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CONTROL ROD PROGRAMMING

PART II

TWO GROUP THEORY OF GENERAL CONFIGURATION

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TABLE OF CONTENTS

LIST OF FIGURES

Control Rod Programming

Part II

Two-Group Theory of General Configuration

Raymond L. Murray

Introduction.

In Part I of the present series of reports*, an investigation of the possibilities of buckling calculations for partly inserted control rods was made on the one-group neutron diffusion model. The results were highly indicative that extension to the two-group model was possible. The study is continued in this report, in which is given a method for determining buckling (and hence reactivity) of a general array of partly inserted control rods in a cylindrical bare equivalent reactor. At this stage of development of the theory, fuel burnup is not yet specifically included; criticality is provided implicitly by variation of the effective number of neutrons per fission. The buckling is found to be calculable from suitable flux averages over the various regions of the operating core.

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* Control Rod Programming, Part I Fundamental Analysis, September 10, 1961

Generalized Buckling Theory

According to the one-group model, the buckling of a reactor is given correctly by the expression

$$B^2 = \frac{\int \phi' M' \phi' \, dV}{\int \phi' \phi' \, dV}$$

where M' is a perturbed operator $- \nabla^2 + P$ and ϕ' is the perturbed flux (and adjoint). In the previous report it was shown that the flux appropriate to fully inserted rods and that for no rods could be employed to obtain an excellent estimate of the B^2 in any partly-inserted rod problem. An extension of this formulation to the two-group model, where the flux is not self-adjoint, is now presented.

Consider the two-group equations for a core in a matrix form,

$$L\vec{\phi} = 0$$

where $\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ is a 2 x 1 matrix, the elements of which are the fast and thermal flux. The 2 x 2 matrix operator L can readily be seen to be

$$L = \begin{bmatrix} D_1 \nabla^2 - \Sigma_1 & \eta f \Sigma_2 \\ p\Sigma_1 & D_2 \nabla^2 - \Sigma_2 \end{bmatrix}$$

The null matrix is $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The two-group equations are elements of the matrix $L\phi = 0$.

It will be convenient to decompose L into the sum $-M + B_0^2$ where

$$M = \begin{bmatrix} -D_1 \nabla^2 & 0 \\ 0 & -D_2 \nabla^2 \end{bmatrix}$$

and

$$B_0^2 = \begin{bmatrix} -\Sigma_1 & \eta f\Sigma_2 \\ p\Sigma_1 & -\Sigma_2 \end{bmatrix}$$

In this notation,

$$M\phi = B_0^2\phi$$

which corresponds formally to the one-group expression. The generalized buckling B_0^2 is a 2 x 2 matrix containing the removal coefficients of the neutron balance equations. The subscript zero refers to the fact that the reactor is in the critical or original state, normally with uniform materials. The analogous expression for the adjoint flux is readily written

$$L^+\phi^+ = 0$$

where $L^+ = \begin{bmatrix} D_1 \nabla^2 - \Sigma_1 & p\Sigma_1 \\ \eta f\Sigma_2 & D_2 \nabla^2 - \Sigma_2 \end{bmatrix}$

or

$$M\phi^+ = B^{2+}\phi^+$$

The solution of both ordinary and adjoint critical equations for a reactor with homogeneous fuel, moderator and structure distribution is straightforward, In a later section, the application to a system with fully inserted rods will be given.

Next consider the equation for a reactor that is perturbed by changes in materials constants such that

$$P = \begin{bmatrix} \delta\Sigma_1 & -\delta(\eta f\Sigma_2) \\ -\delta(p\Sigma_1) & \delta\Sigma_2 \end{bmatrix}$$

and that simultaneously a uniform change in η is imagined that renders the reactor still critical.

$$\delta B^2 = \begin{bmatrix} 0 & \eta f\Sigma_2(d\eta/\eta) \\ 0 & 0 \end{bmatrix}$$

The flux distribution is now ϕ', and the differential equation of the system is

$$(L + P + \delta B^2)\phi' = 0$$

or

$$(M + P)\phi' = (B_0^2 + \delta B^2)\phi'$$

By letting $M' = M + P$ and $B^2 = B_0^2 + \delta B^2$, we obtain the compact expression for the perturbed system

$$M'\phi' = B^2\phi'$$

Multiply the equation through by the 1 x 2 row matrix $\phi^+ = [\phi_1^+\phi_2^+]$ and integrate over the system

$$\int\phi^+M'\phi'\,dV = \int\phi^+B^2\phi'\,dV$$

We are interested in employing this relation to describe any configuration of control rods, some in, some out, some partly out, for which $B^2 = B_0^2 + \delta B_0^2$. Figure 1 shows a typical array. The fluxes are not known and essentially impossible to

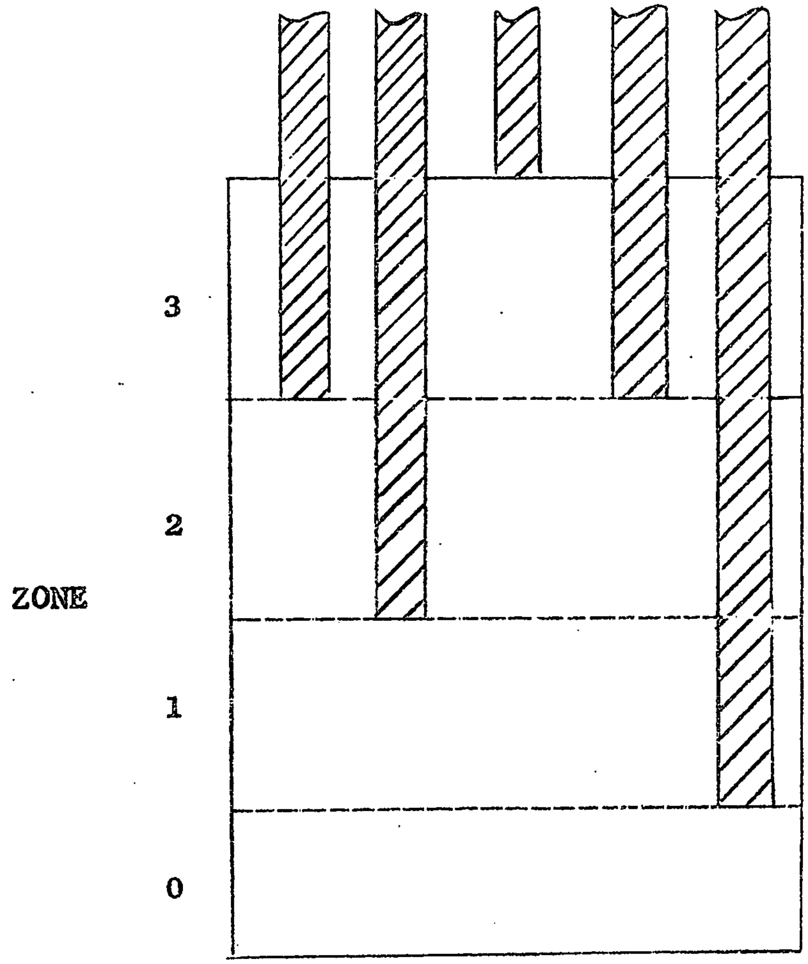

Figure 1. Rod Array

determine precisely by analytical means. If we view the reactor as consisting of several axial zones, four in the Figure, it is seen that the reactor is constructed from a four-rod region, a two-rod and a one-rod, and a region with no rods. Were these zones isolated reactors, it would be possible by analytic methods to determine ordinary and adjoint flux distributions; with fluxes ϕ_0, ϕ_1, ϕ_2, ϕ_3, and critical bucklings B_0^2, B_1^2, B_2^2, B_3^2.

The fundamental assumption is now made, that the integral expression is applicable to the system, using these fluxes for the respective zones. Write

$$\int_{core} \phi^+ M' \phi' \, dV = \int_0 \phi_0' B_0^2 \phi_0' \, dV_0 + \int_1 \phi_1' B_1^2 \phi_1' \, dV_1 + \cdots\cdots$$

and equate to

$$\int_{core} \phi^+ B^2 \phi' \, dV = \int_{core} \phi^+ B_0^2 \phi' \, dV + \int_{core} \phi^+ \delta B^2 \phi' \, dV$$

Then for M zones

$$\left| \delta B^2 \right| = \frac{\sum_{i=0}^{M-1} \phi_i^+ (B_i^2 - B_0^2) \, \phi_i' \, dV}{\sum_{i=0}^{M-1} \phi_i^+ \phi_i' \, dV}$$

This states in words that the total buckling change is equal to the flux-and adjoint-weighted values of the local buckling changes.

The quantity $|\delta B^2|$ is $f\Sigma_2\delta\eta$ from which the change in neutrons per absorption or reactivity $\rho \sim \delta\eta/\eta$ can be computed.

Adjoint Fluxes for System with Rods.

The adjoint fluxes for a bare cylindrical core without rods bear a simple relation to the ordinary fluxes. For that special case $\phi_1 = AX$, $\phi_2 = S_1AX$

$$\phi_1^+ = S_1^+A^+X, \quad \phi_2^+ = A^+X$$

where $X = J_0(\beta r)$. Since the amplitudes A and A^+ are arbitrary for a critical reactor, they may be set equal to 1, thus

$$\frac{\phi_1^+}{\phi_1} = S_1^+ \qquad \frac{\phi_2^+}{\phi_2} = \frac{1}{S_1}$$

There is no a priori evidence that a similar situation may hold for the system with an array of control rods. An investigation of the form of adjoints was thus undertaken.

The first problem was that of establishing appropriate boundary conditions at the rod surfaces. (There appears to be no distinction between the adjoint and ordinary conditions of vanishing flux at the core boundary.) Two equivalent views of rod conditions are available: (a) application of continuity of adjoint fluxes across the core-rod interface, (b) use of extrapolation distances from the core into the rod, defined by

$$d_1^+ = \frac{\phi_1^+}{\phi_1^{+\prime}}, \quad d_2^+ = \frac{\phi_2^+}{\phi_2^{+\prime}}$$

By continuity, $D_1\phi_1^{+\prime} = D_{1r}\phi_{1r}^{+\prime}$, $D_2\phi_2^{+\prime} = D_{2r}\phi_{2r}^{+\prime}$ where the subscript r refers to rod. Now specific solutions for the thermal and fast fluxes in the rod are

$$\phi_{2r}^{+} = G^{+}Z_2, \quad \phi_{1r}^{+} = S_3^{+}G^{+}Z_2 + F^{+}Z_1$$

where $Z_1 = I_0(\bar{\kappa}_{1r}r)$. If diffusion theory is also assumed to hold for thermals*, $Z_2 = I_0(\bar{\kappa}_{2r}r)$. Substitution yields

$$d_2^{+} = \frac{D_2}{D_{2r}\delta}$$

where $$\delta = \frac{[Z_2']}{[Z_2]} = \frac{\bar{\kappa}_{2r}I_1(\bar{\kappa}_{2r}a)}{I_0(\bar{\kappa}_{2r}a)}$$

with a as the rod radius and $\bar{\kappa}_{2r}^2 = \frac{1}{L^2} + \left(\frac{\pi}{H'}\right)^2$.

We note that for $\bar{\kappa}_{2r}a \gg 1$, $I_0 \rightarrow I_1$, $\delta \rightarrow \bar{\kappa}_{2r} \rightarrow \infty$ and $d_2^{+} \rightarrow 0$. The adjoint thermal is seen to obey approximately the same boundary condition as the ordinary thermal flux. Next, the expression for d_1^{+} is formed. It will be

$$d_1^{+} = \frac{D_1}{D_{1r}} \frac{S_3^{+}G[Z_2] + F^{+}[Z_1]}{S_3^{+}G^{+}[Z_2'] + F^{+}[Z_1^{+}]}$$

Invoking the relations≠

$$G^{+}[Z_2] = \beta - \alpha$$

$$F^{+}[Z_1] = S_1^{+}(\beta - \rho_2\delta) + S_2^{+}(\rho_2\delta - \alpha) - S_3^{+}(\beta - \alpha)$$

---

* Normally, a thermal extrapolation distance in the range $\frac{4}{3}\lambda_t$ to $0.71\ \lambda_t$ is employed, based on transport theory.

≠ Raymond L. Murray, "Perturbation Theory and Applications to Reactors", APAE Memo No. 208, 1959.

$$G^{+}[Z_2'] = G^{+}[Z_2]\delta$$

$$F^{+}[Z_1'] = F^{+}[Z_1]\gamma$$

We obtain

$$d_1^{+} = \frac{D_1}{D_{1r}}\left(\frac{N+1}{N\delta+\gamma}\right)$$

where

$$N = \frac{S_3 G^{+}[Z_2]}{F^{+}[Z_1]}$$

An analytical and numerical investigation (not reproduced here) of typical values of the constants was performed, noting that $\bar{\nu} >> \bar{\mu}$. The approximate result was

$$d_1^{+} \simeq \frac{D_1}{D_{1r}\gamma}$$

which is the same as d_1, the fast extrapolation distance for the ordinary fluxes. It thus is shown that use of d_1 and d_2 is appropriate for adjoint extrapolation distances. There is good physical logic behind this result--If the rod has strong thermal absorption and is not fissionable, ϕ_{2r}^{+} is governed by

$$D_{2r}\nabla^2\phi_{2r}^{+} - \phi_{2r}^{+}\Sigma_{2r} = 0$$

for which ϕ_{2r}^{+} drops off very rapidly inside the rod. This gives a negligible fast adjoint source, and thus ϕ_{1r}^{+} is approximately the solution of

$$D_{1r}\nabla^2\phi_{1r}^{+} - \phi_{1r}^{+}\Sigma_{1r} = 0$$

which gives the same solution as for the ordinary flux.

With the above basis, we proceed to the determination of the adjoint flux in the case of a symmetric ring of M rods plus a central rod. Now

$$\phi_1^+ = S_1^+ X^+ + S_2^+ Y^+$$

$$\phi_2^+ = X^+ + Y^+$$

where

$$X^+ = \sum_S A^+ J_S(\bar{\mu} r)\cos S\theta + \sum_{m=0}^{M} B^+_{0m}\, Y_0(\mu r_m)$$

$$Y^+ = \sum_S C^+ I_S(\bar{\nu} r)\cos S\theta + \sum_{m=0}^{M} E^+_{0m}\, K_0(\nu r_m)$$

The procedure described in previous work* was applied in detail (see Appendix) with the surprising result that again $\phi_1^+ = S_1^+\phi_1$, $\phi_2^+ = \phi_2/S_1$. The result is of considerable value in the determination of $|\delta B^2|$, since the products of adjoint and ordinary fluxes develop no new quantities to be integrated. The following products appear:

$$\phi_1^+\phi_1$$

$$\phi_1^+\phi_2$$

$$\phi_2^+\phi_1$$

$$\phi_2^+\phi_2$$

or X^2, XY, and Y^2.

It will be convenient to express X and Y as composed of two types of sums, those containing no singularities, X_A and Y_A

---

* Raymond L. Murray, "Theory of Asymmetric Arrays of Control Rods in Nuclear Reactors", APAE 48 (1959).

and those with singularities X_B and Y_B. For the ith zone of a reactor with general array of rods, these are

$$X_{Ai} = \sum_{S=-\infty}^{\infty} J_{Si}(\bar{\mu}_i r)(A_{Si} \cos S\theta + N_{Si} \sin S\theta)$$

$$Y_{Ai} = \sum_{S=-\infty}^{\infty} I_{Si}(\bar{\nu}_i r)(C_{Si} \cos S\theta + Q_{Si} \sin S\theta)$$

$$X_{Bi} = \sum_{m=0}^{M_i} Y_0(\bar{\mu}_i r_m) B_{Omi}$$

$$Y_{Bi} = \sum_{m=0}^{M_i} K_0(\bar{\nu}_i r_m) E_{Omi}$$

In all cases, the constants A_{Si}, N_{Si}, C_{Si}, Q_{Si}, B_{Omi}, E_{Omi} will be known.

Further breakdown of products then is as follows:

$$X^2 = X_A^2 + 2X_A X_B + X_B^2$$

$$Y^2 = Y_A^2 + 2Y_A Y_B + Y_B^2$$

$$XY = X_A Y_A + X_A Y_B + X_B Y_A + X_B Y_B$$

The number of integrals appears at this stage to be very large. It will be shown that many of these vanish by orthogonality and symmetry properties, and that those remaining are of a rather standard pattern. We consider these according to several classes.

$X_A^2,\ Y_A^2,\ X_AY_A$

The integrals are

$$\int_0^R\int_0^{2\pi}\left[\left(\sum_S J_S(\mu r)\left(A_S\cos S\theta + N_S\sin S\theta\right)\right]^2 r\,dr\,d\theta$$

$$\int_0^R\int_0^{2\pi}\left[\sum_S I_S(\nu r)\left(C_S\cos S\theta + Q_S\sin S\theta\right)\right]^2 r\,dr\,d\theta$$

$$\int_0^R\int_0^{2\pi}\left[\sum_S J_S(\mu r)\left(A_S\cos S\theta + N_S\sin S\theta\right)\right]\left[\sum_S I_S(\nu r)C_S\cos S\theta + Q_S\sin S\theta\right] r\,dr\,d\theta$$

We first note the result of integration over the angular range $0-2\pi$ of the typical products of trigonometric functions.

$$\int_0^{2\pi}\sin^2 S\theta\,d\theta = \int_0^{2\pi}\cos^2 S\theta\,d\theta = \begin{cases}\pi & \text{for } S > 0\\ 2\pi & \text{for } S = 0\end{cases}$$

$$\left.\begin{aligned}\int_0^{2\pi}\sin S\theta\,\sin k\theta\,d\theta &= 0\\ \int_0^{2\pi}\sin S\theta\,\cos k\theta\,d\theta &= 0\\ \int_0^{2\pi}\cos S\theta\,\cos k\theta\,d\theta &= 0\end{aligned}\right\}\quad \text{for all } S \text{ and } k$$

This reveals that all cross product terms vanish. The double sum $\sum_{S=-\infty}^{\infty}$ brings in factors of two multiplying all Bessel functions of order greater than zero. The final result in compact form is

$$\int X_A^2 dS = 2\pi \sum_{S=-\infty}^{\infty} (A_S^2 + N_S^2) \int_0^R J_S^2(\mu r) r dr$$

$$\int Y_A^2 dS = 2\pi \sum_{S=-\infty}^{\infty} (C_S^2 + Q_S^2) \int_0^R I_S^2(\nu r) r dr$$

$$\int X_A Y_A dS = 2\pi \sum_{S=-\infty}^{\infty} (A_S C_S + N_S Q_S) \int_0^R J_S(\mu r) I_S(\nu r) r dr$$

<u>X_B^2, Y_B^2, $X_B Y_B$</u>

These integrals are of the form

$$\int B_{0m}^2 E_{0m} Y_0(\mu r_m) K_0(\nu r_m) dS$$

$$\int B_{0m}^2 Y_0^2(\mu r_m) dS$$

$$\int E_{0m}^2 K_0^2(\nu r_m) dS$$

We make an assumption that is consistent with the basic rod theory, that the flux sinks are large near the rod surface and relatively small far from the rod. The integrations may be performed over the core as if the rod were located at the center, i.e.

$$B_{0m}^2 \int_0^R Y_0^2(\mu r_m) 2\pi r_m dr_m = 2\pi B_{0m}^2 \int_0^R Y_0^2(\mu r) r dr$$

with the corresponding second kind, $2\pi E_{0m}^2 \int_0^R K_0^2(\nu r) r dr$

Great simplification can be effected if a translation of one function into the coordinate system of the other is made, as in applying boundary conditions in the basic rod theory. Since the K_0 function varies most rapidly, we choose to expand approximately

$$Y_0(\mu r_n) \cong \sum_{k=-\infty}^{\infty} (-1)^k Y_k(\mu r_{mn}) J_k(\mu r_m) \cos k\Psi_m$$

then the integral over Ψ_m vanishes except for $k = 0$, when it is 2π. Thus the expression becomes

$$2\pi B_{0m} E_{0m} Y_0(\mu r_{mn}) \int_a^R J_0(\mu r_m) K_0(\nu r_m) r_m dr_m$$

$X_A X_B$, $Y_A Y_B$, $X_A Y_B$, $X_B Y_A$

A typical integral is

$$\int A_S J_S(\mu r) \cos S\theta \; Y_0(\mu r_m) M_{0m} dS$$

Generally, the integral is dominated by the value of the rapidly varying function Y_0, near the rod radius a. Also, for purposes of integration on r_m, we shall allow the range to be $a < r < R$. Expand the ordinary Bessel function as

$$J_S(\mu r) \cos S\theta = \sum_{k=-\infty}^{\infty} (-1)^k J_{S+k}(\mu d_m) J_k(\mu r_m) \cos(S\beta_m - k\Psi_m)$$

and let

$$\cos(S\beta_m - k\Psi_m) = \cos S\beta_m \cos k\Psi_m + \sin S\beta_m \sin k\Psi_m)$$

Typical angular integrals will be $\int_0^{2\pi} \cos k\Psi_m d\Psi_m = 0$ except for $k = 0$, when it is 2π. Also $\int_0^{2\pi} \sin k\Psi_m d\Psi_m = 0$ for all k. The integral becomes

$$2\pi A_S B_{0m} J_S(\mu d_m) \cos S\beta_m \int_a^R J_0(\mu r_m) Y_0(\mu r_m) r_m dr_m$$

For the case involving $\sin S\theta$, replacement of $\cos S\beta_m$ by $\sin S\beta_m$ is made. The integral over r is straightforward. By analogy, all other integrals in this group may be constructed.

Bessel Integrals

The integration of the nine distrinct types of Bessel function products is readily effected using formulas from Watson*, adapted as needed to functions of imaginary argument. The details of the integration appear in the Appendix, with only the summary of working formulas listed in the Table, identified as to location.

---

* G. N. Watson, Theory of Bessel Functions, Cambridge (1944)

TABLE 1

In the table ordinary Bessel functions J_S, Y_S have μR or μa as arguments, while modified Bessel functions I_S, K_S have νR or νa as arguments.

$$X_A^2 : \int_0^R J_S^2(\mu r)\,r\,dr = \frac{R^2}{2}\left[J_S^2 - \frac{2S}{\mu R} J_S J_{S-1} + J_{S-1}^2\right]$$

$$Y_A^2 : \int_0^R I_S^2(\nu r)\,r\,dr = \frac{R^2}{2}\left[I_S^2 + \frac{2S}{\nu R} I_S I_{S-1} - I_{S-1}^2\right]$$

$$X_A Y_A = \int_0^R J_S(\mu r) I_S(\nu r)\,r\,dr = R\left[\frac{\mu J_{S+1} I_S + \nu J_S I_{S+1}}{\mu^2 + \nu^2}\right]$$

$$X_B^2 : \int_a^R Y_0^2(\mu r)\,r\,dr = \left\{\frac{X^2}{2}\left[Y_0^2 + Y_1^2\right]\right\}_a^R$$

$$Y_B^2 : \int_a^R K_0^2(\nu r)\,r\,dr = \left\{\frac{X^2}{2}\left[K_0^2 - K_1^2\right]\right\}_a^R$$

$$\begin{matrix} X_B Y_B \\ \text{and} \\ X_A Y_A \end{matrix} = \int_0^R J_0(\mu r) K_0(\nu r)\,r\,dr = \left\{\frac{X(\mu J_1 K_0 - \nu J_0 K_1)}{\mu^2 + \nu^2}\right\}_a^R$$

$$X_A X_B = \int_0^R Y_0(\mu r) Y_0(\mu r)\,r\,dr = \left\{\frac{X^2}{2}\left[J_0 Y_0 + J_1 Y_1\right]\right\}_a^R$$

$$Y_A Y_B = \int_0^R I_0(\nu r) \quad \nu r)\,r\,dr = \left\{\frac{X^2}{2}\left[I_0 K_0 + I_1 K_1\right]\right\}_a^R$$

$$Y_A X_B = \int_0^R I_0(\nu r) Y_0(\mu r)\,r\,dr = \left\{\frac{X(\mu Y_1 I_0 + \nu Y_0 I_1)}{\mu^2 + \nu^2}\right\}_a^R$$

Flux Amplitude and Integrals

The axial distribution has been taken to be $\sin \frac{\pi z}{H}$ over the whole set of zones. This may be viewed as referring to the distribution of each of the average fluxes. At a given level z in a zone i, the radial fast flux will be of the form

$$\mathcal{a}_{1i} \mathcal{R}_{1i}(r,\theta)$$

where the relative amplitudes of the various functions in $\mathcal{R}_{1i}(r,\theta)$ are correct, and one convenient arbitrary constant has been set equal to 1. We let $\mathcal{a}_{1i}$ have a value for every zone such that $\int \mathcal{a}_{1i} \mathcal{R}_{1i}(r,\theta)dS$ is equal to a constant, which may just as well be unity. Integrating, $\mathcal{a}_i = \left[\int_0^{2\pi} \int_0^R \mathcal{R}_{1i}(r,\theta)dS\right]^{-1}$. The complete flux then becomes

$$\phi_{1i}(r,\theta,z) = \frac{\mathcal{R}_{1i}(r,\theta) \sin \pi z/H}{\int \mathcal{R}_{1i}(r,\theta)dS}$$

A similar approach is applied to the thermal flux. For purposes of buckling calculations, the adjoint fluxes need not be computed, since the relations $\phi_1^+ = S_1^+\phi_1$ and $\phi_2^+ = \phi_2/S_1$ connect them with the ordinary fluxes. The correct physical description would entail imposing continuity of fast and thermal flux and current at each interface between zones. Since the whole problem is basically of the non-separable type, we would expect to employ more relaxed boundary conditions, such as continuity of area flux and current, i.e.

$$\int \phi_i \, dS = \int \phi_j \, dS$$

$$\int J_i \, dS = \int J_j \, dS$$

where the matrices are

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \text{ and } J = \begin{bmatrix} -D_1\phi_1' \\ -D_2\phi_2' \end{bmatrix}$$

In addition we would wish to provide continuity of adjoint thermal and fast flux and current.

$$\int \phi_i^+ \, dS = \int \phi_j^+ \, dS$$

$$\int J_i^+ \, dS = \int J_j^+ \, dS$$

where $\phi^+ = \left[\phi_1^+, \ \phi_2^+\right], \quad J^+ = \left[-D_1\phi_1^{+'}, \ -D_2\phi_2^{+'}\right]$

The normalization adopted satisfies the above boundary conditions at a level h. As now demonstrated consider the fast flux for example.

$$\int \phi_{1i}(r,\theta,h)\,dS = \int \phi_{1j}(r,\theta h)\,dS$$

which is

$$\frac{\sin\frac{\pi h}{H}\int \mathcal{R}_{1i}(r,\theta)\,dS}{\int \mathcal{R}_{1i}(r,\theta)\,dS} = \frac{\sin\frac{\pi h}{H}\int \mathcal{R}_{1j}(r,\theta)\,dS}{\int \mathcal{R}_{1j}(r,\theta)\,dS}$$

This is seen to be an identity. The same proofs obtain for the thermal flux.

Several integrals of the flux functions appearing in

$$\phi_1 = X + Y, \ \phi_2 = S_1X + S_2Y$$

with $X = X_A + X_B, \ Y = Y_A + Y_B$

$$X_A = \sum_S J_S(\mu r)(A_S \cos S\theta + N_S \sin S\theta)$$

$$Y_A = \sum_S I_S(\nu r)(C_S \cos S\theta + Q_S \sin S\theta)$$

$$X_B = \sum_{m=0}^{M} Y_0(\mu r_m) B_{0m}$$

$$Y_B = \sum_{m=0}^{M} K_0(\nu r_m) E_{0m}$$

The integral from 0 to 2π of $\cos S\theta$ vanishes except for $S = 0$, when it is 2π. All integrals of $\sin S\theta$ are zero. Thus

$$\int X_A dS = 2\pi A_0 \int_0^R J_0(\mu r) r dr = \frac{2\pi A_0}{\mu} R J_1(\mu R)$$

$$\int Y_A dS = 2\pi C_0 \int_0^R I_0(\nu r) r dr = \frac{2\pi C_0}{\nu} R I_1(\nu R)$$

$$\int X_B dS \simeq \sum_{m=0}^{M} 2\pi B_{0m} \int_a^R Y_0(\mu r_m) r_m dr_m = 2\pi \sum_{m=0}^{M} \frac{B_{0m}}{\mu} \left[R^2 Y_1(\mu R) - a^2 Y_1(\mu a)\right]$$

$$\int Y_B dS \simeq \sum_{m=0}^{M} 2\pi E_{0m} \int_a^R K_0(\nu r_m) r_m dr_m = -2\pi \sum_{m=0}^{M} \frac{E_{0m}}{\nu} \left[R^2 K_1(\nu R) - a^2 K_1(\nu a)\right]$$

These are readily evaluated.

All of the necessary relations are now available for the computation of the buckling of a reactor with any number of rods having an arbitrary degree of insertion or removal. In a subsequent report, a numerical example will be given to illustrate and emphasize the power of the approach.

APPENDIX

Derivation of Adjoint Fluxes in System with Control Rods.

The adjoint fluxes are given in the case of a symmetric array by

$$\phi_1^+ = S_1^+X^+ + S_2^+Y^+$$

$$\phi_2 = X^+ + Y^+$$

where

$$X^+ = \sum_S S_S^+ \, J_S(\mu r)\cos S\theta + \sum_{m=0}^{M} B_{0m}^+ Y_0(\mu r_m)$$

$$Y^+ = \sum_S C_S^+ \, I_S(\nu r)\cos S\theta + \sum_{m=0}^{M} E_{0m}^+ K_0(\nu r_m)$$

Upon making the transformations of coordinates to the vicinity of the mth rod, as in APAE No. 48, we obtain

$$X^+(r_m) = J_0(\mu r_m)\left[\sum_{n\neq m}^{M} B_{0n}^+ Y_0(\mu r_{mn}) - \sum_{n=0}^{M} B_{0n}^+ \textcircled{H}_{mn}\right] + B_{0m}^+ Y_0(\mu r_m)$$

$$Y^+(r_m) = I_0(\nu r_m)\left[\sum_{n\neq m}^{M} E_{0n}^+ K_0(\nu r_{mn}) - \sum_{n=0}^{M} E_{0n}^+ \textcircled{H}_{mn}\right] + E_{0m}^+ K_0(\nu r_m)$$

The bracket in Y^+ may be neglected and that in X^+ symbolized by [ ] . Boundary conditions are applied at each rod

$$\frac{\phi_1^+}{(\phi_1^+)'} = d_1 \qquad \frac{\phi_2^+}{(\phi_2^+)'}$$

or

$$S_1^+\left(\frac{X^+}{d_1} - X^{+'}\right) + S_2^+\left(\frac{Y^+}{d_1} - Y^{+'}\right) = 0$$

$$\left(\frac{X^+}{d_2} - X^{+'}\right) + \left(\frac{Y^+}{d_2} - Y^{+'}\right) = 0$$

Apply at the specific mth rod of radius a_m. Then

$$S_1^+ \left[\frac{J_0(\mu a_m)}{d_1} + \mu J_1(\mu a_m) \right] [\,] + B_{0m}^+ S_1^+ \left[\frac{Y_0(\mu a_m)}{d_1} + \mu Y_1(\mu a_m) \right]$$
$$+ E_{0m}^+ S_2^+ \left[\frac{K_0(\nu a_m)}{d_1} + \nu K_1(\nu a_m) \right] = 0$$

$$\left[\frac{J_0(\mu a_m)}{d_2} + \mu J_1(\mu a_m) \right] [\,] + B_{0m} \left[\frac{Y_0(\mu a_m)}{d_2} + \mu Y_1(\mu a_m) \right]$$
$$+ E_{0m}^+ \left[\frac{K_0(\nu a_m)}{d_2} + \nu K_1(\nu a_m) \right] = 0$$

Abbreviating in the usual manner,

$$S_1^+ a_{31m} [\,] + B_{0m}^+ S_1^+ a_{32m} + E_{0m}^+ S_2^+ a_{34m} = 0$$

The second equation may be written

$$\frac{a_{41m}}{S_1} [\,] + B_{0m}^+ \frac{a_{42m}}{S_1} + E_{0m}^+ \frac{a_{44m}}{S_2} = 0$$

Multiply the first by a_{41m}/S_1, the second by $S_1^+ a_{31m}$ and subtract. The result is

$$B_{0m}^+ \frac{S_1^+}{S_1} (a_{41m} a_{32m} - a_{31m} a_{42m}) = E_{0m}^+ \left(a_{31m} a_{44m} \frac{S_1^+}{S_1} - a_{41m} a_{34m} \frac{S_2^+}{S_1} \right)$$

or since $\dfrac{S_1^+}{S_2} = \dfrac{S_2^+}{S_1}$,

$$E_{0m}^+ = B_{0m}^+ \frac{S_1^+}{S_2^+} \left(\frac{a_{41m} a_{32m} - a_{31m} a_{42m}}{a_{31m} a_{44m} - a_{34m} a_{41m}} \right)$$

which may be abbreviated

$$E_{0m}^+ B_{0m}^+ \frac{S_1^+}{S_2^+} \gamma_m$$

The corresponding expression for the ordinary flux coefficients* is

$$E_{0m} = B_{0m} \gamma_m$$

* Raymond L. Murray, "Analytical Flux Distributions in Reactors with Multiple Control Rods," RLM-94, April 7, 1960

Now the coefficients A_S^{+} and C_S^{+} are proportional to sums over coefficients B_{0m}^{+} and E_{0m}^{+} as follows:

$$A_S^{+} \sim \sum_{m=0}^{M} a_m B_{0m}^{+}$$

$$C_S^{+} \sim \sum_{m}^{M} c_m E_{0m}^{+}$$

If these are inserted in the expressions for X^{+} and Y^{+}, one obtains, schematically,

$$X^{+} = \sum_m (x) B_{0m}^{+}$$

$$Y^{+} = \sum_m (y) E_{0m}^{+}$$

where (x) and (y) are combinations of Bessel functions. Also,

$$\phi_1^{+} = \sum_m \left[(x) S_1^{+} B_{0m}^{+} + (y) S_2^{+} E_{0m}^{+} \right]$$

$$\phi_2 = \sum_m \left[(x) B_{0m}^{+} + (y) E_{0m}^{+} \right]$$

Now insert the relation between E_{0m}^{+} and B_{0m}^{+} in the fast adjoint flux

$$\phi_1^{+} = \sum_m \left[(x) S_1^{+} B_{0m}^{+} + (y) S_1^{+} B_{0m}^{+} \right]$$

$$\phi_1^{+} = S_1^{+} \sum_m \left[(x) + (y) \right] B_{0m}^{+}$$

With any given one of the B_{0m}^{+} arbitrary, and the rest determined from it in exactly the same way as for the ordinary flux, this may be written

$$\phi_1^{+} = S_1^{+} \sum_m \left[(x) + (y) \right] B_{0m} = S_1^{+} \phi_1$$

Similarly

$$\phi_2^+ = \sum_m \left[(x) B_{0m}^+ + (y) \frac{S_1^+}{S_2^+} B_{0m}^+ \right]$$

However $\frac{S_1^+}{S_2^+} = \frac{S_2}{S_1}$

Rearranging, identifying B_{0m}^+ and B_{0m},

$$\phi_2^+ = \frac{1}{S_1} \sum_m \left[(x) S_1 B_{0m} + (y) S_2 B_{0m} \right] = \frac{\phi_2}{S_1}$$

We thus conclude that the relations

$$\phi_1^+ = S_1^+ \phi_1 \text{ and } \phi_2^+ = \frac{\phi_2}{S_1}$$

hold for the case of a reactor with multiple rods.

APPENDIX: BESSEL FUNCTIONS

$$\mathcal{I}_1 = \int_0^R J_S^2(\mu r) r dr = \frac{1}{2} r^2 \left\{ \left[1 - \frac{S^2}{(\mu r)^2} \right] J_S^2(\mu r) + \left[J_S'(\mu r) \right]^2 \right\}_0^R$$

where the derivative is with respect to the argument.

However, $J_S' = J_{S-1} - J_S$

Substituting,

$$\mathcal{I}_1 = \frac{R^2}{2} \left[J_S^2(\mu R) - \frac{2S}{\mu R} J_{S-1}(\mu R) J_S(\mu R) + J_{S-1}^2(\mu R) \right]$$

For the special case $S = 0$, noting that $J_{-1} = -J_1$, this becomes

$$\mathcal{I}_{10} = \frac{R^2}{2} (J_0^2 + J_1^2)$$

....

$$\mathcal{I}_2 = \int_0^R I_S^2(\nu r) r dr$$

Invoke the relation $I_S(z) = \frac{1}{i^S} J_S(iz)$ and let $i\nu r = x$, $i\nu R = X$

$$\mathcal{I}_2 = \frac{1}{\nu^2 i^{2s}} \int_0^X J_S^2(x) x dx$$

$$= \frac{1}{\nu^2 i^{2s}} \frac{(\nu R)^2}{2} \left[i^{2s} I_S^2(\nu R) - \frac{2S i^{2s-1}}{i\nu R} I_{S-1}(\nu R) I_S(\nu R) + i^{2s-2} I_{S-1}^2(\nu R) \right]$$

$$= \frac{1}{\nu^2 i^{2s}} \frac{X^2}{2} \left[J_S^2(X) - \frac{2S}{X} J_{S-1}(X) J_S(X) + J_{S-1}^2(X) \right]$$

$$\mathcal{I}_2 = \frac{R^2}{2} \left[I_S^2(\nu R) + \frac{2S}{\nu R} I_{S-1}(\nu R) I_S(\nu R) - I_{S-1}^2(\nu R) \right]$$

....

$$\mathcal{I}_3 = \int_0^R J_S(\mu r) I_S(\nu r) r dr$$

From Watson, p. 134, the result for $\int J_S(\mu r) I_S(\nu r) r dr$ is obtained as

$$r \frac{\left[\mu J_{S+1}(\mu r) J_S(\nu r) - \nu J_S(\mu r) J_{S+1}(\nu r)\right]}{\mu^2 - \nu^2}$$

Invoking $I_S(\nu r) = \frac{1}{i^S} J_S(\nu r)$ and substituting yields

$$\mathcal{L}_3 = R\frac{\left[\mu J_{S+1}(\mu R) I_S(\nu R) + \nu J_S(\mu R) I_{S+1}(\nu R)\right]}{\mu^2 + \nu^2}$$

$$\mathcal{L}_4 = \int_a^R Y_0^2(\mu r) r dr.$$

This obeys the same rules as J_0, but with a lower limit a, hence

$$\mathcal{L}_4 = \frac{R^2}{2} \left[Y_0^2(\mu R) + Y_1^2(\mu R)\right] - \frac{a^2}{2} \left[Y_0^2(\mu a) + Y_1^2(\mu a)\right]$$

$$\mathcal{L}_5 = \int_a^R K_0^2(\nu r) r dr = \frac{1}{\nu^2} \int_{\nu a}^{\nu R} K_0^2(x) x dx$$

$$= \frac{x^2}{2\nu^2} \left[K_0^2(x) - K_1^2(x)\right]^{\nu R}$$

$$= \frac{R^2}{2} \left[K_0^2(\nu R) - K_1^2(\nu R)\right] - \frac{a^2}{2} \left[K_0^2(\nu a) - K_1(\nu a)\right]$$

$$\mathcal{L}_6 = \int_a^R J_0(\mu r) K_0(\nu r) r dr$$

This is evaluated by trial and error, setting

$$\int J_0(\mu r) K_0(\nu r) r dr = r \left[A J_0(\mu r) K_1(\nu r) + B K_0(\nu r) J_1(\mu r)\right],$$

differentiating, and determining correct values of A and B to be

$$A = \frac{-\nu}{\mu^2 + \nu^2}, \quad B = \frac{\mu}{\mu^2 + \nu^2}, \text{ thence}$$

$$\mathcal{I}_6 = \left[\frac{-\nu r J_0(\mu r) K_1(\nu r) + \mu r J_1(\mu r) K_0(\nu r)}{\mu^2 + \nu^2} \right]_a^R$$

. . . .
. . . .

$$\mathcal{I}_7 = \int_a^R J_0(\mu r) Y_0(\mu r) r dr$$

From Watson, p. 134, equation (8),

$$\int z C_\mu(kz) \bar{C}_\mu(kz) dz = -\frac{z}{2k} \Big\{ kz C_{\mu+1}(kz) \bar{C}'_\mu(kz) - kz C_\mu(kz) \bar{C}'_{\mu+1}(kz)$$

$$- C_\mu(kz) \bar{C}_{\mu+1}(kz) \Big\}$$

Letting $\bar{C}_\mu(kz) \sim J_0(\mu r)$

$\bar{C}_\mu(kz) \sim Y_0(\mu r)$

This yields

$$-\frac{r}{2\mu} \Big\{ \mu r J_1(\mu r) Y'_0(\mu r) - \mu r J_0 Y'_1(\mu r) - J_0(\mu r) Y'_1(\mu r) \Big\}$$

Then with $Y'_0 = -Y_1$, $Y'_1 = Y_0 - \frac{Y_1}{x}$, it becomes

$$-\frac{r}{2\mu} \Big\{ -\mu r J_1 Y_1 - \mu r J_0 Y_0 + J_0 Y_1 - J_0 Y_1 \Big\} = \frac{r^2}{2} (J_0 Y_0 + J_1 Y_1)_a^R$$

Thus

$$\mathcal{I}_7 = \frac{R^2}{2} \Big[J_0(\mu R) Y_0(\mu R) + J_1(\mu R) Y_1(\mu R) \Big]$$

$$- \frac{a^2}{2} \Big[J_0(\mu a) Y_0(\mu a) + J_1(\mu a) Y_1(\mu a) \Big]$$

. . . .
. . . .

$$\mathscr{I}_7 = \int_a^R I_0(\nu r)K_0(\nu r)\, r\,dr$$

By analogy with $\mathscr{I}_7$, suppose the integral is

$$\frac{r^2}{2}\left[I_0(\nu r)K_0(\nu r) + I_1(\nu r)K_1(\nu r)\right]$$

Differentiation and comparison verifies this assumption.

$$\mathscr{I}_7 = \frac{R^2}{2}\left[I_0(\nu R)K_0(\nu R) + I_1(\nu R)K_1(\nu R)\right] - \frac{a^2}{2}\left[I_0(\nu a)K_0(\nu a) + I_1(\nu a)K_1(\nu a)\right]$$

$$\mathscr{I}_8 = \int_a^R I_0(\nu r)Y_0(\mu r)\, r\,dr$$

Use of the relations $I_0(\nu r) = J_0(i\nu r)$ and $I_1(\nu r) = \frac{1}{i}J_1(i\nu r)$ in conjunction with $\mathscr{I}_6$ yields

$$\mathscr{I}_8 = R\,\frac{\left[\mu Y_1(\mu R)I_0(\nu R) + \nu Y_0(\mu R)I_1(\nu R)\right]}{\mu^2 + \nu^2}$$

$$- a\,\frac{\left[\mu Y_1(\mu a)I_0(\nu a) + \nu Y_0(\mu a)I_1(\nu a)\right]}{\mu^2 + \nu^2}$$