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APPLICATION OF STOCHASTIC DIFFERENTIAL GEOMETRY TO THE TERM STRUCTURE OF INTEREST RATES IN DEVELOPED MARKETS

YURI TARANENKO AND CHRIS BARNES
Los Alamos National Lab, XCM, MS-F645, Los Alamos, NM 87545, USA; email: yuri@lanl.gov

This paper deals with further developments of the new theory that applies stochastic differential geometry (SDG) to dynamics of interest rates. We examine mathematical constraints on the evolution of interest rate volatilities that arise from stochastic differential calculus under assumptions of an arbitrage free evolution of zero coupon bonds and developed markets (*i. e., none of the party/factor can* drive the whole market). The resulting new theory incorporates the Heath-Jarrow-Morton (HJM) model of interest rates and provides new equations for volatilities which makes the system of equations for interest rates and volatilities complete and self consistent. It results in much smaller amount of volatility data that should be guessed for the SDG model as compared to the HJM model. Limited analysis of the market volatility data suggests that the assumption of the developed market is violated around maturity of two years. Such maturities where the assumptions of the SDG model are violated are suggested to serve as boundaries at which volatilities should be specified independently from the model. Our numerical example with two boundaries (two years and five years) qualitatively resembles the market behavior. Under some conditions solutions of the SDG model become singular that may indicate market crashes. More detail comparison with the data is needed before the theory can be established or refuted.

1 Introduction

1.1 Dynamics of Forward Interest Rates in Heath-Jarrow-Morton Model

In the HJM model [*Heath, Jarrow, Morton*, 1992] forward interest rates $F(t,T)$ evolve according to

$$dF(t,T) = \sum_k \nu_k(t,T)\sigma_k(t,T)dt + \sum_k \sigma_k(t,T)dW^k \tag{1}$$

under the equivalent martingale probability measure. Where t is time at which one can contract for a loan at continuously compounded interest rate $F(t,T)$ starting at time T and maturing an instant later, dW^k are independent Brownian motions, $\sigma_k(t,T)$ are corresponding volatilities of forward rates,

$$\nu_k(t,T) = \int_t^T \sigma_k(t,u)du \tag{2}$$

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are corresponding volatilities of pure discount bonds that are given by

$$P(t,T) = \exp\{-\int_t^T F(t,s)ds\} \tag{3}$$

Heath, Jarrow, Morton [1992] have shown that under a simple non-singularity condition on the volatility functions $\sigma_k(t,T)$

$$\int_0^T \sigma_k^2(t,T)dt < +\infty \quad a.e. \quad Q \tag{4}$$

the equivalent martingale measure (for the pure discount bonds) is unique that guarantees unique prices of contingent claims if one can solve equations (1) for forward rates $F(t,T)$. That brings us to the following question.

1.2 What is Needed to Solve HJM Equations?

First, we have to provide the initial term structure of interest rates $F(0,T)$ and the initial term structure of volatility $\sigma_k(0,T)$ for all $0 \leq T \leq T_{max}$ where T_{max} is maximum maturity we are interested in. Such information can be obtained from market observations. Then we have to make some ***guess*** about the future dynamics of volatility. We want to underline that only after making an additional assumption about dynamics of interest rate volatilities $\sigma_k(t,T)$ for $t > 0$ we can solve equations for forward rates and then derive prices of contingent claims.

For example, if we assume that there is only one Wiener process driving the yield curve and that its volatility is independent of time and maturity, that is

$$\sigma(t,T) = \sigma_o = const \tag{5}$$

then we can solve equations (1) for a particular case of first arbitrage free model introduced by *Ho and Lee* [1986]. Let us give another example suggested by *Heath, Jarrow, Morton* [1992]

$$\sigma_k(t,T) = \sigma_k^{ln}(T-t) \times \min(F(t,T), \lambda) \tag{6}$$

where $\sigma_k^{ln}(T-t)$ is a function of time left to maturity that is for forward rate of given maturity this function is constant and λ is a positive constant. For small interest rates $\sigma_k^{ln}(T-t)$ is close to lognormal volatility of interest rates. When we examined market data for normalized perturbations of forward interest rates, that is

$$\frac{F(t+\Delta t,T) - F(t,T)}{F(t,T)\sqrt{(\Delta t)}} \tag{7}$$

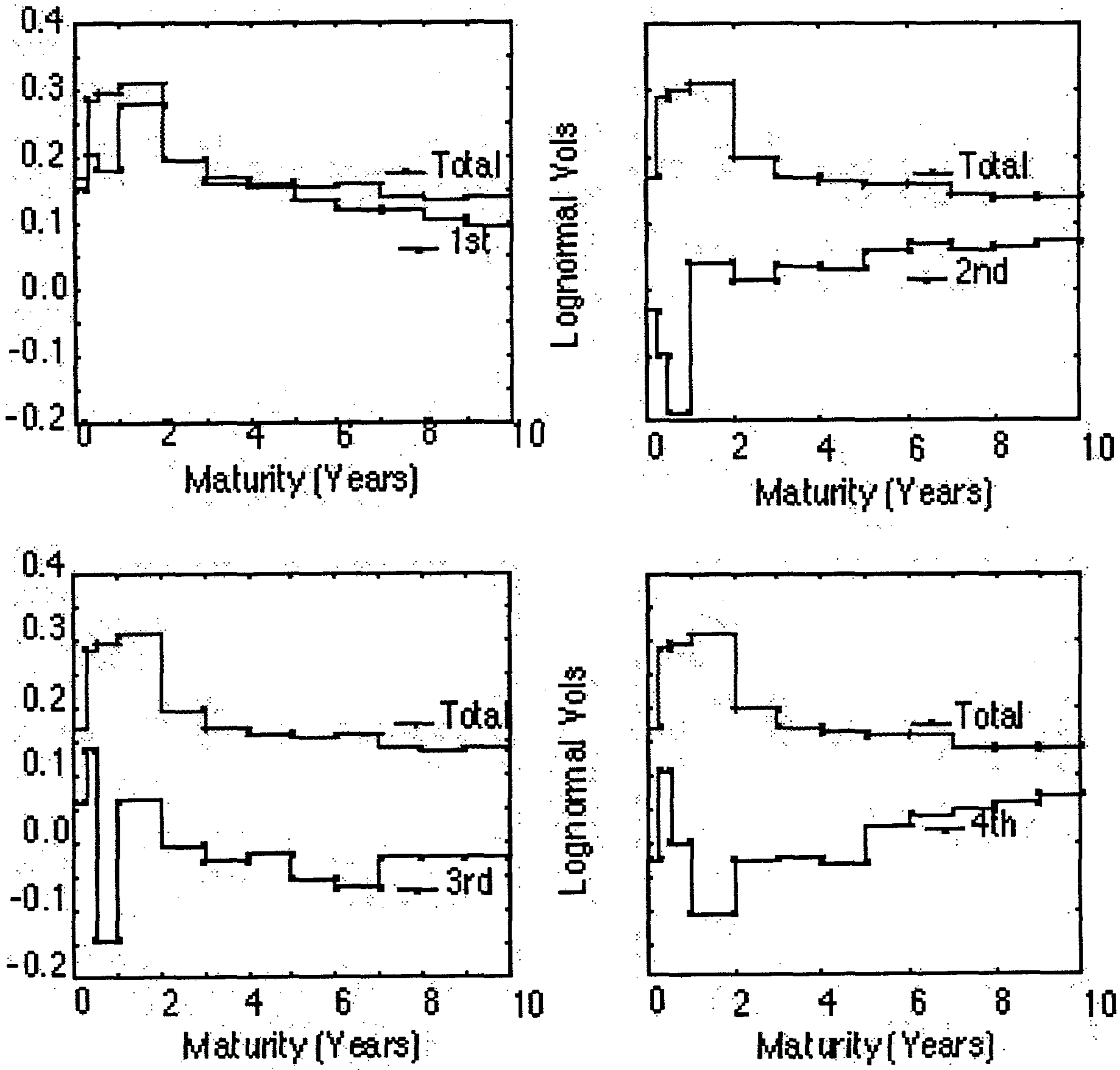

Figure 1: Principal components of lognormal volatilities.

we found that functions $\sigma_k^{ln}(T_{const})$ changes slowly with time showing normalized variations of up to 25 %-per-year. An example of principal components of lognormal volatilities of forward rates is shown in Figure (1).

1.3 How to Do Better on Volatilities?

The volatility assumption (6) discussed in the previous section is a good one to start with, but how can we do better than that? One of the ways to go is to use neural nets or other methods that predict future volatility variations without identifying driving forces for that or as an alternative for curious minds one can try to find fundamental principles that provide dynamic equations for volatility functions. In the rest of the paper we explore one of the latter possibilities.

2 Relating Stochastic Differential Geometry to Interest Rates

In this section we give qualitative introduction to stochastic differential geometry (SDG), show how we conform general SDG equations to the HJM model, and then discuss what is needed to solve them and what advantages they provide in comparison to HJM equations.

2.1 Qualitative Understanding of Stochastic Differential Geometry,

The goal of this section is to describe a Brownian motion on a curved manifold.

We consider an example of a two-dimensional manifold [*Makhankov et. al.*, 1995]. Let us consider a point X_1 on the surface and a patch of a tangential plane in this point, we denote it T_{X_1}. Then we proceed to a neighboring point X_2 and construct T_{X_2}. In such a way, we can cover the entire surface with such patches obtaining a polyhedron. Crucial points of this construction are: *i)* the surface is a manifold, *ii)* the covering is a Euclidean space. A Wiener process (a martingale)

$$dW^q dW^p = \delta^{qp} dt \tag{8}$$

is defined in a Euclidean space. This means that a Wiener process appears on the covering while a particle is moving on the surface. Now we should adjust both phenomena. Consider for that a surface with fluctuating forces (Wiener processes) and a particle in the point X_1 on the surface. This point also belongs to the covering T_{X_1}. Hence the particle undergoes a random shock

$$d\vec{X}_1 = \hat{\sigma}_1 d\vec{W}_1 \tag{9}$$

jumping to a point X_2 on the surface. In this new point it again undergoes a shock

$$d\vec{X}_2 = \hat{\sigma}_2 d\vec{W}_2 \tag{10}$$

and so forth. The matrix $\hat{\sigma}$ defines mobility of a particle. Here we should emphasize that all differentials considered in this section are of the Stratonovich's type, which allows us to use the standard differential calculus [*Gardiner*, 1994].

Now our task is to connect σ_1 and σ_2. Note for that that matrix $\hat{\sigma}$ being in fact a rotating operator, can be constructed of two vectors $\vec{j}_1$ and $\vec{j}_2$

$$\hat{\sigma} = \begin{pmatrix} j_1^x & j_2^x \\ j_1^y & j_2^y \end{pmatrix}$$

which gives a frame of reference for the patch considered. While moving from one patch to another, this frame changes its orientation. Let us recall that the total change of a vector, $\vec{B}$, due to moving from one point to another consists of two parts [*Dubrovin et. al.*, 1992]

$$\delta \vec{B} = d\vec{B} + \Gamma \vec{B} d\vec{X} \tag{11}$$

Where Γ is the connexion. The first term in (11) is the conventional differential

$$d\vec{B} = \frac{\partial \vec{B}}{\partial X^i} dX^i \tag{12}$$

and the second one allows for a change of the frame of reference. Since matrix $\hat{\sigma}$ is constructed of two vectors $\vec{j}_1$ and $\vec{j}_2$ it is transformed following the same rule,

$$\delta \hat{\sigma} = d\hat{\sigma} + \Gamma \hat{\sigma} d\vec{X} \tag{13}$$

Now we make an assumption of fair game or we also call it an assumption of developed markets, which means that none of a single party/factor can influence the whole market. In mathematical terms it means that the total change of $\hat{\sigma}$ should be zero

$$\delta \hat{\sigma} = 0 \tag{14}$$

that results in

$$d\hat{\sigma} = -\Gamma \hat{\sigma} d\vec{X} \tag{15}$$

what along with the equation for the elementary shock

$$d\vec{X} = \hat{\sigma} d\vec{W} \tag{16}$$

gives us the equations of *Stochastic Differential Geometry* on a curved manifold.

2.2 Fitting SDG Equations to HJM Model, [Makhankov et. al., 1995]

To have SDG equations that describe evolution of the interest rates and their volatilities we have to find the connexion Γ that satisfies the arbitrage free model (1) of interest rates. For the purpose of comparing SDG equations with HJM equations for forward interest rates we have to write them in the same form of stochastic differentials. We chose to write SDG equations in the Ito representation that will make them compatible with HJM equations that are given in the Ito form (1). Assuming that the phase space is a Riemannian manifold, the inverse of the Riemannian metric is given by

$$g^{ij} = \sum_q \sigma_q^i \sigma_q^j, \; g_{ij} g^{jk} = \delta_i^k \tag{17}$$

So the Ito form of equations (16) written in a component form is

$$d_I X^i = -\frac{1}{2} \sum_{kj} \Gamma_{kj}^i g^{kj} dt + \sum_q \hat{\sigma}_q^i dW^q \tag{18}$$

Where $d_I X^i$ indicates Ito differentials. Whereas if we go from continuous maturity in HJM equations (1) to finite maturity then they become

$$dF^i = \sum_k \nu_k^i \sigma_k^i dt + \sum_k \sigma_k^i dW^k \tag{19}$$

with index $i = 1, N_m$ where N_m is the number of maturities. If we match volatilities and drifts in both equations then

$$\Gamma_{kj}^i = -\frac{2}{N_m} g_{kj} \sum_q \nu_q^i \sigma_q^i \tag{20}$$

Substituting this expression for Γ_{kj}^i into equation (15) we get equations for volatilities

$$d_S \sigma_j^i = 2\nu_j^i \sum_k \sigma_k^i dW^k \tag{21}$$

where sub-index S underlines that equations are written for Stratonovich differentials. Equations for forward rates are

$$d_S F^i = \sum_k \sigma_k^i dW^k \tag{22}$$

Same equations written in Ito differentials are:

$$d_I F^i = \sum_k \nu_k^i \sigma_k^i dt + \sum_k \sigma_k^i dW^k \tag{23}$$

$$d_I \sigma_j^i = 2 \sum_k \sigma_k^i (\nu_j^i \nu_k^i + \sum_{l \leq i} \sigma_k^l \Delta T_l \nu_j^l) dt + 2\nu_j^i \sum_k \sigma_k^i dW^k \tag{24}$$

where $\nu_j^i(t)$ are

$$\nu_j^i(t) = \sum_{k=1}^{i} \sigma_j^k \Delta T_k + \nu_j^0(t) \tag{25}$$

and ΔT_k is the time interval between maturities for $k > 1$ and $T_1 - t$ for $k = 1$.

Equations (21), (22) is Stratonovich form of SDG equations conformed to the HJM model of interest rates, Ito form of SDG equations for the interest rates and volatilities is given by (21), (22).

2.3 What is Needed to Solve SDG Equations?

Before we give answer to the above question let us quickly examine our set of equations. We note that a set of equations for forward interest rates (23) is exactly the same as in the HJM model (1) which makes the SDG model arbitrage free. In addition to the rate equations the SDG model provides a set of equations (24) for volatilities. This extra set of equations restricts possibilities of volatility changes and consequently guarantees that we have to provide smaller amount of extra information to solve for forward rates and volatilities. In terms of ***initial conditions*** for interest rates and volatilities the SDG model has the same requirements as the HJM model, i.e. $F^i(0)$ and $\sigma_j^i(0)$ are needed. If we assume that the whole yield curve satisfies the developed market condition (i.e., the fair game assumption (14)) and conforms to the SDG model then we do not need anything else to predict volatility and interest rate dynamics for time range up to T_{max} (i.e., the maximum maturity range for which initial conditions were specified). Here we see an advantage of the SDG model as compared to the HJM model as the latter requires additional assumptions (see section (1.2) for more details) about the volatility dynamics before it can solve for interest rates.

On the other hand, the assumption of developed markets may be not satisfied around some particular maturities. If the whole yield curve conforms to the SDG model then on average the dependence of volatilities versus maturity would be given by a smooth curve. However, our analysis of market data indicates a persistent perturbation in volatility functions around the two year term. Such a perturbation is clearly seen in all important principal components shown in Figure (1). We *guess* that such anomaly may be due to high demand for two year borrowing on the part of some strong party (e.g., US government) that drives the whole market. So, what it means for the SDG model? We have to identify maturities around which the SDG model may not hold true and treat volatility and/or interest rate values at those maturities as ***boundary conditions*** when solving SDG equations (23), (24). Again, if we compare this case to the HJM model we see the advantage of the SDG model. The SDG model needs inputs of volatilites $\sigma_k(t, T-t = T_{fixed})$ for very few maturities (e.g., T_{fixed} equal to two years, seven years, and ten years) and then it solves for all other interim maturities whereas the HJM model requires input of future dynamics of all volatilities, e.g., T_{fixed} has to cover zero to ten years with a step equal to one quarter.

3 Solutions of SDG Model

General equations for forward interest rate volatilities (24) are nonlinear and indicate interconnection of volatility functions of different principal components and maturities $\sigma_k^i(t)$. So in general there is little hope for solving such equations analytically and most of the information is expected to be obtained through numerical solutions.

3.1 Analytical Solution for Volatilities in Case of Single Factor Model

Even if it is difficult to find analytical solutions of SDG equations (24) in the general case it turns out that we can find solutions for a case when the interest rates are driven by a single Wiener process [*Makhankov et. al.*, 1995]. In this particular case the volatility equations can be represented as

$$d_S\sigma^i = 2(\nu^{i-1} + \sigma^i \Delta T_i)\sigma^i dW \tag{26}$$

that gives a chain type solution for σ^i

$$\sigma^i(t) = \frac{\sigma^i(0)\exp[2\int_0^t dw_\tau \nu^{i-1}(\tau)]}{1 - 2\sigma^i(0)\int_0^t dw_\tau \Delta T_i \exp[2\int_0^\tau dw_x \nu^{i-1}(x)]} \tag{27}$$

It is seen from the denominator of this expression that the solution may become singular. At the same time, under different initial and boundary conditions the solution may be regular. More investigation is necessary before we can make definite conclusions about the nature of solutions of SDG equations (23), (24).

3.2 Numerical Solution For Constant Boundary Conditions

As it was noted in section (1.2) lognormal volatilities of forward interest rates demonstrate quasi steady behavior. In our following example we concentrate on finding a steady state solution of the SDG equations that might resemble such

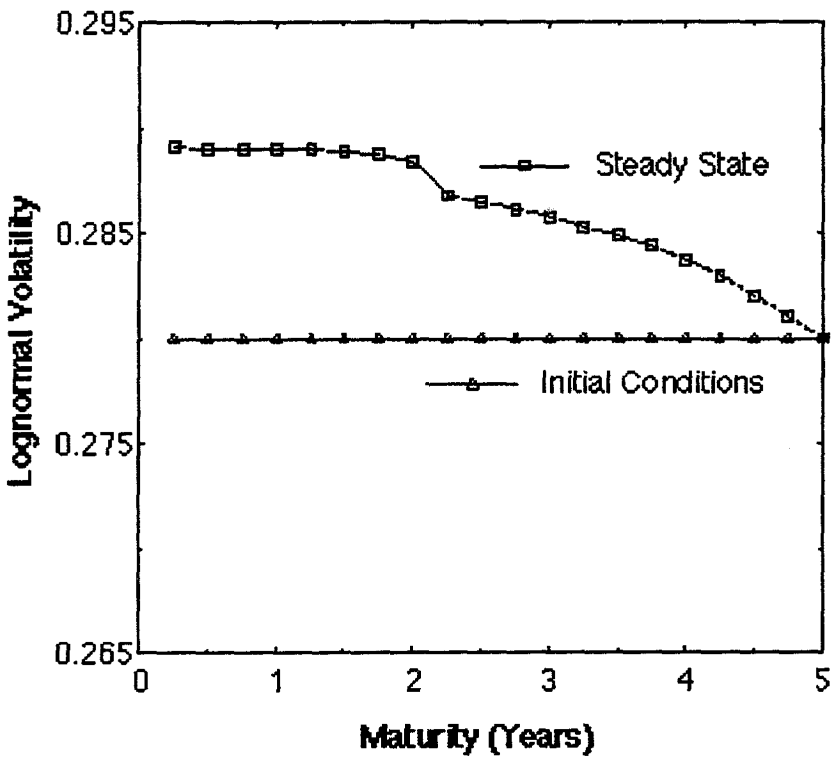

Figure 2: Results of numerical solution.

market behavior. For this purpose we solve SDG equations (24) numerically with the following assumptions: we use a single factor model, the initial term structure of volatility is flat (see Figure (2)), we introduce two "boundaries" (at two years and five years) at which volatilities are kept constant. In Figure (2) we show the initial term structure of lognormal volatility that is flat at 0.28 (i.e., 28 %-per-year) and the steady state that is established after a transition period. The steady state curve is expected value of volatilities obtained from Monte Carlo modeling of equations (24). Comparison of the steady state curve with a typical term structure of volatilites from the market shows qualitative agreement of two curves.

4 Conclusions

We examine mathematical constraints on the evolution of interest rate volatilities that arise from stochastic differential calculus under assumptions of an arbitrage free evolution of zero coupon bonds and developed markets/fair game (i. e., none of the party/factor can drive the whole market). The resulting new theory incorporates the HJM model of interest rates and provides new equations for volatilities which makes the system of equations for interest rates and volatilities complete and self consistent. It results in much smaller amount of volatility data that should be guessed for the SDG model as compared to the HJM model. Limited analysis of the volatility data suggests that the assumption of the developed market is violated around maturity of two years. Such maturities where the assumptions of the model are violated are suggested to serve as boundaries at which volatilities should be specified independently from the SDG model. Our numerical example with two boundaries (two years and five years) qualitatively resembles the market behavior. Under some conditions solutions of the SDG model become singular that may indicate market crashes. The present analysis does not allow us to make strong statements about validity of the theory. More detail comparison with the data is needed before the theory can be established or refuted. Especially valuable should be comparisons for dynamic situations in the market with high volatility and market crashes.

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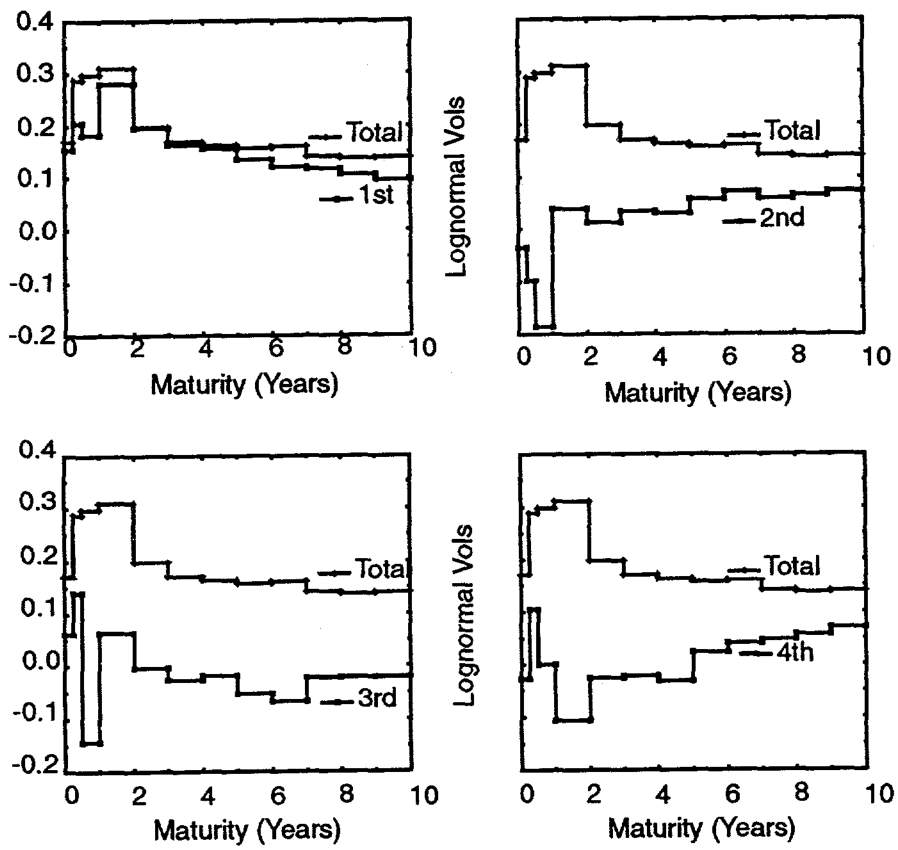

Figure 1: Principal components of lognormal volatilities.