mu\pm}(\vec{R},E,\vec{\Omega}) = \int_{E_1(E)}^{E_2(E)} \int_{\Omega'} F_{\mu\pi}(E',E,\vec{\Omega}'\vec{\Omega})\, Q_{\pi D}(E')\, \Phi_{\pi\pm}(\vec{R},E',\vec{\Omega}')\, d\Omega'\, dE', \qquad (5.4)^*$$

$$F_{\mu\pi}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) = \frac{\frac{1}{2}\left(\frac{m_\pi}{m_\mu}\right)}{U_2}\ \frac{1}{2\pi}\ \frac{\delta\left[\vec{\Omega}'\cdot\vec{\Omega} - k(E',E)\right]}{\sqrt{E'(E' + 2m_\pi c^2)}}, \qquad (5.5)$$

$$k(E',E) = \left[\frac{E + m_\mu c^2}{\sqrt{E(E + 2m_\mu c^2)}}\right]\left[\frac{E' + m_\pi c^2}{\sqrt{E'(E' + 2m_\pi c^2)}}\right]$$

$$- U_1\left(\frac{m_\pi}{m_\mu}\right)\frac{1}{\sqrt{E'(E' + 2m_\pi c^2)}},$$

$$E_2(E) = \frac{m_\pi}{m_\mu}\left\{U_1(E + m_\mu c^2) + U_2\left[E(E + 2m_\mu c^2)\right]^{\frac{1}{2}}\right\} - m_\pi c^2,$$

$$E_1(E) = \frac{m_\pi}{m_\mu}\left\{U_1(E + m_\mu c^2) - U_2\left[E(E + 2m_\mu c^2)\right]^{\frac{1}{2}}\right\} - m_\pi c^2,$$

$$U_1 = \frac{U^*}{m_\mu c^2},$$

$$U_2 = \left[\left(\frac{U^*}{m_\mu c^2}\right) - 1\right]^{\frac{1}{2}},$$

---

*The form of $F_{\mu\pi}$ and the quantities E_1 and E_2 are obtained by assuming that the muons are emitted isotropically in the rest frame of the pion and then transforming into the laboratory system by Lorentz transformation. See B. Rossi, High Energy Particles, Prentice-Hall, Inc., New Jersey (1956), p 191.

$\Phi_{i\mu\pm}(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of primary muons with plus and minus charge, respectively,

$\Phi_{s\mu\pm}(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of secondary muons with plus and minus charge, respectively,

$\Phi_{\pi\pm}(\vec{R},E,\vec{\Omega})$ = angular flux per unit energy range of pions with plus and minus charge, respectively,

$Q_{\mu D}$ = probability per unit distance for the decay of the muon,

S_{μ} = energy loss per unit distance of the muon,

$m_{\pi}c^2$ = rest energy of pion,

$m_{\mu}c^2$ = rest energy of muon,

U^* = total energy of the muon in the rest frame of the pion.

The primary muon flux can, of course, be obtained without reference to the other portions of the cascade and is therefore of very little interest here. Throughout the remainder of the discussion we shall assume that there are no initial muons so

$$\Phi_{i\mu\pm}(\vec{R},E,\vec{\Omega}) = 0.$$

If the pion flux is known, Eq. (5.2) is a special case of Eq. (2.10). It is in principle possible to use the first-order pion flux obtained in the previous sections in Eq. (5.4) and to calculate the muon flux from Eq. (5.2). However, it is rather pointless to treat the muons more exactly than we have treated the pions. It seems consistent with the previous discussion to treat the muons from primary pions exactly but to treat the muons from secondary pions as being emitted in the direction of the decaying pion. To this end, we introduce a first-order muon flux, $\Phi^{(1)}_{s\mu\pm}$, through the equation

$$B_{\mu}\,\Phi^{(1)}_{s\mu\pm}(\vec{R},E,\vec{\Omega}) = s^{(1)}_{\mu\pm}(\vec{R},E,\vec{\Omega}), \tag{5.6}$$

where

$$s^{(1)}_{\mu\pm}(\vec{R},E,\vec{\Omega}) = \int_{E_1(E)}^{E_2(E)} \int_{\Omega'} F_{\mu\pi}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\ Q_{\pi D}(E')\ \Phi_{i\pi\pm}(\vec{R},E',\vec{\Omega})\ d\Omega'\ dE'$$

$$+ \int_{E_1(E)}^{E_2(E)} \int_{\Omega'} F_{\mu\pi}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\Big|_{k=1}\ Q_{\pi D}(E')\ \Phi_{s\pi\pm}(\vec{R},E',\vec{\Omega})\ d\Omega'\ dE',$$

and $F_{\mu\pi}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\big|_{k=1}$ means that k is to be put equal to 1 in Eq. (5.5)

For the case of an infinite beam normally incident on a slab we use the results of Section II in Eqs. (5.6) and (5.7) to obtain

$$\omega \frac{\partial}{\partial z} \Phi^{(1)}_{\mu\pm}(z,E,\omega) + Q_{\mu D}(E)\ \Phi^{(1)}_{\mu\pm}(z,E,\omega)$$

$$- \frac{\partial}{\partial E}\left[S_{\mu}(E)\ \Phi^{(1)}_{\mu\pm}(z,E,\omega)\right] = s^{(1)}_{\mu\pm}(z,E,\omega), \qquad (5.8)$$

$$s^{(1)}_{\mu\pm}(z,E,\omega) = \int_{E_1(E)}^{E_2(E)} f_{\mu\pi}(E',E,\omega)\ Q_{\pi D}(E')\ \Phi_{i\pi\pm}(z,E')\ dE'$$

$$+ \int_{E_1(E)}^{E_2(E)} g_{\mu\pi}(E')\ Q_{\pi D}(E')\ \Phi^{(1)}_{s\pi\pm}(z,E',\omega)\ dE', \qquad (5.9)$$

$$g_{\mu\pi}(E') = \frac{\frac{1}{2}\left(\frac{m_\pi}{m_\mu}\right)}{U_2} \quad \frac{1}{\sqrt{E'(E' + 2m_\pi c^2)}}\ . \qquad (5.10)$$

Equations (5.8) and (5.9) are of the same form as Eqs. (2.17) and (2.18) so the first-order muon flux may be obtained in the same manner as the other particle fluxes. The boundary conditions on the secondary muons are the same as on the other secondary particles, Eq. (2.19), and may, of course, be satisfied exactly as before.

For the case of an infinite beam incident isotropically on a slab, we have from Section III and Eqs. (5.6) and (5.7)

$$\omega \frac{\partial}{\partial z} \Phi^{(1)}_{s\mu\pm}(z,E,\omega) + Q_{\alpha D}(E)\ \phi^{(1)}_{s\mu\pm}(z,E,\omega)$$

$$- \frac{\partial}{\partial E}\left[S_{\mu}(E)\ \Phi^{(1)}_{\mu\pm}(z,E,\omega)\right] = s^{(1)}_{\mu\pm}(z,E,\omega), \qquad (5.11)$$

$$s^{(1)}_{\mu\pm}(z,E,\omega) = \int_{E_1(E)}^{E_2(E)} \int_{\Omega'} F_{\mu\pi}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\ Q_{\pi D}(E')\ \phi_{i\pi\pm}\left(\frac{z}{\omega'},E'\right)\ d\Omega'\ dE'$$

$$+ \int_{E_1(E)}^{E_2(E)} g_{\mu\pi}(E')\ Q_{\pi D}(E')\ \Phi^{(1)}_{s\pi\pm}(z,E',\omega)\ dE'. \qquad (5.12)$$

For the case of a narrow beam incident normally on a slab, Eq. (5.7) becomes

$$s^{(1)}_{\mu\pm}(\vec{R},E,\vec{\Omega}) = \delta(r\ \cos\phi_r)\ \delta(r\ \sin\phi_r) \int_{E_1(E)}^{E_2(E)} F_{\mu\pi}(E',E,\omega)\ Q_{\pi D}\ \phi_{i\pi\pm}(z,E')\ dE'$$

$$+ \frac{(\pm 1)\ \delta\left(\dfrac{r\ \sin\eta}{\cos\phi_\omega}\right) \Theta(\pm r\ \cos\phi_r)}{\cos\phi_\omega} \int_{E_1(E)}^{E_2(E)} g_{\mu\pi}(E')\ Q_{\pi D}(E')\ \chi_{\pi\pm}(r,z,E',\omega)\ dE', \qquad (5.13)$$

and introducing as before the ansatz

$$(1 - \omega^2)^{\frac{1}{2}} \Phi^{(1)}_{\mu\pm}(\vec{R},E,\vec{\Omega}) = \pm \frac{\chi_{\mu\pm}(r,z,E,\omega)}{\cos\phi_\omega} \delta(r \sin\eta/\cos\phi_\omega)\ \delta(\pm r \cos\phi_r), \tag{5.14}$$

we obtain

$$\omega \frac{\partial}{\partial z} \chi_{\mu\pm}(r,z,E,\omega) + Q_{\mu D}(E)\ \chi_{\mu\pm}(r,z,E,\omega) - \frac{\partial}{\partial E} S_\mu(E)\ \chi_{\mu\pm}(r,z,E,\omega)$$

$$= \int_{E_1(E)}^{E_2(E)} g_{\mu\pi}(E')\ Q_{\pi D}(E')\ \chi_{\pi\pm}(r,z,E,\omega)\ dE', \tag{5.15}$$

$$\chi(0,z,E,\omega) = \int_{E_1(E)}^{E_2(E)} F_{\mu\pi}(E',E,\omega)\ Q_{\pi D}(E')\ \phi_{i\pi\pm}(z,E')\ dE'. \tag{5.16}$$

Equations (5.15) and (5.16) are now completely equivalent to Eqs. (4.10) and (4.11) and may be treated in the same manner as these equations were treated.

The lateral structure function for the secondary muons may be written

$$L^{(1)}_{s\mu\pm}(r,z,E) = 2\pi\, \Theta(r) \int_{\text{arc tan}\,\frac{r}{z}}^{\text{arc tan}\,\frac{r}{z-\ell}} \chi_{\mu\pm}(r,z,E,\cos\theta)\ d\theta, \tag{5.17}$$

with

$$L^{(1)}_{s\mu\pm}(0,z,E) = 2\pi \int_0^\pi \left\{ \int_{E_1(E)}^{E_2(E)} F_{\mu\pi}(E',E,\omega)\ Q_{\pi D}(E')\ \phi_{i\pi\pm}(z,E')\ dE' \right\} d\theta. \tag{5.18}$$

Appendix

In the body of this paper the discussion was carried through only to first order. For thick slabs such as those of interest in high-energy accelerator shielding, the first-order computation is quite lengthy and a higher order computation is probably not feasible. However, for the case of an infinite beam incident, either normally or isotropically, on a thin slab it may be possible to carry the computation to higher than first order without involving excessive computing time. In this appendix we indicate how for the infinite beam cases the computation may be carried out to all orders.

By adding and subtracting terms Eq. (2.4) may be put in the form

$$B_\alpha \, \Phi_{s\alpha}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} \int_{\Omega'} g_{\alpha\beta}(E',E) \, \frac{\delta(\vec{\Omega}'\cdot\vec{\Omega} - 1)}{2\pi} \, Q_\beta(E') \, \Phi_{s\beta}(\vec{R},E',\vec{\Omega}') \, d\Omega' \, dE'$$

$$+ \, s_\alpha(\vec{R},E,\vec{\Omega}), \qquad (A.1)$$

where

$$s_\alpha(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_0} \int_{\Omega'} \left[F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) - g_{\alpha\beta}(E',E) \, \frac{\delta(\vec{\Omega}'\cdot\vec{\Omega} - 1)}{2\pi} \right] Q_\beta(E')$$

$$\Phi_{s\beta}(\vec{R},E',\vec{\Omega}') \, d\vec{\Omega}' \, dE' \qquad (A.2)$$

$$+ \sum_\beta \int_E^{E_0} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \, Q_\beta(E') \, \Psi_\beta(\vec{R},E,\vec{\Omega}') \, d\Omega' \, dE'$$

$$- \sum_\beta \int_E^{E_0} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \, Q_\beta(E') \, \Psi_\beta(\vec{R},E,\vec{\Omega}') \, d\Omega' \, dE'$$

$$+ \sum_\beta \int_E^{E_0} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) \, Q_\beta(E') \, \Phi_{i\beta}(\vec{R},E,\vec{\Omega}) \, d\Omega' \, dE'.$$

The quantity Ψ_α which has been introduced in Eq. (A.2) is defined through the equation

$$B_\alpha \, \Psi_\alpha(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_o} \int_{\Omega'} h_{\alpha\beta}(E',E) \, \frac{\delta(\vec{\Omega}'.\vec{\Omega} - 1)}{2\pi}$$

$$Q_\beta(E') \left[\Psi_\beta(\vec{R},E',\vec{\Omega}') + \Phi_{i\beta}(\vec{R},E',\vec{\Omega}') \, d\Omega' \, dE' \right], \qquad (A.3)$$

where $h_{\alpha\beta}$ is a function which is to be defined.

Now let us introduce successive approximations to the flux, $\Phi_s^{(\kappa)}$, through the equations

$$\Phi_{s\alpha}(\vec{R},E,\vec{\Omega}) = \sum_{\kappa=1}^{\infty} \Phi_{s\alpha}^{(\kappa)}(\vec{R},E,\vec{\Omega}), \qquad (A.4)$$

$$B_\alpha \, \Phi_{s\alpha}^{(\kappa)}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_o} g_{\alpha\beta}(E',E) \; Q_\beta(E') \; \Phi_{s\beta}^{(\kappa)}(\vec{R},E',\vec{\Omega}') \, dE' + s_\alpha^{(\kappa)}(\vec{R},E,\vec{\Omega}), \qquad (A.5)$$

$$s_\alpha(\vec{R},E,\vec{\Omega}) = \sum_{\kappa=1}^{\infty} s_\alpha^{(\kappa)}(\vec{R},E,\vec{\Omega}), \qquad (A.6)$$

$$s_\alpha^{(1)}(\vec{R},E,\vec{\Omega}) = \sum_\beta \int_E^{E_o} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'.\vec{\Omega}) \; Q_\beta(E') \; \Psi_\beta(\vec{R},E',\vec{\Omega}') \, d\Omega' \, dE'$$

$$+ \sum_\beta \int_E^{E_o} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'.\vec{\Omega}) \; Q_\beta(E') \; \Phi_{i\beta}(\vec{R},E',\vec{\Omega}') \, dE' \, d\Omega', \qquad (A.7)$$

$$s_{\alpha}^{(2)}(\vec{R},E,\vec{\Omega}) = \sum_{\beta} \int_{E}^{E_o} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\, Q_{\beta}(E)\, \Phi_{s\beta}^{(1)}(\vec{R},E,\vec{\Omega})\, d\Omega'\, dE'$$

$$- \sum_{\beta} \int_{E}^{E_o} g_{\alpha\beta}(E',E)\, Q_{\beta}(E')\, \Phi_{s\beta}^{(1)}(\vec{R},E',\vec{\Omega})\, dE'$$

$$- \sum_{\beta} \int_{E}^{E_o} \int_{\Omega'} G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\, Q_{\beta}(E')\, \psi_{\beta}(\vec{R},E',\vec{\Omega}')\, d\Omega'\, dE', \tag{A.8}$$

$$s_{\alpha}^{(\kappa)}(\vec{R},E,\vec{\Omega}) = \sum_{\beta} \int_{E}^{E_o} \int_{\Omega'} F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega})\, Q_{\beta}(E')\, \Phi_{s\beta}^{(\kappa-1)}(\vec{R},E',\vec{\Omega}')\, d\Omega'\, dE'$$

$$- \sum_{\beta} \int_{E}^{E_o} g_{\alpha\beta}(E',E)\, Q_{\beta}(E')\, \Phi_{s\beta}^{(\kappa-1)}(\vec{R},E',\vec{\Omega})\, dE' \qquad \kappa \geq 3. \tag{A.9}$$

With these definitions Eq. (A.4) is an exact solution to Eq. (A.1) provided that the series converges. It is clear that the first-order equations are the same as those introduced in Section II.

Note that the particles produced by ψ_{α} have been included in the first-order source term and subtracted from the second-order source term. The reason for doing this is, of course, to make the first order as accurate as possible.

In a slab geometry the boundary conditions given in Eq. (2.19) apply and are to be applied in each order. Except for the presence of the source term, $s_{\alpha}^{(2)}$, the equation for $\Phi_{s\alpha}^{(2)}$ is a homogeneous equation subject to zero boundary conditions so the second-order flux will be small (or zero) provided that $s_{\alpha}^{(2)}$ is small (or zero). Therefore, we should like to choose the functions $g_{\alpha\beta}$, $h_{\alpha\beta}$, and $G_{\alpha\beta}$ so as to make $s_{\alpha}^{(2)}$ as small as possible.

It follows immediately from Eq. (A.8) that $s_{\alpha}^{(2)}$ will be zero if

$$\Phi_{s\alpha}^{(1)}(\vec{R},E,\vec{\Omega}) = \Psi_{\alpha}(\vec{R},E,\vec{\Omega}) \tag{A.10}$$

and if

$$G_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) = F_{\alpha\beta}(E',E,\vec{\Omega}'\cdot\vec{\Omega}) - g_{\alpha\beta}(E',E)\,\frac{\delta(\vec{\Omega}'\cdot\vec{\Omega} - 1)}{2\pi}\,. \tag{A.11}$$

Equation (A.10) is just the statement that the first-order flux is exact if the straight-ahead approximation is exact. Since this is not the case, the best one can hope for is that

$$\int_{\Delta_1} \Phi_{s\alpha}^{(1)}(\vec{R},E,\vec{\Omega}')\,d\Omega' = \int_{\Delta_1} \Psi_{\alpha}(\vec{R},E,\vec{\Omega}')\,d\Omega', \tag{A.12}$$

where Δ_1 = some suitably defined solid angle.

Equation (A.12) is not a way of choosing Ψ_{α} but rather is a crude test which can be applied to determine the validity of the first approximation after it has been obtained. The only arbitrariness we have in choosing Ψ_{α} is through the choice of $h_{\alpha\beta}$. On the basis of (A.12) it seems reasonable to choose

$$h_{\alpha\beta}(E',E) = g_{\alpha\beta}(E',E), \tag{A.13}$$

as we have done in the body of the paper, but it must be understood that there is considerable arbitrariness in this choice.

Equation (A.11) could be used as a definition of $G_{\alpha\beta}$. However, as we have stated earlier, this choice is so inconvenient from a computational point of view it seems preferable to use the definition given in the body of the paper.

It is clear that there is, in general, no way of deciding what choice of the functions $g_{\alpha\beta}$, $h_{\alpha\beta}$, and $G_{\beta\alpha}$ will lead to the best first approximation. If the computation is carried out to a sufficiently high order, it is to be expected that the results will be independent of the exact choice made for these functions, but it is clear that the best one can hope for is that the first-order result be approximately independent of the choice made for these functions.

Let us now consider Eqs. (A.4) to (A.9) for the case of an infinite beam incident either normally or isotropically on a slab. By introducing the appropriate form of the primary solution and utilizing Eqs. (2.12), one finds that Eq. (A.4) in all orders is of the form of Eqs. (2.10) and (3.5); i.e., it is of the form of the first-order equations. Therefore, the same code which will give a numerical solution to the first-order equations may be used to obtain a solution in any orders. In principle, then, an exact solution can be obtained by an iteration procedure. The boundary conditions, Eq. (2.19), can, of course, be satisfied exactly in each order.

After the iteration is complete and the exact pion flux has been obtained, the muon flux may be obtained directly from Eqs. (5.2) and (5.4).

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