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MASTER

EVALUATION OF PHASE-SPACE DENSITY MEASUREMENT TECHNIQUES*

A. van Steenberg

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For a particle beam propagating along the z-axis and assuming a simple distribution in the z, p_z projection of the six-dimensional phase-space volume in (x, y, z, p_x, p_y, p_z) , the total beam can be represented by:¹

$$N_{\text{tot}} = \iiint \rho(x, x', y, y') dx dx' dy dy'$$

and the integrated partial density function is given by:

$$D(x, x') = \iint \rho(x, x', y, y') dy dy'$$

Consequently,

$$N_{\text{tot}} = \iint D(x, x') dx dx'$$

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For reasons which will become clear below, use is made here of $x' = p_x/p_z$ and $y' = p_y/p_z$, instead of p_x and p_y ; this does not affect the validity of the foregoing expressions.

The following beam phase-space distribution measurement methods will be considered:

- a) The two pairs of crossed slits method.
- b) The two slit method.
- c) The crossed slits plus photographic film method.

a) The Two Pairs of Crossed Slits Method

This method has been described in detail before,² i.e. the emittance of a 750 keV proton beam was determined with its two-dimensional density distribution. This was done with the object of matching this to the acceptance of the linac, for

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which only the intercepts of the four-dimensional phase-space acceptance volume with the x, x' plane (or y, y' plane) had been calculated. Therefore, only the "specific partial density functions"¹

$$D_{y=0, y'=0}(x, x') \text{ or } D_{x=0, x'=0}(y, y')$$

were determined.

For completeness sake the essentials of the method are given here again.

Two crossed slits are used to sample the beam; their orientation define an orthogonal coordinate system x, y . Beam propagates in the z -direction. The vertical slit, after centering, is kept stationary at $y=0$. After a certain drift space, a second pair of crossed slits is used, with the vertical slit again centered at $y=0$. Therefore, the beam orbits to be examined are limited to the $(x, z)_{y=0}$ plane. For a certain position of $x=x_1$ of the first horizontal slit, the angular spread of the beam sample is determined by scanning vertically with the second horizontal slit. The second set of slits is followed by a Faraday cup. For each value of $x=x_1$ a current distribution I versus x' is obtained, i.e.: $I(x_1, x')_{y=0, y'=0}$. The zero points of these functions define the angular extent of a beam sample as defined by the first pair of crossed slits. The zero points of a set of these functions define an emittance boundary in an x, x' diagram, representing the total beam intensity. Cutting off the $I(x_1, x')_{y=0, y'=0}$ distributions at current values $\neq 0$, say, for example, at a value equal to 10% of the maximum current value ($x=0, x'=0$) results in a new emittance boundary, obtained from the x_1 and associated x_1', x_2' values, as shown in Fig. 1. The so found fractional emittance area represents a beam current value, I_{fr} , given by:

$$I_{fr} = I_{tot} \frac{\int_{-\infty}^{+\infty} dx \int_{x_1'}^{x_2'} I(x, x')_{y=0, y'=0} dx'}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} I(x, x')_{y=0, y'=0} dx'}$$

A family of emittance boundaries may be obtained in this fashion, each of which would "enclose" a certain fraction of the total current providing thereby a density distribution in two-dimensional phase space.

In the following, the limitations of the method are examined.

Consider the four-dimensional density function $\rho(x, x', y, y')$. Experimentally, the number of particles passing through the four slits is determined, therefore

$$\rho(x, x', y, y') = \frac{\Delta N(x, x', y, y')}{\Delta x \Delta x' \Delta y \Delta y'}$$

Using four slits, with identical slit width ΔS one obtains, see Fig. 2:

$$\Delta S = \Delta x = \Delta y$$

$$\frac{|x_2 - x_1| - \Delta S}{L} < |x'| < \frac{\Delta S + |x_2 - x_1|}{L}$$

Consequently:

$$\Delta x' = \Delta y' = \frac{2\Delta S}{L}$$

It follows then, that

$$\rho(x, x', y, y') = \frac{L^2}{4(\Delta S)^4} \Delta N(x, x', y, y')$$

with

$$D_{y=0, y'=0}(x, x') = \int_{-\frac{\Delta S}{2}}^{+\frac{\Delta S}{2}} \int_{-\frac{\Delta S}{L}}^{+\frac{\Delta S}{L}} \rho(x, x', y, y') dy dy'$$

one finds

$$D_{y=0, y'=0}(x, x') = \frac{L}{2(\Delta S)^2} \Delta N(x, x', y, y')_{y=0, y'=0}$$

Further, ΔN (the number of particles per unit time and per ΔE , where ΔE is the total energy spread in the beam) is related to the current measured in the Faraday cup as

$$e \Delta N(x, x', y, y')_{y=0, y'=0} = I(x, x')_{y=0, y'=0}$$

Consequently:

$$D_{y=0, y'=0}(x, x') = CI(x, x')_{y=0, y'=0}$$

with

$$C = \frac{L}{2e(\Delta S)^2}$$

To determine now the "density" distribution in the two-dimensional phase-space intercept it is necessary to consider a fractional emittance area within the total area, defined by the cut-off values of the functions $D_{y=0, y'=0}(x, x')$, as indicated in the foregoing. Integration over these density functions, within the cut-off values will yield the value of the number of particles (or I_{fr}) within the so defined fractional emittance area. This is valid in the approximation that the angular momentum around the z-axis of the particles, propagating in the z-direction, is negligible, i.e., in the approximation $p_\theta \rightarrow 0$ or even $p_\theta \ll p_r$, one has, with the usual notation for the cylindrical coordinate system and axially symmetric beams:

$$D_{y=0, y'=0}(x, x') \cong D_{\eta=0, \theta=0}(r, \xi), \quad \text{for positive}$$

values of x only, and with $\xi = p_r/p_z$ and $\eta = p_\theta/p_z$.

$$\text{Further, } N_{\text{tot}} = \iiint \rho(x, x', y, y') dx dx' dy dy' =$$

$$= \iiint \rho_1(r, \theta, \xi, \eta) J \left(\frac{x, x', y, y'}{r, \theta, \xi, \eta} \right) dr d\theta d\xi d\eta$$

$$= \iiint \rho_2(r, \theta, \xi, \eta) dr d\theta d\xi d\eta \quad ; \text{ when transforming in the usual}$$

way to the cylindrical coordinate system. Here $J \left(\frac{x, x', y, y'}{r, \theta, \xi, \eta} \right)$ is the Jacobian of the transformation and ρ_2 equals $(\rho_1 J)$. Similarly, for the case where $p_\theta \ll p_r$ or $p_\theta \rightarrow 0$, N_{tot}^* will be defined by:

$$N_{\text{tot}}^* = \iiint \rho_2^*(r, \xi, \theta)_{\eta=0} dr d\xi d\theta = 2\pi \iint D_{\eta=0, \theta=0}(r, \xi) dr d\xi .$$

Taking $\theta=0$, and identifying x and x' with r and ξ respectively, it follows then, that

$$N_{\text{tot}}^* = 2\pi \int_0^{+\infty} \int_0^{+\infty} D_{y=0, y'=0}(x, x') dx dx' .$$

Because of symmetry properties in an axial symmetric beam

$$D_{y=0, y'=0}(x, x') = D_{y=0, y'=0}(-x, x') .$$

Therefore,

$$C I_{\text{tot}} = N_{\text{tot}}^* = \pi \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} D_{y=0, y'=0}(x, x') dx' \quad \text{where}$$

$$C \text{ equals } \frac{L}{2e(\Delta S)^2} \quad \text{as in the foregoing.}$$

With this,

$$\frac{I_{\text{fr}}}{I_{\text{tot}}} = \frac{N_{\text{fr}}^*}{N_{\text{tot}}^*} = \frac{\int_0^{\infty} dx \int_{x_1'}^{x_2'} D_{y=0, y'=0}(x, x') dx'}{\int_0^{\infty} dx \int_{-\infty}^{+\infty} D_{y=0, y'=0}(x, x') dx'} = \frac{\int_0^{\infty} dx \int_{x_1'}^{x_2'} I(x, x')_{y=0, y'=0} dx'}{\int_0^{\infty} dx \int_{-\infty}^{+\infty} I(x, x')_{y=0, y'=0} dx'} ;$$

as was used in Ref. 2.

As has been pointed out recently³, in cases where $p_{\theta} \cong p_r$ this method may lead to erroneous values of $I_{\text{fr}}/I_{\text{tot}}$ because $\frac{N_{\text{fr}}^*}{N_{\text{tot}}^*} \equiv \frac{N_{\text{fr}}(y=0, y'=0)}{N_{\text{tot}}(y=0, y'=0)}$ is not necessarily equal to $\frac{N_{\text{fr}}}{N_{\text{tot}}}$, where y' is not limited as $-\frac{\Delta S}{L} < y' < \frac{\Delta S}{L}$. This is only so if the relationship $\frac{L |p_y/p_z|_{y=0}}{\Delta S} \leq 1$ is satisfied. This will be further examined below.

Referring now to the cylindrical coordinate system, there is no obvious reason, when considering an ideal ion source plasma boundary, to assume a distribution in p_r different from the distribution in p_{θ} , i.e. $\Delta p_r \cong \Delta p_{\theta}$, where Δ refers to say, the half maximum width of the p distribution.

However, in practical beam acceleration systems, for example, traversing through a cylindrical optical system with aberrations one finds in general that $\Delta p_r > \Delta p_{\theta}$. More explicitly in practice, the emittance is not determined, in a first approach, by the state of the plasma⁴ (ion and electron temperature, etc.) but more by the aberrations in the system, which for an axial symmetric system, do effectively enlarge the p_r distribution only.

b) The Two Slit Method

To determine the x, x' distribution, with two slits infinitely extended in the y direction and followed by a current measuring device, one obtains¹ the integrated partial density functions $D(x, x')$ directly, i.e., the integrations over y and y' have been automatically carried out. Integration over x and x' yields the total beam current

$$C I_{\text{tot}} = N_{\text{tot}} = \int dx \int D(x, x') dx' .$$

The determination of the density distribution is straightforward because:

$$\frac{I_{\text{fr}}}{I_{\text{tot}}} = \frac{N_{\text{fr}}}{N_{\text{tot}}} = \frac{\int_{-\infty}^{+\infty} dx \int_{x_1'}^{x_2'} D(x, x') dx'}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} D(x, x') dx'} , \text{ where the values } x_1', x_2'$$

again define a fractional emittance area, similarly as was done under Section a).

These arguments apply to the method whereby the sequence of the two slits is followed by a Faraday cup or current transformer. An alternative here is to use at the first location a fixed set of parallel slits and at the second location a photographic film. By selecting the appropriate film material related to beam intensity and particle energy, it is possible to obtain a usable blackness distribution on the film which may be evaluated to obtain the phase-space density distribution. In this case, however, the density function $D(x_1, x')$ is found by integrating densitometer curves, which are obtained by scanning the film image parallel to the slit direction for various values of x' . This is illustrated in Fig. 3 where a single slit at the location $x=x_1$ is considered. With the density functions obtained in this manner the foregoing arguments apply and the density distribution in two-dimensional phase space may be found as indicated above.

c) The Crossed Slit Plus Photographic Film Method

This method³ makes use of crossed slits to define x and y and at a distance L along the z -axis a photographic film is exposed. The density distribution on the film is determined (x', y' distribution).^{*} This method yields the four-dimensional

^{*} It is, of course, also possible to use crossed slits at the location of the film followed by a Faraday cup and to map the distribution by scanning these slits.

phase volume and its density distribution; it provides more complete information on the phase-space distribution, however, it is rather elaborate because of the necessity to determine the distribution in x', y' with a scanning densitometer. From the four-dimensional volume the two-dimensional intercept in x, x' is determined providing also the two-dimensional "density" distribution, similarly as was done under Sections a) and b).

d) Comparison of the Methods

The following approach is an attempt to correlate more closely the density distribution results obtained with the methods under a) and b) and the results from method c).

Consider the density function:

$$D(x, x', y') = \int \rho(x, x', y, y') dy$$

which is the Cartesian coordinate system equivalent (assuming cylindrical symmetry) of the density function³ in cylindrical coordinates; $D(R, R', t)_{\theta=\theta'}$ with $t = R\theta'$. Similarly, as before,

$$D_{y=0}(x, x', y') = \int_{-\frac{\Delta S}{2}}^{+\frac{\Delta S}{2}} \rho(x, x', y, y') dy \quad \text{and}$$

$$\rho(x, x', y, y') = \frac{L^2}{(\Delta S)^2 W^2} \Delta N(x, x', y, y')$$

where W^2 is the area exposed on the film, as a result of beam transmission through a sample area $(\Delta S)^2$.

Consequently:

$$D_{y=0}(x, x', y') = \frac{L^2}{(\Delta S)W^2} \Delta N(x, x', y, y')_{y=0}$$

Further,

$$\frac{(\Delta W)^2}{W^2} \Delta N = C_0 i(x, x', y')_{y=0} \quad \text{where}$$

$i(x, x', y')_{y=0}$ is the densitometer scale reading (proportional to film grain density),

$(\Delta\bar{w})^2$ is densitometer spot size, C_o is a scaling factor, which takes into account film development, densitometer calibration, etc. From this,

$$D_{y=0}(x,x',y') = \frac{L^2 C_o}{(\Delta S)(\Delta w)^2} i(x,x',y')_{y=0}$$

In an axial symmetric beam it is sufficient now to scan only the horizontal slit at the first location, keeping the vertical slit stationary and centered at the beam. Therefore, only the fraction $\frac{(\Delta S)(2R)}{\pi R^2}$ of the total beam intensity is being examined. Consequently, it follows that,

$$C I_{tot} = N_{tot} = \frac{\pi C_o R L^2}{2(\Delta S)^2 (\Delta w)^2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} i(x,x',y')_{y=0} dx' dy'$$

With this, as before,

$$\begin{aligned} \frac{I_{fr}}{I_{tot}} &= \frac{N_{fr}}{N_{tot}} = \frac{\int_{-\infty}^{+\infty} dx \int_{x_1}^{x_2} \int_{-\infty}^{+\infty} i(x,x',y')_{y=0} dx' dy'}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i(x,x',y')_{y=0} dx' dy'} = \\ &= \frac{\int_{-\infty}^{+\infty} dx \int_{x_1}^{x_2} \int_{-\infty}^{+\infty} D_{y=0}(x,x',y') dx' dy'}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_{y=0}(x,x',y') dx' dy'} \end{aligned}$$

Referring now to the two-dimensional phase-space density distribution determination method described under a) whereby the y' values were limited to $|y'| \leq \frac{\Delta S}{L}$, a comparison is made with the above results where y' is not so limited, i.e. y' values are present for which $|y'| > \frac{\Delta S}{L}$ or

$$\frac{L \left| \frac{p_y}{p_z} \right|_{y=0}}{\Delta S} > 1 \quad \text{A hypothetical}$$

$D_{y=0}(x,x',y')$ function is given in Fig. 4 to illustrate the approach. When cutting

off this distribution at $x' = -x_1'$ and $x' = x_2'$ in order to define a fractional two-dimensional emittance boundary, integration, to determine particle content within the fractional boundaries, whereby either $y' \leq \infty$ or $y' \leq \frac{\Delta S}{L}$, respectively, yields the following relationships:

$$\frac{N_{fr}}{N_{tot}} \equiv \frac{\int_0^R dx \int_0^{x_2'} dx' \int_0^{\infty} D_{y=0}(x, x', y') dy'}{\int_0^R dx \int_0^{\infty} dx' \int_0^{\infty} D_{y=0}(x, x', y') dy'} \quad \text{and}$$

$$\frac{N_{fr}^*}{N_{tot}^*} = \frac{\int_0^R dx \int_0^{x_2'} dx' \int_0^{\varphi} D_{y=0}(x, x', y') dy'}{\int_0^R dx \int_0^{\infty} dx' \int_0^{\varphi} D_{y=0}(x, x', y') dy'}$$

where $\varphi = \frac{\Delta S}{L}$. Use is made here of (see the foregoing)

$$\frac{\int_0^R dx \int_0^{x_2'} D_{y=0, y'=0}(x, x') dx'}{\int_0^R dx \int_0^{\infty} D_{y=0, y'=0}(x, x') dx'} \equiv \frac{\int_0^R dx \int_0^{x_2'} \int_0^{\varphi} D_{y=0}(x, x', y') dy'}{\int_0^R dx \int_0^{\infty} \int_0^{\varphi} D_{y=0}(x, x', y') dy'}$$

Also, for simplicity, the assumption has been made here that $|x_1'| = |x_2'|$, which for the present argument means that the phase-space distribution at a beam waist is being considered. As will become evident below x_2' is a function of x in general. The equivalency of the phase-space density distribution methods described above may be investigated now by defining

$$\sigma \equiv \frac{N_{fr}^* / N_{tot}^*}{N_{fr} / N_{tot}}$$

It has been shown in Appendix I that (in general) $\sigma = 1$ if the following density function is used:

$$D_{y=0}(x, x', y') = d_0 e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]}.$$

This is basically an empirical distribution (see Fig. 4, Ref. 2), but is consistent with the Maxwell-Boltzman distribution in x', y' , typically at an ion source plasma boundary, and the experimentally observed result that the current density distribution in a particle beam is in general not homogeneous and may be approximated by a Gaussian (typically, see the experimental results, Ref. 5).

Because $D_{y=0}(x, x', y') \leq 0.02 d_0$ for $x \geq R$, $x' \geq a$, $y' \geq b$, for practical purposes the partial phase-space boundary consistent with this distribution is

$$\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1 \quad \text{with}$$

$$D_{y=0}(x, x', y') = d_0 e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]} \quad \text{within this boundary, and}$$

$$D_{y=0}(x, x', y') \rightarrow 0 \quad \text{outside this boundary.}$$

It is recognized here that in an axially symmetric system for $x=0$ (and $y=0$), a should equal b . To account for aberrations, b is assumed to be of the form

$$b = a \left(1 + C_1 \left(\frac{x}{R}\right)^2 + C_2 \left(\frac{x'}{R}\right)^2 + \dots \right)$$

Actually, as also indicated in Appendix I, also in the case where

$$D_{y=0} = d_0 G(x^2) e^{-4 \left[\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]} \quad \text{with } G(x^2) \rightarrow 0 \text{ for } x > R, \text{ and where } G(x^2) \text{ is}$$

a "well behaved", symmetric, but arbitrary function of x , in the region of $0 < x < R$, it can be shown that $\sigma = 1$.

However, for the case where $b = a \left(1 + C_1 \left(\frac{x}{R}\right)^2 + \dots \right)$ but with $C_1 \gg 1$, i.e. assuming appreciable aberrations, it has been further shown in Appendix I that σ is given by

$$\sigma = 1 + \frac{1}{8} C_1 + \dots \quad \text{and may deviate significantly}$$

from the value 1, consequently the method described under Section a) would yield different density distribution contours in this case as compared with the methods described under Sections b) and c).

Conclusion

It is evident from these results that even though in general it cannot be shown that for any distribution $D_{y=0}(x,x',y')$ the ratio $\sigma = 1$; for a partial density function approaching practical cases σ equals or approximates the value 1, as shown. As a consequence the space-space density distribution methods as described in the foregoing should in general yield similar two-dimensional phase-space contours and density distributions independent of the particular method used.

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APPENDIX I

Evaluation of σ

Consider the partial density function

$$D_{y=0}(x, x', y') = d_0 e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]}$$

For $x=0, x'=0, y'=0$; $D_{y=0}(x, x', y') = d_0$.

Referring now also to the discussion in Section a), then taking a value $D_{y=0}(x, x', y') = f d_0$, where f is a constant and $0 < f < 1$, the cut-off values x_2 for various values of x are determined by:

$$f = \frac{D_{y=0}(x, x', y')}{d_0} = e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x_2'}{a}\right)^2 \right]}$$

This is further illustrated in Fig. 5 where $D_{y=0}(x, x', y')$ is plotted, for various values of $x=x_1$.

This yields then as the upper limits of integration in the expressions for N_{fr} (or N_{fr}^*):

$$x_2' = a \sqrt{\epsilon^2 - \frac{x^2}{R^2}}, \text{ for } x \leq \epsilon R \text{ with}$$

$$\epsilon^2 = -\frac{\log_e f}{4} \text{ . For } x > \epsilon R, x_2' \text{ equals zero.}$$

The integration of $\frac{N_{fr}^*}{N_{tot}^*}$ and $\frac{N_{fr}}{N_{tot}}$ is then as follows:

$$\frac{N_{fr}^*}{N_{tot}^*} = \frac{\int_0^{\epsilon R} dx \int_0^{a \sqrt{\epsilon^2 - \left(\frac{x}{R}\right)^2}} dx' \int_0^{\phi} d_0 e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]} dy'}{\int_0^{\infty} dx \int_0^{\infty} dx' \int_0^{\phi} d_0 e^{-4 \left[\left(\frac{x}{R}\right)^2 + \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 \right]} dy'}$$

with $\varphi/b \ll 1$, it follows

$$\frac{N_{fr}^*}{N_{tot}^*} = \frac{d_o \varphi \frac{a}{2} \int_0^{\epsilon R} e^{-4\left(\frac{x}{R}\right)^2} \left[\int_0^{2\sqrt{\epsilon^2 - \left(\frac{x}{R}\right)^2}} e^{-t^2} dt \right] dx}{d_o \varphi a R \left(\frac{\pi}{16}\right)} = \frac{8}{R\pi} \left[F(x) \Big|_{x=\epsilon R} \right]$$

Similarly for $\frac{N_{fr}}{N_{tot}}$ it is found that

$$\frac{N_{fr}}{N_{tot}} = \frac{8}{R\pi} \left[F(x) \Big|_{x=\epsilon R} \right] \text{ with } b = a \left[1 + C_1 \left(\frac{x}{R}\right)^2 + \dots \right]$$

and taking here, for simplicity, C_1 and higher order terms $\ll 1$.

Consequently, $\sigma = 1$.

It is immediately obvious that $\sigma = 1$ also for the more general case where

$$D_{y=0}(x, x', y') = d_o G(x^2) e^{-4\left[\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2\right]} \text{ where } G(x^2) \text{ is a}$$

"well behaved," but arbitrary, function of x^2 in the region of 0 to $x=R$, and $G(x^2) \rightarrow 0$ for $x > R$.

In this case the integral given by $F(x) \Big|_{x=\epsilon R}$, i.e.

$$\int_0^{\epsilon R} e^{-4\left(\frac{x}{R}\right)^2} \left[\int_0^{2\sqrt{\epsilon^2 - \left(\frac{x}{R}\right)^2}} e^{-t^2} dt \right] dx \text{ becomes}$$

$$\int_0^{x=f(\epsilon)} G(x^2) \left[\int_0^{2\sqrt{\epsilon^2 + \frac{\log_e G(x)}{4}}} e^{-t^2} dt \right] dx = \left[H(x) \Big|_{x=f(\epsilon)} \right],$$

The upper limit $x = f(\epsilon)$ follows from $G(x^2) = e^{-4\epsilon^2}$. The same result

$[H(x) \Big|_{x=f(\epsilon)}]$ occurs both in N_{fr}^* and N_{fr} leading again to $\sigma = 1$.

For the case where

$$b = a [1 + C_1 \left(\frac{x}{R}\right)^2 + C_2 \left(\frac{x}{R}\right)^4 + \dots] \text{ with}$$

C_2 and higher order terms $\ll 1$, but $C_1 \leq 1$, the results are of the form:

$$\sigma = \frac{1 + \frac{1}{8} C_1}{1 + Q(\epsilon) C_1} \quad \text{with}$$

$$Q(\epsilon) = \frac{\int_0^{\epsilon R} e^{-4\left(\frac{x}{R}\right)^2} \left(\frac{x}{R}\right)^2 \left[\int_0^{\sqrt{e^{2-\left(\frac{x}{R}\right)^2}} e^{-t^2} dt \right] dx}{\int_0^{\epsilon R} e^{-4\left(\frac{x}{R}\right)^2} \left[\int_0^{\sqrt{e^{2-\left(\frac{x}{R}\right)^2}} e^{-t^2} dt \right] dx}$$

With the substitutions $u = \frac{t}{2\sqrt{e^{2-\frac{x^2}{R^2}}}}$ and $v = \frac{x}{\epsilon R}$ one gets

$$Q(\epsilon) = \frac{\int_0^1 \int_0^1 v^2 e^{-4\epsilon^2 v^2} e^{-4u^2 \epsilon^2 (1-v^2)} \sqrt{1-v^2} dudv}{\int_0^1 \int_0^1 e^{-4\epsilon^2 v^2} e^{-4u^2 \epsilon^2 (1-v^2)} \sqrt{1-v^2} dudv}$$

In the range $0 \leq f \leq 1$, it follows that $Q(\epsilon) < \epsilon^2$. Limiting f to the range $0.1 < f < 0.9$, which in practice is no restriction, the maximum value of ϵ^2 is approximately 0.6. Consequently,

$$\sigma \cong 1 + \frac{1}{8} C_1 \quad \text{if} \quad C_1 Q(\epsilon) < C_1 \epsilon^2 < 1 \text{ or more explicitly}$$

$C_1 Q(\epsilon) \ll 1$.

Therefore, also in this case (but with the restriction that $C_1 \leq 1$) the value of σ equals 1 to within of the order of 15%. It is evident however, that for the case of appreciable aberration, i.e., $C_1 \gg 1$, σ may deviate significantly from the value 1.

APPENDIX II

Some Comments on the Description of Particle Beams
Related to Phase Space Distribution

In the literature cited and in the "normal" beam treatment by Walsh,⁶ the notation and definitions for beam emittances, etc. vary. Therefore, the following enumeration and relationships might be useful and could serve as a suggestion in the direction of some more generally accepted usage.

Particle distribution in two (x, x') or four (x, x', y, y') dimensional phase space, where x and y are the transverse coordinates, and $x' = p_x/p_z$, $y' = p_y/p_z$:

E_2 , two-dimensional beam emittance in x, x' space.

E_4 , four-dimensional emittance in x, y, x', y' space.

πE_2 or A_2 , two-dimensional phase-space area.

$(\pi^2/2)E_4$ or V_4 , four-dimensional phase-space volume.

The above quantities refer to a fixed p_z value (or simple distribution around p_z). To obtain emittance "invariants" one may normalize the foregoing expressions and obtain p_z independent values:

$$\epsilon_2 = \frac{\beta \gamma A_2}{\pi} = \beta \gamma E_2, \text{ two-dimensional momentum normalized emittance.}$$

$$\epsilon_4 = \frac{\beta^2 \gamma^2 V_4}{(\pi^2/2)} = \beta^2 \gamma^2 E_4, \text{ four-dimensional normalized emittance.}$$

(Similarly, normalized emittance area, $\beta \gamma A_2$, and normalized emittance volume, $\beta^2 \gamma^2 V_4$.)

Actually, it would seem preferable to normalize with the factor $p_z^2 = (m_0 c \beta \gamma)^2$ or $p_z = (m_0 c \beta \gamma)$ in order to obtain an "absolute" value of beam quality or emittance in four-dimensional transverse coordinate and conjugate momentum phase space, or its intercept with the (x, p_x) or (y, p_y) planes, respectively, leading to the two-dimensional emittance $(m_0 c) \epsilon_2$ and four-dimensional emittance $(m_0 c)^2 \epsilon_4$. However, the foregoing expressions for ϵ_2 and ϵ_4 have been more generally used and in effect the assumption is made that $m_0 c \equiv 1$.

A partial description of the particle density in phase space is given by:

$$d_2 = \frac{I}{\pi E_2} = \frac{I}{A_2}, \text{ "density" in two-dimensional } (x, x') \text{ space.}$$

$$\delta_2 = \frac{d_2}{\beta \gamma} = \frac{I}{\pi \epsilon_2}, \text{ normalized "density" in two-dimensional phase space.}$$

Short of a specification of particle density distribution in six-dimensional phase space, a practical expression, if a simple distribution in z, p_z (or E, t) is assumed, is the particle density in the four-dimensional phase-space projection, as follows:

$$B_4 = \frac{I}{V_4} = \frac{I}{(\pi^2/2)E_4}, \text{ density in four-dimensional } (x, y, x', y') \text{ space.}$$

$$\mathcal{B}_4 \text{ (or } \mathcal{B}_4) = \frac{I}{\beta^2 \gamma^2 V_4} = \frac{I}{(\pi^2/2)\epsilon_4}, \text{ momentum normalized density in four-dimensional phase space.}$$

Here, I is the total beam current, the subscript 4 in B_4 and \mathcal{B}_4 seems redundant, however a more complete description of particle beams in accelerators would include the (z, p_z) distributions as well and specify six-dimensional emittance (ϵ_6) and particle density in six-dimensional phase space (Φ_6).

Regarding two and four-dimensional space only it is assumed now that the particles in an axially symmetric beam, propagating along the z -axis collectively, have zero angular momentum around the z -axis. In this case, the two-dimensional emittance area in (x, x') , i.e. the intercept of the $y=0, y'=0$ plane with the four-dimensional volume in (x, y, x', y') is equal to the projection of the four-dimensional volume on the $y=0, y'=0$ plane. Then it follows for a "normal" beam, i.e. identical two-dimensional phase-space ellipses in both transverse planes, that:⁶

$$E_4 = (E_2)^2 \text{ * and}$$

$$V_4 = \frac{1}{2} (A_2)^2$$

Consequently:

$$\epsilon_4 = (\epsilon_2)^2$$

$$B_4 = \frac{2(d_2)^2}{I} = \frac{I}{(\pi^2/2)(E_2)^2} = \frac{2 I}{(A_2)^2} \text{ and}$$

$$\mathcal{B}_4 = \frac{2(\delta_2)^2}{I} = \frac{I}{(\pi^2/2)(\epsilon_2)^2}$$

* In practice one might have to use the more general relationship

$$E_4 = (E_2)_{x,x'} (E_2)_{y,y'}$$

The usefulness of these expressions stems from the fact that in general the maximum beam intensity transport in an accelerator subcomponent is determined by B_4 (or \mathcal{B}_4) of the preceding source, whereas in practice normally the two-dimensional emittance E_2 or A_2/π has been measured.

APPENDIX III

Comparison of Notation Used in the Literature

For the sake of completeness the present notation is related to that used in Ref. 6:

$$(m_0 c) \epsilon_2 = E'$$

$$(m_0 c)^2 \epsilon_4 = E$$

It relates as follows to that used in Ref. 3:

$$E_2 = E$$

$$V_4 = V$$

$$A_2 = F$$

$$\epsilon_2 = U = v$$

$$\epsilon_4 = 2\gamma^*$$

$$\mathcal{B}_4 = N$$

$$\delta_2 = \delta$$

It relates as follows to the notation I used so far:

$$E_2 = E$$

$$A_2 = \iint F(x, \alpha_x) dx d\alpha_x$$

$$\epsilon_2 = v$$

$$\delta_2 = \delta$$

$$d_2 = d$$

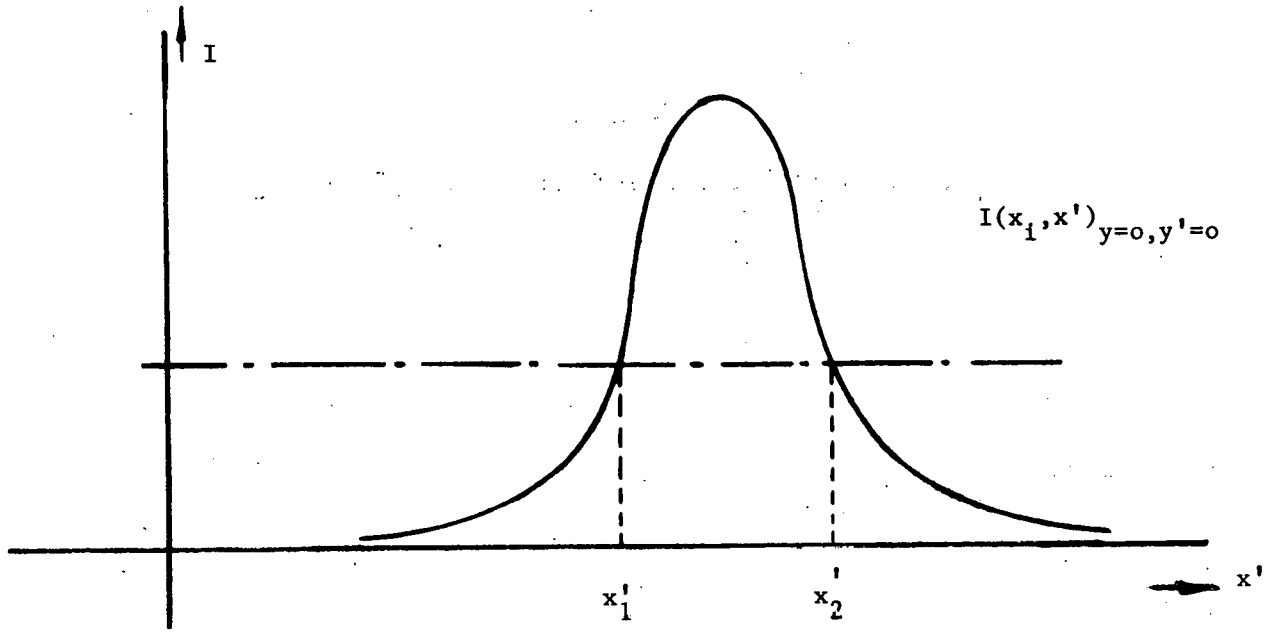
$$\mathcal{B}_4 = 2B^{**}$$

*The factor of 2 enters because the four-dimensional volume has been divided by $\pi^2/2$ to obtain E_4 (see Ref. 6).

**Again the factor of 2 enters for the same reason as under (*). In effect I am redefining here "source brightness", as I used this term before, as

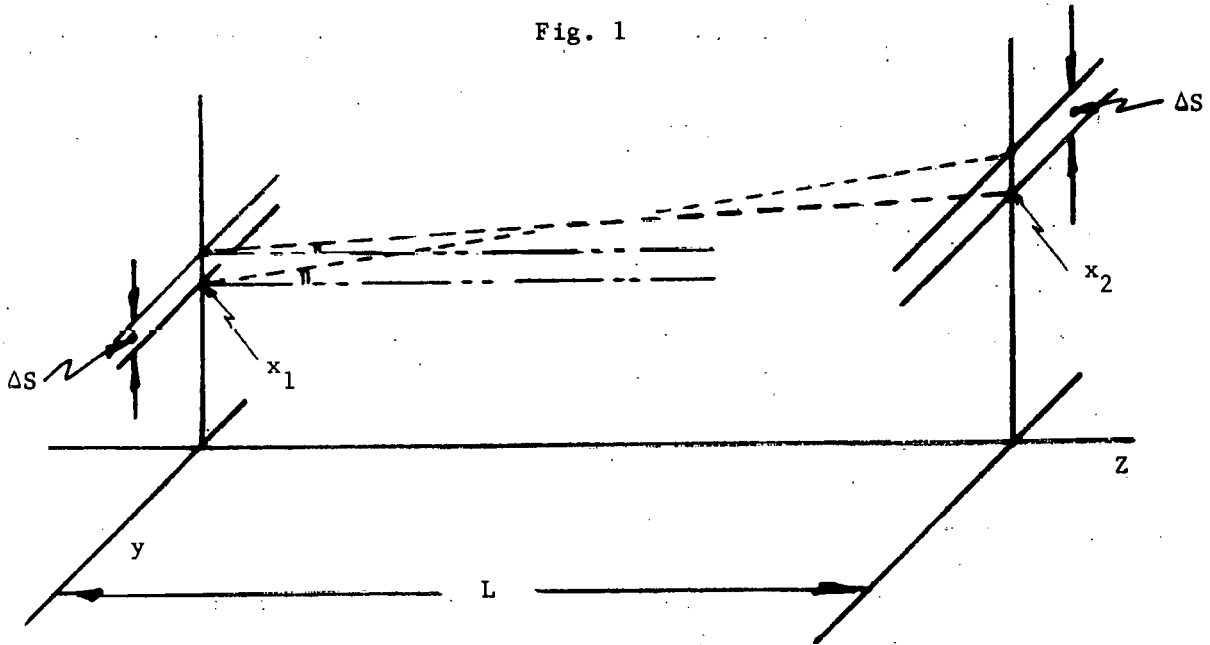
$$\frac{I}{(\pi^2/2)(\epsilon_2)^2} \quad \text{instead of} \quad \frac{I}{\pi^2 v^2} \quad (\text{note } \epsilon_2 = v),$$

in order to be consistent with the four-dimensional treatment.



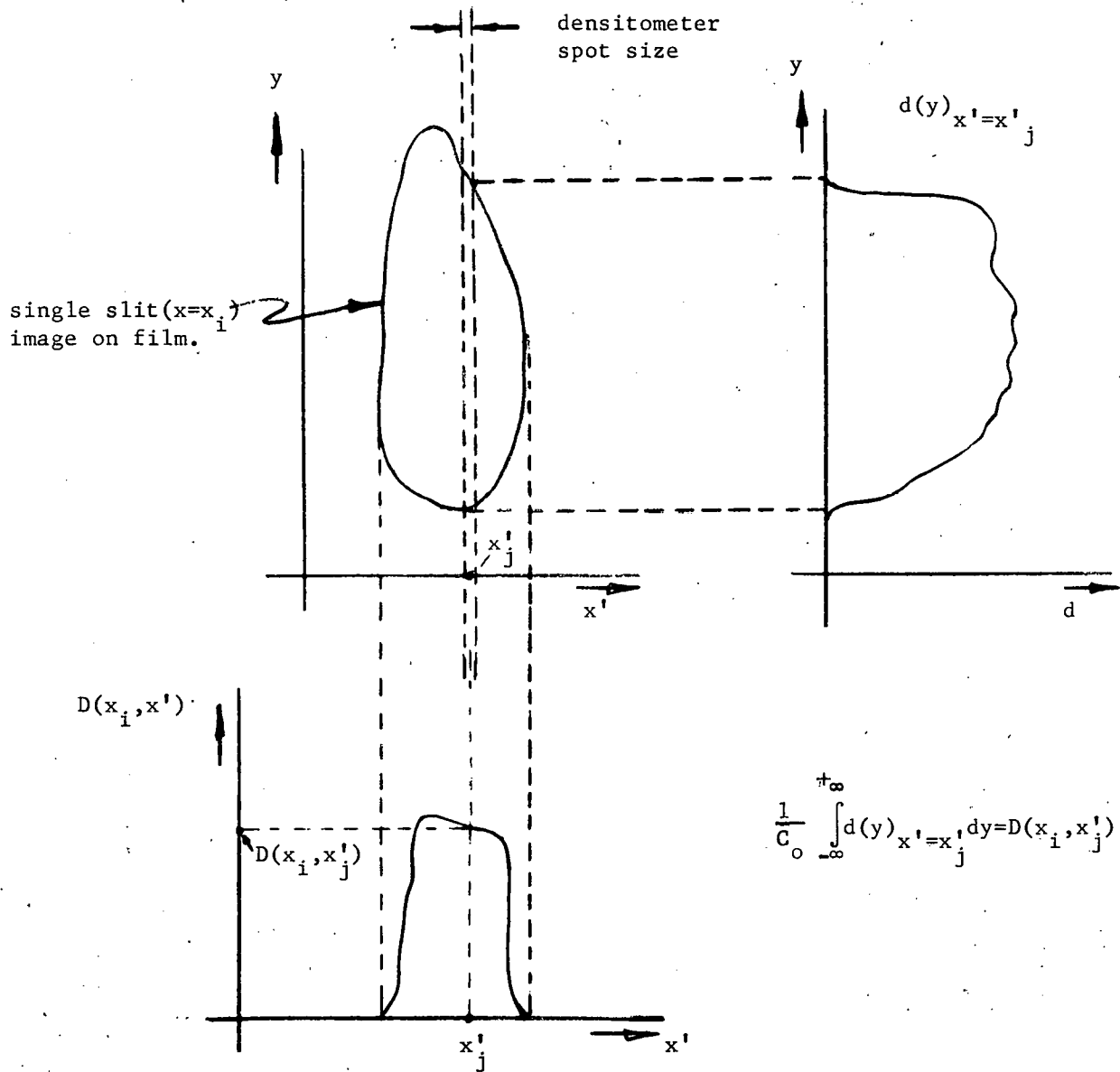
Current Distribution $I=f(x')$ $_{x=x_i}$ for $-\frac{\Delta S}{2} < y < \frac{\Delta S}{2}$ and $-\frac{\Delta S}{L} < y' < \frac{\Delta S}{L}$

Fig. 1



Slit Arrangement (the vertical slits are not shown).

Fig. 2



Evaluation of $D(x_i, x'_j)$ for a fixed set of parallel slits at the first location and a photographic film at the second location. (A single slit image is shown).

Fig. 3

