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Twist Relation, Third Factorization and the  
General Vertex in a Dual Multiparticle Theory with Nonlinear Trajectories\*

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ABSTRACT

The following properties of the dual multiparticle theory with nonlinear trajectories are presented: (I) the twist relation (II) the unsymmetric general vertex obtained directly from third factorization (III) the symmetric general vertex.

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## I. INTRODUCTION

In previous papers,<sup>1</sup> the N-point Born terms of a dual multiparticle theory with nonlinear trajectories<sup>2</sup> were factorized and the vertex involving two external particles with nonzero spins was obtained. In this paper, we will obtain the twisted propagator and the general vertex in the nonlinear model. These are the two basic ingredients in the Kikkawa-Sakita-Virasoro unitarization program.<sup>3</sup> However, further steps in this program have not yet been carried out for the nonlinear model.

In I, we found it convenient to introduce a six-dimensional formalism for purposes of proving factorization. In Section II of this paper, we develop this formalism further by introducing a special choice of kinematic variables which are particularly suited to the study of duality and factorization. Using the formalism developed in Section II, we go on to derive the twist relation in Section III and the unsymmetric general vertex and the symmetric vertex in Section IV.

## II. SIX DIMENSIONAL FORMALISM

In first factorization, we were naturally led to consider six-dimensional "vectors". These "6-vectors" neither satisfy additivity, nor do they scale. Therefore, they do not form a linear vector space. Nevertheless, as a mnemonic device, it is convenient to call them vectors.

Furthermore, we define a six-dimensional "metric," and a "scalar product" in this six dimensional space. The important property of the six-dimensional space is that a pole variable  $\sigma_{ij}$  may be written as a scalar product of two "6-vectors". We will now develop the six-dimensional formalism further.

In I, the pole variable is given by  $\sigma = as + b$  where a and b are constants. In Eq.(14) of I, each 4-vector  $\bar{P}_\mu$  was mapped into a "6-vector"  $P_\alpha$  by

$$\begin{aligned} P_\alpha &= \sqrt{2a} \bar{P}_\alpha & 0 \leq \alpha \leq 3 \\ P_4 &= 1 \\ P_5 &= a \bar{P}^2 + b/2 \end{aligned} \quad (1)$$

A six dimensional metric tensor  $g_{\alpha\beta}$  was defined with the following nonvanishing elements:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = g_{45} = g_{54} = 1$$

If  $\bar{P}_{1\mu}$  and  $\bar{P}_{2\mu}$  are two 4 vectors, then the corresponding 6-vectors satisfy

$$\sigma_{12} \equiv a (\bar{P}_1 + \bar{P}_2)^2 + b = P_{1\sigma} P_2^\sigma \quad (3)$$

where in analogy to the usual Einstein convention, repeated indices imply summation over the 6 components.

For simplicity, in the rest of this paper, we will restrict ourselves to the case  $b = 0$ . We now generalize the six-dimensional formalism by associating with each 4-vector  $\bar{P}_\mu$  a six-dimensional matrix  $\Pi_{\alpha\beta}(\bar{P})$  given by

$$\begin{aligned} \Pi_{\mu\nu} &= g_{\mu\nu} & \text{for } 0 \leq \mu, \nu \leq 3 \\ \Pi_{4\alpha} &= \Pi_{\alpha 4} = 0 & \text{for } \alpha \neq 5 \\ \Pi_{5\alpha} &= \Pi_{\alpha 5} = P_\alpha & \end{aligned} \quad (4)$$

The  $\Pi$  matrices have the following important property:

$$\Pi_{\alpha\beta}(\bar{P}_1 + \bar{P}_2) = \Pi_{\alpha\sigma}(\bar{P}_1) \Pi^\sigma_\beta(\bar{P}_2) \quad (5)$$

For future reference, we will write Eq.(5) in the special case  $\beta = 5$ . We have

$$P_{12\alpha} = \Pi_{\alpha\sigma}(\bar{P}_1) P_2^\sigma \quad (6)$$

where  $P_{12\alpha}$  is the six-vector associated with the four-vector  $(\bar{P}_1 + \bar{P}_2)_\mu$ . Eq.(6) suggests that the  $\Pi$  matrices may be viewed as translation operators in the six-dimensional space. If we set  $\alpha = \beta = 5$  in Eq.(5), we get back Eq.(3). Thus, the result of I is a special case of Eq.(5).

We will now show that the  $\Pi$  matrices enter naturally in off-mass-shell factorization of dual amplitudes.

Consider the interaction of  $N$  spinless particles with

4-momenta  $p_i$   $i = 1, 2, \dots, N$ . A dual amplitude is an explicit function of the pole variables

$$\sigma_{ij} = a (\bar{P}_{ij})^2 \quad (7)$$

where

$$\bar{P}_{ij} = p_i + p_{i+1} + \dots + p_j \quad (8)$$

is the 4-momentum of an internal line. Let the amplitude be denoted by  $B_N(\sigma_{ij})$ . We wish

to consider factorization at a pole, say  $\sigma_{ik}$  (See Figure 1). The residue at a pole is in general a polynomial in the overlapping variables  $\sigma_{ij}$ ,

$k_i \leq k_j$ . Any overlapping variable is a function of both the left and the right momenta of Fig.1. To write the residue of a pole in a factorized form, a direct first step would be to write each overlapping variable in a factorized form. Using Eqs. (7), (8) and (3), we can write

$$\sigma_{ij} = (P_{i-1})^\alpha (P_{j+1N})^\alpha \quad (9)$$

To arrive at Eq.(9), we have used energy-momentum conservation:

$$\begin{aligned} (\bar{P}_{ij})^2 &= (p_i + p_{i+1} + \dots + p_j)^2 = (p_1 + p_2 + \dots + p_{i-1} + p_{j+1} + \dots + p_N)^2 \\ &= (\bar{P}_{i-1} + \bar{P}_{j+1N})^2 \end{aligned}$$

Now in Eq.(9),  $(P_{i-1})_\alpha$  depends only on the vectors  $p_1, p_2, \dots, p_{i-1}$  of the left blob of Figure 1, while  $(P_{j+1})^\alpha$  is a function of vectors of the right blob.

Of course, factorization of each overlapping variable does not imply factorization of the amplitude. However, if the amplitude  $B_N(\sigma_{ij})$  should be factorizable, we will have at the  $\sigma_{1k}$  pole the form

$$B_N(\sigma_{ij}) \sim \frac{\sum_{n=1}^d \bar{A}_{k+1}^n(\sigma_{ij}; P_{1r}) \bar{B}_{N-k+1}^n(\sigma_{ij}; P_{sn})}{\sigma_{1k} - \sigma_0} \quad (10)$$

The amplitude  $\bar{A}_{k+1}^n(\sigma_{ij}; P_{1r})$  corresponds to a process with  $k$  spin-zero particles and one "excited" particle. The kinematic variables which enter explicitly into this amplitude are the  $\sigma_{ij}$  variables and the 6-vectors  $P_{i\alpha}$ ,  $i=2,3,\dots,k-1$ . These 6-vectors correspond to the internal momenta of the multiperipheral graph of Figure 1. Now, we may choose to factorize the amplitude  $\bar{A}_{k+1}^n(\sigma_{ij}; P_{1r})$  at a pole of an internal line of the multiperipheral graph. This process is called second factorization, and, as we have learned in I, all the crossed

kinematic variables in this particular case may be written by Eq.(18) and Eq.(43) of I in factorized form. However, we may also choose to factorize the amplitude  $\bar{A}_{k+1}^n$  at a pole  $\sigma_{ij}$  which does not correspond to the internal lines of Figure 1. (Any pole  $\sigma_{ij}$  with  $1 < i < j < k$  will belong to this category. In Figure 2, we have an example of factorization at such a pole). The residue of such a pole is of course a function of the  $\sigma_{ij}$  variables and 6-vectors  $P_{ir}$   $r = 2, 3, \dots, k-1$ . Following the philosophy of first factorization, if  $\bar{A}_{k+1}^n$  is to be factorized at  $\sigma_{ij}$ , each "crossed" kinematic variable will first be written in a factorized form. Now, it is easy to see from Figure 2 that any 6-vector of the type  $P_{ir}$  where  $i \leq r < j$  will carry momenta of both the left and the right blobs, and is therefore a "crossed" variable. Now, by means of Eq.(6), these 6-vectors can be written in factorized form

$$P_{ir\alpha} = \pi_{\alpha\beta} (\bar{P}_{i,i-1}) (P_{ir})^\beta$$

$\pi_{\alpha\beta}(\bar{P}_{i,i-1})$  depends only on the momenta of the right blob of Figure 2 while  $(P_{ir})^\beta$  depends only on the left lines. If we should succeed in factorizing  $\bar{A}_{k+1}^n$  at  $\sigma_{ij}$ , the resultant amplitude describing the process of the right blob of Figure 2 will depend explicitly on the  $\pi$  matrices. If we wish to perform any further factorization, Eq.(5)

may be used to factorize the  $\Pi$  matrices. Thus, the  $\Pi$ -matrices with their property Eq.(5) allow factorization of crossed variables in any channel. In the derivation of the twist relation and the three-point vertex in the following sections, we will demonstrate explicitly how the  $\Pi$  matrices come into the nonlinear theory.

We now introduce one more operator in the six-dimensional space, the inversion operator, defined by

$$I_{\alpha}^{\beta} = \begin{bmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \quad (11)$$

In first and second factorization, we introduced 6-vectors  $P'_{\alpha}$  given by

$$\begin{aligned} P'_{\alpha} &= -P_{\alpha} & \text{for } 0 \leq \alpha \leq 3 \\ P'_{\alpha} &= P_{\alpha} & \text{for } \alpha = 4, 5 \end{aligned} \quad (12)$$

Now using the inversion operator (11), we can write

$$P'_{\alpha} = I_{\alpha}^{\beta} P_{\beta} \quad (13)$$

For the studies of the twist relation and the three-point vertex which follow, it will also be convenient to define

another matrix  $\Pi'_{\alpha}{}^{\beta}(\bar{P})$  given by

$$\Pi'_{\alpha}{}^{\beta}(\bar{P}) = I_{\alpha}^{\gamma} \Pi_{\gamma}{}^{\beta}(\bar{P}) \quad (14)$$

### III. TWIST RELATION

The "twisted" propagator was first introduced by Kikkawa, Sakita, and Virasoro<sup>3</sup> as a necessary element for the construction of the most general higher order diagram. In this section, we will first briefly indicate how the "twisted" propagator enters into a dual theory. We will then indicate why a twist relation should exist as a consequence of the symmetry of the original amplitude. Finally, we will derive the twist relation in the case of the dual multiparticle theory with nonlinear trajectories.

The physical amplitude for  $N$  identical particles is symmetric under the permutation of any number of particles. This property must also be present in the Born terms of a crossing symmetric theory. The  $N$ -point tree graphs of the dual theory have cyclic symmetry and reflection symmetry. The entire  $N$ -point Born term is then given as the sum of all tree graphs not related by cyclic or reflection symmetry. If one starts with a cyclically symmetric tree graph

$$B_N (p_1 p_2 \dots p_{l-1} p_l p_{l+1} \dots p_k p_{k+1} \dots p_N) ,$$

one way of arriving at a tree graph with a different ordering of external lines is to invert the ordering of a subset of external lines, i.e. we make the inversion transformation

$$p_{k+i} \rightarrow p_{l-i} \quad \text{for } i = 0, 1, \dots, l-k \text{ and where } k < l. \text{ This}$$

$$\text{gives the amplitude } B_N (p_1 p_2 \dots p_{l-1} p_l p_{l-1} \dots p_k p_{k+1} p_{k+2} \dots p_N) .$$

One can arrive at any ordering of external lines by a finite number of such transformations (each time inverting a different subset of external lines). For the particular case  $l = N$ ,

the version operation transforms the amplitude  $B_N(p_1 \dots p_R, p_{R+1} \dots p_N)$  [A] depicted in Figure 3a into the amplitude

$B_N(p_1 \dots p_R, p_N \dots p_{R+1})$  [B] depicted in Figure 4a. Because of the cyclic symmetry of  $B_N$ ,

any inversion operation with  $l < N$  is equivalent to an inversion with  $l = N$ . The reason for introducing the inversion operation is that the higher spin amplitudes obtained by factorizing the tree graphs (A) and (B) at  $\sigma_{1k}$  are linearly related. To be more specific, suppose (A) and (B) factorize at  $\sigma_{1k}$  then, we have, as in Eq. (10)

$$B_N(p_1 \dots p_R, p_{R+1} \dots p_N) \sim \frac{R(p_1 \dots p_R, p_{R+1} \dots p_N)}{\sigma_{1k} - \sigma_0}$$

where

$$R(p_1 \dots p_R, p_{R+1} \dots p_N) = \sum_{n=1}^d \bar{A}_{R+1}^n(p_1 \dots p_R) \bar{B}_{N-R+1}^n(p_{R+1} \dots p_N) \quad (15)$$

and

$$B_N(p_1 \dots p_R, p_N \dots p_{R+1}) \sim \frac{R(p_1 \dots p_R, p_N \dots p_{R+1})}{\sigma_{1k} - \sigma_0}$$

where

$$R(p_1 \dots p_R, p_N \dots p_{R+1}) = \sum_{n=1}^d \bar{A}_{R+1}^n(p_1 \dots p_R) \bar{B}_{N-R+1}^n(p_N \dots p_{R+1}) \quad (16)$$

we will show there exists a  $d \times d$  matrix  $\Omega^{nm}$

depending only on  $\bar{P}_{R+1, N} = p_{R+1} + p_{R+2} + \dots + p_N$

such that

$$\bar{B}_{N-k+1}^n(p_N \cdots p_{k+1}) = \sum_{m=1}^d \Omega^{nm}(\bar{P}_{k+1, N}) \bar{B}_{N-k+1}^m(p_{k+1} \cdots p_N) \quad (17)$$

Substituting Eq.(17) into Eq.(16), we have

$$B_N(p_1 \cdots p_R p_{k+1} \cdots p_N) = \sum_{m, n=1}^d \bar{A}_{k+1}^n(p_1 \cdots p_R) \frac{\Omega^{nm}}{\sigma_{1k} - \sigma_0} \bar{B}_{N-k+1}^m(p_{k+1} \cdots p_N) \quad (18)$$

Comparing Eq.(18) to Eq.(15), we see that the transformation

of one diagram to another with a different ordering may be carried out by inserting the new propagator  $\frac{\Omega^{nm}}{\sigma_{1k} - \sigma_0}$

in place of  $\frac{1}{\sigma_{1k} - \sigma_0}$ . The relation Eq.(17)

is known as the twist relation.  $\frac{\Omega^{nm}}{\sigma_{1k} - \sigma_0}$  is commonly

called the twist operator, and  $\frac{\Omega^{nm}}{\sigma_{1k} - \sigma_0}$  is

the twisted propagator. The insertion of the twisted propagator in higher order Feynman-like diagrams produce a similar effect of generating other diagrams. The twist relation Eq.(17) is a consequence of the symmetry properties of  $B_N$ . To see this, we consider for simplicity the special case  $k = N-k$ , i.e. we have the same number of lines on the left and right of  $\sigma_{1k}$  in Figure 3.

The amplitude  $B_N(p_1 \cdots p_R p_{k+1} \cdots p_N)$  has cyclic symmetry and reflection symmetry. This implies that the residue  $R(p_1 \cdots p_R p_{k+1} \cdots p_N)$  in Eq.(15) has the following symmetry properties:

(i) Reflection symmetry

$$R(p_1 p_2 \cdots p_N) = R(p_N \cdots p_1) \quad (19)$$

(ii) The special cyclic symmetry

$$R(p_1 \cdots p_R p_{k+1} \cdots p_N) = R(p_{k+1} \cdots p_N p_1 \cdots p_R)$$

Suppose

$$\bar{A}_{k+1}^n(p_1 \cdots p_R)$$

in Eq.(15) are the following linear combinations of  $\bar{B}_{k+1}^m(p_1 \dots p_k)$ :

$$\bar{A}_{k+1}^n(p_1 \dots p_k) = \sum_{m=1}^d \bar{B}_{k+1}^m(p_1 \dots p_k) N^{mn}(\bar{P}_{1k}) \quad (21)$$

where the  $d \times d$  matrix  $N^{mn}(\bar{P}_{1k})$

depends only on  $\bar{P}_{1k} = p_1 + p_2 + \dots + p_k$ ,

and has the property

$$N^{mn}(\bar{P}_{1k}) = N^{nm}(-\bar{P}_{1k}) \quad (22)$$

then, from Eqs.(21) and (15), we have

$$R(p_1 \dots p_k p_{k+1} \dots p_N) = \sum_{m,n=1}^d \bar{B}_{k+1}^m(p_1 \dots p_k) N^{mn}(\bar{P}_{1k}) \bar{B}_{N-k+1}^n(p_{k+1} \dots p_N). \quad (23)$$

The reflection symmetry property Eq.(19) of R follows

immediately from the representation (23) of R and the

property (22) of  $N^{mn}$ . Likewise, if  $\bar{A}_{k+1}^n(p_1 \dots p_k)$

in Eq. (15) are the following linear combinations

of  $\bar{B}_{k+1}^m(p_1 \dots p_k)$ :

$$\bar{A}_{k+1}^n(p_1 \dots p_k) = \sum_{m=1}^d \bar{B}_{k+1}^m(p_1 \dots p_k) (\Omega N)^{mn}(\bar{P}_{1k}) \quad (24)$$

where the  $d \times d$  matrix  $(\Omega N)^{mn}(\bar{P}_{1k})$  depends

only on  $\bar{P}_{1k}$ , and has the property

$$(\Omega N)^{mn}(\bar{P}_{1k}) = (\Omega N)^{nm}(-\bar{P}_{1k}) \quad (25)$$

then, from Eqs.(24) and (15), we have

$$R(p_1 \dots p_k p_{k+1} \dots p_N) = \sum_{m,n=1}^d \bar{B}_{k+1}^m(p_1 \dots p_k) (\Omega N)^{mn}(\bar{P}_{1k}) \bar{B}_{N-k+1}^n(p_{k+1} \dots p_N) \quad (26)$$

The special cyclic symmetry Eq.(20) of R follows from the

representation (26) of  $R$  and the property (25) of  $(\Omega N)^{mn}(\bar{P}_{1k})$ .

Furthermore, from Eqs.(21) and (24), we have

$$\bar{B}_{k+1}^n(p_1 \dots p_k) = \sum_{m=1}^d \bar{B}_{k+1}^m(p_1 \dots p_k) \Omega^{mn}(\bar{P}_{1k}) \quad (27)$$

where

$$\Omega^{mn} = \sum_{i=1}^d (\Omega N)^{mi} (N^{-1})^{in}$$

From the properties of  $N^{mn}$  and  $\Omega N^{mn}$  in Eqs. (22) and (25), it follows that

$$\Omega^{mn}(\bar{P}_{1k}) = \Omega^{nm}(-\bar{P}_{1k}). \quad (28)$$

Since  $\Omega^{mn}$  is a function only of  $\bar{P}_{1k} = p_1 + p_2 + \dots + p_k$ ,

Eq. (26) also implies

$$\bar{B}_{k+1}^n(p_1 \dots p_k) = \sum_{m=1}^d \bar{B}_{k+1}^m(p_k \dots p_1) \Omega^{mn}(\bar{P}_{1k}) \quad (29)$$

Substituting Eq.(29) into Eq.(27), we see that

$$\sum_{i=1}^d \Omega^{mi} \Omega^{in} = \delta^{mn}$$

Finally, if we perform the transformation

$$p_i \rightarrow p_{k+i}, \quad i=1, 2, \dots, k \quad \text{on Eq. (26) using}$$

the property (28) of  $\Omega^{mn}$ , we arrive immediately at the twist relation of Eq.(17). In Eq.(22) of I, we derived a factorized form of the  $N$ -point tree graph of the nonlinear model which has the reflection

symmetric form of Eq.(23). The N matrix, which is  $N^{\alpha_1, \alpha_2 \dots \alpha_L, \beta_1, \beta_2 \dots \beta_L}$  defined in Eq.(19) of I, is independent of  $\bar{P}_{ik}$  and depends only on the parameter  $g$ . It is a symmetric matrix, consistent with Eq.(22). The twist operator we will derive will have explicitly the properties of Eqs.(28) and (30). It is interesting that  $\Omega^{mn}$  of the theory depends on  $\bar{P}_{ik}$ , but is independent of the parameter  $g$ .

The various different factorized forms of the tree graphs are summarized in Figs. 3 and 4. The three figures, 3a, 3b, and 3c, are different representations of the amplitude

$$B_N(p_1 \dots p_k p_{k+1} \dots p_N).$$

Figures 3a and 3b have poles at different locations, but are related by duality. Figure 3a corresponds to the reflection symmetric form of Eq.(23), whereas Figure 3b corresponds to the special cyclically symmetric form of Eq.(26).

Figure 3c is by definition equal to Figure 3b. In the form of Figure 3c, it is clear why  $\Omega^{mn}$  is called the twist operator. Figure 4 gives the corresponding representations for the amplitude  $B_N(p_1 \dots p_k p_N \dots p_{k+1})$ .

It is clear that all graphs in Figure 4 can be built out of different combinations of the same propagators and excited amplitudes occurring in Figure 3.

We now go on to derive the twist relation in the dual model with nonlinear trajectories. The  $N$ -point amplitude with one excited particle, corresponding to Figure 5, is given by<sup>1</sup>

$$B_N^{\alpha_1, \alpha_2, \dots, \alpha_l}(p_1 \dots p_N) = \sum_{n_i | l} B_N(\sigma_{ij} \delta^{\sum_{r=1}^{j-1} n_r}) \frac{P_{11}^{\alpha_1} \dots P_{11}^{\alpha_{n_1}}}{f(n_1)} \frac{P_{12}^{\alpha_{n_1+1}} \dots P_{12}^{\alpha_{n_1+n_2}}}{f(n_2)} \dots \frac{P_{1N-2}^{\alpha_{2+n_1+n_2}} \dots P_{1N-2}^{\alpha_2}}{f(n_{N-2})} \quad (31)$$

The 4-vectors  $\bar{P}_{i\mu}$  associated with  $P_{i\alpha}$  of Eq. (31)

are related to the external momenta  $p_j$  by

$$\bar{P}_{i\mu} = (p_1 + p_2 + \dots + p_i)_\mu \quad 0 \leq \mu \leq 3 \quad (32)$$

The twisted amplitude is given by making the transformation

$$p_i \rightarrow p_{N-i} \quad i = 1, 2, \dots, N-1$$

Thus, corresponding to Figure 6, we have the twisted amplitude

$$B_N^{\alpha_1, \alpha_2, \dots, \alpha_l}(p_{N-1} \dots p_1) = \sum_{m_i | l} B_N(\sigma_{ij} \delta^{\sum_{k=i+1}^j m_k}) \frac{P_{N-1N-1}^{\alpha_1} P_{N-1N-1}^{\alpha_2} \dots P_{N-1N-1}^{\alpha_{m_{N-1}}}}{f(m_{N-1})} \\ \times \frac{P_{N-2N-1}^{\alpha_{m_{N-1}+1}} \dots P_{N-2N-1}^{\alpha_{m_{N-1}+m_{N-2}}}}{f(m_{N-2})} \dots \frac{P_{2N-1}^{\alpha_{l-m_{N-1}+1}} \dots P_{2N-1}^{\alpha_l}}{f(m_2)} \quad (33)$$

where

$$\bar{P}_{iN-1\mu} = (p_i + p_{i+1} + \dots + p_{N-1})_\mu \quad 0 \leq \mu \leq 3 \quad (32')$$

Now, by energy-momentum conservation

$$\begin{aligned} \overline{P}_{i+1 \dots N-1 \mu} &= (p_{i+1} + \dots + p_{N-1})_{\mu} = -(p_N + p_1 + p_2 + \dots + p_i) \\ &= -(p_N + \overline{P}_{i2})_{\mu} \quad 0 \leq \mu \leq 3 \end{aligned} \quad (34)$$

Therefore, by Eqs.(12), (13), (6), and (14) of the previous section, we have

$$\begin{aligned} P_{i+1 \dots N-1 \alpha} &= I_{\alpha}^{\beta} \Pi_{\beta}^{\gamma}(p_N) P_{i2 \gamma} \\ &= \Pi'_{\alpha}{}^{\gamma} P_{i2 \gamma} \quad 0 \leq \alpha, \beta, \gamma \leq 5 \end{aligned} \quad (35)$$

Applying Eq. (35) to every vector  $P_{i+1 \dots N-1}$  in Eq. (32), and making the trivial change  $m_i \rightarrow n_{i-1}$ , we have

$$B_N^{(\alpha_1 \alpha_2 \dots \alpha_L)}(p_{N-1} \dots p_1) = B_N^{(\beta_1 \beta_2 \dots \beta_L)}(p_1 \dots p_{N-1}) \Omega_{\beta_1 \beta_2 \dots \beta_L}^{\alpha_1 \alpha_2 \dots \alpha_L}(p_N) \quad (36)$$

where the twist operator  $\Omega$  is given by

$$\Omega_{\beta_1 \beta_2 \dots \beta_L}^{\alpha_1 \alpha_2 \dots \alpha_L}(p_N) = \Pi'^{\alpha_1}_{\beta_1}(p_N) \Pi'^{\alpha_2}_{\beta_2}(p_N) \dots \Pi'^{\alpha_L}_{\beta_L}(p_N), \quad (37)$$

Eq.(36) is the twist relation of the theory. We verify easily that the twist operator  $\Omega_{\beta_1 \beta_2 \dots \beta_L}^{\alpha_1 \alpha_2 \dots \alpha_L}$

satisfy the properties Eqs.(28) and (29).

#### IV. THIRD FACTORIZATION AND THE GENERAL THREE-POINT VERTEX

By second factorization, we arrived at the N-point

Born term with two nonzero spins given by Eq.(34) of I.

$$B_N^{(\beta_1 \beta_2 \dots \beta_L)}{}^{\alpha_1 \alpha_2 \dots \alpha_L}$$

denotes the amplitude for the process in Fig. 7. We wish to obtain the general vertex with three nonzero spins. To do this, we note that Fig. 7 is equivalent to Fig. 8 by duality. Thus, to extract the general vertex, we need only to consider factorization at  $\sigma_2^{N-1}$  as in Figure 9.

The procedure of third factorization is identical to first and second factorization. First, to exhibit the pole  $\sigma_2^{N-1}$ , we sum over the index  $n_2^{N-1}$ . We then separate all the remaining kinematic variables and summation indices into three groups, the "left" group, the "right" group, and the "crossed" group. Finally, we apply the factorization of the crossed polynomial as in Eqs.(20) and (49) of I to bring about total factorization.

From Eq.(34) of I, we see that there are three kinds of kinematic variables appearing in  $B_N^{\beta_1 \beta_2 \dots \beta_{N-2} \alpha_1 \alpha_2 \dots \alpha_{N-2}}$  i.e.  $\sigma_{ij}$  ( $1 \leq i < j \leq N-1$ ),  $P_i^\alpha$  ( $1 \leq i \leq N-2$ ) and  $P_i'^\beta$  ( $2 \leq i \leq N-1$ ).

The  $\sigma$  variables may be grouped into the "left", the "right" and the "crossed" sets as in first and second factorization.  $P_1^\alpha$  and  $P_{N-1}'^\beta$  carry the momenta and spin indices of the right hand side of Figure 9. They are therefore "right" variables. All other six-vectors are crossed variables. The tensor

$$N^{-1} \beta_1 \beta_2 \dots \beta_{N-2} \alpha_1 \alpha_2 \dots \alpha_{N-2}$$

in Eq.(34) of I is independent of momentum, but it carries

the spin indices of the spinning particles on the right hand side of Figure 9. It may therefore be classified as a "right variable."

The crossed  $\sigma$  variables may be decomposed into a scalar product of two six-vectors by Eq. (9) of the present paper. The  $P_i^\alpha$  and  $P_i'^\beta$  6-vectors  $2 \leq i \leq N-2$  may be decomposed, using Eqs (6), (13) and (14) of the present paper. Noting that the coefficients that enter the crossed polynomial are just those of C in Eq. (14) of I, we arrive at once at the factorized form of  $B_N^{\beta_1 \beta_2 \dots \beta_{N-2} \alpha_1 \alpha_2 \dots \alpha_{N-2}}$  given by

$$B_N^{\beta_1 \beta_2 \dots \beta_{N-2} \alpha_1 \alpha_2 \dots \alpha_{N-2}} = \sum_{l''=0}^{\infty} B_{N-1}^{\delta_1 \delta_2 \dots \delta_{N-1} l''} \frac{N_{\delta_1 \delta_2 \dots \delta_{N-1} l''} \delta_{\alpha_1 \beta_1} \dots \delta_{\alpha_{N-2} \beta_{N-2}}}{G(\sigma_{2N-1} l'')} \tilde{V}^{\delta \beta \alpha} \quad (38)$$

where  $B_{N-1}^{\delta_1 \delta_2 \dots \delta_{N-1} l''}$  is just the (N-1)-point amplitude with one excited particle, corresponding to the left hand side of Figure 9, and given by Eq. (31) of the present paper. The general three-point vertex is given by

$$\begin{aligned} \tilde{V}^{\delta \beta \alpha} = & \sum (N-1 \beta_1 \beta_2 \dots \beta_d \alpha_1 \alpha_2 \dots \alpha_d) \left( \frac{P_1^{\alpha_{d+1}} P_1^{\alpha_{d+2}} \dots P_1^{\alpha_{d+n_2}}}{f(n_2)} \right) \\ & \times \left( \frac{P_{N-1}'^{\beta_{d+1}} P_{N-1}'^{\beta_{d+2}} \dots P_{N-1}'^{\beta_{d+n_1}}}{f(d_{N-1})} \right) \left( \frac{P_1^{\delta_{v_3+1}} P_1^{\delta_{v_3+2}} \dots P_1^{\delta_{v_3+u_2}}}{f(v_2)} \right) \\ & \times (N-1 \beta_2 \dots \beta_{v_3+1} \dots \beta_{v_3} \delta_1 \dots \delta_{v_3} \pi_{\delta_1}^{\delta_{v_3}}(P_1) \pi_{\delta_2}^{\delta_{v_3-1}}(P_1) \dots \pi_{\delta_{v_3}}^{\delta_1}(P_1)) \end{aligned}$$

$$\begin{aligned}
& \times (N^{-1} \alpha_{d+n_2+1} \cdots \alpha_l \rho_1 \rho_2 \cdots \rho_{v_1} \prod_{\rho_1}^{\delta_{l''-v_1+1}}(P_1) \prod_{\rho_2}^{\delta_{l''-v_1+2}}(P_1) \cdots \prod_{\rho_{v_1}}^{\delta_{l''}}(P_1)) \\
& \times \sum_{\delta}^{n_2 d_{N-1} + n_2 v_3 + n_2 v_2 + d_{N-1} v_1 + d_{N-1} v_2 + d v_2}
\end{aligned}
\tag{39}$$

where the sum is performed over  $d, d_{N-1}, n_2, v_1, v_2, v_3$  subject to the conditions

$$\begin{aligned}
d + n_2 + v_1 &= l \\
d + d_{N-1} + v_3 &= l' \\
v_1 + v_2 + v_3 &= l''
\end{aligned}
\tag{40}$$

(39)

The vertex/is not symmetric with respect to the three particles. The reason is that we had extracted the vertex from an unsymmetric factorized form of the tree graph corresponding to Figure 10. It is relatively easy to see that by the first, second, and third factorization procedures up to this point, we have arrived at a factorized form of the tree graph in which the left and upper middle blobs of Figure 10 are related by the cyclic transformation of Eq.(20) but they are in turn related to the right blob by the reflection symmetry of Eq.(19). Thus, the three-point vertex which links the three blobs is naturally unsymmetric. In order to have a cyclically symmetric vertex,

we want the right blob in a form which is related to the left and upper middle blobs by the cyclic transformation of Eq.(20). This can be achieved by performing a twist on the right hand blob. The symmetric form obtained by twisting the right-hand graph is graphically represented in Figure 11.

We wish now to derive the symmetric three-point vertex  $V^{\delta\beta\alpha}$ . Let us consider only the lower right-hand corner of Figures 10 and 11. Corresponding to Figure 10, we have

$$\sum_{l=0}^{\infty} V^{\delta\beta\alpha} \frac{N_{\alpha_1, \alpha_2, \dots, \alpha_l} \delta_1 \delta_2 \dots \delta_l}{G(\sigma_{k+1} N g^l)} B_N^{\delta_1 \delta_2 \dots \delta_l} (P_N P_{N-1} \dots P_{k+1}) \quad (41)$$

Corresponding to Figure 7, we have

$$\sum_{l=0}^{\infty} V^{\delta\beta\alpha} \frac{N_{\alpha_1, \alpha_2, \dots, \alpha_l} \delta_1 \delta_2 \dots \delta_l}{G(\sigma_{k+1} N g^l)} B_N^{\delta_1 \delta_2 \dots \delta_l} (P_{k+1} P_{k+2} \dots P_N) \quad (42)$$

The vertex  $V^{\delta\beta\alpha}$  can be found by requiring that the expressions (41) and (42) be equal.

We will first perform the twist on  $B_N^{\delta_1 \delta_2 \dots \delta_l} (P_N \dots P_{k+1})$

in Eq.(41). Eqs.(36) and (37) give us the operator which twists a "left" graph. We want the operator which twists a "right" graph. Eqs.(20) and (17) allow us to go from the twist relation for a "left" graph to the twist relation for a "right" graph. Thus, we can write down

$$B_N^{\delta_1 \delta_2 \dots \delta_l} (P_N \dots P_{k+1}) = \Omega \frac{\delta_1 \delta_2 \dots \delta_l}{\beta_1 \beta_2 \dots \beta_l} (-\bar{P}_{k+1} N) B_N^{\beta_1 \beta_2 \dots \beta_l} (P_{k+1} \dots P_N). \quad (43)$$

Substituting the twist relation of Eq.(43) in (41), and equating (41) with (42), we have

$$V^{\delta\beta\alpha} N_{\alpha_1\alpha_2\cdots\alpha_\ell} \delta_1\cdots\delta_\ell = \tilde{V}^{\delta\beta\alpha} N_{\alpha_1\alpha_2\cdots\alpha_\ell} p_1 p_2 \cdots p_\ell \Omega_{\delta_1\delta_2\cdots\delta_\ell}^{p_1 p_2 \cdots p_\ell} (-\bar{P}_{R+1} N) \quad (43)$$

where the twist operator  $\Omega(-\bar{P}_{R+1} N)$  is defined in Eq.(37). It can be easily proved that the twist operator commutes with the tensor  $N$  i.e.,

$$N_{\alpha_1\alpha_2\cdots\alpha_\ell} p_1 p_2 \cdots p_\ell \Omega_{\delta_1\delta_2\cdots\delta_\ell}^{p_1 p_2 \cdots p_\ell} = \Omega_{\alpha_1\alpha_2\cdots\alpha_\ell}^{p_1 p_2 \cdots p_\ell} N_{\delta_1\delta_2\cdots\delta_\ell} \quad (44)$$

Using Eq.(44) in Eq.(43), we have

$$V^{\delta\beta\alpha} = \tilde{V}^{\delta\beta\delta} \Omega_{\delta_1\delta_2\cdots\delta_\ell}^{\alpha_1\alpha_2\cdots\alpha_\ell} \quad (45)$$

Substituting into Eq.(45), the expression  $\tilde{V}^{\delta\beta\delta}$  in Eq.(39), we arrive finally at the general symmetric three-point vertex corresponding to Figure 12 given by

$$\begin{aligned} V^{\delta\beta\alpha} = & \sum_{c_i, d_i} \left( \prod_{p_1}^{\alpha_\ell} \prod_{p_2}^{\alpha_{\ell-1}} \cdots \prod_{p_{c_1}}^{\alpha_{\ell-c_1+1}} N^{-1} p_1 p_2 \cdots p_{c_1} \delta_1 \delta_2 \cdots \delta_{c_1} \right) \\ & \times \left( \prod_{\mu_1}^{\beta_{c_1}} \prod_{\mu_2}^{\beta_{c_1-1}} \cdots \prod_{\mu_{c_2}}^{\beta_{c_2-c_1+1}} N^{-1} \mu_1 \mu_2 \cdots \mu_{c_2} \alpha_1 \alpha_2 \cdots \alpha_{c_2} \right) \\ & \times \left( \prod_{\nu_1}^{\delta_{c_2}} \prod_{\nu_2}^{\delta_{c_2-1}} \cdots \prod_{\nu_{c_3}}^{\delta_{c_3-c_2+1}} N^{-1} \nu_1 \nu_2 \cdots \nu_{c_3} \beta_1 \beta_2 \cdots \beta_{c_3} \right) \\ & \times \frac{P_1^{\delta_{c_1+1}} P_1^{\delta_{c_1+2}} \cdots P_1^{\delta_{c_1+d_1}} P_2^{\alpha_{c_2+1}} P_2^{\alpha_{c_2+2}} \cdots P_2^{\alpha_{c_2+d_2}}}{f(d_1) f(d_2)} \\ & \times \frac{P_3^{\beta_{c_3+1}} P_3^{\beta_{c_3+2}} \cdots P_3^{\beta_{c_3+d_3}}}{f(d_3)} \\ & \times \delta^{(c_2 d_1 + c_3 d_2 + c_1 d_3 + d_1 d_2 + d_2 d_3 + d_3 d_1)} \quad (50) \end{aligned}$$

where  $\sum_{c_i, d_i}$  implies summation over  $c_1, c_2, c_3, d_1, d_2, d_3$

subject to the conditions

$$c_1 + c_2 + d_2 = l_1$$

$$c_2 + c_3 + d_3 = l_2$$

$$c_3 + c_1 + d_1 = l_3$$

47  
(51)

and  $\pi'^{\alpha}_{\beta}(P)$  is defined in  
Eqs. (14) and (11).

#### V. CONCLUSION

In this paper, we have derived the twist relation, and the general unsymmetric and symmetric vertices in the nonlinear dual model. While the existence of these objects is expected in any factorizable dual theory, it is interesting to note that both the results as well as the techniques used to arrive at these results are quite different in the Veneziano and in the nonlinear model.

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4. In the case  $\sigma = as + b$ ,  $b \neq 0$

we can define a 7-vector  $P_\alpha$  by

$$P_\alpha = \sqrt{2a} \bar{P}_\mu \quad \text{for } 0 \leq \alpha \leq 3$$

$$P_4 = 1$$

$$P_5 = \sigma = a\bar{P}^2 + b$$

$$P_6 = \sqrt{b}$$

A 7-dimensional metric is defined with the only nonvanishing elements given by

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = g_{45} = g_{54} = -g_{66} = 1$$

The  $\Pi$  matrices then become seven-dimensional matrices given by Eq.(4) plus the definition

$$\Pi_{6\alpha} = \Pi_{\alpha 6} = 0 \quad \text{for } \alpha \neq 5$$

then, the multiplicative property of the  $\Pi$  matrices, given by Eq.(5) still holds.

## FIGURE CAPTION

Figure 1. Symbolic factorization of the N-point tree graph with N spin-zero external particles at  $\sigma_{ik}$ .

Figure 2. Symbolic factorization of the k+1-point tree graph with k spin-zero external particle, and one "excited" external particle with momentum  $-\bar{P}_{ik}$  at a pole  $\sigma_{ij}$ .

Figure 3. Three different representations of the N-point amplitude

$$B_N(p_1 p_2 \dots p_k p_{k+1} \dots p_N)$$

described in the text.

Figure 4. Three different representations of the N-point amplitude

$$B_N(p_1 p_2 \dots p_k p_N p_{N-1} \dots p_{k+1})$$

described in the text.

Figure 5. Tree graph with N-1 spin-zero external lines having momenta  $p_1, p_2, \dots, p_{N-1}$

and another external line with momentum  $p_N$   
The momentum labelling  $\bar{P}_{11}, \bar{P}_{12}, \dots, \bar{P}_{1N-2}$

is used in the text.

- Figure 6. Tree graph which is related to the graph in Figure 5 by a twist.
- Figure 7. Tree graph with two external lines having nonzero spin.
- Figure 8. Tree graph related to the graph of Figure 7 by duality.
- Figure 9. Symbolic factorization of the graph in Figure 8 into an amplitude, "propagator," and a general vertex.
- Figure 10. Configuration of the tree graph from which the nonsymmetric general vertex is obtained by third factorization.
- Figure 11. Tree graph configuration obtained by twisting Figure 10. The symmetric general vertex is obtained by factorization of this graph.
- Figure 12. Symbolic form of the symmetric general vertex indicating the association of tensor indices and particles.

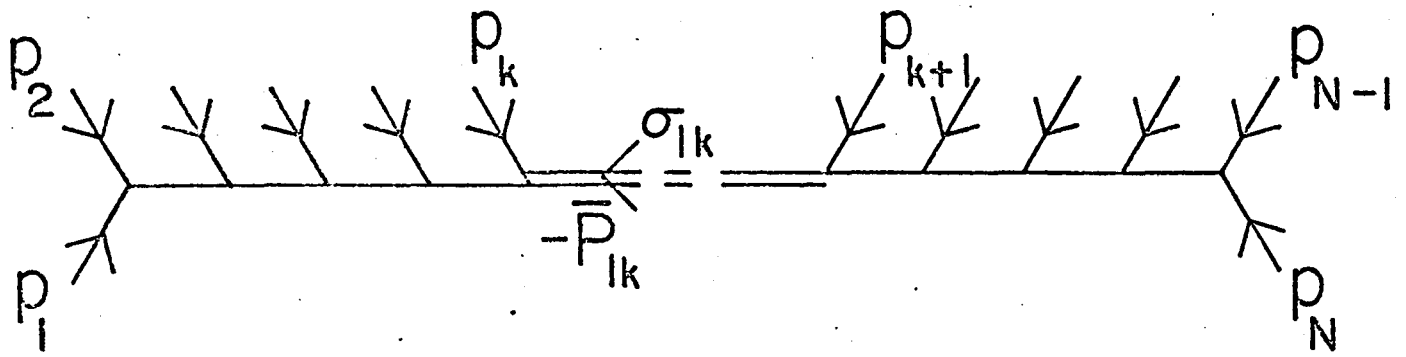


Figure 1

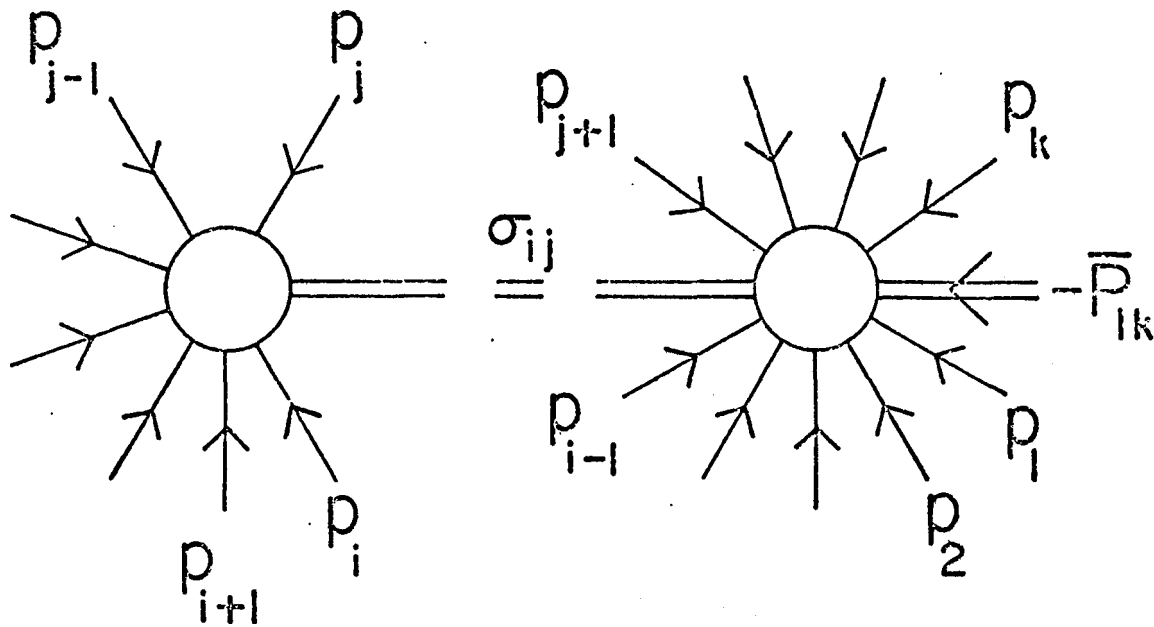
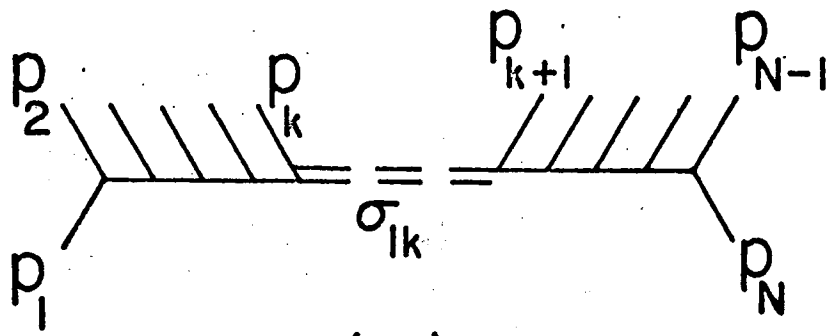
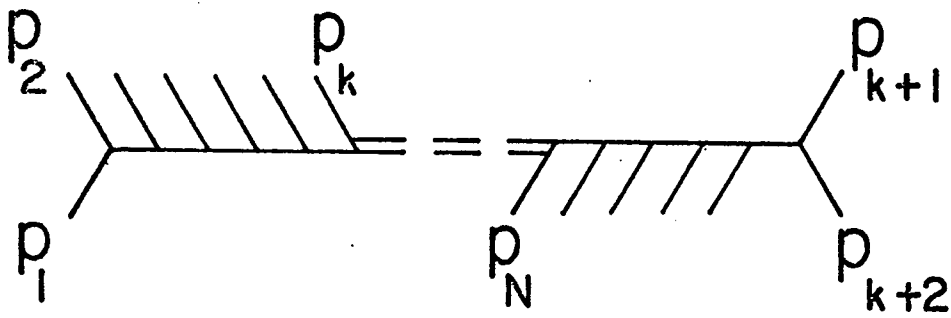


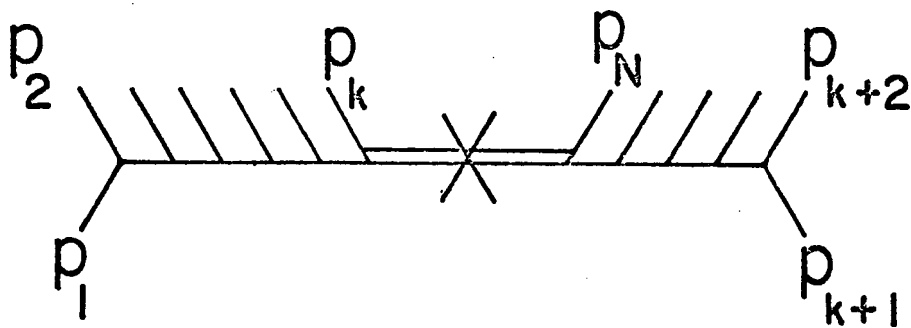
Figure 2



(a)

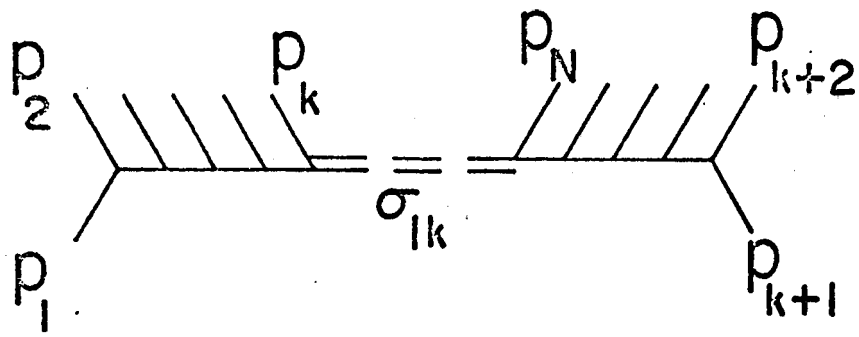


(b)

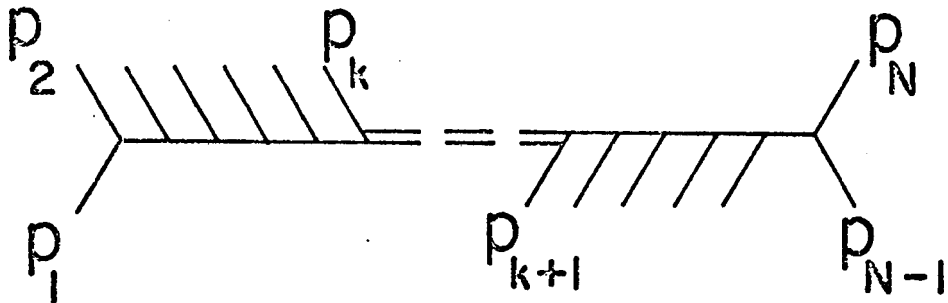


(c)

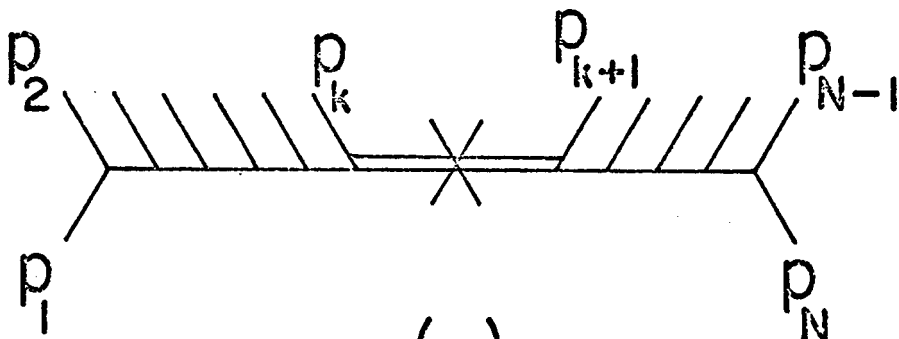
Figure 3



(a)



(b)



(c)

Figure 4

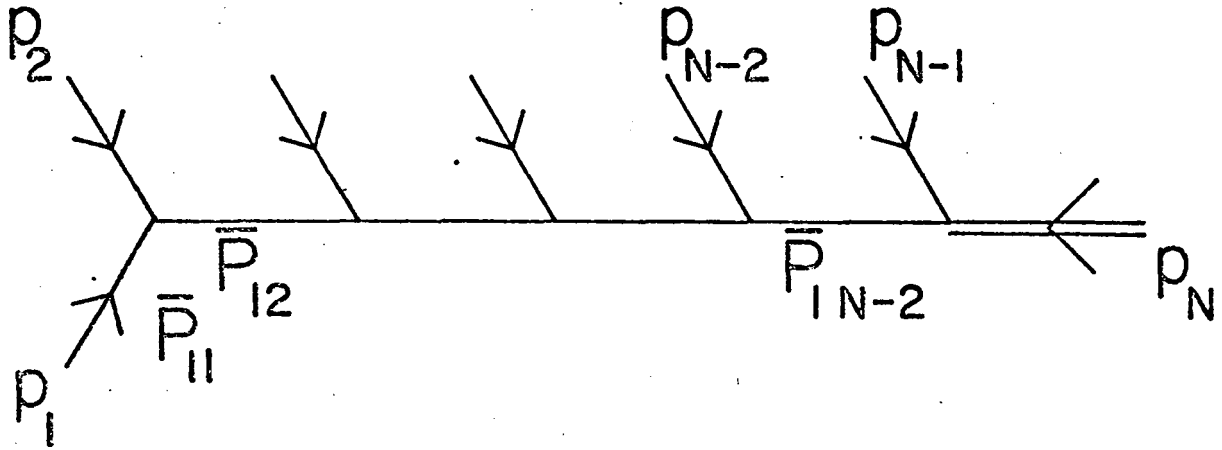


Figure 5

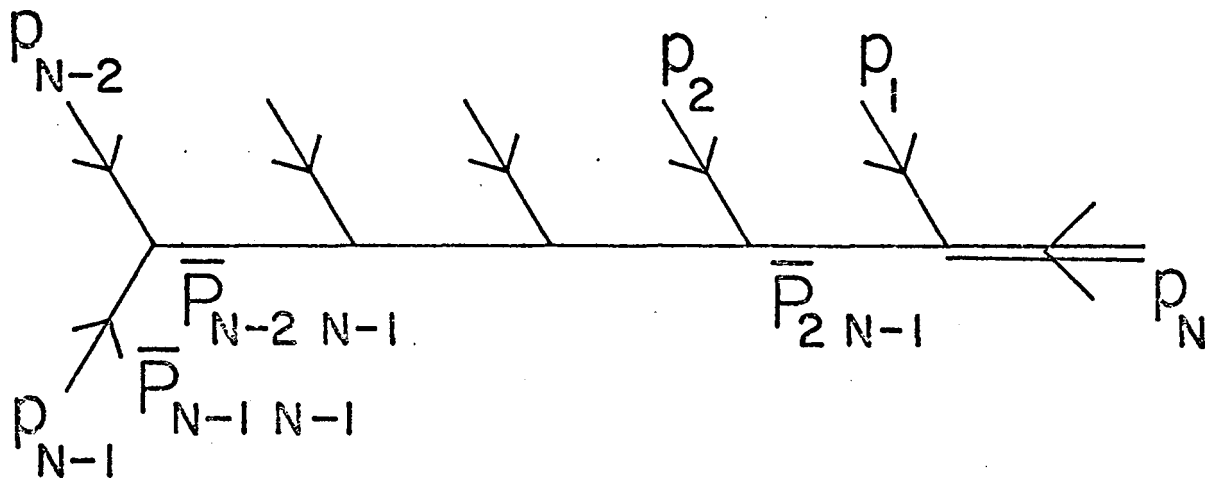


Figure 6

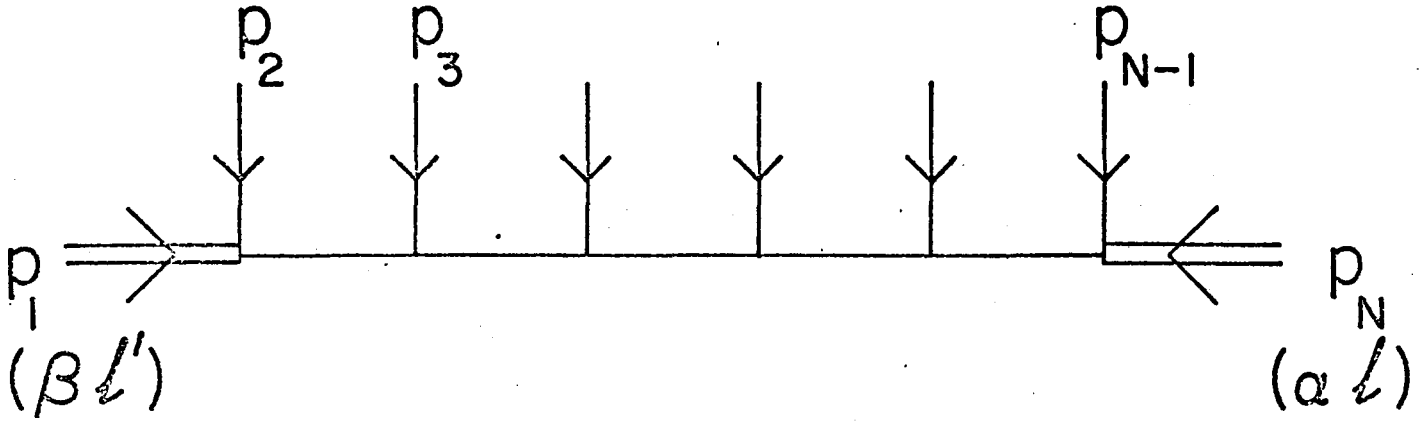


Figure 7

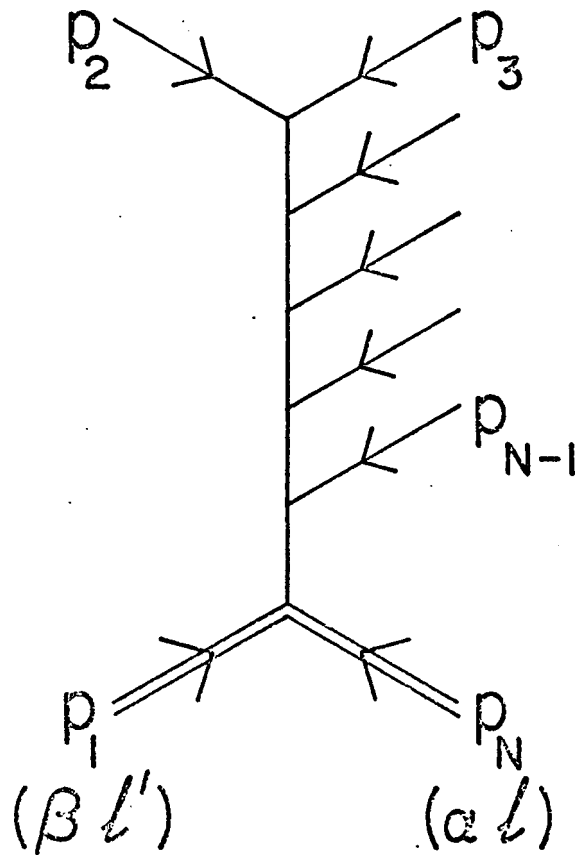


Figure 8

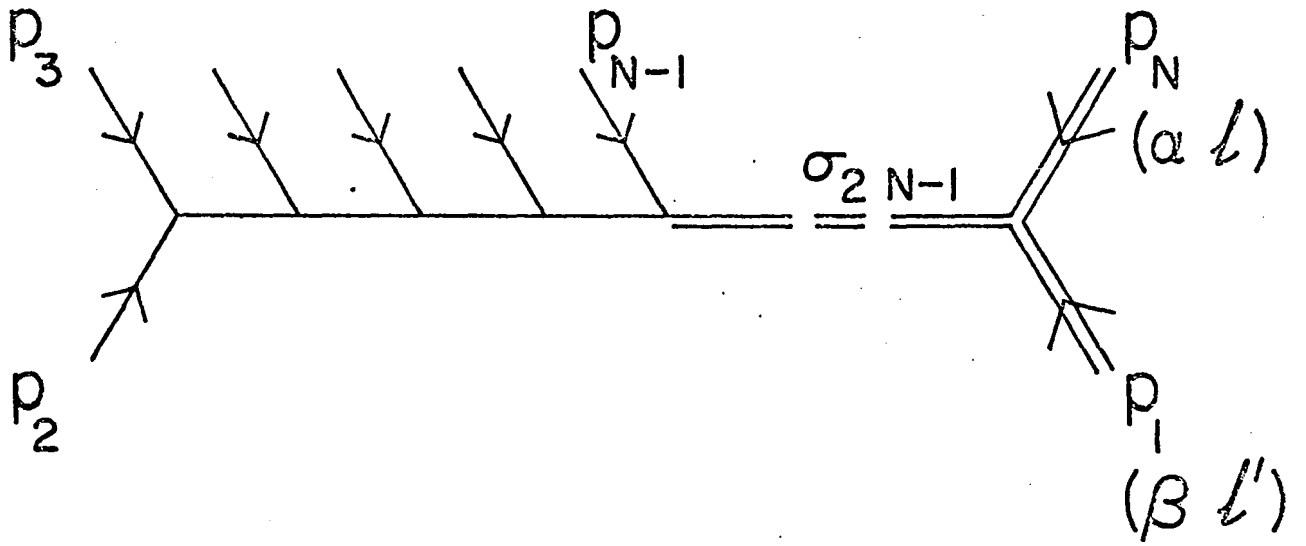


Figure 9

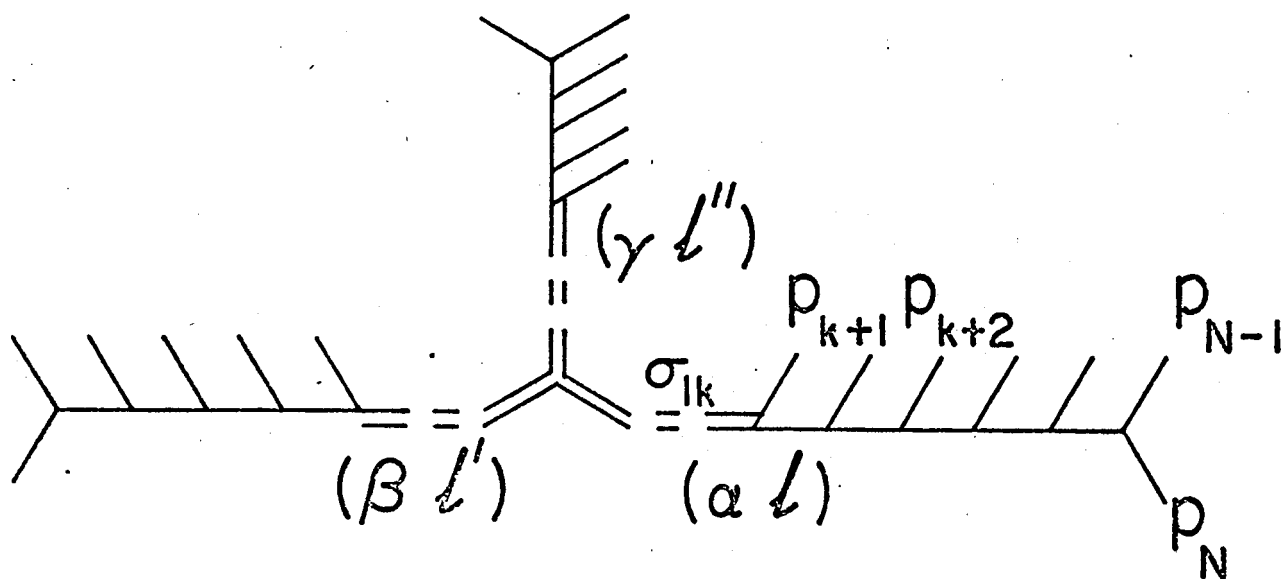


Figure 10

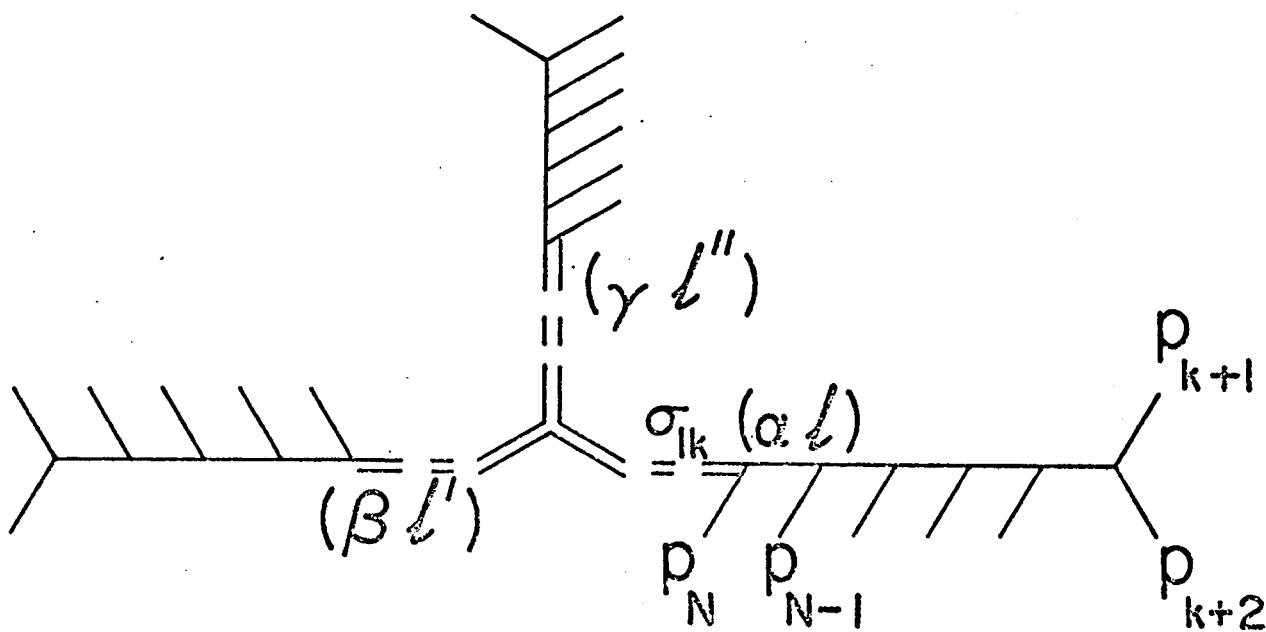


Figure 11

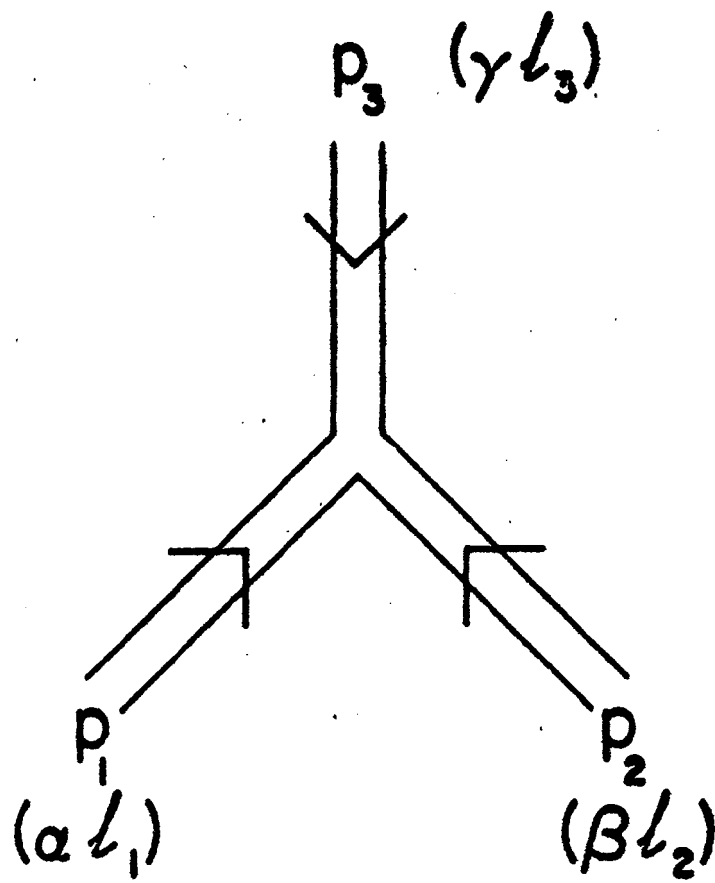


Figure 12