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A SIMPLE GRAPHICAL METHOD IN THE ANALYSIS OF SU_3

by

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ABSTRACT

The commutation relations of the algebra of SU_3 are used to construct a simple graphical picture of the irreducible representations. The graphs are useful in the reduction of products of irreducible representations. A simple systematic method for the calculation of the generalized Clebsch-Gordan coefficients is also presented.

I. INTRODUCTION

In the last few years the attention of elementary-particle physicists has been strongly drawn to the group of three-dimensional unitary unimodular matrices SU_3.[1] It appears that the "elementary" particles as well as an ever increasing number of resonances can rather conveniently be classified according to representations of this group, in particular the 8-dimensional representations (pseudoscalar mesons π, K, $\overline{K}$, η^0; vector ρ, M, $\overline{M}$, ω; baryons Σ, N, Ξ, Λ^0) and the 10-dimensional one [N_{33}^*, Y_1^*, Ξ^*, Z(?)].[2] In contrast to the simpler group SU_2, which plays a role in the phenomenon of charge independence and which is almost identical with the familiar rotation group, SU_3 and its connection with the symmetric group[3] is not well known to many physicists. Thus an ever increasing number of papers which take the mathematical background for granted appear somewhat mysterious to those unfamiliar with the group structure. A recent review article on Lie groups by Behrends *et al.*[4] has done much to remove this gap in mathematical background; but this paper, covering as it does many groups of potential interest and many techniques, still leaves something to be desired in the specific treatment of SU_3. This paper in a certain sense complements the BDFL paper: the problem of obtaining explicit representations and of reducing products of representations is treated by

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[1] M. Ikeda, S. Ogawa, and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) 22, 715 (1959); M. Gell-Mann, "The Eightfold Way," CTSL Report No. 20 (1961); Phys. Rev. 125, 1067 (1962); Y. Ne'eman, Nucl. Phys. 26, 222 (1961); Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto), Suppl. No. 11 (1959); A. Salam and J. Ward, Nuovo cimento 20, 419 (1961); J. Wess, Nuovo cimento 10, 15 (1960).

[2] S. Glashow and J. J. Sakurai, Nuovo cimento 25, 337 (1962); *ibid.* 26, 622 (1962).

[3] For example, J. Wess, Nuovo cimento 10, 15 (1960).

[4] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Revs. Modern Phys. 34, 1 (1962).

generalizing the "raising" and "lowering" operator technique familiar from the simple textbook treatment of angular momentum. For the sake of simplicity, only this technique is worked out in detail; no attempt is made to give a complete proof of a number of results obtained on the way. The technique could, of course, be adapted for the treatment of other Lie groups, but in view of the great interest in SU_3 to the exclusion of any of the other groups, no attempt at generalization is made.

The group is defined by eight generators and, as established by Lie,[5] only the commutators of these generators are necessary to study the properties of the group. For example, the rotation group is (almost) completely defined by the commutation relations

$$[J_i, J_k] = i\,\epsilon_{ik\ell}\,J_\ell$$

among the generators of rotations J_i. In what follows, we shall assume it to be proved that for SU_3, too, the algebra of its generators (i.e., the commutation relations) are all that are needed, and that the representations of the algebra yield the representations of the group. Aside from the gap—not proving the above statement and not filling in the mathematical background—the paper is self-contained and follows the lines laid out in Sec. II, which reviews i-spin.

II. THE ALGEBRA OF SU_2

This section briefly reviews the familiar algebra of i-spin and its representations, to prepare us for the somewhat more complicated case of SU_3. As was mentioned in the introduction, the algebra is defined by the commutation relations among the operators, and in order to find (i) all the

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[5] See for example G. Racah, "Group Theory and Spectroscopy," Institute for Advanced Study Lecture Notes, Princeton, N. J. (1951).

operators and (ii) the defining commutation relations of the algebra, we start with the simplest physical realization of the algebra—the one which, in fact, made us interested in SU_2 in this connection. We assume that there is an entity, the "nucleon," which has two states, the "proton" state and the "neutron" state, which we shall describe by the two-component wave function $\begin{pmatrix} p \\ n \end{pmatrix}$. We can construct an operator which converts the neutron to a proton, namely, the raising operator

$$\tau_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has the property that

$$\tau_+ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} .$$

Its Hermitian adjoint

$$\tau_- \equiv \tau_+^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a lowering operator, since

$$\tau_- \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} .$$

We can form the commutator of these, and find that

$$[\tau_+, \tau_-] = \tau_3 , \qquad \text{(II - 1)}$$

where

$$\tau_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We also find that

$$[\tau_3, \tau_+] = 2\tau_+ , \qquad \text{(II - 2)}$$

$$[\tau_3, \tau_-] = -2\tau_- . \qquad \text{(II - 3)}$$

Equations (II - 1), (II - 2), and (II - 3) complete the commutation relations; all further commutators such as $\left[\tau_+, [\tau_+, \tau_-]\right]$, $\left[\tau_3, [\tau_3, \tau_-]\right]$, and the like can be expressed in terms of the operators τ_+, τ_-, τ_3, which are therefore the basic set. In terms of the 2 $\times$ 2 matrices, there are also properties such as

$$\tau_+^{\,2} = 0,$$

$$\tau_-^{\,2} = 0,$$

but these are peculiar to the two-dimensional representation only. To obtain other representations of the algebra, we must use only the defining commutation relations (II - 1, 2, 3).

It is clear from these relations that only one of the Hermitian operators τ_3, $(\tau_+ + \tau_-)$, and $i(\tau_- - \tau_+)$ can be diagonalized. The representation of the space in which these operators act will therefore be chosen such that τ_3 is diagonal—as turned out to be the case in the two-dimensional representation with which we started. We label the states only by the eigenvalue m of τ_3 for the time being, i.e.,

$$\tau_3 |m\rangle = m|m\rangle . \qquad \text{(II - 4)}$$

From Eq. (II - 2) it follows that

$$\tau_3\tau_+|m\rangle = (\tau_+\tau_3 + 2\tau_+)|m\rangle = (m + 2)\,\tau_+|m\rangle \qquad \text{(II - 5)}$$

so that $\tau_+|m\rangle$ is an eigenstate of τ_3 with eigenvalue $(m + 2)$—hence the appellation "raising operator" for τ_+. Similarly $\tau_-|m\rangle$ can be shown to be an eigenstate of τ_3 with eigenvalue $(m - 2)$. If we start with any state in a given representation, we can generate states with higher and higher eigenvalues by repeated application of τ_+ until we reach the state with maximum eigenvalue $|M\rangle$. This state has the property that

$$\tau_+|M\rangle = 0 . \qquad \text{(II - 6)}$$

Let us now start with this "highest weight" state, which is unique for an irreducible representation,[5] and generate the whole sequence of states. We have

$$\tau_-|M\rangle = \lambda_1|M-2\rangle . \qquad \text{(II - 7)}$$

If the states are chosen to be normalized to unity, i.e., if

$$\langle m|m\rangle = 1 , \qquad \text{(II - 8)}$$

then

$$\begin{aligned} \lambda_1^2 &= \langle M|\tau_+\tau_-|M\rangle \\ &= \langle M|[\tau_+, \tau_-] + \tau_-\tau_+|M\rangle \\ &= \langle M|\tau_3|M\rangle = M . \end{aligned} \qquad \text{(II - 9)}$$

The phases have been chosen such that λ_1 is real. Note that

$$\langle M-2|\tau_-|M\rangle = \lambda_1 = \langle M|\tau_+|M-2\rangle^*.$$

Hence

$$\tau_+ |M-2\rangle = \lambda_1 |M\rangle . \qquad \text{(II - 10)}$$

Next consider

$$\tau_- |M-2\rangle = \lambda_2 |M-4\rangle . \qquad \text{(II - 11)}$$

Again

$$\begin{aligned} \lambda_2{}^2 &= \langle M-2 | \tau_+ \tau_- | M-2 \rangle \\ &= \langle M-2 | \tau_- \tau_+ | M-2 \rangle + M-2 \\ &= \lambda_1{}^2 + M-2 . \end{aligned} \qquad \text{(II - 12)}$$

In general, if

$$\tau_- |M-2(p-1)\rangle = \lambda_p |M-2p\rangle , \qquad \text{(II - 13)}$$

we see that

$$\lambda_p{}^2 = \lambda_{p-1}{}^2 + M-2(p-1) . \qquad \text{(II - 14)}$$

We thus find that

$$\lambda_p{}^2 = p(M - p + 1) . \qquad \text{(II - 15)}$$

The "minimum state" is reached when $\lambda_p = 0$, i.e., when

$$p = M + 1 . \qquad \text{(II - 16)}$$

This is the multiplicity of the irreducible representation in terms of the maximum eigenvalue of τ_3. The states may be pictured as forming a linear array (Fig. 1) and the operators $\tau_\pm$ represent steps from one point to

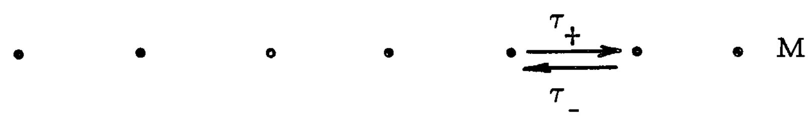

Fig. 1. The states of an irreducible representation of SU_2 and the action of "shift" operators on them.

another. There is only one state with a given value of m; and furthermore only one irreducible representation is generated from a given maximum state $|M\rangle$. We therefore expect that there exists one independent operator constructed out of $\tau_\pm, \tau_3$ which commutes with all the τ's and which serves to distinguish irreducible representations.[6] Such an operator is, for example, the square of the i-spin. We write it as

$$\mathbb{C} = \tfrac{1}{2}(\tau_+\tau_- + \tau_-\tau_+) + \tfrac{1}{4}\tau_3^2 . \qquad \text{(II - 17)}$$

Since

$$[\mathbb{C}, \tau_i] = 0,$$

it follows that

$$\mathbb{C}|m\rangle = c|m\rangle \qquad \text{(II - 18)}$$

for all m, i.e.,

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[6] The relation between the number of invariant operators and the number of parameters necessary to characterize an irreducible representation is mentioned by L. C. Biedenharn, Phys. Letters 3, 69 (1962).

$$\langle m|\mathbb{C}|m\rangle = c = \langle M|\mathbb{C}|M\rangle \,. \qquad \text{(II - 19)}$$

Now use of the commutation relations yields

$$\mathbb{C} = \tfrac{1}{4}\tau_3{}^2 + \tfrac{1}{2}\tau_3 + \tau_-\tau_+ \,. \qquad \text{(II - 20)}$$

Hence

$$c = \langle M|\mathbb{C}|M\rangle = (\tfrac{1}{4}M^2 + \tfrac{1}{2}M)$$
$$= \tfrac{1}{2}M(\tfrac{1}{2}M + 1) \equiv t(t+1) \qquad \text{(II - 21)}$$

and the multiplicity is

$$p = (M+1) = 2t+1 \,, \qquad \text{(II - 22)}$$

where $2t+1$ is an integer. This is a well-known result. The following discussion of the representations of the algebra of SU_3 will imitate the one above as far as possible.

III. THE ALGEBRA OF SU_3

The generalization from SU_2 to SU_3 consists in enlarging the set of operators $\tau_\pm$, τ_3 of Sec. II.to one which "shifts" the positions of three objects, say the proton, neutron, and lambda (as in the symmetric Sakata model)[1] in the wave function $\begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$. The simplest representation of the shift operators by 3×3 matrices is the set

$$E_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{-2} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \text{(III - 1)}$$

$$E_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{-3} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The factor $1/\sqrt{6}$ is inserted in order to make the notation conform to the canonical notation of Behrends *et al.*[4]

If we introduce the matrices

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \text{(III - 2)}$$

we can easily derive the set of commutation relations which completely defines the algebra. These are

$$[E_1, E_2] = 0, \qquad [E_{-1}, E_{-2}] = 0,$$

$$[E_1, E_{-2}] = -\frac{1}{\sqrt{6}} E_{-3}, \qquad [E_{-1}, E_2] = \frac{1}{\sqrt{6}} E_3,$$

$$[E_1, E_3] = \frac{1}{\sqrt{6}} E_2, \qquad [E_{-1}, E_{-3}] = -\frac{1}{\sqrt{6}} E_{-2},$$

$$\text{(III - 3)}$$

$$[E_1, E_{-3}] = 0, \qquad [E_{-1}, E_3] = 0,$$

$$[E_2, E_3] = 0, \qquad [E_{-2}, E_{-3}] = 0,$$

$$[E_2, E_{-3}] = \frac{1}{\sqrt{6}} E_1, \qquad [E_{-2}, E_3] = -\frac{1}{\sqrt{6}} E_{-1}.$$

Further

$$[E_1, E_{-1}] = \frac{1}{\sqrt{3}} H_1 ,$$

$$[E_2, E_{-2}] = \frac{1}{2\sqrt{3}} H_1 + \tfrac{1}{2} H_2 , \qquad \text{(III - 4)}$$

$$[E_3, E_{-3}] = -\frac{1}{2\sqrt{3}} H_1 + \tfrac{1}{2} H_2 ,$$

and

$$[H_1, E_1] = \frac{1}{\sqrt{3}} E_1 , \qquad [H_2, E_1] = 0,$$

$$[H_1, E_2] = \frac{1}{2\sqrt{3}} E_2 , \qquad [H_2, E_2] = \tfrac{1}{2} E_2 , \qquad \text{(III - 5)}$$

$$[H_1, E_3] = -\frac{1}{2\sqrt{3}} E_3 , \qquad [H_2, E_3] = \tfrac{1}{2} E_3 ,$$

$$[H_1, H_2] = 0.$$

Note that

$$E_{-\alpha} = E_{\alpha}^{\dagger} . \qquad \text{(III - 6)}$$

The relations (III - 3)—(III - 6) completely characterize the algebra. Repeated commutators, e. g., $\left[E_{\alpha}, [E_{\beta}, H_i]\right]$ and the like do not lead to any new operators.

Our task now is to construct representations of the algebra. We first note that we may choose our representation such that H_1 and H_2, which commute with each other, are simultaneously diagonal in it. We label the states in a given irreducible representation by the eigenvalues of H_1 and H_2, and denote them by $|m_1, m_2\rangle$. Then

$$H_1 \; |m_1, m_2\rangle = m_1 \; |m_1, m_2\rangle$$
$$H_2 \; |m_1, m_2\rangle = m_2 \; |m_1, m_2\rangle \; . \qquad \text{(III - 7)}$$

As is suggested by the explicit form of the special representation in Eqs. (III - 1), the operators $\sqrt{6}\, E_1$, $\sqrt{6}\, E_{-1}$, and $2\sqrt{3}\, H_1$ obey the same commutation relations as τ_+, τ_-, and τ_3 and thus form a sub-algebra. The invariant $\mathbb{C}$ takes the form

$$\vec{T}^2 = 3\,(E_1\, E_{-1} + E_{-1}\, E_1) + 3H_1^{\;2} \; ; \qquad \text{(III - 8)}$$

and it is easily verified that

$$[\vec{T}^2, E_{\pm 1}] = 0,$$
$$[\vec{T}^2, H_1] = 0, \qquad \text{(III - 9)}$$
$$[\vec{T}^2, H_2] = 0,$$

whereas

$$[\vec{T}^2, E_{\pm 2}] \neq 0,$$
$$[\vec{T}^2, E_{\pm 3}] \neq 0.$$

Thus $\vec{T}^2$ is not an invariant operator. It may, however, be diagonalized simultaneously with H_1 and H_2; and the states $|m_1, m_2\rangle$ should actually be written as $|m_1, m_2; t\rangle$ with

$$\vec{T}^2 \; |m_1, m_2; t\rangle = t(t+1) \; |m_1, m_2; t\rangle \, . \qquad \text{(III - 10)}$$

The form of the eigenvalue will become evident later. For the time being we will suppress the dependence on t, and only use it later to classify states that may, even within a given irreducible representation, have the same values of m_1 and m_2.

It is clear from Eqs. (III - 5) that since

$$H_1 E_1 = E_1 \left(H_1 + \frac{1}{\sqrt{3}}\right),$$
$$H_2 E_1 = E_1 H_2,$$

E_1 raises the value m_1 by $1/\sqrt{3}$ but leaves m_2 unchanged. The commutation relations for H_1, H_2 with $E_{\pm 2}$, $E_{\pm 3}$ imply the properties exhibited in Table I.

TABLE I. The "shifting" properties of the generating operators $E_{\pm\alpha}$.

Operation by	m_1 eigenvalue	m_2 eigenvalue
E_1	raised by $\frac{1}{\sqrt{3}}$	unchanged
E_{-1}	lowered by $\frac{1}{\sqrt{3}}$	unchanged
E_2	raised by $\frac{1}{2\sqrt{3}}$	rasied by $\frac{1}{2}$
E_{-2}	lowered by $\frac{1}{2\sqrt{3}}$	lowered by $\frac{1}{2}$
E_3	lowered by $\frac{1}{2\sqrt{3}}$	raised by $\frac{1}{2}$
E_{-3}	raised by $\frac{1}{2\sqrt{3}}$	lowered by $\frac{1}{2}$

The properties may be exhibited graphically on a plot (Fig. 2) with coordinates m_1 and m_2. Because of the canonical normalization, the operators act along the sides of equilateral triangles. This graphical representation of the operators will be used extensively.

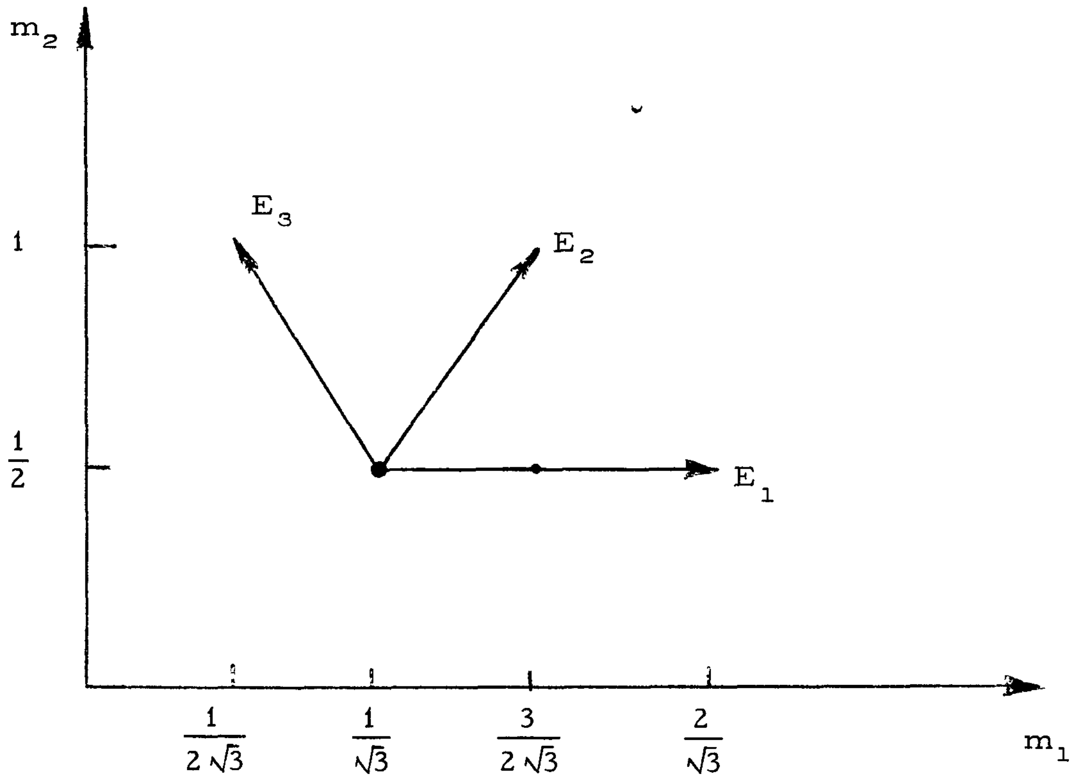

Fig. 2. Graphical representation of "shift" operators in $m_1 m_2$ plane.

A final remark is in order: one can easily check that the commutation relations are invariant under the replacement

$$
\begin{aligned}
E_1{}' &= E_{-2}, \qquad H_1{}' = -\tfrac{1}{2} H_1 - \frac{\sqrt{3}}{2} H_2, \\
E_2{}' &= E_{-3}, \qquad H_2{}' = \frac{\sqrt{3}}{2} H_1 - \tfrac{1}{2} H_1, \qquad\qquad \text{(III - 11)} \\
E_3{}' &= E_1 ,
\end{aligned}
$$

i.e., under a rotation of Fig. 2 through 120°. The implications of this are that not only do $[E_1, E_{-1}, (1/\sqrt{3}) H_1]$ form a sub-algebra (SU_2) but so also do (E_{-2}, E_2, $-\frac{1}{2} H_1 - \frac{1}{2}\sqrt{3}\ H_2$) and the set obtained by a 240° rotation. A linear combination of E_1 and E_2 may also be used to generate a sub-algebra.[7]

[7] B. D'Espagnat and J. Preutki, Nuovo cimento 24, 497 (1962).

IV. THE REPRESENTATION SPACE

In this section, a graphical method will be used to exhibit some of the properties of the representation space, and in particular, the possible values, m_1 and m_2, of the eigenvalues of H_1 and H_2. As was shown in Sec. III, the operators $E_{\pm 1}$, $E_{\pm 2}$, and $E_{\pm 3}$ are shift operators which transform a given state into one with eigenvalues such that it is displaced by a distance $1/\sqrt{3}$ along one of the three sides of an equilateral triangle. The states therefore must lie on the sites shown as solid circles in Fig. 3. The direction AB will be called the "1 line," AC the "2 line," and AD the "3 line."

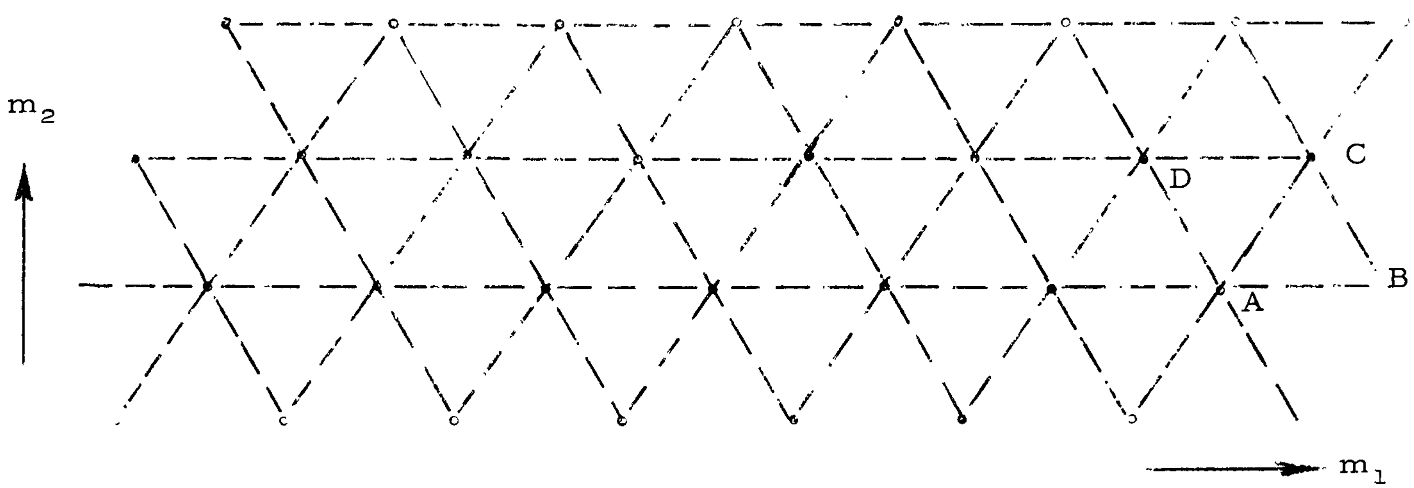

Fig. 3. Possible sites of states in the $m_1 m_2$ plane.

For finite-dimensional representations, only a finite number of sites will be occupied, and our first task will be to learn something about the distribution of occupied sites. As in our discussion of SU_2, we note that there will be one or more points with maximum value of m_1—points of dominant weight. If there are several of them, we choose the one with the highest value of m_2—the point of highest weight. Let the "coordinates" of this point be denoted by (M_1, M_2). We shall assume that attached to the point of highest weight there is only one state[5] $|M_1, M_2\rangle$. Then the condition that it is the state farthest to the right implies that

$$E_1 \,|M_1M_2\rangle = E_2 \,|M_1M_2\rangle = E_{-3} \,|M_1M_2\rangle = 0. \qquad \text{(IV - 1)}$$

Suppose that in addition

$$E_{-1} \,|M_1M_2\rangle = 0; \qquad \text{(IV - 2)}$$

then it follows that

$$E_{-2} \,|M_1M_2\rangle = \sqrt{6}\,[E_{-3}, E_{-1}] \,|M_1M_2\rangle = 0 \qquad \text{(IV - 3)}$$

and

$$E_{+3} \,|M_1M_2\rangle = \sqrt{6}\,[E_{-1}, E_2] \,|M_1M_2\rangle = 0. \qquad \text{(IV - 4)}$$

Thus $E_{\pm\alpha} \,|M_1M_2\rangle = 0$ for all α and from the commutation relations for $[E_\alpha, E_{-\alpha}]$ it follows that

$$M_1 = M_2 = 0. \qquad \text{(IV - 5)}$$

Thus (IV - 2) further restricts the state to be one that is annihilated by all operators. The state must therefore be the unitary singlet state.

If

$$E_{-1} \,|M_1M_2\rangle \neq 0, \qquad \text{(IV - 6)}$$

there are at least two points on a horizontal line (the 1 line). It can now be shown that either

$$E_3 \,|M_1M_2\rangle \neq 0 \qquad \text{(IV - 7)}$$

or

$$E_{-2} \,|M_1M_2\rangle \neq 0 \qquad \text{(IV - 8)}$$

or both are true. This is done by noting that

$$E_{-1} \,|M_1M_2\rangle = \sqrt{6}\,[E_3, E_{-2}] \,|M_1M_2\rangle \neq 0, \qquad \text{(IV - 9)}$$

which could not be true if both E_3 and E_{-2} annihilated $|M_1M_2\rangle$. Hence the boundary of the distribution of sites near the state of maximum weight must be one of the three cases shown in Fig. 4.

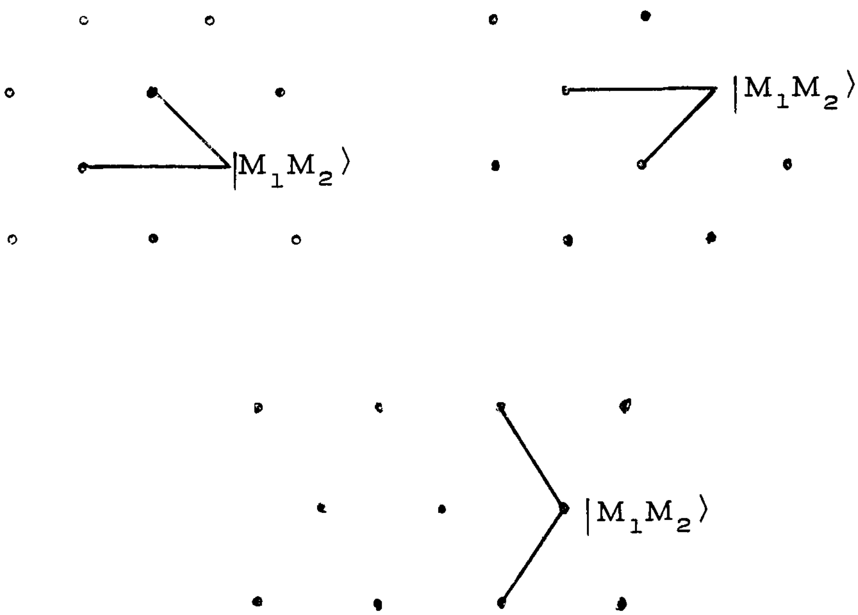

Fig. 4. Boundary of the distribution of sites near the state of maximum weight.

Next we would like to show that the boundary of the set of sites cannot be concave. The occurrence of a situation such as exhibited in Fig. 5

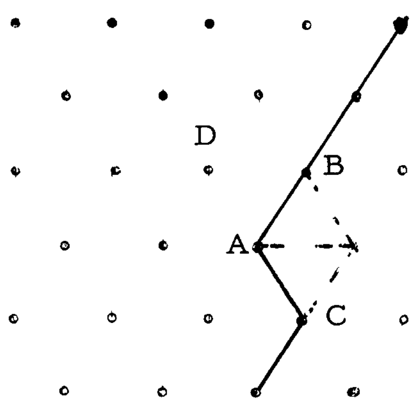

Fig. 5. A concave boundary in the m_1m_2 plane.

would imply that

$$E_1 \left| A \right\rangle = 0. \qquad \text{(IV - 10)}$$

Now let

$$E_{-2} |B\rangle = \lambda |A\rangle, \qquad \text{(IV - 11)}$$

$$E_3 |C\rangle = \sigma |A\rangle. \qquad \text{(IV - 12)}$$

Then

$$\sigma\lambda = \langle B|E_2E_3|C\rangle = \langle B|E_3E_2|C\rangle = 0 \qquad \text{(IV - 13)}$$

so that either λ or σ (or both) vanish, provided $|A\rangle$ is nondegenerate. Since the boundary assumed in Fig. 5 leads to a contradiction, it follows that such a concavity cannot occur in the boundary.

Suppose now that at the site A there are two states such that

$$E_{-2} |B\rangle = \lambda |A'\rangle \qquad \text{(IV - 14)}$$

and

$$\langle A|A'\rangle = 0. \qquad \text{(IV - 15)}$$

Then

$$E_2 |A'\rangle = 0 \qquad \text{(IV - 16)}$$

and of course

$$E_1 |A'\rangle = 0. \qquad \text{(IV - 17)}$$

The next step is to argue that this implies $|B\rangle$ can never be reached from $|A'\rangle$; and this contradicts the assumption that $|A'\rangle$ belongs to the same irreducible representation as $|B\rangle$. First we note that the simplest non-direct way from A' to B is via D. However,

$$E_1E_3 |A'\rangle = (E_3E_1 + \frac{1}{\sqrt{6}} E_2)|A'\rangle = 0 \qquad \text{(IV - 18)}$$

by (IV - 16) and (IV - 17). It remains to be proved that all paths reduce to the one via D. This is done with the help of (i) the relations

$$[E_1,E_2] = [E_1,E_{-3}] = [E_2,E_3] = 0 \qquad \text{(IV - 19)}$$

which, in terms of pictures, imply the deformability of paths like ABC into ADC as in Fig. 6, (ii) the other commutation relations whose graphical implication is that, for example, MNO is equivalent to MPO plus MO, and (iii) the fact that

$$E_{\alpha}E_{-\alpha}\,|m_1m_2\rangle = \lambda_{\alpha}(m_1m_2)\,|m_1m_2\rangle \qquad \text{(IV - 20)}$$

whose graphical meaning is that, say, a step ST followed by TS is equivalent to never leaving S at all. The reader can easily convince himself with the help of (i), (ii), and (iii) that all paths linking A to B in Fig. 5 reduce to ADB.

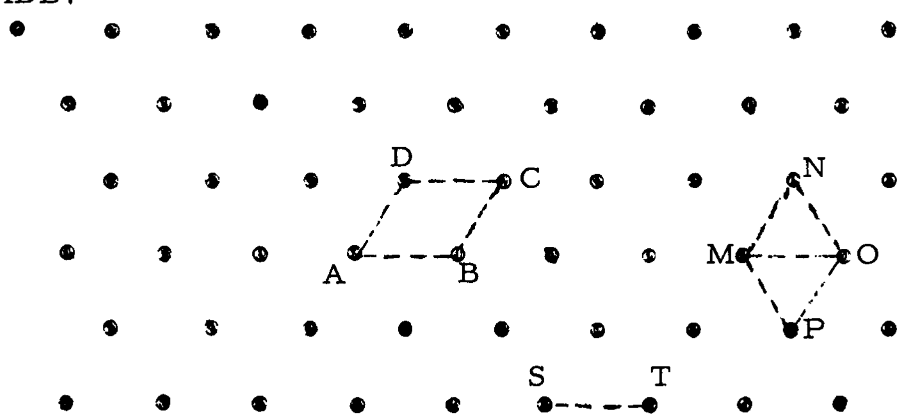

Fig. 6. Paths in the m_1m_2 plane.

For the purposes of this paper, this is enough "proof" that the boundary must be concave. Figure 7 shows some possible boundary shapes.[8] We shall show later that all such polygonal boundaries must be invariant under a rotation of 120°, a result which is not surprising in view of the

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[8] After this work was finished, it was pointed out to me that similar considerations appear in a paper by E. P. Wigner, Phys. Rev. 51, 106 (1937). I wish to thank Prof. G. C. Wick for this observation, which was communicated to me by Prof. J. J. Sakurai.

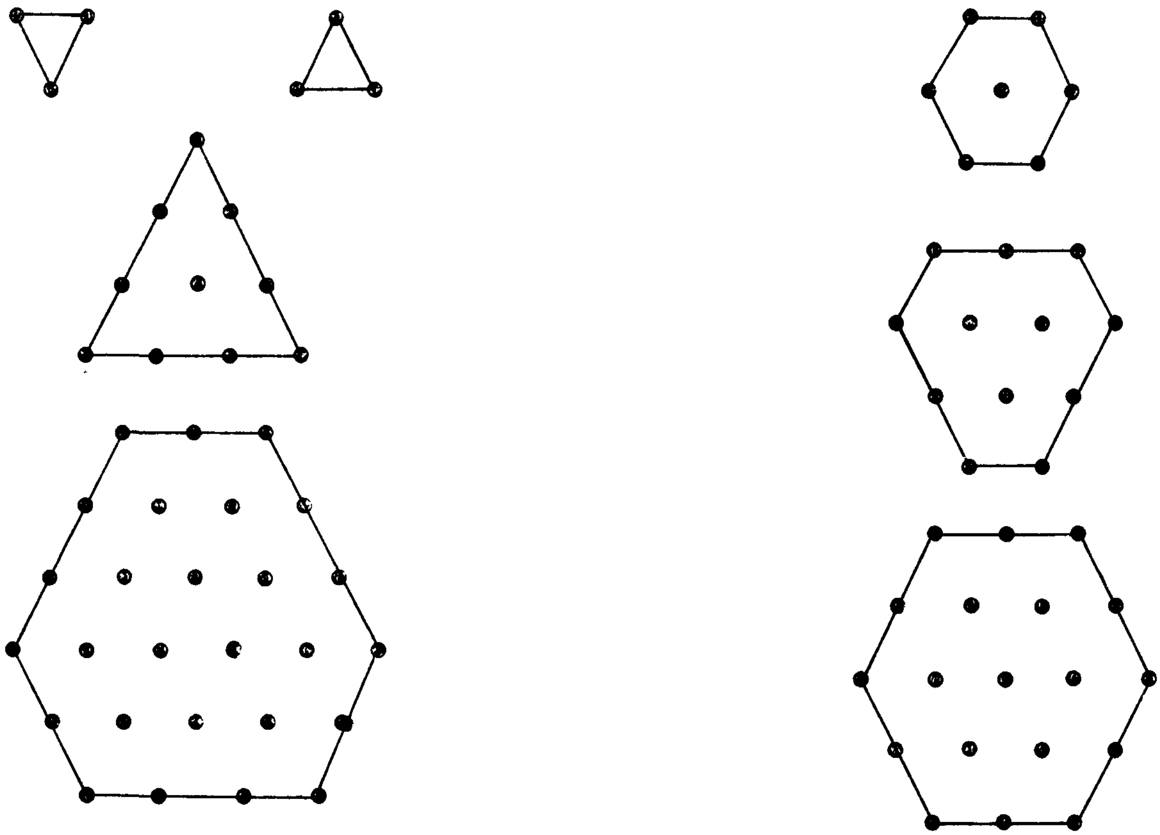

Fig. 7. Possible shapes of the boundary of the distribution of sites.

corresponding symmetry of the algebra. It is clear that all interior points represent states; a "hole" in the middle is excluded by the argument used to prove the convexity of the boundary.

We have not yet discussed the multiplicity of states (in a given irreducible representation) at a given site in the (m_1, m_2) plane. We shall first show that at the boundary points the multiplicity is one, if it is assumed there is only one state of highest weight. For the situation pictured in Fig. 8, we have

$$\lambda_1 |B\rangle \equiv E_{-2} |M\rangle. \qquad \text{(IV - 21)}$$

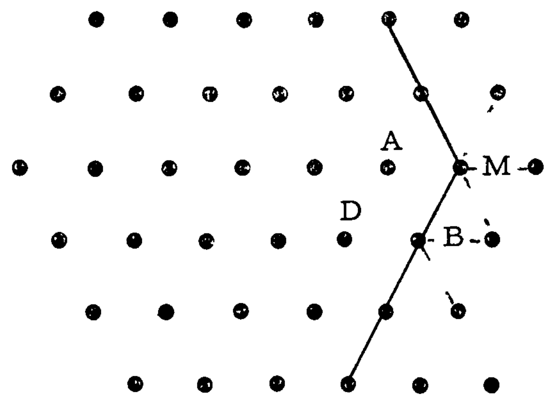

Fig. 8. Some of the forbidden steps are denoted by dotted lines.

If there is to be another state $|B'\rangle$ at the same site as $|B\rangle$, we must be able to reach it from $|M\rangle$ by another path, i.e., for example $E_{-3}E_{-1}|M\rangle$ should not be equal to $\lambda_2 |B\rangle$. However, if $|B'\rangle \equiv E_{-3}E_{-1}|M\rangle$, then it follows that

$$E_{-3}E_{-1}|M\rangle = (E_{-1}E_{-3} + \frac{1}{\sqrt{6}} E_{-2})|M\rangle = \frac{1}{\sqrt{6}} E_{-2}|M\rangle = \frac{\lambda_1}{\sqrt{6}} |B\rangle. \quad (IV - 22)$$

This implies that $|B\rangle$ is a multiple of $|B'\rangle$. Similarly, another path from M to B could be represented by

$$|B''\rangle \equiv E_1E_{-2}E_{-1}|M\rangle.$$

However, it is also true that

$$\begin{aligned} E_1E_{-2}E_{-1}|M\rangle &= E_1E_{-1}E_{-2}|M\rangle \\ &= \lambda_1 E_1E_{-1}|B\rangle \\ &= \lambda_1 [E_1, E_{-1}]\,|B\rangle \qquad (IV - 23) \end{aligned}$$

which is proportional to $|B\rangle$ and again implies that $|B''\rangle$ is a multiple of $|B\rangle$.

To prove the nondegeneracy of a boundary point in a less exhaustive way, we could assume that at the site B there are two states $|B_1\rangle$ and $|B_2\rangle$, with

$$E_2 \, |B_1\rangle = c_1 \, |M\rangle,$$
$$E_2 \, |B_2\rangle = c_2 \, |M\rangle. \qquad \text{(IV - 24)}$$

However, from this it follows that

$$|\Phi\rangle = c_2 \, |B_1\rangle - c_1 \, |B_2\rangle \qquad \text{(IV - 25)}$$

has the property that

$$E_2 \, |\Phi\rangle = 0. \qquad \text{(IV - 26)}$$

It is also true, however, for any state at the site B that

$$E_1 \, |\Phi\rangle = E_{-3} \, |\Phi\rangle = 0. \qquad \text{(IV - 27)}$$

Conditions (IV - 27) and (IV - 26), however, imply that $|\Phi\rangle$ is a state of maximum weight; and this contradicts the assertion that $|M\rangle$ is the only such state. This argument can be carried out for all boundary points, since by "carrying" the assumed degeneracy along the boundary in a counterclockwise direction to the site B, we reduce all such cases to the one discussed above.

The multiplicity of interior points need not be one, but it turns out to be one whenever the boundary is triangular. Consider the situation exhibited in Fig. 9. We have

$$|A\rangle = E_{-2}E_{-1} \, |M\rangle; \qquad \text{(IV - 28)}$$

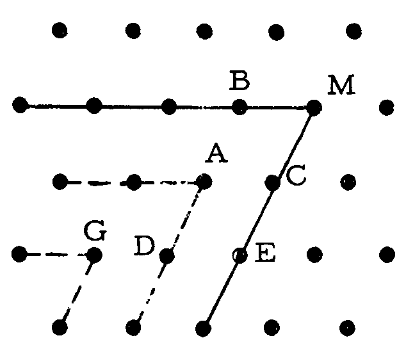

Fig. 9. Points in the vicinity of the maximum-weight point for a triangular boundary.

and since

$$[E_{-1}, E_{-2}] = 0,$$

this is equal to $E_{-2}E_{-1} |M\rangle$, so that whether we go along MBA or MCA we get the same state. In the same manner we can convince ourselves that no matter what path we choose, it always leads to the same state $|A\rangle$.

It is not hard to use this information to prove that the state $|D\rangle$ is unique, and similarly for the points on the next layer of points (lying on the dotted line in Fig. 9). We can then start all over again to prove the statement about the next layer beyond that, etc., etc.

If the boundary is not triangular (see Fig. 10), the next layer of points has two states at each site. To show this, define

$$|A_1\rangle = E_{-1} |M\rangle,$$
$$|A_2\rangle = E_{-2}E_3 |M\rangle. \quad \text{(IV - 29)}$$

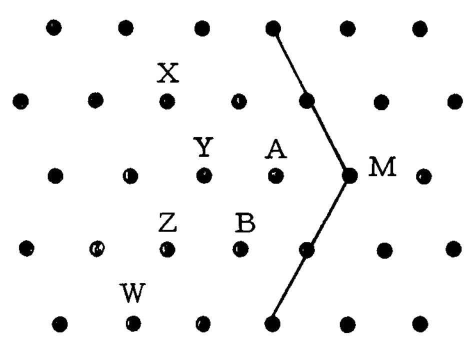

Fig. 10. Points near maximum point for a non-triangular boundary.

With the help of the commutation relations we readily show that

$$\langle A_1 |A_1\rangle = \frac{1}{\sqrt{3}} M_1,$$
$$\langle A_1 |A_2\rangle = \frac{1}{2\sqrt{6}} \left(M_2 - \frac{1}{\sqrt{3}} M_1\right),$$
$$\langle A_2 |A_2\rangle = \frac{1}{4}\left(\frac{1}{\sqrt{3}} M_1 - M_2\right)\left(\frac{1}{\sqrt{3}} M_1 + M_2 + \frac{1}{3}\right). \quad \text{(IV - 30)}$$

The states $|A_1\rangle$ and $|A_2\rangle$ will be linearly dependent if and only if there exists an α such that

$$|\Phi\rangle \equiv \cos\alpha\,|A_1\rangle + \sin\alpha\,|A_2\rangle = 0. \qquad \text{(IV - 31)}$$

There will be a solution provided

$$\langle A_2|A_2\rangle\,\langle A_1|A_1\rangle - \langle A_1|A_2\rangle^2 = 0, \qquad \text{(IV - 32)}$$

i.e., provided

$$(M_1 + \sqrt{3}\,M_2)(M_1 - \sqrt{3}\,M_2)(M_1 + \sqrt{3}) = 0. \qquad \text{(IV - 33)}$$

Now

$$M_1 = \pm\sqrt{3}\,M_2$$

corresponds to the triangular cases. To show this, note that for a triangular case $E_{\pm 2}|M\rangle = 0$ or $E_{\pm 3}|M\rangle = 0$; i.e., the requirement $\pm(M_1/2\sqrt{3}) + \frac{1}{2}M_2 = 0$ holds for one sign of the first term. If $M_1 \neq +\sqrt{3}\,M_2$, there is no solution ($M_1 \geqslant 0$). Hence there are <u>at least</u> two independent states.

To show that there are no more than two, note that if one of the states is

$$|A_1\rangle \equiv E_{-1}|M\rangle \qquad \text{(IV - 34)}$$

then the other may be chosen as

$$|A_1'\rangle = (\lambda\,E_{-2}E_3 + E_{-1})\,|M\rangle \qquad \text{(IV - 35)}$$

so that it is orthogonal to $|A_1\rangle$, i.e.,

$$\langle A_1'|A_1\rangle = 0. \qquad \text{(IV - 36)}$$

On the other hand

$$\langle A_1|A_1'\rangle = \langle M|E_1|A_1'\rangle.$$

Also

$$E_1|A_1'\rangle = \rho\,|M\rangle.$$

Hence

$$\rho = 0,$$

i.e.,

$$E_1|A_1'\rangle = 0. \qquad \text{(IV - 37)}$$

If there is a third independent state, it may be chosen orthogonal to both $|A_1\rangle$ and $|A_1'\rangle$. If this state is denoted by $|A_1''\rangle$, it obeys

$$\langle A_1|A_1''\rangle = \langle A_1'|A_1''\rangle = 0. \qquad \text{(IV - 38)}$$

This implies that

$$E_1|A_1''\rangle = 0,$$

$$E_{-3}E_2|A_1''\rangle = 0,$$

and consequently

$$E_2E_{-3}|A\ ''\rangle = 0.$$

This, however, implies that

$$E_1|A_1''\rangle = E_2|A_1''\rangle = E_{-3}|A_1''\rangle = 0 \qquad \text{(IV - 39)}$$

which is the condition that $|A_1''\rangle$ be a maximum state. Since $|M\rangle$ is the only such state, there cannot be more than a two-fold degeneracy at the site A.

We may now start at the site A and by moving along the line parallel to the boundary (Fig. 10) show that at the site B there are exactly two states. This can be continued all around the layer one step in from the boundary.

It is only a little more tedious to show that the multiplicity at the sites of the next layer (X, Y, Z, W in Fig. 10) is three. In fact the multiplicity increases at each step until the layer has a triangular shape,[8] after which the multiplicity ceases to increase. A simple way to visualize this is to view the boundary as the base of a truncated pyramid and the successive layers as contour lines on it. Then the top of the pyramid is always a triangle, and the multiplicity increases with altitude.

These last statements have been proved here only for special cases. They should be easy to prove quite generally by use of some powerful techniques due to Weyl,[9] but such departures from a purely practical discussion of the representations is outside the scope of this paper.

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[9] See references in the review paper of R. Behrends et al. (reference 4). See also the reference quoted in footnote 8.

The degenerate states may now be labeled by their i spin, i.e., by the quantum number $\vec{T}^2$ of Eq. (III - 8) which may be written

$$\begin{aligned}\vec{T}^2 &= 3(E_1E_{-1} + E_{-1}E_1) + 3H_1^{\,2} \\ &= 6E_{-1}E_1 + 3H_1^{\,2} + \sqrt{3}\ H_1 \\ &= 6E_{-1}E_1 + \sqrt{3}\ H_1(\sqrt{3}\ H_1 + 1).\end{aligned} \qquad \text{(IV - 40)}$$

The state $|M\rangle$ is an eigenstate of $\vec{T}^2$. In fact, since

$$E_1|M\rangle = E_1\ |M_1M_2\rangle = 0,$$

it follows that

$$\vec{T}^2\ |M_1M_2\rangle = \sqrt{3}\ M_1\ (\sqrt{3}\ M_1 + 1)\ |M_1M_2\rangle. \qquad \text{(IV - 41)}$$

The states $E_{-1}|M_1M_2\rangle$, $(E_{-1})^2\ |M_1M_2\rangle$, $\cdots$, etc., are (aside from normalization constants) the other states that belong to the i-spin multiplet of which $|M_1M_2\rangle$ is the highest member. Clearly the multiplicity is 2t + 1; i.e., in terms of M_1 it is $(2\sqrt{3}\ M_1 + 1)$. The state $|A_1\rangle$ is part of this multiplet. The state $|A_1{'}\rangle$, which is orthogonal to $|A_1\rangle$ and which satisfies

$$E_1\ |A_1{'}\rangle = 0,$$

is also an eigenstate of $\vec{T}^2$. The value of m_1 at the site A is $M_1 - 1/\sqrt{3}$, so that $|A_1{'}\rangle$ is the highest member of an i-spin multiplet of multiplicity $2[\sqrt{3}(M_1 - 1/\sqrt{3})] + 1 = 2\sqrt{3}\ M_1 - 1$.

The general features are perhaps best illustrated by examples (Fig. 11). In the next section we shall work out a formula for the multiplicity once the shape of the boundary is given. Note that there is only one triangular boundary on a given pyramid—the triangle is the top surface of the pyramid. For the hexagonal figures, e.g., cases (b) and (d), the triangle reduces to a point. Thus in all cases there is only one T = 0 state per

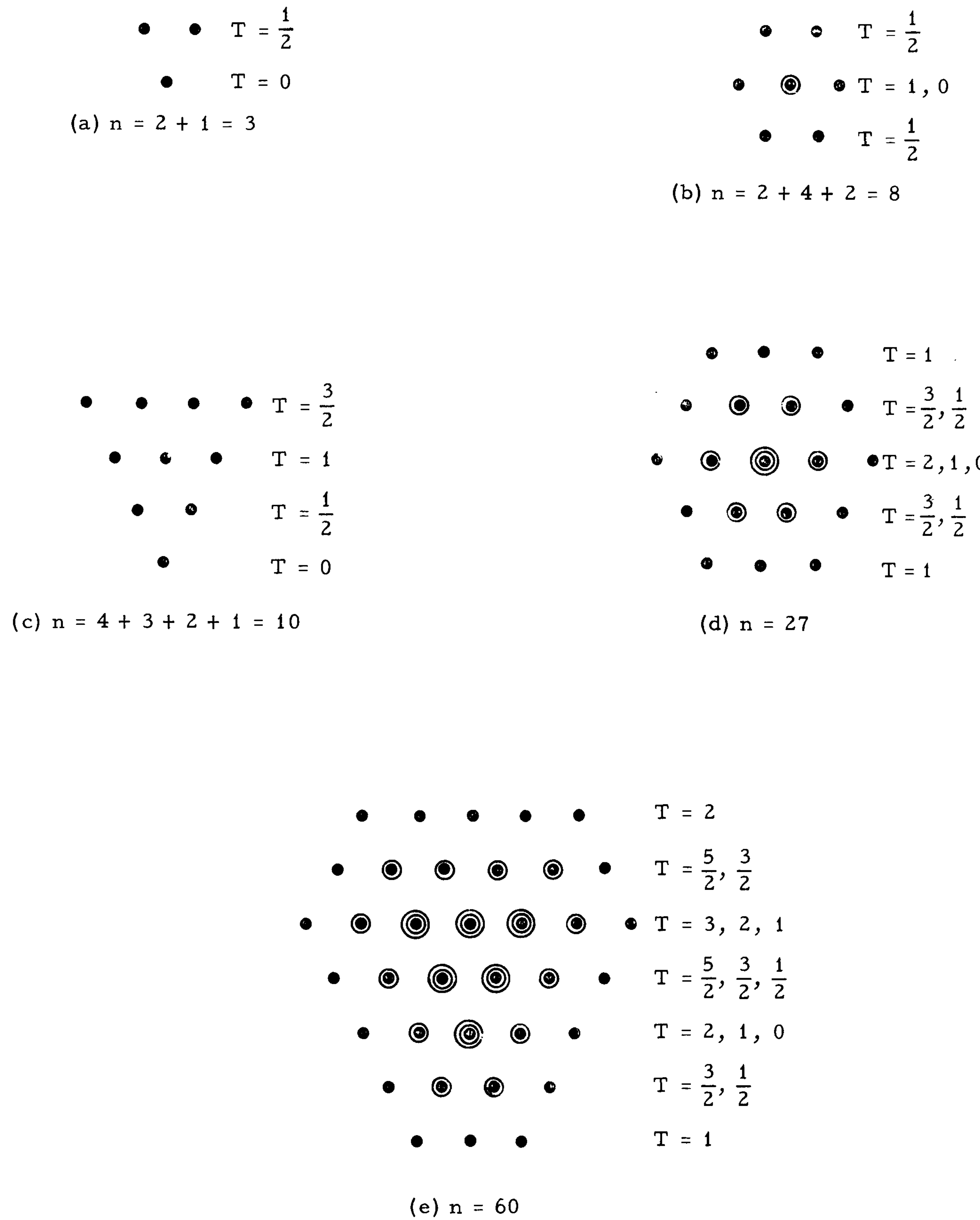

Fig. 11. Some examples of patterns of irreducible representations. Points represent multiplicity one, points with one circle represent multiplicity two, etc.

irreducible representation,[10] this state lying at the tip of the triangle (along the vertical axis). We shall return to this point later.

V. THE MULTIPLICITY OF AN IRREDUCIBLE REPRESENTATION

We begin by considering the simplest representations corresponding to the two graphs shown in Fig. 12. In each case, the tip of the triangle

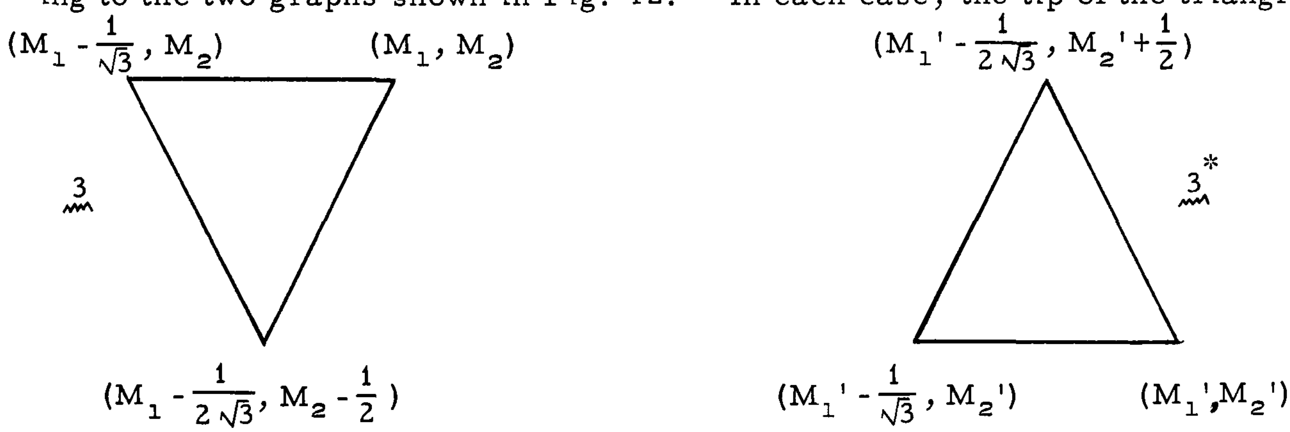

Fig. 12. The representations $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$.

corresponds to an iso-singlet state, i.e.,

$$t = M_1 - \frac{1}{2\sqrt{3}} = 0. \qquad \text{(V - 1)}$$

Hence

$$M_1 = M_1' = \frac{1}{2\sqrt{3}} .$$

In the first case (representation labeled by $\underset{\sim}{3}$) we have

$$E_{\pm 3} \left| M_1 M_2 \right\rangle = 0, \qquad \text{(V - 2)}$$

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[10] The fact that there is only one T = 0 state per irreducible representation was pointed out to me by Prof. M. Bolsterli.

i.e.,

$$[E_3, E_{-3}] \, |M_1 M_2\rangle = \left(- \frac{1}{2\sqrt{3}} M_1 + \tfrac{1}{2} M_2 \right) |M_1 M_2\rangle = 0.$$

Hence

$$M_2 = \frac{1}{6} \,. \qquad \text{(V - 3)}$$

In the second (representation labeled by $\underset{\sim}{3}^*$) we have

$$E_{\pm 2} \, |M_1 M_2\rangle = 0, \qquad \text{(V - 4)}$$

i.e.,

$$[E_2, E_{-2}] \, |M_1' M_2'\rangle = \left(\frac{1}{2\sqrt{3}} M_1' + \tfrac{1}{2} M_2' \right) |M_1' M_2'\rangle = 0.$$

Hence

$$M_2' = -\frac{1}{6} \,. \qquad \text{(V - 5)}$$

Just as all representations of SU_2 can be obtained from the reduction of products of the two-dimensional representation (all i-spin states can be built out of products of doublets), so can all representations of the algebra of SU_3 be built up out of products of $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$. In a product of the form

$$(\underset{\sim}{3})^a \otimes (\underset{\sim}{3}^*)^b \qquad \text{(V - 6)}$$

there will be many irreducible representations, but the one with highest weight will have

$$M_1 = \frac{a + b}{2\sqrt{3}}$$

$$M_2 = \frac{a - b}{6} \,. \qquad \text{(V - 7)}$$

The irreducible representation with this as the highest weight is the one that we want to study.

For a given (a, b), what is the shape of the polygon? The i-spin multiplet to which the state of highest weight belongs has

$$t = \sqrt{3}\, M_1 = \frac{a + b}{2} \qquad \text{(V - 8)}$$

so that there are (a + b + 1) points on that line.

How far up can we go? We have

$$E_{-3}\left|\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right\rangle = 0 \qquad \text{(V - 9)}$$

and write

$$E_{3}\left|\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right\rangle = \lambda_1\left|\frac{a+b-1}{2\sqrt{3}}, \frac{a-b+3}{6}\right\rangle. \qquad \text{(V - 10)}$$

Then

$$\lambda_1^2 = \left\langle\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right|E_{-3}E_3\left|\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right\rangle$$

$$= \left\langle\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right|[E_{-3},E_3]\left|\frac{a+b}{2\sqrt{3}}, \frac{a-b}{6}\right\rangle$$

$$= \frac{b}{6}.$$

Next we write

$$E_{3}\left|\frac{a+b-1}{2\sqrt{3}}, \frac{a-b+3}{6}\right\rangle = \lambda_2\left|\frac{a+b-2}{2\sqrt{3}}, \frac{a-b+6}{6}\right\rangle.$$

Then

$$\lambda_2^2 = \left\langle\frac{a+b-1}{2\sqrt{3}}, \frac{a-b+3}{6}\right|\ [E_{-3},E_3] + E_3E_{-3}\left|\frac{a+b-1}{2\sqrt{3}}, \frac{a-b+3}{6}\right\rangle$$

$$= \lambda_1^2 + \frac{b-2}{6}.$$

Similarly

$$\lambda_3^2 = \lambda_2^2 + \frac{b-4}{6},$$

$$\vdots$$

$$\lambda_p^2 = \lambda_{p-1}^2 + \frac{b-2(p-1)}{6}.$$

Hence

$$\lambda_p^2 = \frac{1}{6}[b + (b-2) + \cdots + (b-2p+2)] = \frac{p(b-p+1)}{6}. \qquad \text{(V - 11)}$$

Hence $\lambda_p^2 = 0$ when $p = b+1$, so that we can go upward b steps.

In the same way we can show that we can go down a steps along the 2 line. At the top line

$$m_1 = \frac{a + b - b}{2\sqrt{3}} = \frac{a}{2\sqrt{3}},$$

i.e., there are (a + 1) sites. At the lowest line

$$m_1 = \frac{a + b - a}{2\sqrt{3}} = \frac{b}{2\sqrt{3}},$$

i.e., there are (b + 1) sites.

Similarly, we can show that the figure has symmetry under a 120° rotation. Figure 13 below shows the shape of the figure. Let us take $a \geq b$.

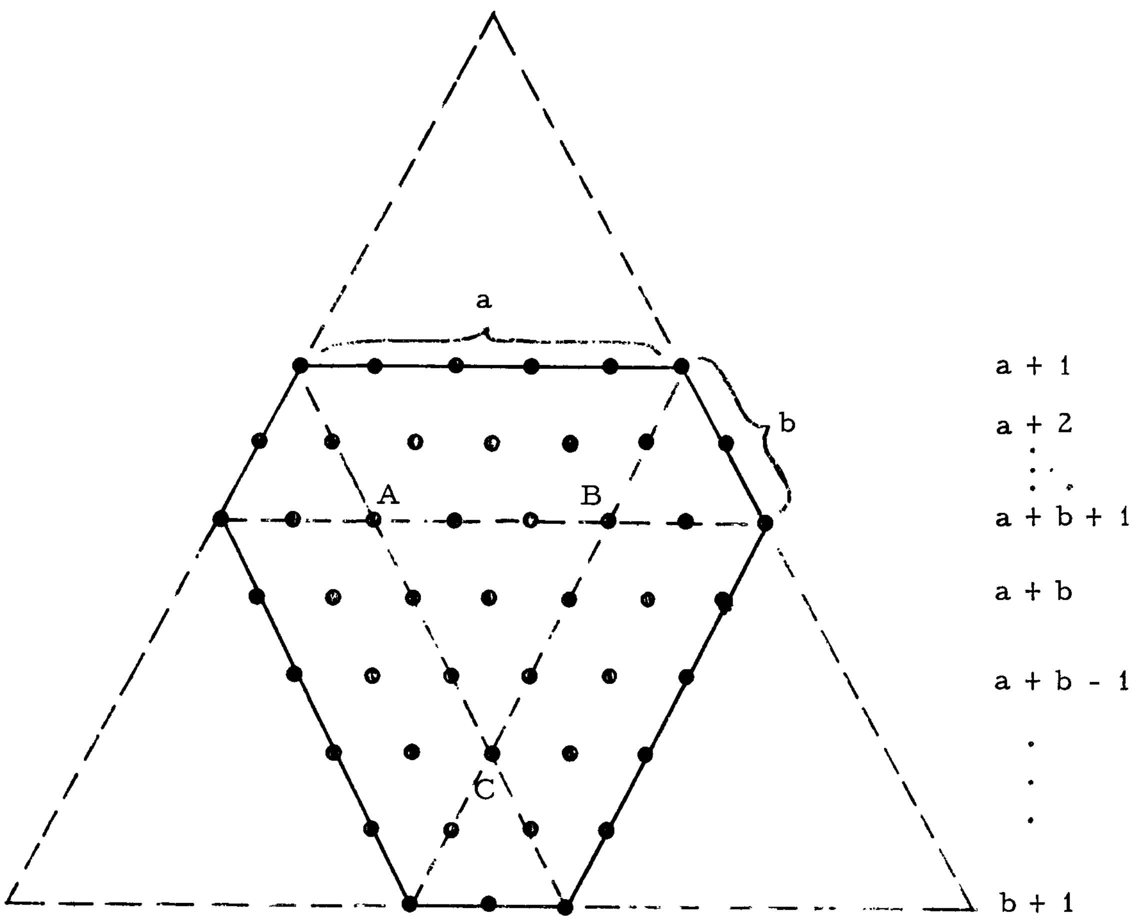

Fig. 13. The polygon for the representation (a, b). The top of the pyramid is the equilateral triangle ABC.

(The reverse choice would be just the same figure upside down.) At the lowest level of the pyramid, the number of states is

$$(a+1)+(a+2)+(a+3)+\cdot\cdot\cdot+(a+b)+(a+b+1)+(a+b)+\cdot\cdot\cdot+(b+3)+(b+2)+(b+1).$$

At the next level, the number is

$$a+(a+1)+\cdot\cdot\cdot+(a+b-2)+(a+b-1)+(a+b-2)+\cdot\cdot\cdot+(b+1)+b.$$

And at the next,

$$(a-1)+\cdot\cdot\cdot+(a+b-4)+(a+b-3)+(a+b-4)+\cdot\cdot\cdot+(b-1),$$

etc. The sequence ends with a triangle when the last number in the sum is unity, corresponding to the tip of the triangle. This occurs after $N = b + 1$ lines.

The number of states in the first line is

$$\frac{(a+b+1)(a+b+2)}{2}+\frac{(a+b)(a+b+1)}{2}-\frac{a(a+1)}{2}-\frac{b(b+1)}{2}$$

$$=(a+b+1)^2-\frac{a(a+1)}{2}-\frac{b(b+1)}{2}.$$

The number of states in the second line is

$$(a+b-1)^2-\frac{(a-1)a}{2}-\frac{(b-1)b}{2}.$$

The number of states in the nth line is

$$(a+b+3-2n)^2-\frac{(a+1-n)(a+2-n)}{2}-\frac{(b+1-n)(b+2-n)}{2}$$

$$=3n^2-3(a+b+3)n+(a+b+3)^2-\frac{(a+1)(a+2)}{2}-\frac{(b+1)(b+2)}{2}$$

$$\equiv S_n. \qquad (V-12)$$

The multiplicity is

$$\sum_{n=1}^{b+1} S_n = \tfrac{1}{2}(b+1)(a+1)(a+b+2), \qquad \text{(V-13)}$$

a well-known formula for the dimensionality of a representation labeled by (a, b).

Table II gives the isotopic content in an (a, b) representation. The

TABLE II. The isotopic content of an irreducible representation.

Y	Values of T
$\frac{a-b}{3}+b$	$\frac{a}{2}$
$\frac{a-b}{3}+b-1$	$\frac{a+1}{2}, \frac{a-1}{2}$
$\vdots$	$\vdots$
$\frac{a-b}{3}+2$	$\frac{a+b-2}{2}, \frac{a+b-4}{2}, \frac{a+b-6}{2}, \ldots, \frac{a-b+2}{2}$
$\frac{a-b}{3}+1$	$\frac{a+b-1}{2}, \frac{a+b-3}{2}, \frac{a+b-5}{2}, \ldots, \frac{a-b+1}{2}$
$\frac{a-b}{3}$	$\frac{a+b}{2}, \frac{a+b-2}{2}, \frac{a+b-4}{2}, \ldots, \frac{a-b}{2}$
$\frac{a-b}{3}-1$	$\frac{a+b-1}{2}, \frac{a+b-3}{2}, \frac{a+b-5}{2}, \ldots, \frac{a-b-1}{2}$
$\frac{a-b}{3}-2$	$\frac{a+b-2}{2}, \frac{a+b-4}{2}, \ldots, \frac{a-b-2}{2}$
$\vdots$	$\vdots$
$\frac{a-b}{3}-a$	$\frac{b}{2}$

table uses the notation

$$Y = 2m_2. \qquad \text{(V - 14)}$$

This checks with the requirements that (i) for a unitary singlet ($M_2 = 0$) the hypercharge should be zero and (ii) the operators $E_{\pm 2}$, $E_{\pm 3}$ change the hypercharge by unity.

This is the form for $a \geqslant b$. Note the asymmetry about the line $Y = (a - b)/3$ on which the state of highest weight lies. As one reads up from this line, the number of multiplets decreases; but as one reads down, it remains constant until the iso-singlet is reached. Thereafter the number of multiplets decreases until there is only a single one, as illustrated for another particular case in Fig. 14. It will be shown in Sec. VII that the iso-singlet always

$Y = 3 \quad T = \frac{5}{2}$

$Y = 2 \quad T = 3, 2$

$Y = 1 \quad T = \frac{7}{2}, \frac{5}{2}, \frac{3}{2}$

$Y = 0 \quad T = 3, 2, 1$

$Y = -1 \quad T = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$

$Y = -2 \quad T = 2, 1, 0$

$Y = -3 \quad T = \frac{3}{2}, \frac{1}{2}$

$Y = -4 \quad T = 1$

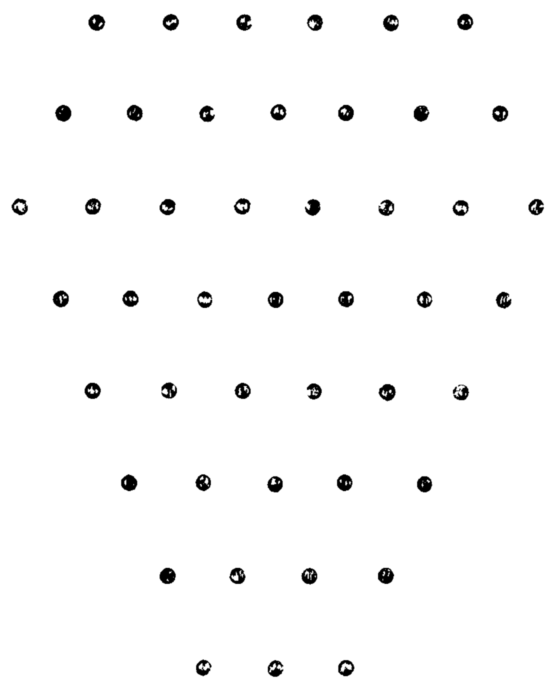

Fig. 14. Multiplicities for the case $a = 5$, $b = 2$ ($N = 81$).

occurs at

$$Y_0 = -\frac{2}{3}(a - b). \qquad \text{(V - 15)}$$

It is also easy to see that the "height" of the truncated pyramid is given by b and the side of the equilateral triangle which forms its top has length a - b (see Fig. 13).

VI. THE REDUCTION OF PRODUCTS OF REPRESENTATIONS

In physical applications, the reduction of products of irreducible representations is particularly important in obtaining decay widths.[11] We shall first develop the general method and then illustrate it by carrying out some reductions, in particular the $\underset{\sim}{8} \otimes \underset{\sim}{8}$. We shall obtain the wave functions found by Glashow and Sakurai[2] and also obtain matrix representations of the E_α in the $\underset{\sim}{8}$- and $\underset{\sim}{10}$-dimensional representations as an illustration of the present method.

Consider first the basic states $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$ (Fig. 15). It is rather

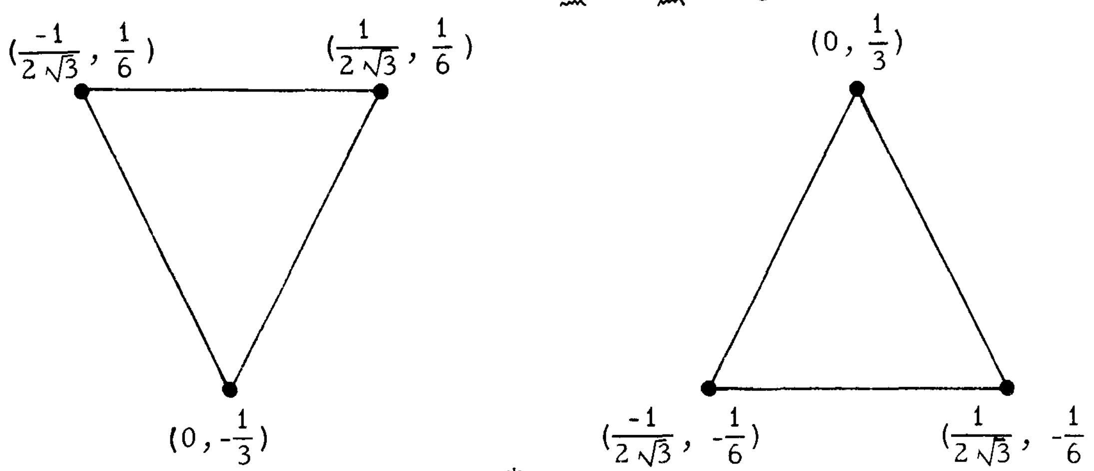

Fig. 15. The states $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$ with their $m_1 m_2$ labels.

simple to work out the effects of the operations of $E_{\pm\alpha}$ on the various states. Since

$$E_{-1} \left| \frac{1}{2\sqrt{3}}, \pm \frac{1}{6} \right\rangle = \lambda \left| -\frac{1}{2\sqrt{3}}, \pm \frac{1}{6} \right\rangle, \qquad \text{(VI - 1)}$$

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[11] See reference 2; also S. Glashow and A. Rosenfeld, Phys. Rev. Letters 10, 192 (1963).

it follows that

$$\lambda^2 = \left\langle \frac{1}{2\sqrt{3}}, \pm\frac{1}{6} \middle| E_1 E_{-1} \middle| \frac{1}{2\sqrt{3}}, \pm\frac{1}{6} \right\rangle$$

$$= \left\langle \frac{1}{2\sqrt{3}}, \pm\frac{1}{6} \middle| [E_1, E_{-1}] \middle| \frac{1}{2\sqrt{3}}, \pm\frac{1}{6} \right\rangle = \frac{1}{6}. \qquad \text{(VI - 2)}$$

Hence

$$\lambda = \pm \frac{1}{\sqrt{6}}.$$

The value of λ is the same for all legs of the triangle, because of the symmetry under 120° rotations.

To determine the signs, note that for $\underset{\sim}{3}$

$$\lambda^3 \left| \frac{1}{2\sqrt{3}}, \frac{1}{6} \right\rangle = E_2 E_{-3} E_{-1} \left| \frac{1}{2\sqrt{3}}, \frac{1}{6} \right\rangle$$

$$= E_2 \left(E_{-1} E_{-3} + \frac{1}{\sqrt{6}} E_{-2} \right) \left| \frac{1}{2\sqrt{3}}, \frac{1}{6} \right\rangle$$

$$= \frac{1}{\sqrt{6}} \lambda^2 \left| \frac{1}{2\sqrt{3}}, \frac{1}{6} \right\rangle. \qquad \text{(VI - 3)}$$

Hence

$$\lambda^{(3)} = \frac{1}{\sqrt{6}}. \qquad \text{(VI - 4)}$$

For $\underset{\sim}{3}^*$,

$$\lambda^3 \left| \frac{1}{2\sqrt{3}}, -\frac{1}{6} \right\rangle = E_1 E_{-2} E_3 \left| \frac{1}{2\sqrt{3}}, -\frac{1}{6} \right\rangle$$

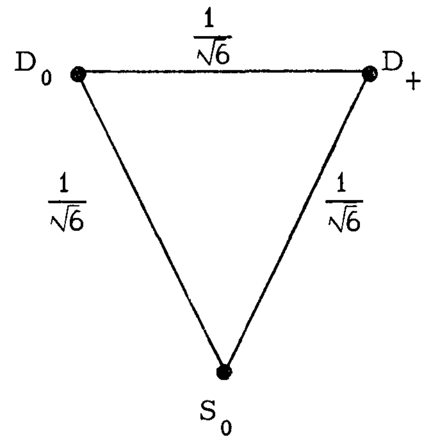
$$= E_1 \left(E_3 E_{-2} - \frac{1}{\sqrt{6}} E_{-1} \right) \left| \frac{1}{2\sqrt{3}}, -\frac{1}{6} \right\rangle$$

$$= -\frac{1}{\sqrt{6}} \lambda^2 \left| \frac{1}{2\sqrt{3}}, -\frac{1}{6} \right\rangle. \qquad \text{(VI - 5)}$$

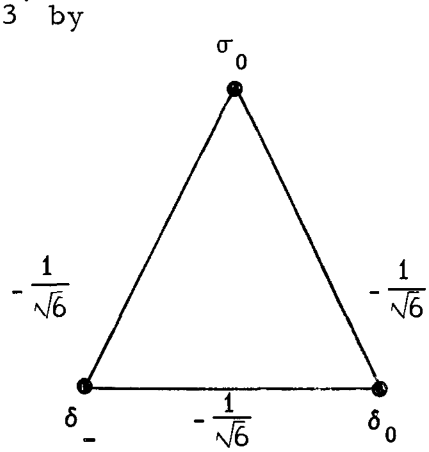
Hence

$$\lambda^{(3^*)} = -\frac{1}{\sqrt{6}}.$$

The matrix elements of $E_{\pm\alpha}$ in $\underset{\sim}{3}$ will be represented by



in which the states are simply labeled[12] by (D_+, D_0, S_0) (doublet, singlet) and the matrix elements in 3^* by



Thus, for example,

$$E_2 = \left(-\frac{1}{\sqrt{6}}\right) \begin{array}{c|ccc} & \delta_0 & \delta_- & \sigma_0 \\ \hline \delta_0 & 0 & 0 & 1 \\ \delta_- & 0 & 0 & 0 \\ \sigma_0 & 0 & 0 & 0 \end{array} \qquad \text{(VI - 6)}$$

etc.

---

[12] We use the notation of J. J. Sakurai, Proceedings of the International Summer School of Physics at Varenna (to be published).

If we take the direct product of two representations, the wave functions will be

$$|m_1, m_2\rangle = |m_1^{(1)}, m_2^{(1)}\rangle \times |m_1^{(2)}, m_2^{(2)}\rangle, \qquad \text{(VI - 7)}$$

where

$$\begin{aligned} m_1 &= m_1^{(1)} + m_1^{(2)}, \\ m_2 &= m_2^{(1)} + m_2^{(2)}. \end{aligned} \qquad \text{(VI - 8)}$$

A simple way to obtain the sites of the product wave functions is to draw lines from the origin to the states in one representation and vectorially add the same "weight diagram" of the second representation to each state of the first one. The weight diagrams for $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$ are shown in Fig. 16 and the addition of the weight diagrams in $\underset{\sim}{3} \otimes \underset{\sim}{3}$ is shown in Fig. 17.

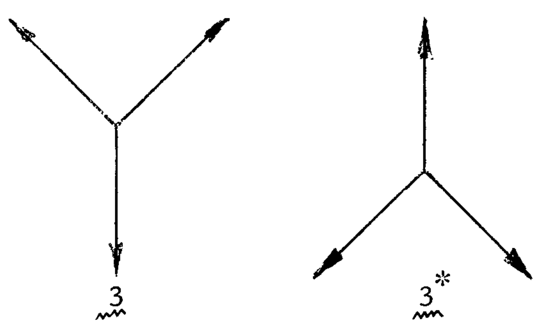


Fig. 16. Weight diagrams for $\underset{\sim}{3}$ and $\underset{\sim}{3}^*$.

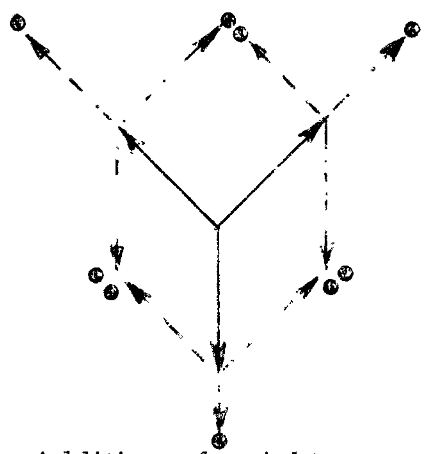


Fig. 17. Addition of weight diagrams in $\underset{\sim}{3} \otimes \underset{\sim}{3}$.

It is clear that the product is not irreducible, because for an irreducible representation the boundary points have multiplicity 1. In this example the reduction is easy to carry out if we first take the boundary points and see what is left over. The remainder is shown in Fig. 18; i.e.,

$$\underset{\sim}{3} \otimes \underset{\sim}{3} = \underset{\sim}{6} \oplus \underset{\sim}{3}^* . \tag{VI - 9}$$

The remainder for $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$ is shown in Fig. 19. Again, as may be seen from our

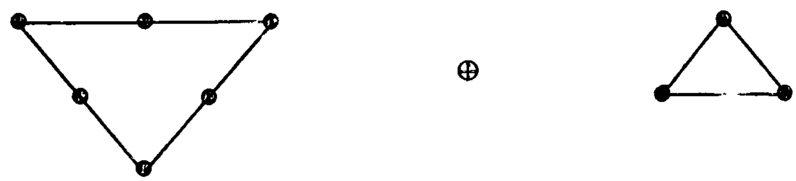


Fig. 18. Reduction of $\underset{\sim}{3} \otimes \underset{\sim}{3}$.

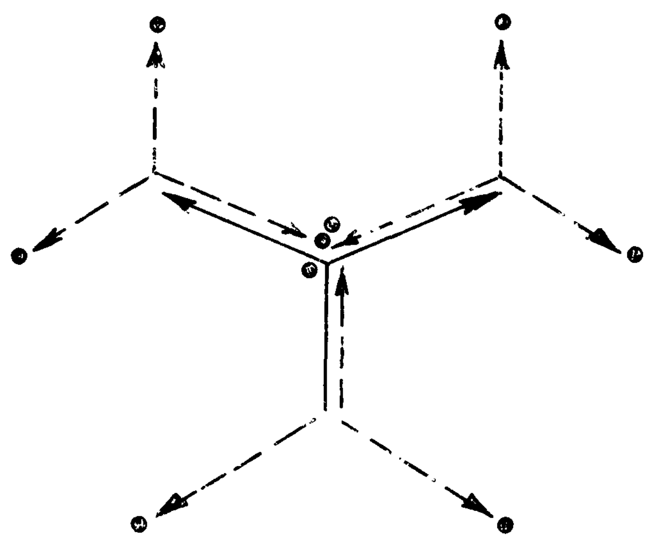


Fig. 19. The reduction of $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$.

discussion of multiplicities, there cannot be three points in the middle for an irreducible representation: two belong with the boundary points and one remains. Thus

$$\underset{\sim}{3} \otimes \underset{\sim}{3}^* = \underset{\sim}{8} \oplus \underset{\sim}{1} . \tag{VI - 10}$$

Before working out $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$ in detail, we note that $\underset{\sim}{8}$ has two states at Y = 0. Hence all Y values in $\underset{\sim}{8}$ are integral. The same must be true in all states that occur in $\underset{\sim}{8} \otimes \underset{\sim}{8} \otimes \underset{\sim}{8} \otimes \cdots$. However, since the value Y for the state of highest weight in (a, b) is

$$Y = 2M_2 = \frac{a - b}{3} , \qquad \text{(VI - 11)}$$

we see that

$$a = b \pmod 3 \qquad \text{(VI - 12)}$$

for all irreducible representations contained in $\underset{\sim}{8} \otimes \underset{\sim}{8} \otimes \underset{\sim}{8} \otimes \cdots$.

Let us now proceed to the explicit reduction of $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$ (Fig. 20). Let

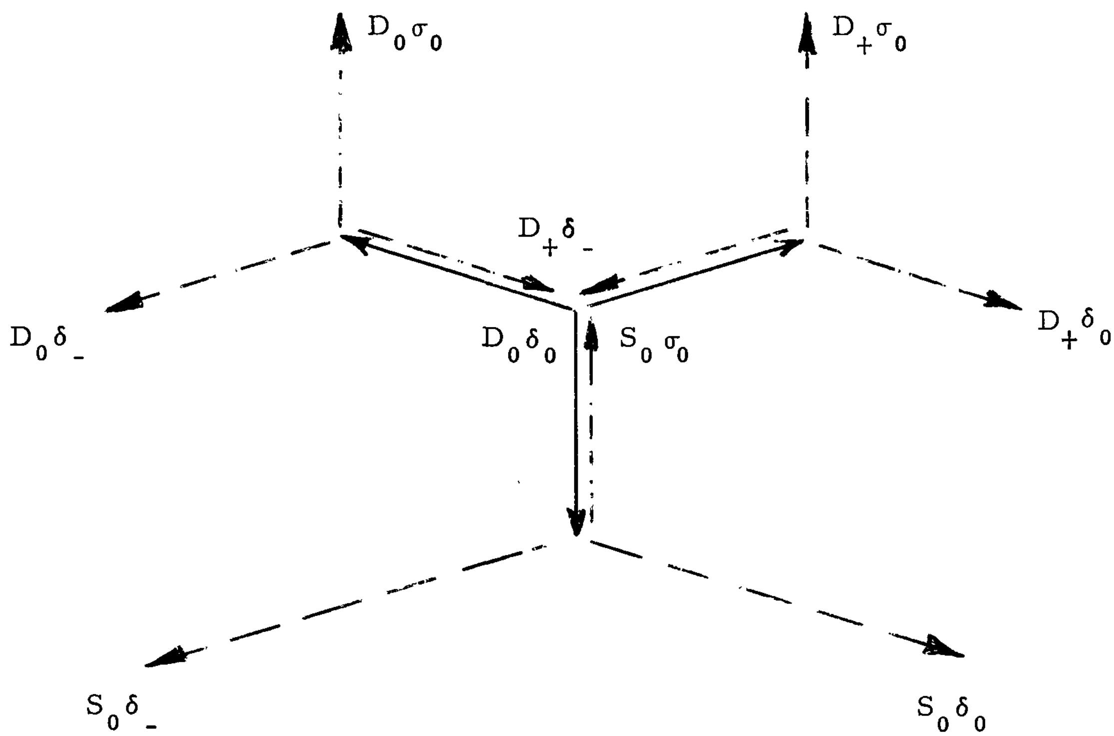


Fig. 20. The states in $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$.

$$|D_+ \delta_0\rangle \equiv |\Sigma_+\rangle,$$
$$|D_+ \sigma_0\rangle \equiv |P\rangle,$$
$$|D_0 \sigma_0\rangle \equiv |N\rangle, \qquad \text{(VI - 13)}$$
$$|D_0 \delta_-\rangle \equiv |\Sigma^-\rangle,$$
$$|S_0 \delta_-\rangle \equiv |\Xi^-\rangle,$$
$$|S_0 \delta_0\rangle \equiv |\Xi^0\rangle.$$

Then

$$E_{-1}|\Sigma^+\rangle = E_{-1}\,|D_+ \delta_0\rangle = \frac{1}{\sqrt{6}}\,|D_0 \delta_0\rangle - \frac{1}{\sqrt{6}}\,|D_+ \delta_-\rangle$$

$$= \frac{1}{\sqrt{3}} \quad \frac{|D_0 \delta_0\rangle - |D_+ \delta_-\rangle}{\sqrt{2}} \,. \qquad \text{(VI - 14)}$$

Writing

$$\frac{1}{\sqrt{2}}\,(|D_0 \delta_0\rangle - |D_+ \delta_-\rangle) \equiv |\Sigma_0\rangle \qquad \text{(VI - 15)}$$

leads to

$$E_{-1}\,|\Sigma_+\rangle = \frac{1}{\sqrt{3}}\,|\Sigma_0\rangle. \qquad \text{(VI - 16)}$$

Again

$$E_{-1}|\Sigma_0\rangle = \frac{1}{\sqrt{2}}\left\{-\frac{1}{\sqrt{6}}\,|D_0 \delta_-\rangle - \frac{1}{\sqrt{6}}\,|D_- \delta_0\rangle\right\}$$

$$= -\frac{1}{\sqrt{3}}\,|\Sigma_-\rangle, \qquad \text{(VI - 17)}$$

$$E_3|\Sigma^+\rangle = E_3|D_+ \delta_0\rangle = -\frac{1}{\sqrt{6}}\,|D_+ \sigma_0\rangle = -\frac{1}{\sqrt{6}}|P\rangle, \qquad \text{(VI - 18)}$$

$$E_{-2}|\Sigma^+\rangle = E_{-2}|D_+ \delta_0\rangle = \frac{1}{\sqrt{6}}\quad S_0 \delta_0\rangle = \frac{1}{\sqrt{6}}\,|\Xi^0\rangle, \qquad \text{(VI - 19)}$$

$$E_{-1}|P\rangle = E_{-1}|D_+ \sigma_0\rangle = \frac{1}{\sqrt{6}}\,|D_0 \sigma_0\rangle = \frac{1}{\sqrt{6}}\,|N\rangle, \qquad \text{(VI - 20)}$$

$$E_{-1}|\Xi^0\rangle = E_{-1}|S_0\,\delta_0\rangle = -\frac{1}{\sqrt{6}}\,|S_0\,\delta_-\rangle = -\frac{1}{\sqrt{6}}\,|\Xi^-\rangle, \qquad \text{(VI - 21)}$$

$$E_{-2}|N\rangle = E_{-2}|D_0\,\sigma_0\rangle = -\frac{1}{\sqrt{6}}\,|D_0\,\delta_-\rangle = -\frac{1}{\sqrt{6}}\,|\Sigma^-\rangle, \qquad \text{(VI - 22)}$$

$$E_3|\Xi^-\rangle = E_3|S_0\,\delta_-\rangle = \frac{1}{\sqrt{6}}\,|D_0\,\delta_-\rangle = \frac{1}{\sqrt{6}}\,|\Sigma^-\rangle. \qquad \text{(VI - 23)}$$

Next consider

$$E_{-2}|P\rangle = E_{-2}|D_+\,\sigma_0\rangle = \frac{1}{\sqrt{6}}\,(|S_0\,\sigma_0\rangle - |D_+\,\delta_-\rangle).$$

This state is not identical with $|\Sigma^0\rangle$. It must therefore be a linear combination of $|\Sigma^0\rangle$ and another state at that point, the iso-singlet in the $\underset{\sim}{8}$ which we shall call $|\Lambda^0\rangle$. If we write

$$|\Lambda_0\rangle \propto |S_0\,\sigma_0\rangle + \alpha\,|D_0\,\delta_0\rangle + \beta\,|D_+\,\delta_-\rangle$$

and determine α and β from

$$\langle\Lambda^0|\Sigma^0\rangle = 0, \qquad \text{(VI - 24)}$$

we find $\alpha - \beta = 0$ so that, after normalization,

$$\Lambda_0 = a\,|S_0\,\sigma_0\rangle + \frac{\sqrt{1 - a^2}}{\sqrt{2}}\,(|D_0\,\delta_0\rangle + |D_+\,\delta_-\rangle). \qquad \text{(VI - 25)}$$

The state is not yet determined. If we take the symmetric combination

$$|X\rangle = \frac{1}{\sqrt{3}}\,(|D_+\,\delta_-\rangle + |D_0\,\delta_0\rangle + |S_0\,\sigma_0\rangle), \qquad \text{(VI - 26)}$$

we observe that

$$E_{\pm 1}|X\rangle = E_{\pm 2}|X\rangle = E_{\pm 3}|X\rangle = 0, \qquad \text{(VI - 27)}$$

i.e., $|X\rangle$ is the $\underset{\sim}{1}$ state in the decomposition

$$\underset{\sim}{3} \otimes \underset{\sim}{3}^* = \underset{\sim}{8} \oplus \underset{\sim}{1}.$$

We also require that

$$\langle X | \Lambda^0 \rangle = 0. \quad \text{(VI - 28)}$$

This fixes $\underline{a}$ such that

$$|\Lambda_0\rangle = \frac{1}{\sqrt{6}} (|D_0 \delta_0\rangle + |D_+ \delta_-\rangle - 2 |S_0 \sigma_0\rangle). \quad \text{(VI - 29)}$$

We can now calculate the remaining matrix elements:

$$\langle \Sigma^0 | E_{-2} | P \rangle = \frac{1}{2\sqrt{3}},$$

$$\langle \Lambda_0 | E_{-2} | P \rangle = -\tfrac{1}{2};$$

and similarly

$$\langle \Sigma_0 | E_{-3} | N \rangle = -\frac{1}{2\sqrt{3}},$$

$$\langle \Lambda_0 | E_{-3} | N \rangle = -\tfrac{1}{2},$$

$$\langle \Sigma^0 | E_2 | \Xi^- \rangle = -\frac{1}{2\sqrt{3}},$$

$$\langle \Lambda^0 | E_2 | \Xi^- \rangle = \tfrac{1}{2},$$

$$\langle \Sigma^0 | E_3 | \Xi^0 \rangle = \frac{1}{2\sqrt{3}},$$

$$\langle \Lambda_0 | E_3 | \Xi^0 \rangle = \tfrac{1}{2}.$$

We can conveniently summarize the matrix elements by labeling the lines connecting different states on the $\underset{\sim}{8}$ weight diagram. Figure 21 shows the results for $\underset{\sim}{8}$; the dotted lines indicate the transitions to the Λ_0 state.

These matrix elements are essential in the reduction of $\underset{\sim}{8} \otimes \underset{\sim}{8}$ which we now carry out. The elements of one of the $\underset{\sim}{8}$ will be labeled by (Σ^+, Σ^0, Σ^-, P, N, Λ^0, Ξ^-, Ξ^0), those of the other by (π^+, π^0, π^-, K^+, K^0, η^0, K^-, $\overline{K}^0$). The "vector addition" of weight diagrams is shown in

Fig. 22, in which the dots indicate the original $\underset{\sim}{8}$, the crosses the states in $\underset{\sim}{8} \otimes \underset{\sim}{8}$.

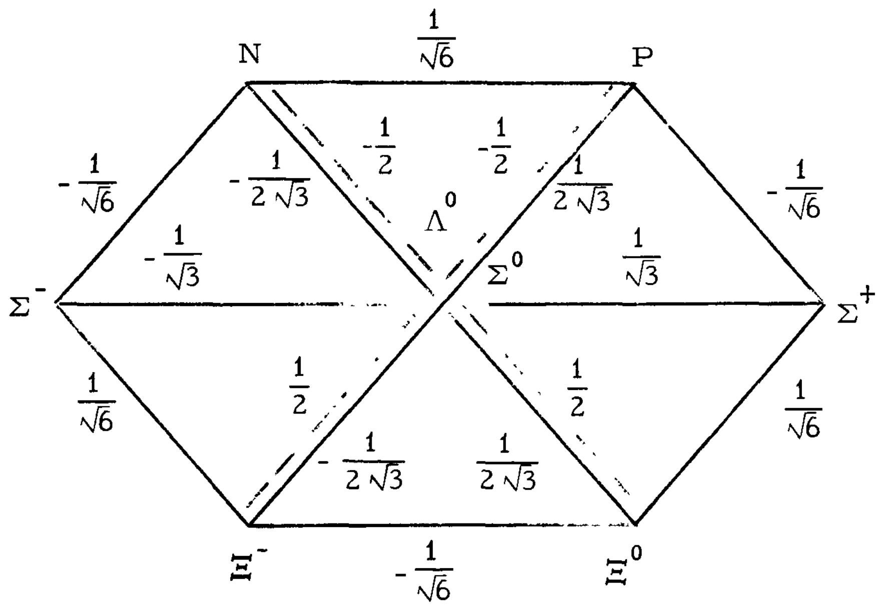


Fig. 21. The matrix elements of the $E_{\pm\alpha}$ in the representation $\underset{\sim}{8}$.

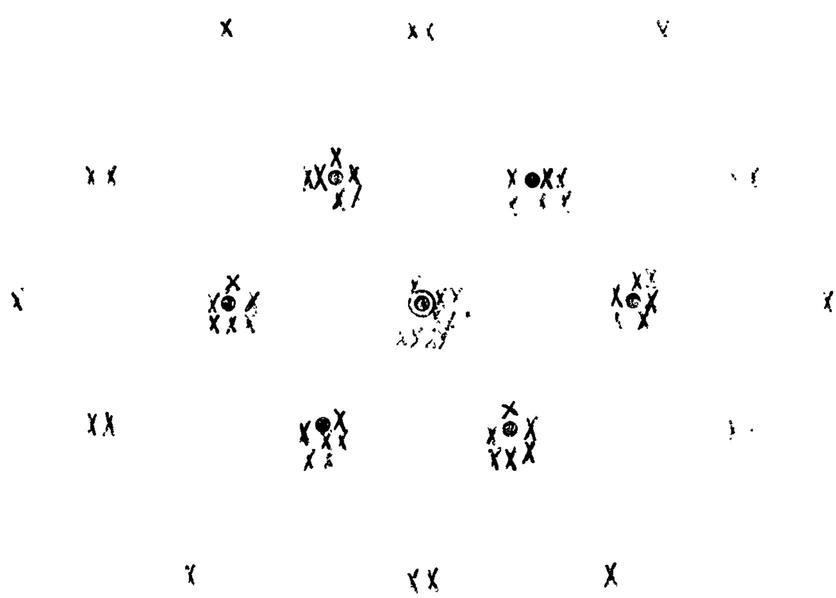


Fig. 22. The multiplicities in $\underset{\sim}{8} \otimes \underset{\sim}{8}$.

We note that the corner points are nondegenerate. They must belong to the largest representation. Since for $\underset{\sim}{8}$ we have (a, b) = (1, 1) this must be the representation (2, 2) whose dimensionality is 27. This is the first one we reduce out.

We start with the state of highest weight, $|\Sigma^+\pi^+\rangle$. It is clear from Fig. 22 that repeated application of E_{-1} will yield a quintet. The quintet lies on the $M_2 = 0$ line and therefore is associated with hypercharge Y = 0. The results, starting with $|\Sigma^+, \pi^+\rangle$, are

$$E_{-1}|\Sigma^+\pi^+\rangle = \frac{1}{\sqrt{3}}|\Sigma^0\pi^+\rangle + \frac{1}{\sqrt{3}}|\Sigma^+\pi^0\rangle = \sqrt{\frac{2}{3}}\left(\frac{|\Sigma^0\pi^+\rangle + |\Sigma^+\pi^0\rangle}{\sqrt{2}}\right),$$

$$E_{-1}\left(\frac{|\Sigma^0\pi^+\rangle + |\Sigma^+\pi^0\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}|\Sigma^-\pi^+\rangle + \frac{1}{\sqrt{3}}|\Sigma^0\pi^0\rangle + \frac{1}{\sqrt{3}}|\Sigma^0\pi^0\rangle - \frac{1}{\sqrt{3}}|\Sigma^+\pi^-\rangle\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{2}{3}}|\Sigma^0\pi^0\rangle - \frac{1}{\sqrt{6}}|\Sigma^+\pi^-\rangle - \frac{1}{\sqrt{6}}|\Sigma^-\pi^+\rangle\right),$$

etc., where in each case the normalized wave functions have been enclosed in the parentheses. The full 64 combinations will not be worked out. Instead, the results will be listed and the not-entirely-trivial steps will be pointed out along the way.

The states with T = 2, Y = 0 are

$$|\Sigma^+\pi^+\rangle,$$

$$\frac{1}{\sqrt{2}}(|\Sigma^0\pi^+\rangle + |\Sigma^+\pi^0\rangle),$$

$$\sqrt{\frac{2}{3}}|\Sigma^0\pi^0\rangle - \frac{1}{\sqrt{6}}|\Sigma^+\pi^-\rangle - \frac{1}{\sqrt{6}}|\Sigma^-\pi^+\rangle,$$

$$\frac{1}{\sqrt{2}}(|\Sigma^0\pi^-\rangle + |\Sigma^-\pi^0\rangle),$$

$$|\Sigma^-\pi^-\rangle.$$

To get the first state of the quartet for Y = 1, we apply E_3 to $|\Sigma^+\pi^+\rangle$ and then repeatedly apply E_{-1}. The results for $T = \frac{3}{2}$, Y = 1 are

$$\frac{1}{\sqrt{2}} (|\Sigma^+ K^+\rangle + |P \pi^+\rangle),$$

$$\frac{1}{\sqrt{3}} (|\Sigma^0 K^+\rangle + |\pi^0 P\rangle) + \frac{1}{\sqrt{6}} (|\Sigma^+ K^0\rangle + |N \pi^+\rangle),$$

$$\frac{1}{\sqrt{3}} (|\Sigma^0 K^0\rangle + |\pi^0 N\rangle) - \frac{1}{\sqrt{6}} (|\Sigma^- K^+\rangle + |\pi^- P\rangle),$$

$$\frac{1}{\sqrt{2}} (|\Sigma^- K^0\rangle + |\pi^- N\rangle).$$

The quartet with hypercharge $T = \frac{3}{2}$, $Y = -1$, obtained in an analogous way, is

$$\frac{1}{\sqrt{2}} (|\Sigma^+ \overline{K}^0\rangle + |\pi^+ \Xi^0\rangle),$$

$$\frac{1}{\sqrt{3}} (|\Sigma^0 \overline{K}^0\rangle + |\pi^0 \Xi^0\rangle) - \frac{1}{\sqrt{6}} (|\Sigma^+ K^-\rangle + |\pi^+ \Xi^-\rangle),$$

$$\frac{1}{\sqrt{3}} (|\Sigma^0 K^-\rangle + |\pi^0 \Xi^-\rangle) + \frac{1}{\sqrt{6}} (|\Sigma^- \overline{K}^0\rangle + |\pi^- \Xi^0\rangle),$$

$$\frac{1}{\sqrt{2}} (|\Sigma^- K^-\rangle + |\pi^- \Xi^-\rangle).$$

We observe that the two quartets (which are symmetric with respect to each other about the origin) can be obtained from each other; the states which make up one quartet are obtained by reflecting the other quartet in the origin, i.e., by the transformation

$$\Sigma^+ \leftrightarrow \Sigma^-, \qquad \pi^+ \leftrightarrow \pi^-,$$

$$P \leftrightarrow \Xi^-, \qquad K^+ \leftrightarrow K^-,$$

$$N \leftrightarrow \Xi^0, \qquad K^0 \leftrightarrow \overline{K}^0.$$

This transformation is called the hypercharge-conjugation transformation, or the R transformation. It will be used to simplify our work.

The (T = 1, Y = 2) triplet, obtained by starting with $|PK^+\rangle$, is

$$|PK^+\rangle,$$

$$\frac{1}{\sqrt{2}}(|PK^0\rangle + |NK^+\rangle),$$

$$|NK^0\rangle.$$

Then applying the R transformation to this yields the (T = 1, Y = -2) triplet

$$|\Xi^0\overline{K}^0\rangle,$$

$$\frac{1}{\sqrt{2}}(|\Xi^0 K^-\rangle + |\Xi^-\overline{K}^0\rangle),$$

$$|\Xi^- K^-\rangle.$$

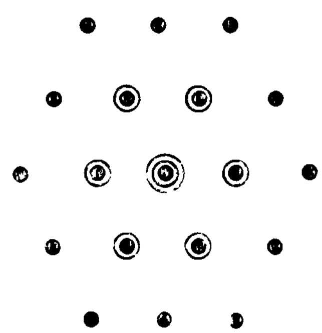


Fig. 23. States left after 19 of the 27 have been removed.

We have obtained 19 of the 27 states in the highest representation contained in $\underset{\sim}{8} \otimes \underset{\sim}{8}$. In the pattern of points (Fig. 23), the degeneracy of the points in the layer immediately inside the boundary layer leads us to expect another triplet with Y = 0 and two doublets with Y = ± 1.

This leaves one state, which must be an isosinglet with Y = 0. To find the first doublet (with Y = 1) we construct

$$E_{-2}\ |PK^+\rangle = \frac{1}{2\sqrt{3}}\ (|P\pi^0\rangle + |K^+\Sigma^0\rangle) - \frac{1}{2}\ (|P\eta^0\rangle + |K^+\Lambda^0\rangle)$$

and note that this is not orthogonal to the state in $T = \frac{3}{2}$, Y = 1 at the same point. A state orthogonal to that one may however be obtained by taking an appropriate linear combination of $E_{-2}\,|PK^+\rangle$ with the third of the states ($T = \frac{3}{2}$, Y = 1). The state and its partner are

$$\frac{1}{\sqrt{60}}\ (|P\pi^0\rangle + |K^+\Sigma^0\rangle) - \frac{3}{\sqrt{20}}\ (|P\eta^0\rangle + |K^+\Lambda^0\rangle) - \frac{1}{\sqrt{30}}\ (|\Sigma^+K^0\rangle + |\pi^+N\rangle),$$

$$\frac{1}{\sqrt{60}}\ (|N\pi^0\rangle + |K^0\Sigma^0\rangle) + \frac{3}{\sqrt{20}}\ (|N\eta^0\rangle + |K^0\Lambda^0\rangle) + \frac{1}{\sqrt{30}}\ (|P\pi^-\rangle + |\Sigma^-K^+\rangle).$$

Then for $T = \frac{1}{2}$, Y = -1, the R operation yields

$$\frac{1}{\sqrt{60}}\ (|\Xi^0\pi^0\rangle + |\overline{K}^0\Sigma^0\rangle) + \frac{3}{\sqrt{20}}\ (|\Xi^0\eta^0\rangle + |\overline{K}^0\Lambda^0\rangle) + \frac{1}{\sqrt{30}}\ (|\Xi^-\pi^+\rangle + |\Sigma^+K^-\rangle),$$

$$\frac{1}{\sqrt{60}}\ (|\Xi^-\pi^0\rangle + |K^-\Sigma^0\rangle) - \frac{3}{\sqrt{20}}\ (|\Xi^-\eta^0\rangle + |K^-\Lambda^0\rangle) - \frac{1}{\sqrt{30}}\ (|\Sigma^-\overline{K}^0\rangle + |\pi^-\Xi^0\rangle).$$

To construct the triplet, we take a linear combination of $E_{-2}(|\Sigma^+K^+\rangle + |\pi^+P\rangle)$ and the corresponding (T = 2, Y = 0) state, such that it is orthogonal to the latter. The (T = 1, Y = 0) triplet obtained in this way is

$$\frac{1}{\sqrt{5}}\ (|P\overline{K}^0\rangle + |K^+\Xi^0\rangle - \sqrt{\frac{3}{10}}\ (|\Sigma^+\eta^0\rangle + |\pi^+\Lambda^0\rangle),$$

$$\frac{1}{\sqrt{10}}\ (|N\overline{K}^0\rangle + |K^0\Xi^0\rangle) - \frac{1}{\sqrt{10}}\ (|PK^-\rangle + |\Xi^-K^+\rangle) - \sqrt{\frac{3}{10}}(|\Sigma^0\eta^0\rangle + |\Lambda^0\pi^0\rangle),$$

$$-\frac{1}{\sqrt{5}}\ (|NK^-\rangle + |K^0\Xi^-\rangle) + \sqrt{\frac{3}{10}}\ (|\Sigma^-\eta^0\rangle + |\pi^-\Lambda^0\rangle).$$

The set of 27 states in the representation is completed by taking a linear combination of $E_{-2}|T = \frac{1}{2}, T_z = \frac{1}{2}, Y = 1\rangle$ and the $T_z = 0$ states with (T = 2, Y = 0) and (T = 1, Y = 0) such that it is orthogonal to the last two. The result, for which T = 0, Y = 0, is simply

$$\frac{1}{\sqrt{120}}|\Sigma^0\pi^0\rangle + \sqrt{\frac{27}{40}}|\Lambda^0\eta^0\rangle - \sqrt{\frac{3}{40}}(|PK^-\rangle + |K^+\Xi^-\rangle)$$

$$- \sqrt{\frac{3}{40}}(|N\bar{K}^0\rangle + |K^0\Xi^0\rangle) + \frac{1}{\sqrt{120}}(|\Sigma^+\pi^-\rangle + |\Sigma^-\pi^+\rangle).$$

To go on from here, we note that we started the Y = 1 quartet with the symmetric linear combination $(1/\sqrt{2})(|\Sigma^+K^+\rangle + |P\pi^+\rangle)$. Use of the antisymmetric combination orthogonal to it would clearly lead to another representation — the 10-dimensional representation with $T = \frac{3}{2}$, Y = 1 for which

$$\frac{1}{\sqrt{2}}(|\Sigma^+K^+\rangle - |P\pi^+\rangle),$$

$$\frac{1}{\sqrt{6}}(|\Sigma^+K^0\rangle - |N\pi^+\rangle) + \frac{1}{\sqrt{3}}(|\Sigma^0K^+\rangle - |\pi^0P\rangle),$$

$$-\frac{1}{\sqrt{6}}(|\Sigma^-K^+\rangle - |\pi^-P\rangle) + \frac{1}{\sqrt{3}}(|\Sigma^0K^0\rangle - |\pi^0N\rangle),$$

$$-\frac{1}{\sqrt{2}}(|\Sigma^-K^0\rangle - |\pi^-N\rangle).$$

Note that the results are always antisymmetric combinations, so that all of them will be orthogonal to the $\underset{\sim}{27}$ — which is as it should be.

To get the next iso-multiplet we apply E_{-2} to the first of the above states and get the (T = 1, Y = 0) triplet

$$\frac{1}{2\sqrt{3}}(|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) - \frac{1}{\sqrt{6}}(|P\bar{K}^0\rangle - |\Xi^0K^+\rangle) - \frac{1}{2}(|\Sigma^+\eta^0\rangle - |\Lambda^0\pi^+\rangle),$$

$$-\frac{1}{2\sqrt{3}}(|\Sigma^+\pi^-\rangle - |\Sigma^-\pi^+\rangle) + \frac{1}{2\sqrt{3}}(|PK^-\rangle - |\Xi^-K^+\rangle) - \frac{1}{2\sqrt{3}}(|N\bar{K}^0\rangle - |\Xi^0K^0\rangle)$$

$$-\frac{1}{2}(|\Sigma^0\eta^0\rangle - |\Lambda^0\pi^0\rangle),$$

$$-\frac{1}{2\sqrt{3}}(|\Sigma^-\pi^0\rangle - |\Sigma^0\pi^-\rangle) + \frac{1}{\sqrt{6}}(|NK^-\rangle - |\Xi^-K^0\rangle) + \frac{1}{2}(|\Sigma^-\eta^0\rangle - |\Lambda^0\pi^-\rangle).$$

Next apply E_{-2} to the first of these to get the $(T = \frac{1}{2},\ Y = -1)$ states

$$\frac{1}{2\sqrt{3}}\left(|\Xi^0\pi^0\rangle - |\Sigma^0\overline{K}^0\rangle\right) - \frac{1}{\sqrt{6}}\left(|\Sigma^+K^-\rangle - |\Xi^-\pi^+\rangle\right) - \frac{1}{2}\left(|\Xi^0\eta^0\rangle - |\Lambda^0\overline{K}^0\rangle\right),$$

$$\frac{1}{2\sqrt{3}}\left(|\Xi^-\pi^0\rangle - |\Sigma^0K^-\rangle\right) + \frac{1}{\sqrt{6}}\left(|\Sigma^-\overline{K}^0\rangle - |\Xi^0\pi^-\rangle\right) + \frac{1}{2}\left(|\Xi^-\eta^0\rangle - |\Lambda^0K^-\rangle\right).$$

Finally by inspection, we obtain the $(T = 0,\ Y = -2)$ state

$$\frac{1}{\sqrt{2}}\left(|\Xi^-\overline{K}^0\rangle - |\Xi^0K^-\rangle\right).$$

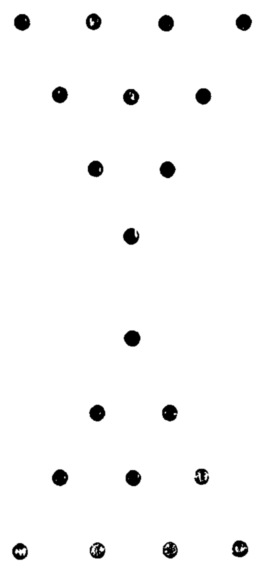


Fig. 24. The patterns for $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$.

The resulting pattern is shown in Fig. 24. This is the $\underset{\sim}{10}$ representation. It is clear that by starting with the antisymmetric combination

$$\frac{1}{\sqrt{2}}\left(|PK^0\rangle - |NK^+\rangle\right)$$

we can generate such a "triangle" upside down by application of E_{-3}. This will be the $\underset{\sim}{10}^*$ representation, which may be obtained from the $\underset{\sim}{10}$ by hypercharge reflection. The states, listed for completeness, are the following.

For $T = 0$, $Y = 2$:

$$\frac{1}{\sqrt{2}}(|PK^0\rangle - |NK^+\rangle).$$

For $T = \frac{1}{2}$, $Y = 1$:

$$\frac{1}{2\sqrt{3}}(|P\pi^0\rangle - |\Sigma^0K^+\rangle) + \frac{1}{\sqrt{6}}(|\Sigma^+K^0\rangle - |N\pi^+\rangle) + \frac{1}{2}(|P\eta^0\rangle - |\Lambda^0K^+\rangle),$$

$$\frac{1}{2\sqrt{3}}(|N\pi^0\rangle - |\Sigma^0K^0\rangle) - \frac{1}{\sqrt{6}}(|\Sigma^-K^+\rangle - |P\pi^-\rangle) - \frac{1}{2}(|N\eta^0\rangle - |\Lambda^0K^0\rangle).$$

For $T = 1$, $Y = 0$:

$$\frac{1}{2\sqrt{3}}(|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) + \frac{1}{\sqrt{6}}(|\Xi^0K^+\rangle - |P\overline{K}^0\rangle) + \frac{1}{2}(|\Sigma^+\eta^0\rangle - |\Lambda^0\pi^+\rangle),$$

$$-\frac{1}{2\sqrt{3}}(|\Sigma^-\pi^+\rangle - |\Sigma^+\pi^-\rangle) + \frac{1}{2\sqrt{3}}(|\Xi^-K^+\rangle - |PK^-\rangle) - \frac{1}{2\sqrt{3}}(|\Xi^0K^0\rangle - |N\overline{K}^0\rangle),$$

$$-\frac{1}{2}(|\Sigma^0\eta^0\rangle - |\Lambda^0\pi^0\rangle),$$

$$\frac{1}{2\sqrt{3}}(|\Sigma^-\pi^0\rangle - |\Sigma^0\pi^-\rangle) - \frac{1}{\sqrt{6}}(|\Xi^-K^0\rangle - |NK^-\rangle) - \frac{1}{2}(|\Sigma^-\eta^0\rangle - |\Lambda^0\pi^-\rangle).$$

And finally for $T = \frac{3}{2}$, $Y = -1$:

$$\frac{1}{\sqrt{2}}(|\Sigma^+\overline{K}^0\rangle - |\Xi^0\pi^+\rangle),$$

$$-\frac{1}{\sqrt{6}}(|\Sigma^+K^-\rangle - |\Xi^-\pi^+\rangle) + \frac{1}{\sqrt{3}}(|\Sigma^0\overline{K}^0\rangle - |\Xi^0\pi^0\rangle),$$

$$\frac{1}{\sqrt{6}}(|\Sigma^-\overline{K}^0\rangle - |\Xi^0\pi^-\rangle) + \frac{1}{\sqrt{3}}(|\Sigma^0K^-\rangle - |\Xi^-\pi^0\rangle),$$

$$\frac{1}{\sqrt{2}}(|\Sigma^-K^-\rangle - |\Xi^-\pi^-\rangle).$$

A glance at Fig. 22 shows that there still are many states on the $Y = 0$ axis. As we go one step in from the boundary there are six products (the terms with $T_z = 1$, and $Y = 0$ are $|\pi^+\Sigma^0\rangle$, $|\pi^0\Sigma^+\rangle$, $|\Lambda^0\pi^+\rangle$, $|\Xi^0K^+\rangle$, $|P\overline{K}^0\rangle$, $|\Sigma^+\eta^0\rangle$) of which we have found four: the quintet and triplet in $\underset{\sim}{27}$ and the two triplets in $\underset{\sim}{10}$ and $\underset{\sim}{10}^*$. Thus two more triplets are expected. We now try to construct a state orthogonal to the states that have already appeared, viz.

$$\frac{1}{\sqrt{2}} (|\Sigma^0\pi^+\rangle + |\Sigma^+\pi^0\rangle),$$

$$\frac{1}{\sqrt{5}} (|P\overline{K}^0\rangle + |\Xi^0 K^+\rangle) - \sqrt{\frac{3}{10}} (|\Sigma^+\eta^0\rangle + |\Lambda^0\pi^+\rangle),$$

$$\frac{1}{2\sqrt{3}} (|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) - \frac{1}{\sqrt{6}} (|P\overline{K}^0\rangle - |\Xi^0 K^+\rangle) - \frac{1}{2} (|\Sigma^+\eta^0\rangle - |\Lambda^0\pi^+\rangle),$$

$$\frac{1}{2\sqrt{3}} (|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) - \frac{1}{\sqrt{6}} (|P\overline{K}^0\rangle - |\Xi^0 K^+\rangle) + \frac{1}{2} (|\Sigma^+\eta^0\rangle - |\Lambda^0\pi^+\rangle).$$

A little algebra shows that the states orthogonal to these must have the general form

$$a_1\{\sqrt{2}(|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) + (|P\overline{K}^0\rangle - |\Xi^0 K^+\rangle)\}$$

$$+ a_2\{\sqrt{\frac{3}{2}}(|P\overline{K}^0\rangle + |\Xi^0 K^+\rangle) + (|\Sigma^+\eta^0\rangle + |\Lambda^0\pi^+\rangle)\}.$$

The two parts have different symmetries and are orthogonal to each other, so that it is natural to use each part separately to generate an octet each (as will be seen).

Starting with the antisymmetric part, we obtain the normalized ($T = 1$, $Y = 0$) states

$$\frac{1}{\sqrt{3}}(|\Sigma^+\pi^0\rangle - |\Sigma^0\pi^+\rangle) + \frac{1}{\sqrt{6}} (|P\overline{K}^0\rangle - |\Xi^0 K^+\rangle),$$

$$-\frac{1}{\sqrt{3}}(|\Sigma^+\pi^-\rangle - |\Sigma^-\pi^+\rangle) - \frac{1}{2\sqrt{3}} (|PK^-\rangle - |\Xi^- K^+\rangle) + \frac{1}{2\sqrt{3}} (|N\overline{K}^0\rangle - |\Xi^0 K^0\rangle),$$

$$\frac{1}{\sqrt{3}}(|\Sigma^-\pi^0\rangle - |\Sigma^0\pi^-\rangle) - \frac{1}{\sqrt{6}} (|NK^-\rangle - |\Xi^- K^0\rangle).$$

The remaining two doublets and the isosinglet are obtained in the standard way. They are listed below for completeness.

For $T = \frac{1}{2}$, $Y = 1$:

$$-\frac{1}{\sqrt{6}}(|\Sigma^+K^0\rangle - |N\pi^+\rangle) - \frac{1}{2\sqrt{3}} (|P\pi^0\rangle - |\Sigma^0 K^+\rangle) + \frac{1}{2} (|P\eta^0\rangle - |\Lambda^0 K^+\rangle),$$

$$\frac{1}{\sqrt{6}}(|P\pi^-\rangle - |\Sigma^- K^+\rangle) + \frac{1}{2\sqrt{3}} (|N\pi^0\rangle - |\Sigma^0 K^0\rangle) + \frac{1}{2} (|N\eta^0\rangle - |\Lambda^0 K^0\rangle).$$

For $T = \frac{1}{2}$, $Y = -1$:

$$\frac{1}{\sqrt{6}}(|\Xi^-\pi^+\rangle - |\Sigma^+K^-\rangle) + \frac{1}{2\sqrt{3}}(|\Xi^0\pi^0\rangle - |\Sigma^0\overline{K}^0\rangle) + \frac{1}{2}(|\Xi^0\eta^0\rangle - |\Lambda^0\overline{K}^0\rangle),$$

$$-\frac{1}{\sqrt{6}}(|\Sigma^-\overline{K}^0\rangle - |\Xi^0\pi^-\rangle) - \frac{1}{2\sqrt{3}}(|\Xi^-\pi^0\rangle - |\Sigma^0K^-\rangle) + \frac{1}{2}(|\Xi^-\eta^0\rangle - |\Lambda^0K^-\rangle).$$

For $T = 0$, $Y = 0$:

$$\frac{1}{2}(|PK^-\rangle - |\Xi^-K^+\rangle) + \frac{1}{2}(|N\overline{K}^0\rangle - |\Xi^0K^0\rangle).$$

For $T = 1$, $Y = 0$, the symmetric octet is

$$\frac{\sqrt{2}}{\sqrt{10}}(|\Lambda^0\pi^+\rangle + |\Sigma^+\eta^0\rangle) + \sqrt{\frac{3}{10}}(|P\overline{K}^0\rangle + |\Xi^0K^+\rangle),$$

$$\frac{\sqrt{2}}{\sqrt{10}}(|\Sigma^0\eta^0\rangle + |\Lambda^0\pi^0\rangle) + \sqrt{\frac{3}{20}}(|N\overline{K}^0\rangle + |\Xi^0K^0\rangle) - \sqrt{\frac{3}{20}}(|PK^-\rangle + |\Xi^-K^+\rangle),$$

$$\sqrt{\frac{2}{10}}(|\Sigma^-\eta^0\rangle + |\Lambda^0\pi^-\rangle) + \sqrt{\frac{3}{10}}(|NK^-\rangle + |\Xi^-K^0\rangle).$$

For $T = \frac{1}{2}$, $Y = 1$:

$$\sqrt{\frac{1}{20}}(|P\eta^0\rangle + |\Lambda^0K^+\rangle) + \sqrt{\frac{3}{20}}(|P\pi^0\rangle + |\Sigma^0K^+\rangle) - \sqrt{\frac{3}{10}}(|\Sigma^+K^0\rangle + |N\pi^+\rangle),$$

$$\sqrt{\frac{1}{20}}(|N\eta^0\rangle + |\Lambda^0K^0\rangle) - \sqrt{\frac{3}{20}}(|N\pi^0\rangle + |\Sigma^0K^0\rangle) - \sqrt{\frac{3}{10}}(|P\pi^-\rangle + |\Sigma^-K^+\rangle).$$

For $T = \frac{1}{2}$, $Y = -1$:

$$\sqrt{\frac{1}{20}}(|\Xi^0\eta^0\rangle + |\Lambda^0\overline{K}^0\rangle) - \sqrt{\frac{3}{20}}(|\Xi^0\pi^0\rangle + |\Sigma^0\overline{K}^0\rangle) - \sqrt{\frac{3}{10}}(|\Xi^-\pi^+\rangle + |\Sigma^+K^-\rangle),$$

$$\sqrt{\frac{1}{20}}(|\Xi^-\eta^0\rangle + |\Lambda^0K^-\rangle) + \sqrt{\frac{3}{20}}(|\Xi^-\pi^0\rangle + |\Sigma^0K^-\rangle) - \sqrt{\frac{3}{10}}(|\Sigma^-\overline{K}^0\rangle + |\Xi^0\pi^-\rangle).$$

For T = 0, Y = 0:

$$\frac{1}{\sqrt{5}} |\Sigma^0 \pi^0\rangle - \frac{1}{\sqrt{5}} |\Lambda^0 \eta^0\rangle + \frac{1}{\sqrt{5}} (|\Sigma^+ \pi^-\rangle + |\Sigma^- \pi^+\rangle) - \frac{1}{2\sqrt{5}} (|PK^-\rangle + |\Xi^- K^+\rangle) - \frac{1}{2\sqrt{5}} (|N\overline{K}^0\rangle + |\Xi^0 K^0\rangle).$$

We have now

$$27 + 10 + 10 + 8 + 8 = 63$$

states, and there clearly remains only the unitary singlet. Its wave function is totally symmetric, namely,

$$\frac{1}{2\sqrt{2}} \{|\Sigma^0 \pi^0\rangle + |\Lambda^0 \eta^0\rangle + |\Sigma^+ \pi^-\rangle + |\Sigma^- \pi^+\rangle + |PK^-\rangle + |\Xi^- K^+\rangle + |N\overline{K}^0\rangle + |\Xi^0 K^0\rangle\}.$$

This concludes our reduction of $\underset{\sim}{8} \otimes \underset{\sim}{8}$. This method can be used for other reductions. Once the ideas are clear, short cuts can be taken with confidence. For example, consider the reduction of $\underset{\sim}{8} \otimes \underset{\sim}{10}$. The addition of weight graphs yields the pattern shown in Fig. 25. The boundary is

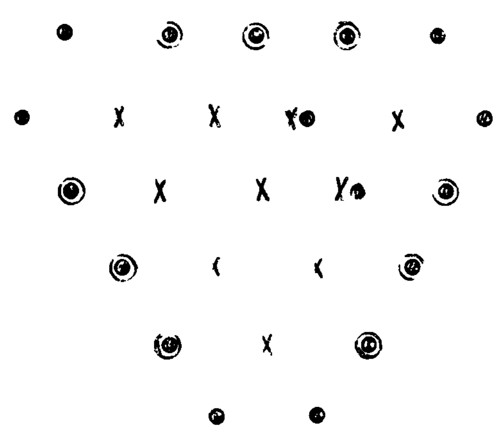


Fig. 25. The states for $\underset{\sim}{8} \otimes \underset{\sim}{10}$.

seen to have a maximum-weight point with a = 4, b = 1. Hence the multiplicity of that representation is 35. The boundary points other than the corners are doubly degenerate, and removing these corners leaves a hexagon with two "steps" on each side. This just corresponds to the representation 27. Inside that we can only inscribe 10 and 8 (Fig. 26) if we take note of the fact that all representations have a = b (mod 3) as shown before. Thus

$$8 \otimes 10 = 35 + 27 + 10 + 8.$$

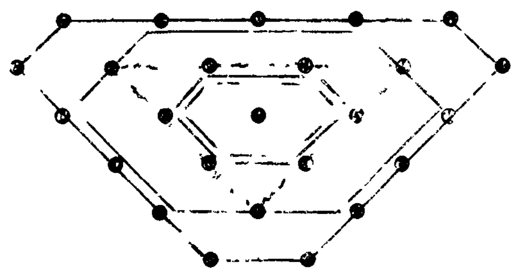


Fig. 26. The reduction of 8 ⊗ 10.

A very useful observation[10] is that in each irreducible representation there is only one isosinglet state. We can thus learn how many irreducible representations there are in the decomposition and something about the value of $-\frac{2}{3}(a_i - b_i)$ for these representations, labeled by the unknown numbers (a_i, b_i). Thus 8 ⊗ 10 has the isotopic content:

8: $Y = 1,\ T = \frac{1}{2}$; $Y = 0,\ T = 0, 1$; $Y = -1,\ T = \frac{1}{2}$;

10: $Y = 1,\ T = \frac{3}{2}$; $Y = 0,\ T = 1$; $Y = -1,\ T = \frac{1}{2}$; $Y = -2,\ T = 0$.

The isosinglets come from $(Y = 1,\ T = \frac{1}{2})_8 \oplus (Y = -1,\ T = \frac{1}{2})_{10}$, $(Y = 0,\ T = 0)_8 \oplus (Y = -2,\ T = 0)_{10}$, $(Y = 0,\ T = 1)_8 \oplus (Y = 0,\ T = 1)_{10}$, and $(Y = -1,\ T = \frac{1}{2})_8 \oplus (Y = -1,\ T = \frac{1}{2})_{10}$. Thus there are <u>four</u> irreducible representations in (1, 1) ⊗ (3, 0), one of which must be (4, 1). The four are such that for two of them $-\frac{2}{3}(a-b) = 0$ and for two it is -2.

The wave functions which we have obtained allow us to find the matrix elements of $E_{\pm\alpha}$ in the $\underset{\sim}{10}$, $\underset{\sim}{10}^*$, and $\underset{\sim}{27}$ representations in a manner analogous to the way in which we found them for the $\underset{\sim}{8}$ representation. The matrix elements for $\underset{\sim}{10}$ are exhibited in Fig. 27. Those of $\underset{\sim}{10}^*$ have all signs changed.

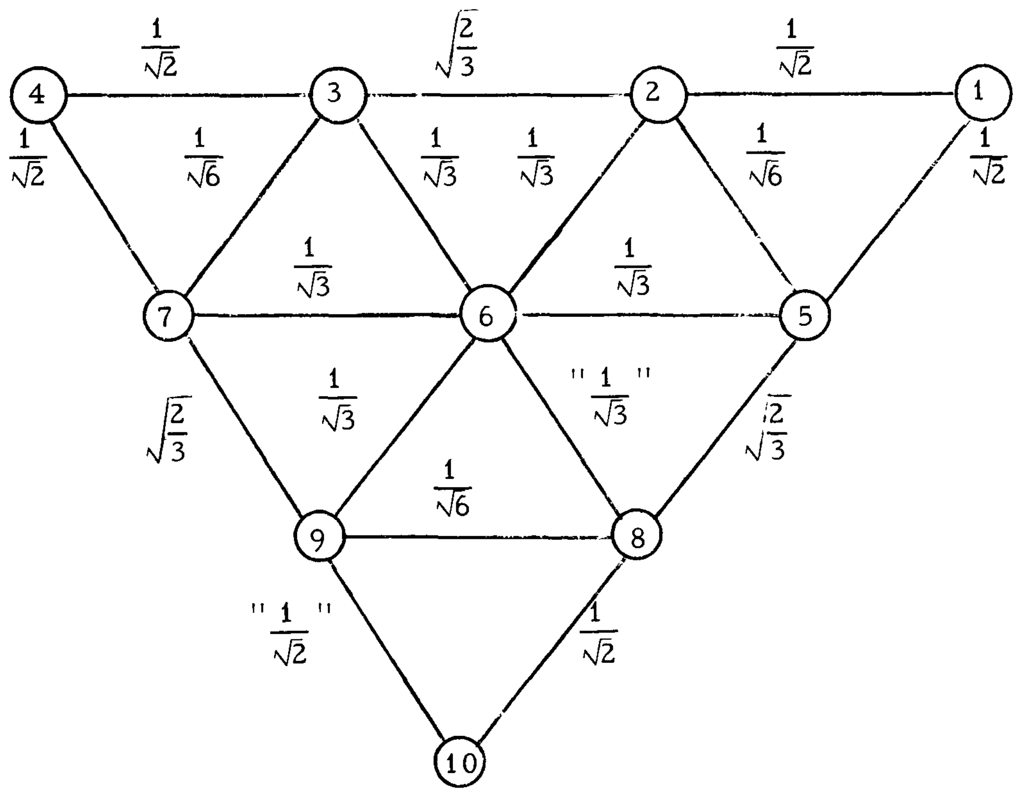


Fig. 27. The matrix elements of $E_{\pm\alpha}$ in the representation $\underset{\sim}{10}$.

The techniques developed in this section will now be used to prove a very important theorem: In the reduction of the product $\underset{\sim}{8} \otimes \underset{\sim}{N}$, where $\underset{\sim}{N}$ is some irreducible representation, $\underset{\sim}{N}$ will occur no more than twice.

This theorem will be proved diagrammatically. Figure 28 shows the right-hand top corner of the weight diagram of some representation $\underset{\sim}{N}$, with the multiplicities at each site of $\underset{\sim}{N}$ indicated. When the weight of $\underset{\sim}{8}$ is added vectorially to this graph, the result is the points labeled with crosses in Fig. 28. The graph is not complete but the multiplicities at the points A (six), B (two), C (one), and D (two) are correct, regardless of N, provided N is not triangular. (If it were, there would be only one state belonging to N at the site E, instead of two, and no state at F.)

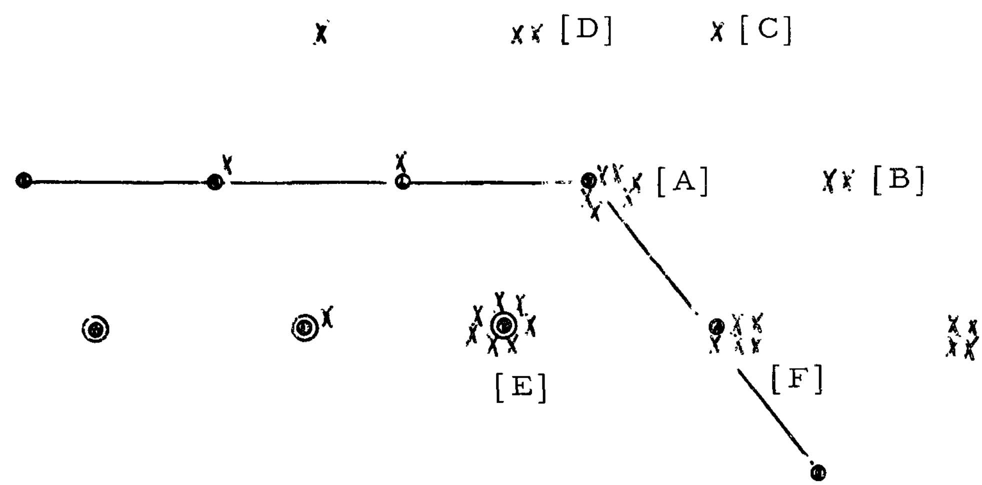


Fig. 28. Some multiplicities in 8 ⊗ N. The line denotes the original boundary of N.

Now the maximum state in N ⊗ 8 will contain C, one each of the states at B and D, and (from our multiplicity rules) two states at A. The remaining states at B and D are contained in two irreducible representations in N ⊗ 8, since a single one would have to have a concave boundary DAB··· . Each of these irreducible representations "uses up" one of the states at A. This leaves two states at A, and either could belong to an N. This establishes the theorem. If N is triangular, the fact that it cannot occur more than once in 8 ⊗ N can be shown in an analogous way.

With the help of the powerful "Young tableau" techniques, it can be shown that actually N occurs exactly twice in 8 ⊗ N unless it is "triangular" (a or b = 0); in a triangular case it occurs exactly once. With a little more attention to the states at the points A, B, C, · · ·, the methods developed above could undoubtedly be used to prove this stronger result. But for our purposes (Sec. VII) the theorem proved above is sufficient.

VII. THE INVARIANT OPERATORS

In our brief discussion of SU_2 (Sec. II), we noted that each irreducible representation was characterized by a single number, the eigenvalue of τ_3 for the state of highest weight. This one-parameter characterization is reflected in the existence of one invariant operator ($\mathbb{C}$ in Eq. II-17) that commutes with all the elements of the algebra.

For SU_3, we have seen that two integers (a, b) completely characterize an irreducible representation; and by the same token we expect to be able to construct two independent invariant operators $\mathbb{C}_1$ and $\mathbb{C}_2$, each of which commutes with all the elements $E_{\pm\alpha}$, H_1, H_2 of the algebra and (of course) with each other. In order to construct these operators, we follow a method outlined by Okubo,[13] whose paper includes a direct proof that there are only two operators.

We introduce the notation

$$E_1 = \frac{1}{\sqrt{6}} M_{12}, \qquad E_{-1} = \frac{1}{\sqrt{6}} M_{21},$$
$$E_2 = \frac{1}{\sqrt{6}} M_{13}, \qquad E_{-2} = \frac{1}{\sqrt{6}} M_{31}, \qquad \text{(VII - 1)}$$
$$E_3 = \frac{1}{\sqrt{6}} M_{23}, \qquad E_{-3} = \frac{1}{\sqrt{6}} M_{32}.$$

In the 3×3 matrix representation, M_{AB} is seen to be the matrix that has unity in the AB position and zeros elsewhere. From this observation, it is obvious that the commutation relations in that representation, and hence generally, are

$$[M_{AB}, M_{CD}] = \delta_{BC} M_{AD} - \delta_{DA} M_{CB}. \qquad \text{(VII - 2)}$$

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[13] S. Okubo, Progr. Theoret. Phys. (Kyoto) 27, 949 (1962).

We can then identify H_1 and H_2 from the relations

$$\frac{1}{\sqrt{3}} H_1 = [E_1, E_{-1}] = \frac{1}{6} [M_{12}, M_{21}] = \frac{1}{6} (M_{11} - M_{22}), \qquad \text{(VII - 3)}$$

and

$$\frac{1}{2\sqrt{3}} H_1 + \frac{1}{2} H_2 = [E_2, E_{-2}] = \frac{1}{6} [M_{13}, M_{31}] = \frac{1}{6} (M_{11} - M_{33}). \qquad \text{(VII - 4)}$$

Similarly

$$-\frac{1}{2\sqrt{3}} H_1 + \frac{1}{2} H_2 = [E_3, E_{-3}] = \frac{1}{6} (M_{22} - M_{33}). \qquad \text{(VII - 5)}$$

Hence

$$H_2 = \frac{1}{6} (M_{11} + M_{22} - 2M_{33}), \quad H_1 = \frac{1}{2\sqrt{3}} (M_{11} - M_{22}). \qquad \text{(VII - 6)}$$

On the other hand, M_{11}, M_{22}, M_{33} are not defined completely in terms of H_1 and H_2. If we impose the condition that

$$M_{11} + M_{22} + M_{33} = 0, \qquad \text{(VII - 7)}$$

we find

$$M_{11} = H_2 + \sqrt{3} H_1, \quad M_{22} = H_2 - \sqrt{3} H_1, \quad M_{33} = -2 H_2. \qquad \text{(VII - 8)}$$

Now we consider the operator

$$X_{AB} = \sum_F M_{AF} M_{FB}. \qquad \text{(VII - 9)}$$

This operator satisfies the commutation relations

$$[X_{AB}, M_{CD}] = \sum_F M_{AF} (\delta_{BC} M_{FD} - \delta_{FD} M_{CB})$$
$$= \sum_F (\delta_{FC} M_{AD} - \delta_{AD} M_{CF}) M_{FB}$$
$$= \delta_{BC} X_{AD} - \delta_{AD} X_{CB}. \qquad \text{(VII - 10)}$$

From this it follows that

$$\mathbb{C}_1 = \sum_A X_{AA} = \sum_{AB} M_{AB} M_{BA} \qquad \text{(VII - 11)}$$

commutes with all M_{CD}; i.e., it is an invariant operator. It is also easy to see that

$$\mathbb{C}_2 = \sum_{AFG} M_{AF} M_{FG} M_{GA} \qquad \text{(VII - 12)}$$

is also an invariant operator. Of course the operators

$$\mathbb{C}_3 = \sum_{ABCD} M_{AB} M_{BC} M_{CD} M_{DA},$$

etc., will also be invariant, but we only need two to classify the irreducible representation.

If we write

$$\mathbb{M} \equiv \begin{pmatrix} H_2 + \sqrt{3} H_1 & \sqrt{6} E_1 & \sqrt{6} E_2 \\ \sqrt{6} E_{-1} & H_2 - \sqrt{3} H_1 & \sqrt{6} E_3 \\ \sqrt{6} E_{-2} & \sqrt{6} E_{-3} & -2H_2 \end{pmatrix}, \qquad \text{(VII - 13)}$$

we have

$$\mathbb{C}_1 \equiv \mathrm{Tr}\ \mathbb{M}^2, \qquad \text{(VII - 14)}$$
$$\mathbb{C}_2 = \mathrm{Tr}\ \mathbb{M}^3. \qquad \text{(VII - 15)}$$

Both are easily written down in terms of $E_{\pm\alpha}$, H_1, and H_2. The first is

$$\mathbb{C}_1 = 6\,(H_2^2 + H_1^2 + E_1 E_{-1} + E_{-1} E_1 + E_2 E_{-2} + E_{-2} E_2 + E_3 E_{-3} + E_{-3} E_3), \qquad \text{(VII-16)}$$

and using the commutation relations, we can write this as

$$\mathbb{C}_1 = 6\,(H_2^2 + H_1^2 + 2E_{-1} E_1 + 2E_{-2} E_2 + 2E_{-3} E_3 + \frac{2}{\sqrt{3}} H_1). \qquad \text{(VII-17)}$$

The usefulness of these operators may be illustrated by using them to find the position of an isosinglet (on the assumption that it exists) in an irreducible representation. The singlet state must satisfy

$$E_{\pm 1} \mid \rangle = 0. \qquad \text{(VII-18)}$$

Hence

$$\frac{1}{\sqrt{3}} H_1 \mid \rangle = 0,$$

i.e.,

$$m_1 = 0. \qquad \text{(VII-19)}$$

The value of m_2 will also be desired. For the representation in question, whose highest weight is (M_1, M_2), we have

$$C_1 \equiv \langle M_1 M_2 | \mathbb{C}_1 | M_1 M_2 \rangle = 6\,(M_1^2 + M_2^2 + \frac{2}{\sqrt{3}} M_1). \qquad \text{(VII-20)}$$

For the singlet state,

$$\begin{aligned} C_1 \,|\, 0, m_2 \rangle &= \mathbb{C}_1 \,|\, 0, m_2 \rangle \\ &= 6\,(m_2^2 + 2E_{-2} E_2 + 2E_3 E_{-3}) \,|\, 0, m_2 \rangle. \end{aligned} \qquad \text{(VII-21)}$$

We also calculate

$$C_2 = \langle M_1 M_2 \,|\, \mathbb{C}_2 \,|\, M_1 M_2 \rangle. \qquad \text{(VII-22)}$$

It turns out to be

$$C_2 = -6M_2^3 + 18\,M_1^2 M_2 + 9M_1^2 + 9M_2^2 + 12\sqrt{3}\,M_1 M_2 + 6\sqrt{3}\,M_1 + 6\,M_2 . \qquad \text{(VII - 23)}$$

Also one finds that

$$\mathbb{C}_2 \,|0, m_2\rangle = [\,-6\,m_2^3 + 9\,m_2^2 - 3\,m_2 + 18\,(1 - m_2)(E_{-2}E_2 + E_3 E_{-3})]\,|0, m_2\rangle. \qquad \text{(VII - 24)}$$

Hence the condition on m_2 is

$$\frac{C_1 - 6\,m_2^2}{12} = \frac{C_2 + 6\,m_2^3 - 9\,m_2^2 + 3\,m_2}{18\,(1 - m_2)}$$

which, after a little algebra, can be written

$$(m_2 + 2M_2)(m_2 - M_2 - \sqrt{3}\,M_1 - 1)(m_2 - M_2 + \sqrt{3}\,M_1 + 1) = 0. \qquad \text{(VII - 25)}$$

Moreover the singlet must lie between the top and bottom lines of the graph, i.e., since

$$-\frac{1}{2}\,(\sqrt{3}\,M_1 + M_2) \leqslant m_2 \leqslant \frac{1}{2}\,(\sqrt{3}\,M_1 - M_2). \qquad \text{(VII - 26)}$$

Then by use of

$$\sqrt{3}\,M_1 - 3M_2 \equiv b \geqslant 0,$$

it is easy to see that only the root

$$m_2 = -2M_2 \qquad \text{(VII - 27)}$$

is acceptable, a result quoted earlier in Sec. IV.

In conclusion, for completeness, we briefly discuss the mass formula of Gell-Mann and Okubo.[14] In Eq. (VII - 10) we found quantity X_{AB}

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[14] The considerations which follow cannot be made completely satisfactory without going into the analysis of tensors of SU_3, which we do not want to do. The discussion leans rather heavily on some (here unproved) similarity to the rotation group.

which had the same commutation relations with M_{AB} as M_{AB} does itself. Any quantity like this will be called a vector operator in analogy with the SU_2 (or the rotation group) definition of a vector operator as one that obeys the commutation relations

$$[V_i, J_k] = i \epsilon_{ik\ell} V_\ell \ . \qquad \text{(VII - 28)}$$

The vector operators in the rotation group are just a special type of irreducible tensor operators; and the more general irreducible tensor operators in SU_3 could be constructed in the same way. We do not do this, however, because such a program would take us somewhat beyond the simple goals set for this paper.

We will, however, spend a little time on the vector operators — in particular on their matrix elements between the states of a given irreducible representation, i.e., on $\langle \underset{\sim}{N}, m_1' m_2' | X_{AB} | \underset{\sim}{N}, m_1 m_2 \rangle$. The analogous problem in the case of the rotation group is solved with the help of the Wigner-Eckart theorem. There one finds that

$$\langle j, m | V_k | j, m' \rangle = f(j) \langle j, m | J_k | j, m' \rangle . \qquad \text{(VII - 29)}$$

The proportionality of $\langle j, m | V_k | j, m' \rangle$ to $\langle j, m | J_k | j, m' \rangle$ follows from the facts that $\vec{V}$ transforms like the $j = 1$ representation of the rotation group and that, in the reduction of $D(1) \otimes D(j)$, the irreducible representation $D(j)$ occurs only once. In the case of SU_3, X_{AB} transforms like the $\underset{\sim}{8}$ representation. We have already shown that in the reduction of $\underset{\sim}{8} \otimes \underset{\sim}{N}$, $\underset{\sim}{N}$ ordinarily occurs twice. Thus there will be two terms on the right-hand side of the generalization of (VII - 29) to SU_3. The form of the Wigner-Eckart theorem must be

$$\langle a, b; m_1' m_2' | X_{AB} | a, b; m_1 m_2 \rangle$$

$$= \langle a, b; m_1' m_2' | M_{AB} | a, b; m_1 m_2 \rangle f(a, b)$$

$$+ \langle a, b; m_1' m_2' | \sum_C M_{AC} M_{CB} | a, b; m_1 m_2 \rangle g(a, b). \qquad \text{(VII - 30)}$$

In his discussion of the symmetry-breaking interactions, Gell-Mann[1] proposed that these terms have the transformations characteristic of what we called a vector; i.e., the symmetry violation has the transformation properties of the H_2 operator in the $\underset{\sim}{8}$ representation. We thus want

$$\langle a, b; m_1 m_2 | X_{33} | a, b; m_1 m_2 \rangle$$
$$= f'(a,b) \langle a, b; m_1 m_2 | H_2 | a, b; m_1 m_2 \rangle$$
$$+ g(a,b) \langle a, b; m_1 m_2 | \sum_C M_{3C} M_{C3} | a, b; m_1 m_2 \rangle. \quad \text{(VII - 31)}$$

Now

$$\sum_C M_{3C} M_{C3} = 6 (E_{-2}E_2 + E_{-3} E_3 \frac{2}{3} H_2^2 \equiv \mathcal{H}_2 .$$

But

$$\mathbb{C}_1 = 6 (H_1^2 + H_2^2 + \{E_1, E_{-1}\} + 2E_{-2}E_2 + 2E_{-3} E_3 + H_2),$$

i.e.,

$$\frac{1}{6} \mathbb{C}_1 = \frac{1}{3} \mathcal{H}_2 + \frac{1}{3} (\vec{T}^2 - H_2^2) + H_2 .$$

Thus

$$\mathcal{H}_2 = \frac{1}{2} \mathbb{C}_1 - (\vec{T}^2 - H_2^2) - 3 H_2 .$$

This leads to the result that

$$\langle a, b; m_1 m_2 | X_{33} | a, b; m_1 m_2 \rangle$$
$$= A(a,b) + B(a,b) Y$$
$$+ C(a,b) [T(T+1) - \frac{1}{4} Y^2] ,$$

since $H_2 = \frac{1}{2} Y$. This yields the mass formula of Gell-Mann and Okubo.[15]

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[15] The above derivation is just a synopsis of the content of reference 13.

It should be noted that for bosons, invariance under charge conjugation demands that the mass spectrum be invariant under $Y \rightarrow -Y$. Furthermore, the boson polarization operator is what would be calculated in field theory and it gives the corrections to the square of the boson mass. Hence, for bosons we are justified in writing

$$\mu^2 = \mu_0^2 + \delta\mu_1^2 \left[T(T+1) - \frac{1}{4} Y^2\right].$$

For nucleons,

$$M = M_0 + \delta M_1 Y + \delta M_2 \left[T(T+1) - \frac{1}{4} Y^2\right].$$

And for the triangular representations, for which

$$T = \pm \frac{1}{2} Y + 1,$$

we get

$$M = M_0' + \delta M_1' Y.$$

In all cases in which it has been possible to associate a set of i-spin multiplets with a single representation of SU_3, the mass formula of Gell-Mann and Okubo has worked very well.[11] The only deviation is in the prediction of the ω mass in terms of the ρ and K^* masses. In this situation, however, as pointed out by Sakurai,[16] an important perturbing influence may be the possible existence of the unitary singlet vector φ, evidence for whose existence is beginning to come in.

These remarks conclude our discussion of the structure of SU_3. Current work in the exploration of SU_3 in elementary-particle physics is more dynamical in character;[17] papers on this subject do not dwell on the purely kinematic aspects of the subject. It is hoped that this paper will serve to make advances in this field more accessible to a larger group of physicists.

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[16] J. J. Sakurai, Phys. Rev. Letters 9, 472 (1962).

[17] See for example R. Cutkosky, J. Kalckar, and P. Tarjanne, Phys. Letters 1, 93 (1962).

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