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A Relativity Notebook for Monte Carlo Practice

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A Relativity Notebook for Monte Carlo Practice

by

C. J. Everett

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FOREWORD

This "notebook" is the result of an attempt to organize the basic ideas and formulas of special relativity into a form convenient for Monte Carlo treatment of relativistic systems of point particles.

Some mathematical sidelights have been included which may interest the beginner, as well as rouse the ire of physicists, who certainly should not take them seriously.

ACKNOWLEDGMENT

The author is deeply indebted to Barbara Powell for all the fine figures, so much clearer than the text.

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CHAPTER I

DYNAMICS OF A POINT PARTICLE

1. The parameters of a particle. In an inertial frame Σ of events (R,t), a definite type of particle (i) is assigned a constant characteristic mass $m_i \geqq 0$. For a "material" particle (electron, meson, nucleon, $\cdots$) m_i is positive, being the "rest mass," or mass of the particle at rest in Σ, whereas an "immaterial" particle (photon, neutrino, $\cdots$) is assigned a ch. mass $m_i = 0$.

A particle moving on a trajectory $(R_i(t),t)$ has velocity $V_i = \dot{R}_i$, and speed $v_i = |V_i| \geqq 0$. If $m_i = 0$, v_i has the constant value $c \cong 3 \times 10^{10}$ cm/sec, but is confined to the range $0 \leqq v_i < c$ if $m_i > 0$. It is customary to write $\beta \equiv v_i/c$, and, for $m_i > 0$, also $\gamma_i = 1/(1-\beta_i^{\,2})^{\frac{1}{2}}$.

At time t, every particle has a positive mass $M_i = M_i(t)$. For a material particle, M_i is speed-dependent, being by definition

$$M_i = m_i\gamma_i \geqq m_i > 0. \tag{1}$$

However, the mass M_i of an immaterial particle is an independent parameter, which may have any positive value.

The energy of a particle is defined to be $E_i \equiv M_i c^2 > 0$, its characteristic energy ("rest energy" if $m_i > 0$) being $e_i \equiv m_i c^2$. The excess $k_i \equiv E_i - e_i$ of E_i over e_i is called the kinetic energy of the particle.

The momentum of a particle is defined as the vector $P_i = M_i V_i$, of magnitude $p_i = |P_i| = M_i v_i$.

A particle with $v_i > 0$ at time t has a well-defined direction, namely, the unit vector $\Psi_i = v_i^{-1} V_i = p_i^{-1} P_i$, of magnitude $|\Psi_i| = 1$. If the spatial trajectory $R_i = R_i(t)$ is so parameterized by arc-length s_i that $ds_i/dt > 0$, then the relation

$$v_i \Psi_i = V_i = (dR_i/ds_i)\cdot(ds_i/dt)$$

shows that $v_i = ds_i/dt$, and $\Psi_i = dR_i/ds_i$, the latter being the geometric direction of the trajectory tangent.

Finally, every particle is assigned a frequency $\nu_i = E_i/h$, and (if $v_i > 0$) a wave-length $\lambda_i = h/p_i$, where h is Planck's constant. Note here that

$$\lambda_i \nu_i = E_i/p_i = M_i c^2/M_i v_i = c/\beta_i \geqq c$$

with equality ($\lambda_i \nu_i = c$) in case $m_i = 0$.

We are able to treat particles of both kinds ($m_i \geqq 0$) in a uniform way by virtue of the following basic

Theorem 1. A number M and vector P are possible values for the mass and momentum of a particle of ch. mass m_i if and only if they satisfy the "validity condition"

$$M > 0 \quad \& \quad P^2 = c^2(M^2 - m_i^2). \tag{2}$$

Proof. Case 1. ($m_i > 0$) By (1), the mass $M_i = m_i\gamma_i > 0$ of such a particle satisfies the equation $M_i^2\beta_i^2 = M_i^2 - m_i^2$; hence $P_i^2 \equiv M_i^2 v_i^2 = c^2(M_i^2 - m_i^2)$ as in (2). Conversely, (2) implies that a velocity V, <u>defined</u> by P = MV, has magnitude $v = |V| < c$, and that $M = m_i/(1-v^2c^{-2})^{\frac{1}{2}}$ as required by (1).

Case 2. ($m_i = 0$) Here, condition (2) is equivalent to $M > 0$ & $|P| = Mc$, which obviously obtains for the mass M_i and momentum $P_i \equiv M_i V_i$ of an immaterial particle. Conversely, defining V by P = MV for such a pair M, P, we have $M > 0$ and $|V| = c$, which is all that is required of an immaterial particle.

The net <u>force</u> acting on a particle is by definition $F_i \equiv \dot{P}_i$, a <u>free</u> particle being one with $F_i \equiv 0$, hence with P_i, M_i, and V_i constant on its straight line trajectory $R_i(t) = R_i^o + V_i t$.

The parameters (P_i, M_i, m_i), although dependent, as required by the validity condition (2), do completely characterize a free point particle at an event (R,t). They are adopted because of their intuitive relation to classical mechanics, and the simplicity of their transformation to other inertial frames.

The closely related "energy-parameters" (cP_i, E_i, e_i), expressed in any convenient energy unit, are preferable in computations, and may be used interchangeably, by an obvious scaling.

For example, one may verify the basic relations:

$$E_i = e_i + k_i \tag{3}$$

$$E_i^2\beta_i^2 = (cp_i)^2 = E_i^2 - e_i^2$$

$$\gamma_i = E_i/e_i \qquad (e_i > 0)$$

$$\Psi_i = cP_i/cp_i \qquad (cp_i \neq 0)$$

Notes 1.

1. The work done by a force F_i on a particle between points 1 and 2 of its trajectory is

$$w \equiv \int_1^2 F_i \cdot \Psi_i ds_i = \int_1^2 F_i \cdot \dot{R}_i dt \qquad (ds_i/dt > 0) .$$

Differentiating (2), one has $\dot{P}_i \cdot P_i = M_i \dot{M}_i c^2$. Since $\dot{P}_i \equiv F_i$ and $P_i \equiv M_i \dot{R}_i$ with $M_i > 0$, this implies

$$F_i \cdot \dot{R}_i = \dot{M}_i c^2 \tag{4}$$

Consequently

$$w = \int_1^2 \dot{M}_i c^2 dt = \Delta(M_i c^2) = \Delta(k_i).$$

Classically, for a force $F = -\text{grad}\ \Phi(R)$, we have also

$$w = \int_1^2 F_i \cdot \dot{R}_i dt = -\int_1^2 \dot{\Phi}(R_i(t)) dt = -\Delta\Phi.$$

In such a case, it follows that $E + \Phi$ and $k + \Phi$ are constant on the trajectory.

2. A positive electron ϵ^+ of charge q esu (TABLE I) in an electrostatic field due to a potential $\varphi(R)$ esu volts is subject to a force $F = -\text{grad}\, q\varphi(R)$ dyne. If the difference in potential between points 1, 2 of its resulting trajectory is $10^8/c$ esu volts ($\equiv$ 1 practical volt), the corresponding k.e. increase is

$$\Delta k = -\Delta(q\varphi) = q(10^8/c) = 1.602095 \times 10^{-12} \text{ erg}$$

a unit of energy called the <u>electron-volt</u> (ev)

$$1 \text{ Kev} \equiv 10^3 \text{ ev} \qquad 1 \text{ Mev} \equiv 10^6 \text{ ev} \qquad 1 \text{ Bev} \equiv 1 \text{ Gev} \equiv 10^9 \text{ ev}.$$

3. From $cp_i\lambda_i = ch$ and Eq. (3), one obtains the numerical relation

$$\left(\sqrt{E_i^2 - e_i^2} \text{ Bev}\right) \times (\lambda_i \text{ in fermi}) = 1.239806.$$

For example, a 1.24 Bev electron has wave-length $\lambda_i \cong 1$ fermi ($\equiv 10^{-13}$ cm).

4. The <u>Compton wave-length</u> λ_c of a material particle of rest-mass $m_i > 0$ is an intrinsic parameter, defined as the wave-length of a photon having energy equal to the rest-energy of the particle: $h(c/\lambda_c) \equiv e_i = m_i c^2$ i.e., $\lambda_c = hc/e_i = = h/m_i c$. One may show that

$$\lambda_c \lesseqgtr \lambda_i \qquad \text{as} \qquad \beta_i \lesseqgtr 1/\sqrt{2}$$

where λ_i is the wave-length of the moving particle.

The value of λ_c may be obtained from the relation (e_i in Bev) $\times$ (λ_c in fermi) = 1.239806. The Compton wave-length λ_ϵ of the electron is given in TABLE I, as well as its mass m_ϵ in grams and its ch. energy $e_\epsilon = m_\epsilon c^2$ in Mev.

5. For the k.e. of a material particle, one has the convergent series

$$k_i = m_i c^2(\gamma_i - 1) = m_i c^2 \left\{ \frac{1}{2} \beta_i^{\,2} + \frac{3}{8} \beta_i^{\,4} + \cdots \right\} = \frac{1}{2} m_i v_i^{\,2} \left\{ 1 + \frac{3}{4} \beta_i^2 + \cdots \right\}$$

from which one may see that $k_i > \frac{1}{2} m_i v_i^{\,2}$ unless $v_i = 0$.

Mathematical side-light: The inequality $m_i c^2(\gamma_i - 1) > \frac{1}{2} m_i v_i^{\,2}$ may be written in the form

$$(1-\beta_i^{\,2})\ (1+\tfrac{1}{2}\beta_i^{\,2})\ (1+\tfrac{1}{2}\beta_i^{\,2}) < 1$$

and so deduced from the classical inequality (with $n = 3$):

For $a_j \geqq 0$, $(\Pi_1^{\,n} a_j)^{1/n} < (\Sigma_1^n\, a_j)/n$ unless all a_j are equal.

6. If a particle of rest energy e_i = 500 Mev has k.e. k_i = 800 Mev, its other scalar parameters may be obtained as follows:

$E_i = e_i + k_i$ = 1300 Mev $\qquad \gamma_i = E_i/e_i = 13/5$

$cp_i = (E_i^{\,2} - e_i^{\,2})^{\frac{1}{2}}$ = 1200 Mev $\qquad \beta_i = cp_i/E_i = 12/13$

λ_i = 1.03 f. (Note 3) $\qquad \lambda_c$ = 2.48 f. (Note 4)

If required, $v_i = \beta_i c = 2.77 \times 10^{10}$ cm/sec, and the conversion 1 Mev $\equiv$ 1.602095×10^{-6} erg, with the appropriate constants of TABLE I, yield

$$\nu_i = E_i/h = 3.14 \times 10^{23}\ \text{sec}^{-1}$$

$$p_i = cp_i/c = 6.41 \times 10^{-14}\ \text{gm cm/sec}$$

Note that $\lambda_i \nu_i = 3.25 \times 10^{10}$ cm/sec > c and k_i = 800 Mev > $\frac{1}{2} m_i v_i^2$ $\equiv \frac{1}{2} e_i \beta_i^2$ = 213 Mev.

7. (This, and its "application" in Notes 8, 9, are mathematical "recreations." Any resemblance to physics is purely coincidental.)

A particle of ch. mass m starts from R = 0 at t = 0 with initial momentum $P_o = p_o \Psi_o$, $p_o = M_o v_o$, and is subject therefore to a "friction" F = - HP where H > 0 is a constant. Then, for $t \geqq 0$,

$$P = P_o e^{-Ht} = p_o e^{-Ht} \Psi_o$$

and, since P = MV = M(dR/ds) (ds/dt), (ds/dt > 0) we have $R = s\Psi_o$ and $p = Mv = p_o e^{-Ht}$ where v = ds/dt.

(a) If m > 0, the relation $M = m(1-v^2c^{-2})^{-\frac{1}{2}}$ then implies

$v/c = 1/\{1+(mc/p_o)^2 e^{2Ht}\}^{\frac{1}{2}} \equiv 1/f(t) \to 0$ whence $M \to m$ and $k \to 0$. Integration yields

$s = (c/2H)\, \ln(f(0)+1)(f(t)-1)/(f(0)-1)(f(t)+1)$

with limit $(c/2H)\, \ln(f(0)+1)/(f(0)-1)$ as $t \to \infty$.

(b) If m = 0, then $v \equiv c$, s = ct, and $Mc = M_o c e^{-Ht}$, whence $E = h\nu = h\nu_o e^{-Hs/c} \to 0$.

8. (Hubble's law.) In astrophysics, the absolute magnitude M of a point light source G of luminosity $\mathfrak{L}$ erg/sec is defined by $(\mathfrak{L}/4\pi 10^{-10})/\phi_0 = 10^{-\frac{2}{5}M}$, where ϕ_0 is a standard flux (2.4×10^{44} erg/Mpc^2 sec). If ϕ is its observed flux, then G is said to have apparent magnitude m, where $\phi/\phi_0 \equiv 10^{-\frac{2}{5}m}$, and luminosity distance D(Mpc), where $\phi \equiv \mathfrak{L}/4\pi D^2$. These parameters are therefore related by the identity

$$m = M + 25 + 5 \log D. \tag{5}$$

A friction $F = -HP$ (Note 7) acting on the N_0 photons, of average energy $h\nu_0$, emitted per second by a source G at (constant) distance s, would imply a flux $\phi = N_0 h\nu_0 e^{-Hs/c}/4\pi s^2 \equiv \mathfrak{L}/4\pi s^2 e^{s/R}$ $\quad$ ($R \equiv c/H$ Mpc) and hence a luminosity distance $D = se^{s/2R}$. Moreover, the light received would exhibit a "red shift"

$$z \equiv (\lambda - \lambda_0)/\lambda_0 = e^{s/R} - 1 = s/R + \frac{1}{2!}(s/R)^2 + \cdots. \tag{6}$$

In terms of z, we see that $D = R(1+z)^{\frac{1}{2}} \ln(1+z)$, so that (5) gives

$$m = M + 25 + 5 \log R + \frac{5}{2} \log (1+z) + 5 \log \log (1+z) + 1.811 \tag{7}$$

for the apparent magnitude m of a motionless source of absolute magnitude M showing a red shift $z > 0$.

In reality, an approximate relation (Hubble's law)

$$Z_H \equiv (\lambda - \lambda_0)/\lambda_0 \cong s/R. \tag{8}$$

($R \equiv c/H \cong 3000$ Mpc, $H \cong 100$ (Km/sec)/Mpc is found to exist between the observed red shift Z_H of light, of normal wave length λ_0, received from an "average" galaxy ($M = -20.3$), now at estimated distance s. Measurement of z is relatively clear cut, whereas the estimation of s is very ambiguous. For distant galaxies, m and z are the observables, the function (M. L. Humason, et al., Astron. J., 61, 1956, p. 149) $m = 5 \log cz - 1.18z - 5.81$ having been "fitted" to the observations for $3 < w \equiv \log cz < 5$, ($c = 3 \times 10^5$). The function $m = m(z(w))$ in (7) has the same general features ($dm/dw \leqq 5$; $dm/dw \to 5$ as $z \to 0$), and numerical agreement is surprisingly good for the orthodox values $R = 3000$, $M = 20.3$, i.e. for

$$m = 23.9 + 5\{\tfrac{1}{2} \log(1+z) + \log \log (1+z)\}. \tag{9}$$

For the quasar 3C9, with reported $m = 18.2$, $z = 2.012$! (J. B. Oke, Astrophys. J., 145, 1966, p. 669), we find from (6) and (7) a distance $s = R \ln(1+z) = 3300$ Mpc, and an absolute magnitude $M = -25.6$.

9. (Olbers' paradox.) Suppose infinite Euclidean space has a uniform density of n_o motionless point galaxies per cm^3, each of luminosity $\mathcal{L} = N_o h\nu_o$ erg/sec, as in Note 8. Such a galaxy, at distance s from earth, produces a flux $\phi = \mathcal{L}/4\pi s^2$ erg/cm^2 sec (in the simplest model), of which the earth, of radius r_E, receives πr_E^2 times this. Multiplying by the number $n_o(4\pi s^2 ds)$ of galaxies in the "s-shell" about earth, integration on $r_E < s < \infty$, and division by the surface area $4\pi r_E^2$ of earth gives the infinite result

$$\varphi = \int_{r_E}^{\infty} \tfrac{1}{4}\, n_o \mathcal{L} ds \qquad \text{erg/cm}^2 \text{ sec.}$$

Assuming the friction of Note 7(b), we should use instead the galactic flux $\phi = (\mathcal{L}/4\pi s^2)e^{-Hs/c}$ of (8), with the final result

$$\varphi = \int_{r_E}^{\infty} \tfrac{1}{4}\, \mathcal{L} n_o e^{-Hs/c} ds = \tfrac{1}{4}\, n_o \mathcal{L} \left(\frac{c}{H}\right) .$$

With $n_o = 10^{-75}$ galaxies/cm^3, $\mathcal{L} = 4 \times 10^{43}$ erg/sec (M = - 20.3), and $H = 3.24 \times 10^{-18}$ sec^{-1} (C. W. Allen, <u>Astrophysical Quantities</u>, Univ. of London, Athlone Press, 2nd Ed'n. 1963), one finds $\varphi \cong 10^{-4}$ erg/cm^2 sec, which is far too great (just visible flux $\sim 10^{-7}$, m = 6).

10. The relativistic parameters of a particle are exploited (rather naively) in Appendix I to obtain physically correct properties of a "gas" of such particles.

<u>2. The Lorentz transformation</u>. Let Σ and Σ' be inertial frames, the position space of Σ' having <u>constant</u> velocity U_o of magnitude $u_o \equiv |U_o| < c$, relative to Σ. For such frames, the <u>Lorentz transformation</u> defines a one-to-one correspondence

$$(R,t) \sim (R',t')$$

between all events of Σ and Σ', corresponding "four vectors" being regarded as the "same event," as it appears in the two frames.

Parallel spatial axes $\mathcal{S}, \mathcal{S}'$ may always be chosen so that the spatial origin O' of Σ' moves on the X-axis of Σ in the positive direction, its velocity U_o having components $(u_o, 0, 0)_{\mathcal{S}}$; and the time so measured that O' coincides with O at $t = 0 = t'$. For this standard configuration, indicated in Fig. 2.1, the Lorentz transformation assumes the form

$$\text{(L)} \qquad x = \gamma_o(x' + u_o t') \qquad y = y' \qquad z = z'$$

$$t = \gamma_o(u_o c^{-2} x' + t')$$

where $\beta_o \equiv u_o/c < 1$, and $\gamma_o \equiv 1/(1-\beta_o{}^2)^{\frac{1}{2}} \geq 1$.

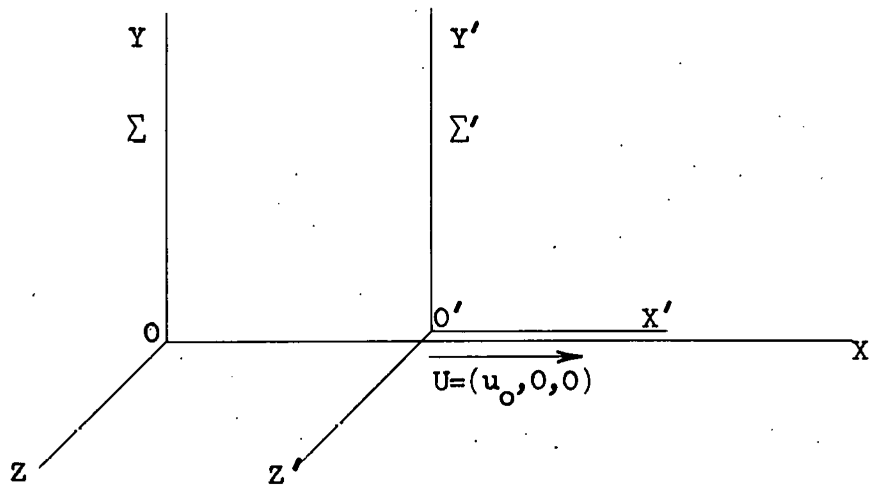


FIG. 2.1

The inverse of (L) and other transformations derived from it results from the formal substitution $- u_o \rightarrow u_o$, and interchange of primed and unprimed variables.

From this simple form, all physically meaningful inferences may be drawn. The most important one is the invariance relation

$$R^2 - c^2t^2 = R'^2 - c^2t'^2 \tag{1}$$

existing between corresponding events, from which we conclude

$$R^2 = c^2t^2 \text{ if and only if } R'^2 = c^2t'^2 \tag{2}$$

and $$R^2 < c^2t^2 \text{ " " " " } R'^2 < c^2t'^2. \tag{3}$$

Moreover, if for an event (R',t') we have $R'^2 \leqq c^2t'^2$ and $t' > 0$, the same relations govern the event (R,t). For, in the formula $t = \gamma_o(u_o c^{-2}x'+t')$, we see that $|x'| \leqq |R'| \leqq ct'$ while $u_o < c$. Hence $|u_o c^{-2}x'| < t'$, and $t > 0$. In this way we establish the further result

$$R^2 \leqq c^2t^2 \;\&\; t > 0 \text{ if and only if } R'^2 \leqq c^2t'^2 \;\&\; t' > 0. \tag{4}$$

These are formal properties of the transformation itself, which will assume added significance in §4.

Of immediate relevance here is the "time dilatation" effect, with its implications for the life time of a moving particle. Consider two times $t_1' < t_2'$ at the same point R_o' in Σ', observed in Σ as times t_1 and t_2. The situation is schematized in Fig. 2. From (L) we find, for the corresponding events

$$(R_1,t_1) \sim (R_o',t_1') \quad \text{and} \quad (R_2,t_2) \sim (R_o',t_2')$$

that $$x_2 - x_1 = \gamma_o u_o(t_2'-t_1') \qquad y_2 - y_1 = 0 \qquad z_2 - z_1 = 0$$

$$t_2 - t_1 = \gamma_o(t_2'-t_1').$$

From this we conclude that a free <u>material</u> particle, moving with speed u_o in Σ, and having an intrinsic life time τ_i' in its own rest-frame Σ', appears in Σ with a life time

$$\tau_i = \gamma_o \tau_i' \geqq \tau_i'$$

and to travel a distance

$$\delta_i = u_o \tau_i \equiv \beta_o \gamma_o c \tau_i' .$$

Note the identification of the particle parameters v_i, β_i, γ_i with the transformation parameters u_o, β_o, γ_o, and the significant proportion $\tau_i/\tau_i' = \gamma_o = E_i/e_i$. (Energy is the secret of longevity?)

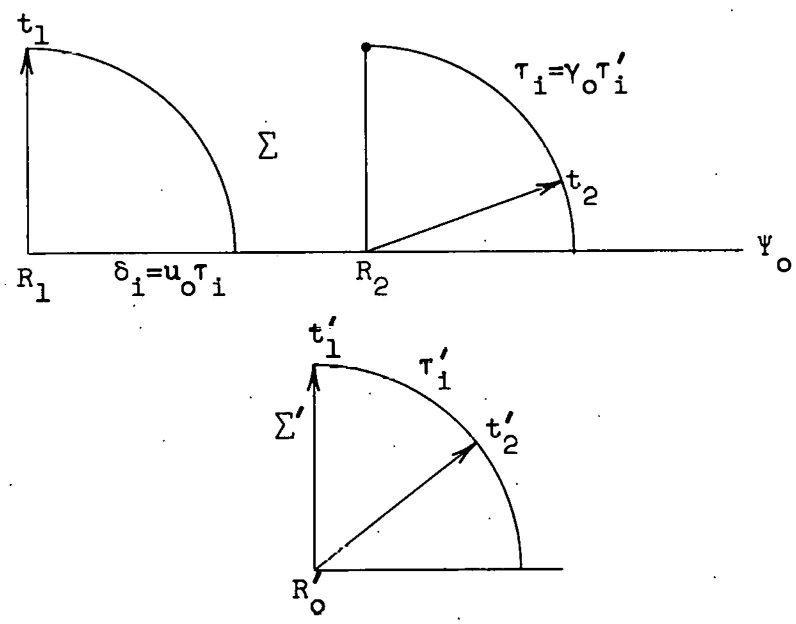


FIG. 2.2

Notes 2.

1. From (2) it appears that $|R'| = c|t'|$ implies $|R| = c|t|$, picturesquely, "the transformation (L) takes the light cone into the light cone." This may be regarded as the basic feature of the Lorentz transformation. Indeed, it is well known that an arbitrary non-singular linear transformation of two 4-spaces with this formal property assumes the simple form (L) when spatial axes are properly aligned by rotations and units suitably standardized. For a generalization, see Appendix II.

2. In practice, given spatial axes are usually not in the standard configuration of Fig.2.1, and auxiliary rotations are required in order to apply the simple transformation (L). These are discussed in Appendix III, which will be referred to when necessary. An alternative device is afforded by the "vector form" of the transformation, which may be derived in the following way.

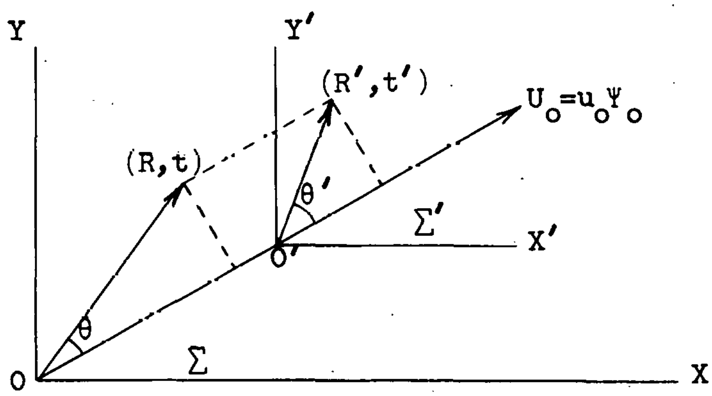


FIG. 2.3

If R' is a vector in Euclidean space, and Ψ_o is an arbitrary unit vector, the projection of R' on Ψ_o is

$$|R'| \cos\theta' = |R'| \; (R'\cdot\Psi_o)/|R'|\cdot|\Psi_o| = R'\cdot\Psi_o \; .$$

The component $R'_{\parallel}$ of R', parallel to Ψ_o, is therefore

$$R'_{\parallel} = (R'\cdot\Psi_o)\Psi_o \qquad (*)$$

and the vector $R'_{\perp} \equiv R' - R'_{\parallel}$ is then orthogonal to Ψ_o since $R'_{\perp}\cdot\Psi_o = 0$.

Now, if $(R,t) \sim (R',t')$ are corresponding events in the two frames Σ,Σ' of this section, and $U_o = u_o\Psi_o$ $(u_o > 0)$, Ψ_o being the direction of relative motion, we may resolve R and R' relative to Ψ_o as indicated above. From (L) we then see

that $\qquad R\cdot\Psi_o = \gamma_o\{(R'\cdot\Psi_o) + u_o t'\}$

whence $\qquad R_{\parallel} = (R\cdot\Psi_o)\Psi_o = \gamma_o R'_{\parallel} + \gamma_o u_o t'\Psi_o$

while $\qquad R_{\perp} = R'_{\perp}$.

Hence $\qquad R = R_{\perp} + R_{\parallel} = R'_{\perp} + \gamma_o R'_{\parallel} + \gamma_o u_o t'\Psi_o$

$$= R'_{\perp} + R'_{\parallel} + (\gamma_o - 1)R'_{\parallel} + \gamma_o u_o t'\Psi_o$$

or $\qquad R = R' + \{(\gamma_o - 1)(R'\cdot\Psi_o) + \gamma_o u_o t'\}\Psi_o \qquad$ (L)

while $\qquad t = \gamma_o\{u_o c^{-2}(R'\cdot\Psi_o) + t'\}$.

In applications of this <u>vector form</u> of (L), components of R,R′ are specified on arbitrary <u>parallel</u> axes $\mathcal{G},\mathcal{G}'$ in Σ,Σ'. Its inverse, and that of other transformations derived from it (Notes, §§3,4) results upon the substitution $-\Psi_o \to \Psi_o$ and interchange of primed and unprimed variables.

3. A particle of rest energy e_i and energy E_i, which travels a distance δ_i in Σ between birth and decay, has an intrinsic life time

$$\tau_i' = (\delta_i/c)/\{(E_i/e_i)^2-1\}^{\frac{1}{2}}$$

N.B. $\beta_o\gamma_o \equiv (\gamma_o{}^2-1)^{\frac{1}{2}}$ and $\gamma_o = E_i/e_i$.

For example, a Ξ^o particle (TABLE III) of rest energy 1314 Mev and k.e. 545 Mev travelling 3 cm in its Σ life time has an intrinsic life time $\tau_i' \cong 10^{-10}$ sec.

4. In order to cover a distance $\delta_i > 0$ in Σ during an intrinsic life time τ_i', a particle must have speed

$$u_o = c/\{1+(c\tau_i'/\delta_i)^2\}^{\frac{1}{2}}$$

and energy $$E_i = e_i\{1+(\delta_i/c\tau_i')^2\}^{\frac{1}{2}} .$$

Thus an earth dweller with 40 Y to live, who wishes to visit α-Centauri, 4 LY away, must travel at the modest speed $c/\sqrt{101}$, with a k.e.

$$k_i = e_i\{(1+10^{-2})^{\frac{1}{2}} -1\} \cong e_i/200.$$

If his rest-mass is $m_1 = 10^5$ grams (neglecting the ship!) then $k_1 = 500\ c^2$ erg (1 kilo-ton high explosive yields $\sim 4 \times 10^{19}$ erg).

5. If a source of photons at $0'$ in Σ' (Fig. 2.1) emits two photons, in direction $\Psi' = (-1, 0, 0)$, at times $t_2' > t_1' > 0$ resp., their times $T_2 > T_1 > 0$ of arrival at 0 in Σ will have the difference

$$T_2 - T_1 = (t_2 - t_1) + c^{-1}(x_2 - x_1) = \gamma_o(1+\beta_o)(t_2' - t_1').$$

Hence if the source emits N' such photons/sec, the number N per sec received at 0 is

$$N = N'/\gamma_o(1+\beta_o) \equiv N'\gamma_o(1-\beta_o).$$

This is precisely the formula governing the energy degradation of each photon, but is quite an independent effect. (Cf. §5.)

6. A statement dual to that connoting time dilatation reads: Consider two points R_1', R_2' at the same time t_o' in Σ', observed as the points R_1, R_2 in Σ. For the corresponding events

$$(R_1, t_1) \sim (R_1', t_o') \qquad (R_2, t_2) \sim (R_2', t_o')$$

we obtain from (L) in this case

$$x_2 - x_1 = \gamma_o(x_2' - x_1') \qquad y_2 - y_1 = y_2' - y_1' \qquad z_2 - z_1 = z_2' - z_1'.$$

$$t_2 - t_1 = \gamma_o u_o c^{-2}(x_2' - x_1').$$

When interpreted as referring to simultaneous observations in Σ' of the ends of a rod in its own rest frame Σ, we find that

$$x_2' - x_1' = \gamma_o^{-1}(x_2 - x_1) \leq (x_2 - x_1)$$

i.e., the dimension in the direction of relative motion appears less by factor γ_o^{-1} in Σ'. (Fitzgerald contraction.)

3. <u>The velocity transformation</u>. If $(R_i(t),t) \sim (R_i'(t'),t')$ are trajectories in Σ,Σ', with component events correlated as in §2, then, via (2L), the system

$$x_i = \gamma_o\{x_i'(t') + u_o t'\} \qquad y_i = y_i'(t') \qquad z_i = z_i'(t')$$

$$t = \gamma_o\{u_o c^{-2} x_i'(t') + t'\}$$

defines a t'-parameterization of the trajectory $(R_i(t),t)$. Since $dR_i/dt = (dR_i/dt')/(dt/dt')$, we obtain the (non-linear!) transformation

(V)
$$v_{ix} = (v_{ix}' + u_o)/d_i'$$

$$v_{iy} = v_{iy}'/\gamma_o d_i' \qquad\qquad d_i' \equiv (u_o c^{-2} v_{ix}' + 1) > 0$$

$$v_{iz} = v_{iz}'/\gamma_o d_i'$$

for the instantaneous velocities $V_i = dR_i/dt$, $V_i' = dR_i'/dt'$ (referred to standard axes) at corresponding events on the two trajectories. Note that $u_o < c$, and $|v_{ix}'| \leqq |V_i'| \leqq c$ (<u>assumed</u> in Σ') insures $d_i' > 0$.

We now obtain from (V) the transformation for speed v. From (V1) we find (suppressing i for the moment)

$$d'^2(v_x{}^2-c^2)\ ) = (v_x'+u_o)^2 - (u_o c^{-1} v_x'+c)^2 = (v_x'^2-c^2)/\gamma_o{}^2$$

or $$v_x{}^2 - c^2 = (v_x'^2-c^2)/\gamma_o{}^2 d'^2.$$

Combining this with (V2,3) results in

$$v_x{}^2 + v_y{}^2 + v_z{}^2 - c^2 = (v_x'^2+v_y'^2+v_z'^2-c^2)/\gamma_o{}^2 d'^2.$$

Hence, the speeds $v_i \equiv |V_i|$ and $v_i' \equiv |V_i'|$ satisfy

(v) $$v_i{}^2 - c^2 = (v_i'^2-c^2)/\gamma_o{}^2 d_i'^2.$$

We conclude that $v_i' \leqq c$ in Σ' <u>implies</u> $v_i \leqq c$ in Σ, and moreover, $v_i = c$ if and only if $v_i' = c$ at corresponding events of <u>arbitrary</u> trajectories. In case $v_i' < c$, $v_i < c$, we obtain from (v) the law

(v*) $$1/(1-v_i^2 c^{-2})^{\frac{1}{2}} = \gamma_o d_i'/(1-v_i'^2 c^{-2})^{\frac{1}{2}}.$$

In view of the relations $V_i = v_i\Psi_i$ and $V_i' = v_i'\Psi_i'$ holding for trajectory directions where $v_i v_i' \neq 0$, the transformation for their components relative to the standard axes $\mathbb{S},\mathbb{S}'$ (Fig. 2.1) follows at once from (V). Writing

$$\Psi_i = (a_{ix}, a_{iy}, a_{iz})_{\mathbb{S}} \text{ and } \Psi_i' = (a_{ix}', a_{iy}', a_{iz}')_{\mathbb{S}'}$$

we obtain

(Ψ) $$a_{ix} = (a_{ix}'+\rho_i')/D_i'$$

$$a_{iy} = a_{iy}'/\gamma_o D_i'$$

$$a_{iz} = a_{iz}'/\gamma_o D_i'$$

where we have set $\rho_i' \equiv u_o/v_i'$

and $$D_i' \equiv d_i' v_i/v_i', \qquad d_i' \equiv u_o c^{-2} v_i' a_{ix}' + 1.$$

For v_i we must of course use its evaluation

$$v_i = \{c^2 + (v_i'^2 - c^2)/\gamma_o^2 d_i'^2\}^{\frac{1}{2}}$$

in terms of Σ' parameters, as provided by (v).

This involved transformation is by-passed in most problems by a device given in §4. It is required for transformation of differential cross sections however, and for that reason we include the explicit evaluation

$$D_i' = \{(a_{ix}' + \rho_i')^2 + \gamma_o^{-2}(1 - a_{ix}'^2)\}^{\frac{1}{2}} \tag{1}$$

in terms of the essential parameters a_{ix}' and $\rho_i' \equiv u_o/v_i'$.

To verify (1), we note (suppressing i)

$$\begin{aligned} D'^2 \equiv d'^2 v^2/v'^2 &= (d'c/v')^2 + \gamma_o^{-2}(1 - c^2 v'^{-2}) \\ &= (u_o c^{-1} a_x' + c v'^{-1})^2 + \gamma_o^{-2} - \gamma_o^{-2} c^2 v'^{-2} \\ &= \beta_o^2 a_x'^2 + 2\rho' a_x' + c^2 v'^{-2} + \gamma_o^{-2} - (1 - u_o^2 c^{-2}) c^2 v'^{-2} \\ &= (1 - \gamma_o^{-2}) a_x'^2 + 2\rho' a_x' + \gamma_o^{-2} + \rho'^2 \end{aligned}$$

which yields (1).

For the application referred to, we require the relation

(Ψ1) $$\cos\psi_i = (\cos\psi_i' + \rho_i')/D_i'$$

which gives the important relation between $a_{ix} \equiv \cos \psi_i$ and $a'_{ix} \equiv \cos \psi'_i$ for the angles ψ_i, ψ'_i made by a trajectory with the X,X′ axes of Fig. 2.1, i.e., with the direction of relative motion of the frames.

Using the value of D'_i in (1), we find, for ρ'_i fixed, the derivative

$$da_{ix}/da'_{ix} = \gamma_o^{-2}(1+\rho'_i a'_{ix})/D'^{\,3}_i \ . \tag{2}$$

When $v_i = c = v'_i$ (as is the case for the trajectories of an immaterial particle), these results become greatly simplified, since then

$$D'_i = d'_i = \beta_o a'_{ix} + 1 \quad \text{and} \quad \rho'_i = \beta_o.$$

Specifically, in this case,

$$(\Psi c) \qquad a_{ix} = (a'_{ix}+\beta_o)/(\beta_o a'_{ix}+1)$$

$$a_{iy} = a'_{iy}/\gamma_o(\beta_o a'_{ix}+1)$$

$$a_{iz} = a'_{iz}/\gamma_o(\beta_o a'_{ix}+1)$$

and $$da_{ix}/da'_{ix} = \gamma_o^{-2}/(\beta_o a'_{ix}+1)^2. \tag{2c}$$

Finally, we derive the (expected) inverse of (V), namely

$$(V)^{-1} \qquad v'_{ix} = (v_{ix}-u_o)/d_i$$

$$v'_{iy} = v_{iy}/\gamma_o d_i \qquad\qquad d_i \equiv (-u_o c^{-2} v_{ix}+1) > 0$$

$$v'_{iz} = v_{iz}/\gamma_o d_i .$$

From (V1),

$$- u_o c^{-2}(v_{ix} d_i') = - u_o c^{-2} v_{ix}' - u_o^2 c^{-2}$$

$$= - (u_o c^{-2} v_{ix}' + 1) + (1 - u_o^2 c^{-2})$$

$$= - d_i' + \gamma_o^{-2}.$$

Therefore, $(-u_o c^{-2} v_{ix} + 1) d_i' = \gamma_o^{-2}.$

It follows that $d_i d_i' = \gamma_o^{-2}$, $d_i > 0$ and

(d) $$(\gamma_o d_i)(\gamma_o d_i') = 1$$

where d_i is defined as in $(V)^{-1}$. With this established, the last two equations of $(V)^{-1}$ follow trivially from those of (V) while the first results simply from solving (V1) for v_{ix}', using the formula for d_i' in (V).

Notes 3.

1. The "vector form" of the velocity transformation (V) follows from Note 2.2. For,

$$dR_i/dt' = V_i' + \{(\gamma_o - 1)(V_i' \cdot \Psi_o) + \gamma_o u_o\}\Psi_o$$

$$dt/dt' = \gamma_o\{u_o c^{-2}(V_i' \cdot \Psi_o) + 1\} \equiv \gamma_o d_i'$$

so

(V) $$V_i = (\gamma_o d_i')^{-1}[V_i' + \{(\gamma_o - 1)(V_i' \cdot \Psi_o) + \gamma_o u_o\}\Psi_o]$$

We have as before $$(\gamma_o d_i)(\gamma_o d_i') = 1 \qquad \text{(d)}$$

where $d_i \equiv \{- u_o c^{-2}(V_i \cdot \Psi_o) + 1\}$.

2. From (V) and the relations $V_i = v_i \Psi_i$, $V_i' = v_i' \Psi_i'$ one obtains the vector form of

$$(\Psi) \qquad \Psi_i = (\gamma_o D_i')^{-1}[\Psi_i' + \{(\gamma_o - 1)(\Psi_i' \cdot \Psi_o) + \gamma_o \rho_i'\}\Psi_o]$$

where $\rho_i' \equiv u_o/v_i'$, $D_i' \equiv d_i' v_i/v_i'$, and d_i' is defined in (V) above.

Explicitly,

$$D_i' = \{[\Psi_i' \cdot \Psi_o + \rho_i']^2 + \gamma_o^{-2}[1 - (\Psi_i' \cdot \Psi_o)^2]\}^{\frac{1}{2}}$$

which reduces, for $v_i = c = v_i'$, $\rho_i' = \beta_o$, to

$$D_i' = d_i' = \beta_o(\Psi_i' \cdot \Psi_o) + 1.$$

3. One easily obtains the relation

$$\tan \psi_i = \gamma_o^{-1} \sin \psi_i'/(\cos \psi_i' + \rho_i')$$

between ψ_i and ψ_i' by division of

$$\sin \psi_i = \left(a_{iy}^2 + a_{iz}^2\right)^{\frac{1}{2}} = \left(a_{iy}'^2 + a_{iz}'^2\right)^{\frac{1}{2}} \Big/ \gamma_o D_i' = \gamma_o^{-1} \sin \psi_i'/D_i'$$

by $\cos \psi_i = (\cos \psi_i' + \rho_i')/D_i'$.

4. <u>The momentum-mass transformation</u>. The preceding results, involving no reference to mass, may be regarded as purely kinematic aspects of the Lorentz transformation. Suppose now that a <u>particle</u> of ch. mass mass $m_i' \geqq 0$, mass $M_i' = M_i'(t') > 0$ moves on the trajectory $(R_i'(t'),t')$ in Σ', appearing in Σ as a particle of ch. mass m_i, mass $M_i = M_i(t)$, on the corresponding trajectory $(R_i(t),t)$, the frames being related as in Fig. 2.1. We shall <u>assume</u> (Cf. Note 5) that

(m) $$m_i = m_i' \qquad \text{and}$$

(M) $$M_i = M_i'\gamma_o d_i', \qquad d_i' \equiv u_o c^{-2} v_{ix}' + 1$$

are the laws of transformation for the ch. mass, and mass, of an arbitrary particle. Note that (3d) provides the inverse of (M).

The relation (M), combined with the (non-linear) velocity transformation (3V), yields a transformation for the momentum-mass 4-vector P_i, M_i of a particle which is not only linear, but has the same matrix as (2L) itself. Thus,

(PM)
$$p_{ix} = M_i v_{ix} = \gamma_o d_i' M_i' \cdot (v_{ix}' + u_o)/d_i' = \gamma_o(p_{ix}' + u_o M_i')$$

$$p_{iy} = M_i v_{iy} = \gamma_o d_i' M_i' \cdot v_{iy}'/\gamma_o d_i' = p_{iy}'$$

$$p_{iz} = M_i v_{iz} = \gamma_o d_i' M_i' \cdot v_{iz}'/\gamma_o d_i' = p_{iz}'$$

$$M_i = \gamma_o d_i' M_i' = \gamma_o(u_o c^{-2} v_{ix}' + 1)M_i' = \gamma_o(u_o c^{-2} p_{ix}' + M_i').$$

The formal properties of (2L) derived in §2 imply here that, at corresponding events on the trajectories of a particle, one has

$$P_i^{\,2} - c^2M_i^{\,2} = P_i'^{\,2} - c^2M_i'^{\,2} \tag{1}$$

$$P_i^{\,2} = c^2M_i^{\,2} \text{ if and only if } P_i'^{\,2} = c^2M_i'^{\,2} \tag{2}$$

$$P_i^{\,2} < c^2M_i^{\,2} \quad \text{" " " "} \quad P_i'^{\,2} < c^2M_i'^{\,2} \tag{3}$$

$$P_i^{\,2} \leqq c^2M_i^{\,2} \;\&\; M_i > 0 \quad \text{" " " "} \quad P_i'^{\,2} \leqq c^2M_i'^{\,2} \;\&\; M_i' > 0. \tag{4}$$

It follows from (1) and (4) that <u>the validity condition of Th. 1.1 is preserved by the momentum--mass transformation</u>. For example, if P_i and M_i are "valid" parameters for the ch. mass m_i ($\geqq 0$) in Σ, then P_i' and M_i', as computed from $(PM)^{-1}$, are valid for the <u>same</u> m_i in Σ'. The import of (2) and (3) for the speed of a particle is manifest.

The transformation makes it clear that a particle is "free" in Σ (P_i constant) if and only if it is free in Σ'.

An obvious scaling of (PM) yields the analogous transformation

(cPE) $\quad cp_{ix} = \gamma_o(cp_{ix}' + \beta_o E_i') \qquad cp_{iy} = cp_{iy}' \qquad cp_{iz} = cp_{iz}'$

$$E_i = \gamma_o(\beta_o cp_{ix}' + E_i')$$

for the energy parameters of a particle. The inverse results from the usual interchange of variables and the substitution $-\beta_o \to \beta_o$.

The latter equations afford a simple means of computing

$$cp_i = (E_i^{\,2} - e_i^{\,2})^{\frac{1}{2}}$$

and the direction relations (3Ψ) for $\Psi_i = cP_i/cp_i$, $\Psi_i' = cP_i'/cp_i'$ in the form $\quad a_{ix} = cp_{ix}/cp_i \quad a_{iy} = cp_{iy}/cp_i \quad a_{iz} = cp_{iz}/cp_i$. $\qquad$ (Ψ)

In particular, $\cos\psi_i = \gamma_o(cp'_i \cos\psi'_i + \beta_o E'_i)/cp_i$. (Ψ1)

Explicit evaluation leads of course to the previous result (§3) where note $\beta_o E'_i/cp'_i \equiv u_o/v'_i = \rho'_i$.

By eliminating $cp'_i \cos\psi'_i \equiv cp'_{ix}$ between the above form of (Ψ1) and the formula for E_i in (cPE), one easily obtains

$$\cos\psi_i = (E_i - \gamma_o^{-1}E'_i)/\beta_o(E_i{}^2 - e_i{}^2)^{\frac{1}{2}} \tag{5}$$

showing the direct dependence of $\cos\psi_i$ on E_i, for a given Σ' energy E'_i. For an immaterial particle, $e_i = 0$, and (5) has the simpler form

$$\cos\psi_i = \{1 - (E'_i/\gamma_o E_i)\}/\beta_o$$

or

$$E_i = E'_i\gamma_o^{-1}/(1-\beta_o\cos\psi_i). \tag{5c}$$

(Cf. §6 for a geometric interpretation.)

Notes 4.

1. The "vector form" of (PM) follows from Note 3.1, and the "coordinate free" version of the mass transformation:

(M) $$M_i = M'_i\gamma_o d'_i \qquad d'_i \equiv u_o c^{-2}(V'_i \cdot \Psi_o) + 1.$$

Thus,

(PM) $$P_i = M_i V_i = M'_i(\gamma_o d'_i)V_i$$

$$= P'_i + \{(\gamma_o - 1)(P'_i \cdot \Psi_o) + \gamma_o u_o M'_i\}\Psi_o$$

$$M_i = \gamma_o\{u_o c^{-2}(P'_i \cdot \Psi_o) + M'_i\}.$$

Scaling yields the energy parameter form

$$(\mathrm{cPE}) \qquad cP_i = cP_i' + \{(\gamma_o-1)(cP_i'\cdot\Psi_o) + \gamma_o\beta_o E_i'\}\Psi_o$$

$$E_i = \gamma_o\{\beta_o(cP_i'\cdot\Psi_o) + E_i'\}.$$

This will be useful in computation. For the inverse, set $-\Psi_o \to \Psi_o$ and interchange (cP_i, E_i) and (cP_i', E_i'). The spatial relations involved were introduced originally in Note 2.2.

2. The "vector form" of the <u>force transformation</u> is obtained from (PM) of Note 1. By definition, $F_i' \equiv dP_i'/dt'$, and

$$F_i \equiv dP_i/dt = (dP_i/dt')/(dt/dt').$$

Just as in Note 3.1, we find

$$dt/dt' = \gamma_o d_i' \qquad \text{where} \quad d_i' \equiv u_o c^{-2}(V_i'\cdot\Psi_o) + 1$$

and, from Note 1.1, the relation (1.4), interpreted in Σ', yields

$$dM_i'/dt' = c^{-2}F_i'\cdot V_i'.$$

Computing dP_i/dt' from (PM) and using these results, we have

$$(\mathrm{F}) \qquad F_i = (\gamma_o d_i')^{-1}[F_i' + \{(\gamma_o-1)(F_i'\cdot\Psi_o) + \gamma_o\beta_o c^{-1}(F_i'\cdot V_i')\}\Psi_o].$$

For components on the standard axes of Fig. 2.1, we need only set $\Psi_o = (1,0,0)$ to obtain

$$F_{ix} = \{F_{ix}' + \beta_o c^{-1}(F_i'\cdot V_i')\}/d_i' \qquad\qquad d_i' = \beta_o c^{-1}v_{ix}' + 1$$

$$F_{iy} = F_{iy}'/\gamma_o d_i' \qquad\qquad F_{iz} = F_{iz}'/\gamma_o d_i'.$$

3. (Biot-Savart cum Lorentz law) In a long but straightforward way, without reference to electromagnetic fields, we derive a relativistic version of the vector product force law for two moving charges, assuming as primitive only the electrostatic force. This derivation would seem to indicate an analogous force law for gravity.

For $i = 0,1$, let i be a particle of charge q_i (esu) in Σ. We suppose q_o moves with constant velocity $U_o = u_o\Psi_o$ on its trajectory $(R_o(t),t)$, (essentially) unperturbed by q_1, which travels with velocity $V_1(t)$ on $(R_1(t),t)$. In the rest frame Σ' of q_o, q_1 appears on a corresponding trajectory $(R_1'(t'),t')$, which we suppose determined solely by the Coulomb force exerted on q_1 by q_o fixed at R_o', namely

$$dP_1'/dt' = F_1' = C'\Delta R' \quad (6)$$

where $\Delta R' = R_1'(t') - R_o'$ and $C' = q_1 q_o/|\Delta R'|^3$ (7)

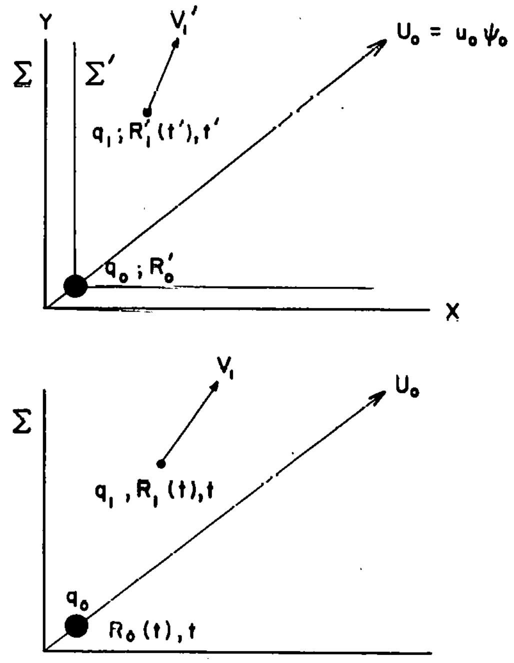


FIG. 4.1

From Note 2, the force of q_o on q_1 at $R_1(t),t$ in Σ is

$$F_1 = (\gamma_o d_1')^{-1}\left[F_1' + \left\{\delta_o F_1'\cdot\Psi_o + \gamma_o\beta_o c^{-1}F_1'\cdot V_1'\right\}\Psi_o\right], \quad \delta_o \equiv \gamma_o - 1 \tag{8}$$

where F_1', V_1' are evaluated at the corresponding event $R_1'(t'),t'$ in Σ'.

Applying $(L)^{-1}$ of Note 2.2 to the Σ-events $(R_1(t),t)$ and $(R_o(t),t)$, we see that

$$\Delta R' \equiv R_1'(t') - R_o' = \Delta R + \delta_o(\Delta R\cdot\Psi_o)\Psi_o \tag{9}$$

where $\Delta R \equiv R_1(t) - R_o(t)$. Substituting (6) in (8), recalling that $(\gamma_o d_1')^{-1} = \gamma_o d_1$, and using (9), we find

$$F_1 = C'\gamma_o\left[d_1\Delta R + \left\{d_1\delta_o(\Delta R+\Delta R')\cdot\Psi_o + \beta_o c^{-1}(\Delta R'\cdot V_1'\gamma_o d_1)\right\}\Psi_o\right]. \tag{10}$$

We will write $\overline{U}_o = U_o/c$ and $\overline{V}_1 = V_1/c$. Then, by definition,

$$d_1 = -u_o c^{-2}(V_1\cdot\Psi_o) + 1 = 1 - \overline{V}_1\cdot\overline{U}_o$$

so that, in (10),

$$d_1\Delta R = \Delta R - \overline{V}_1\cdot\overline{U}_o\Delta R. \tag{11}$$

Turning to the bracket in (10), we have for the first term, using (9)

$$d_1\delta_o\{2\Delta R + \delta_o(\Delta R\cdot\Psi_o)\Psi_o\}\cdot\Psi_o$$

$$= d_1\delta_o\{2\Delta R\cdot\Psi_o + \delta_o\Delta R\cdot\Psi_o\} = d_1(\gamma_o^2-1)\Delta R\cdot\Psi_o \tag{12}$$

$$= d_1\beta_o^2\gamma_o^2\Delta R\cdot\Psi_o = d_1\beta_o\gamma_o^2\Delta R\cdot\overline{U}_o.$$

From (9), and $(V)^{-1}$ of Note 3.1, the final bracket term in (10) is

$$\beta_o[\Delta R + \delta_o(\Delta R\cdot\Psi_o)\Psi_o]\cdot[\overline{V}_1 + \{\delta_o\overline{V}_1\cdot\Psi_o - \gamma_o\beta_o\}\Psi_o]$$

$$= \beta_o\Delta R\cdot\overline{V}_1 - d_1\gamma_o^2\beta_o\Delta R\cdot\overline{U}_o.$$

Combining this with (12) and (11) we have finally, from (10)

$$F_1 = C'\gamma_o[\Delta R - (\overline{V}_1\cdot\overline{U}_o)\Delta R + (\overline{V}_1\cdot\Delta R)\overline{U}_o]$$

$$= \frac{q_1q_o\gamma_o}{|\Delta R'|^3}[\Delta R + \overline{V}_1 \times (\overline{U}_o \times \Delta R)]$$

where $\Delta R = R_1(t) - R_o(t)$ and $\Delta R' = \Delta R + (\gamma_o - 1)(\Delta R\cdot\Psi_o)\Psi_o$.

By the same argument, we should obtain, on the basis of a primitive gravitational force in Σ', namely

$$dP_1'/dt' = F_1' = N'\Delta R'$$

where $N' = -GM_1'm_o/|\Delta R'|^3$, the apparent force in Σ

$$F_1 = \frac{-GM_1M_o\gamma_o}{|\Delta R'|^3}(1-\overline{V}_1\cdot\overline{U}_o)\ [\Delta R + \overline{V}_1 \times (\overline{U}_o \times \Delta R)].$$

For, at the final step, we should have $-GM_1'm_o$ in place of q_1q_o, where $M_1' = M_1\gamma_od_1$ and $m_o = M_o\gamma_o^{-1}$.

As before, the second part of the force lies in the plane of U_o and $R_1 - R_o$, and (however weak) would tend to produce rotation of M_1 about a much greater mass M_o.

4. If $(cP_i, E_i) \sim (cP_i', E_i')$, $i=1,\ldots,I$ under the (cPE) transformation, then formally, $(\Sigma cP_i, \Sigma E_i) \sim (\Sigma cP_i', \Sigma E_i')$ also, by virtue of its linearity. Moreover, we know $(cP)^2 - E^2 = (cP')^2 - E'^2$ for <u>every</u> such corresponding pair. It follows that not only are all the $(cP_i)^2 - E_i^2$ invariant, but $(\Sigma cP_i)^2 - (\Sigma E_i)^2$ as well. In particular, when I = 2, we have the identity $(cP_1+cP_2)^2 - (E_1+E_2)^2 \equiv (cP_1)^2 - E_1^2 + (cP_2)^2 - E_2^2 + 2(cP_1 \cdot cP_2 - E_1E_2)$ and conclude that the function $cP_1 \cdot cP_2 - E_1E_2$ is an invariant too.

Such invariants often allow elegant derivations of parameter values. For example, to obtain the energy E_1' of a particle 1, in the rest-frame Σ' of a second, material particle 2, these having known parameters cP_i, E_i in Σ, one need only note that, since $cP_2' = 0$ and $E_2' = e_2$ in Σ', one must have

$$cP_1 \cdot cP_2 - E_1E_2 = 0 - E_1'e_2$$

so that $E_1' = (E_1E_2 - cP_1 \cdot cP_2)/e_2$.

The whole story may of course be obtained from $(cPE)^{-1}$ of Note 1, where $i = 1$, and $\beta_o, \gamma_o, \Psi_o$ are the Σ-parameters of particle 2, namely $\beta_o = cp_2/E_2$, $\gamma_o = E_2/e_2$, $\Psi_o = cP_2/cp_2$. (The case $cp_2 = 0$ is trivial.)

Note. The relativistic invariants $R^2 - c^2t^2$ and $(cP)^2 - E^2$ involve the same parameters as the Heisenberg uncertainty principles

$$|\Delta R|\,|\Delta P| \cong h/2\pi \cong \Delta t \Delta E.$$

For the significance of this and the invariance status of the principles in relativistic quantum mechanics, I have no reference.

5. The laws of transformation (m) and (M) are not derivable from (2L) alone. It is clear from the properties of particles assumed in §1, and the kinematic results of §3 concerning speeds, that at least the "materiality" of a particle must be an invariant, so certainly $m_i = m'_i$ (= 0) for an immaterial one. If one regards the rest mass of a material particle as an intrinsic property, with $m_i = m'_i$ axiomatic, then the mass law (M) follows at once from the definition (§1) of the masses M_i, M'_i and the speed relation (3v*). However, we cannot deduce (M) for the completely independent mass of an immaterial particle, and consider it an additional assumption, warranted by all experimental evidence.

It is interesting (mathematically) to note the following consequences of the "axiom" (M).

(a) Combined with (3v) it implies directly the formal invariance of $P_i^2 - c^2 M_i^2$. This, together with (m) insures the invariance of "validity."

(b) For a particle with $m'_i > 0$ in Σ', (M) and (3v*) <u>imply</u> that $m_i = m'_i$, and (m) is entirely redundant.

(c) As shown, (M) implies the transformation (PM) for all particles. This enters in a fundamental way in transmutations involving particles of both kinds, and leads to no contradiction with experiment.

(d) The law (M) implies the energy transformation

(E) $$E_i = E'_i \gamma_o d'_i$$

and hence also (assuming h = h'!) the transformation (ν) $\nu_i = \nu_i'\gamma_o d_i'$

for the frequency parameter $\nu_i \equiv E_i/h$ of an arbitrary particle. The significance of this for photons is next considered.

5. The Doppler effect. A free immaterial particle, with direction Ψ_i' and energy E_i', in a frame Σ' related to Σ in the standard way (Fig. 2.1), appears as such a particle in the latter frame with direction (§3)

(Ψ_c) $a_{ix} = (a_{ix}'+\beta_o)/d_i' \qquad a_{iy} = a_{iy}'/\gamma_o d_i' \qquad a_{iz} = a_{iz}'/\gamma_o d_i'$

and energy (§4)

(E) $E_i = E_i'\gamma_o d_i' \qquad d_i' \equiv \beta_o a_{ix}' + 1.$

Thus a photon of energy $h\nu_i'$, moving in Σ' in the transverse direction $\Psi_i' = (0,1,0)_s$, has energy $h\nu_i = h\nu_i'\gamma_o \geqq h\nu_i'$ and direction $\Psi_i = (\beta_o,\gamma_o^{-1},0)_s$ in Σ.

On the other hand, if its direction in Σ' is opposite to the motion of Σ' in Σ, then $\Psi_i' = (-1,0,0) = \Psi_i$, and

$$h\nu_i = h\nu_i'\gamma_o(1-\beta_o) \leqq h\nu_i'.$$

Moreover, N' such photons/sec emitted at O' in Σ' arrive at O in Σ at a rate $N = N'\gamma_o(1-\beta_o) \leqq N'$

satisfying the same formal transformation (Note 2.5).

Notes 5.

1. In certain experiments, a "plane monochromatic light wave," of wave frequency f' (in Σ') acts like a beam of particles (photons) each of energy hf'. Such a wave is described at (R',t') in terms of a function

$$A' \sin 2\pi c^{-1} f'(R'\cdot\Psi_i' - ct')$$

where $R'\cdot\Psi_i'$ is the projection of R' on the direction Ψ_i' of wave motion. It follows easily from the "kinematic" relations $(2L)^{-1}$, $(3d)$, $(\Psi_c)^{-1}$ alone that the angle here involved appears as

$$2\pi c^{-1} f(R\cdot\Psi_i - ct)$$

at the corresponding event (R,t) in Σ, where

$$\text{(f)} \qquad f = f'\gamma_o d_i'$$

(relativistic Doppler equation). Granting that the wave in Σ behaves as a beam of photons of energy hf, one concludes that photon energy E_i <u>must</u> transform as in (E). The situation for neutrinos is perhaps less convincing.

2. It appears from the final remarks of this section that the energy flux in a <u>beam</u> of photons such as described there should transform according to the relation

$$\mathcal{E} = \mathcal{E}'\gamma_o^2(1-\beta_o)^2 \quad \text{erg/cm}^2 \text{ sec.}$$

In the wave picture, $\mathcal{E}$ may be related to the average magnitude of the Poynting vector (wave intensity), which is proportional to the square of the "amplitude" A of the wave. It may be shown (Einstein, Ann. Phys., Lpz., 17, 1905, p. 891) that

$$A^2 = A'^2 \gamma_o^2 (1-\beta_o)^2$$

is the transformation law for A^2, although this is not so "elementary" as the frequency result.

3. ("Flux now-distance now" relation.) In the standard configuration of §2, an isotropic point source at O' in its own rest frame Σ' constantly emits N' photons/sec of energy $h\nu_1'$, and is receding radially from O in Σ with speed $u_o > 0$. Photons emitted at $t_1 > 0$, $x_1 = u_o t_1$ are received at O at a later time $T_1 > t_1$, when O' has reached $X_1 = u_o T_1$ in Σ. From $(2L)^{-1}$, we see that the Σ events of emission (x_1, t_1) and arrival $(0, T_1)$ appear in Σ' as $(0, \gamma_o^{-1} t_1)$, and $(-\gamma_o X_1, \gamma_o T_1)$. The numerical flux in Σ' at time $\gamma_o T_1$ is uniform over the sphere of radius $\gamma_o X_1$ about O', namely

$$N'/4\pi(\gamma_o X_1)^2 \quad \text{photons/cm}^2 \text{ sec.}$$

Due to the rate diminution of Note 2.5 and the energy degradation of (E) in the direction $\Psi_i' = (-1,0,0)_{g'}$, the energy flux observed at O in Σ is

$$\phi = \{N'\gamma_o(1-\beta_o)/4\pi\gamma_o^2 X_1^2\} h\nu_i' \gamma_o (1-\beta_o)$$

$$= (\mathcal{L}/4\pi X_1^2)(1-\beta_o)^2 \quad \text{erg/cm}^2 \text{ sec} \qquad (1)$$

where $\mathcal{L} \equiv N'h\nu_i'$ is the intrinsic luminosity of the source, and X_1 is its distance from O in Σ at the time its flux is observed at O.

The energy degradation results in a Doppler red shift

$$Z_D \equiv (\lambda-\lambda')/\lambda' = \gamma_o(1+\beta_o) - 1 = \beta_o + \tfrac{1}{2}\beta_o^2 + \cdots \qquad (2)$$

in the wave length of the light received.

N.B. There has been some controversy about the flux formula (1). The result is usually given in the form $\phi = (\mathcal{L}/4\pi\rho^2)\gamma_o^2(1-\beta_o)^2$, where ρ should be the Σ' radius $\gamma_o X_1$. (Cf. H. P. Robertson, Zeitschrift für Astrophysik, 15, 1938, p. 77.)

4. If the observed red shift of Hubble's law (Note 1.8) is due to radial recession of galaxies, with speeds u_o constant in time, one infers that $u_o/c = \beta_o \cong Z_D = Z_H \cong Hs/c$ whence $u_o = u_1(s) \cong s/\tau$ is the speed of a galaxy now at distance $s(\tau = 1/H \simeq 10^{10}$ Y). This implies a "big bang" around 10^{10} Y ago, and of course the decrease of Hubble's "constant" with time.

Within the framework of special relativity, the assumption of a Doppler shift $Z = \gamma_o(1+\beta_o) - 1$, together with the flux formula (1), with average $\mathcal{L}$ assumed: $\phi = \mathcal{L}/4\pi s^2(1-\beta_o)^{-2}$, $s \equiv X_1$, implies a luminosity distance (Note 1.8) $D = s(1-\beta_o)^{-1} = s(1+\zeta)/2$, where $\zeta \equiv (1+z)^2$, and hence a relation $m = M + 25 - 5 \log 2 + 5 \{\log s + \log (1+\zeta)\}$, determining s as a function of Z (or u_o) via the observables m and z. If $z \equiv Hs/c$, the argument of Note 1.9 leads to an Olbers' flux

$\varphi = .125\ (\pi-2) n_o \mathfrak{L}(c/H)$, n_o = density now.

6. The momentum ellipsoid. In passing from Σ' to Σ, we saw in §4 that the momenta of a particle i, referred to the standard axes, are in the relation

$$p_{ix} = \gamma_o u_o M_i' + \gamma_o p_{ix}' \qquad p_{iy} = p_{iy}' \qquad p_{iz} = p_{iz}'. \tag{1}$$

If $p_i' \equiv |P_i'|$ is fixed, then all possible momenta P_i' terminate on a sphere of radius $b \equiv p_i'$ about the origin O' of Σ' momentum space. The following geometric construction for the momentum P_i, corresponding to a given such $P_i' \equiv (p_{ix}', p_{iy}', p_{iz}')_{s'}$, with $p_{ix}' \equiv p_i' \cos \psi_i'$, helps in visualizing the nature of the transformation.

A second concentric sphere of radius $a \equiv \gamma_o p_i' > p_i'$ is imagined, and the given vector P_i' represented in the plane of the paper as in Fig. 1. An associated point Q_i' is next located, by the construction shown, with

$$q_{ix}' \equiv (\gamma_o p_i') \cos \psi_i' \equiv \gamma_o p_{ix}', \quad q_{iy}' \equiv p_{iy}', \quad q_{iz}' \equiv p_{iz}' \tag{2}$$

as its coordinates. From (2), one sees that Q_i' ranges over an ellipsoid of revolution in Σ' momentum space, namely

$$(\mathcal{E}) \qquad \frac{q_{ix}'^{\,2}}{(\gamma_o p_i')^2} + \frac{q_{iy}'^{\,2}}{p_i'^{\,2}} + \frac{q_{iz}'^{\,2}}{p_i'^{\,2}} = 1.$$

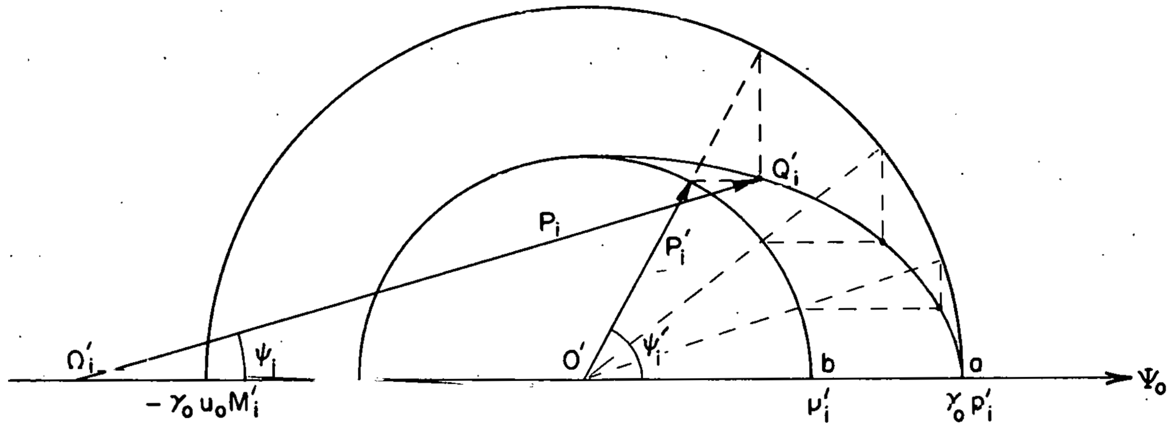


FIG. 6.1

If we now set up, with parallel axes, a momentum space for Σ, with origin O_i' at $(-\gamma_o u_o M_i', 0, 0)$, it is clear that the point Q_i', referred to <u>these</u> axes, has the components required of P_i in (1), and hence the vector from O_i' to Q_i' represents P_i.

The formulas (Ψ) of §§3,4 govern the directions of P_i' and P_i in the figure. In particular, the dependence of cos ψ_i on cos ψ_i' is given by (Ψ1), where the ρ_i' involved may be written in any of the forms

$$\rho_i' \equiv u_o/v_i' = M_i'u_o/p_i' = \gamma_o u_o M_i'/\gamma_o p_i' = \gamma_o \beta_o E_i'/\gamma_o c p_i' = O_i'O'/a \qquad (3)$$

The ratio $\gamma_o \beta_o E_i'/\gamma_o c p_i'$ is indicated for use with energy parameters, in a figure <u>similar</u> to Fig. 1, its momenta

$$\gamma_o u_o M_i' \quad p_i' \quad \gamma_o p_i' \quad P_i' \quad P_i$$

being replaced by the corresponding energies (cf. Fig. 2)

$$\gamma_o\beta_o E_i' \qquad cp_i' \qquad \gamma_o cp_i' \qquad cP_i' \qquad cP_i.$$

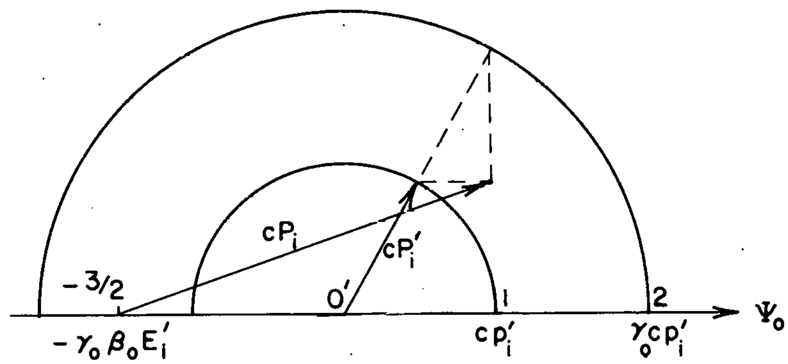


FIG. 6.2

It is clear from the figure that, as ψ_i' varies (in any Σ' plane) from 0° to 180°, the behavior of ψ_i in Σ will depend upon the position of O_i' with respect to the ellipsoid $\mathcal{E}$. Since $\rho_i' = u_o/v_i' = O_i'O'/a$, the following cases arise:

Case I. ($u_o < v_i'$, O_i' inside $\mathcal{E}$) ψ_i also ranges from 0° to 180°.

Case II. ($u_o = v_i'$, O_i' on $\mathcal{E}$) ψ_i ranges from 0° to a limiting 90° at the tangent plane.

Case III. ($u_o > v_i'$, O_i' outside $\mathcal{E}$) ψ_i ranges from 0° to a maximal value $\overline{\psi}_i$ (opening of tangent cone) and thence back to 0°. In this case, each angle $\psi_i < \overline{\psi}_i$ arises from two distinct ψ_i' and therefore appears with two different values of p_i.

These three cases are also distinguished by the inequalities

$$\beta_o^2 E_i'^2 \lesseqqgtr (cp_i')^2 = E_i'^2 - e_i^2 \qquad (4)$$

or

$$e_i\gamma_o \lesseqqgtr E_i' \qquad (\gamma_o \lesseqqgtr \gamma_i' \text{ if } e_i > 0). \qquad (5)$$

Note throughout that u_o, β_o, γ_o pertain to the relative velocity $U_o = u_o\gamma_o$ of the frames, while all parameters with subscript i are those of the particle i.

Notes 6.

1. What is wrong with Fig. 2? (Deduce the value of e_i.)

2. An ellipse with semi-axes $a > b > 0$ may be described as the locus of a point which maintains a fixed ratio ϵ (eccentricity, $0 < \epsilon < 1$) between its distances from a fixed point F (focus) and a fixed line D (directrix). If $\delta > 0$ is the distance from F to D, a polar coordinate equation of the curve is therefore $\rho/(\delta + \rho \cos\theta) = \epsilon$, or

$$\rho = \delta\epsilon/(1-\epsilon\cos\theta).$$

The relation between δ,ϵ and a,b is given by

$$\epsilon = (1-b^2a^{-2})^{\frac{1}{2}} \qquad \delta = b^2/a\epsilon$$

and $f = a\epsilon$ is the distance from F to the center.

For the elliptic section of Fig. 1, $b = p_i'$, $a = \gamma_o p_i'$, and hence $\epsilon = \beta_o$, $\delta = p_i'/\gamma_o\beta_o$, and $f = \gamma_o\beta_o p_i' = \gamma_o u_o M_i'(v_i'/c) \leqq \gamma_o u_o M_i' = O_i'O'$.

The origin O_i' of momenta P_i is therefore never closer to the center O' than the (left) focus F, and coincides with it if and only if $v_i' = c$ ($m_i = 0$). In such a case, the dependence of p_i on ψ_i is that of ρ on θ in the above polar form, namely, $p_i = p_i'\gamma_o^{-1}/(1-\beta_o \cos \psi_i)$ and therefore also $E_i = E_i'\gamma_o^{-1}/(1-\beta_o \cos \psi_i)$. (Cf. (5c) of §4.)

3. It is a simple exercise (not requiring calculus) to show that the maximal angle $\overline{\psi}_i$ in Case III is given by

$$\tan \overline{\psi_i} = 1/\gamma_o(\rho_i'^2-1)^{\frac{1}{2}}, \qquad \rho_i' \equiv u_o/v_i' > 1.$$

4. If Fig. 1, scaled for energy parameters, has $cp_i' = 3$, $\gamma_o cp_i' = 5$, $\gamma_o\beta_o E_i' = 20/3$, one may infer from this alone that $\gamma_o = 5/3$, $\beta_o = 4/5$, $E_i' = 5$, $e_i = 4$. Note the necessity, in all such figures, of the relation $E_i' \geqq cp_i'$, as reflected by the condition $O_i'O'/b = \gamma_o\beta_o E_i'/cp_i' \geqq \gamma_o\beta_o \equiv (\gamma_o^2-1)^{\frac{1}{2}}$, equivalently, $O_i'O'/f \geqq 1$, as in Note 2.

5. Note from the figure the obvious relation $\sin \psi_i'/\sin \psi_i = p_i/p_i'$ and compare with the value given in Note 3.3, $\gamma_o D_i' = \gamma_o d_i' v_i/v_i' = \gamma_o d_i' M_i' v_i/M_i' v_i' = M_i v_i/M_i' v_i'$.

6. Since the matrices in the (R,t) Lorentz transformation and the momentum-mass transformation are identical, a figure <u>similar</u> to Fig. 1 relates the corresponding <u>displacements</u> $(\Delta R_i, \Delta t_i) \sim (\Delta R_i', \Delta t_i')$ of a free particle with the momenta shown, the basic parameters

$$\gamma_o u_o M_i' \qquad p_i' \qquad \gamma_o p_i' \qquad P_i' \qquad P_i$$

being replaced by the distances

$$\gamma_o u_o \Delta t_i' \quad |\Delta R_i'| \quad \gamma_o |\Delta R_i'| \quad \Delta R_i' \quad \Delta R_i .$$

All proportions in the two figures are identical, in particular

$$p_i'/p_i = \sin \psi_i / \sin \psi_i' = |\Delta R_i'|/|\Delta R_i| .$$

CHAPTER II

CLASSES OF SYSTEMS

7. Systems of particles. A system $S = S(m_1,\ldots,m_I)$ in a frame Σ is defined as a set of $I \geqq 1$ triads $(P_i,\ M_i,\ m_i)$, $i = 1,\ldots,I$, of constant valid parameters, together with a specific set of trajectories $R_i(t) = R_i^{\,o} + V_i t$, $-\infty < t < \infty$ $(V_i \equiv M_i^{-1}P_i)$, and is considered to represent a set of I free physical particles in the time interval Δt of its duration. The "mathematical object" S may thus be regarded as "real" during Δt, and "virtual" otherwise.

The total momentum, mass, and ch. mass of S are denoted by

$$P_o \equiv \Sum P_i \qquad M_o \equiv \Sigma M_i \qquad m_S = \Sigma m_i .$$

Similarly, cP_o, $E_o \equiv M_o c^2$, and $e_S \equiv m_S c^2$ designate the corresponding energy-parameter totals of S. Its total kinetic energy is therefore $k_S \equiv E_o - e_S$.

The center of mass (CM) of the system is the point

$$R_S \equiv M_o^{-1} \Sigma M_i R_i .$$

Since the M_i are constant, its velocity is

$$\dot{R}_S = M_o^{-1} \Sigma M_i V_i = M_o^{-1} P_o .$$

A system S with all velocities V_i identical we call coherent, immaterial if the common speed is c (hence all $m_i = 0$), or material if it is less (hence having all $m_i > 0$). Obviously the common velocity V_C of a coherent system is that of its CM:

$$V_C = \dot{R}_S = M_o^{-1} P_o.$$

Concerning systems, we prove the following fundamental

Theorem 1. The totals P_o, M_o, m_S of a system $S(m_i)$ satisfy the conditions

$$M_o > 0 \qquad \& \qquad P_o^2 \leqq c^2(M_o^2 - m_S^2).$$

Moreover, $P_o^2 = c^2(M_o^2 - m_S^2)$ if and only if S is coherent.

Proof. Clearly $M_o \equiv \Sigma M_i > 0$. The classical "polygon" and "Cauchy" inequalities insure that

$$|P_o| \leqq \Sigma |P_i| = c \Sigma (M_i - m_i)^{\frac{1}{2}} (M_i + m_i)^{\frac{1}{2}} \leqq$$

$$c\{\Sigma (M_i - m_i) \Sigma (M_i + m_i)\}^{\frac{1}{2}} = c(M_o^2 - m_S^2)^{\frac{1}{2}}.$$

The known properties of the inequalities cited imply that equality holds between the above extremes if and only if both

(a) all P_i (equivalently, all $M_i^{-1} P_i \equiv V_i$) are unidirectional, and

(b) $M_i - m_i = C(M_i + m_i)$, $i = 1, \ldots, I$, for some constant $C \geqq 0$.

Clearly, (b) is true if and only if all ratios m_i/M_i are equal. But, for each i, $M_i^2 V_i^2 \equiv P_i^2 = c^2(M_i^2 - m_i^2)$, so (b) is also equivalent to the identity of all magnitudes $|V_i|$. It follows that $|P_o| = c(M_o{}^2-m_S{}^2)^{\frac{1}{2}}$ if and only if all velocities V_i are identical, i.e., S is coherent.

Corollary 1. For the CM velocity of a system S, one has always $|\dot{R}_S| \leqq c$, with equality if and only if S is coherent-immaterial.

Proof. This is apparent from Th. 1, and the relations $|\dot{R}_S| = M_o^{-1}P_o$, $P_o{}^2 \leqq c^2(M_o{}^2-m_S{}^2) \leqq c^2M_o{}^2$.

Note 7.

1. The basic inequality underlying Th. 1 may be stated in various ways, none very "elegant" in form. For example:

If $W_1,\ldots,W_I$ are vectors of an inner-product space, with all $|W_i| \leqq 1$, and $\alpha_1,\ldots,\alpha_I$ are positive members of sum $\Sigma\,\alpha_i = 1$, then

$$(\Sigma\,\alpha_i W_i)^2 + \left\{\Sigma\,\alpha_i\left(1-W_i{}^2\right)^{\frac{1}{2}}\right\}^2 \leqq 1.$$

Equality holds if and only if all W_i are identical.

For the Theorem, one takes $W_i \equiv V_i/c$ in Euclidean 3-space, and $\alpha_i \equiv M_i/M_o$.

A second version reads: For numbers $e_i \geqq 0$ and vectors Q_i,

$$(\Sigma\,e_i)^2 + (\Sigma\,Q_i)^2 \leqq \left\{\Sigma\left(e_i{}^2+Q_i{}^2\right)^{\frac{1}{2}}\right\}^2,$$

where the condition for equality is that all $Q_i/\left(e_i{}^2+Q_i{}^2\right)^{\frac{1}{2}}$ be identical. (The terms in the above bracket are assumed positive.)

8. The class of a system. Since all transmutations A → S conserve the total momentum and mass of the systems involved, it is convenient to define the class $\{P_o, M_o\} \equiv \{cP_o, E_o\}$ of all systems A,S,... having the same total momentum P_o and mass M_o, regardless of the number and nature of their individual ch. masses. A (concurrent) system A being given, its totals P_o, M_o define its class, and the possible systems S which may result from its transmutation are all those in the class having the same point of concurrence. The present chapter, which studies the totality of systems belonging to a given class is therefore of immediate relevance for Ch. III, which deals with transmutations as such, stressing the rôle of the initial system A.

For the class parameters P_o, M_o there is first of all the simple "validity condition" of

Theorem 1. A number M_o and vector P_o are possible values for the total mass and momentum of a system S, and so define a non-empty class $\{P_o, M_o\}$, if and only if $M_o > 0$, and $P_o^2 \leq c^2 M_o^2$, i.e., $|M_o^{-1} P_o| \leq c$.

Proof. If S is a system with totals P_o, M_o, and total ch. mass m_S, then by Th. 7.1, we know $M_o > 0$ and $P_o^2 \leq c^2(M_o^2 - m_S^2) \leq c^2 M_o^2$.

Conversely, a pair M_o, P_o with the stated properties may serve as system "totals" for any system $S_o(m_o)$ of I = 1 particle, with ch. mass $m_o \geq 0$ defined by the validity condition for particle parameters (Th. 1.1)

$$P_o^2 = c^2(M_o^2 - m_o^2).$$

It is therefore clear that a non-empty class $\{P_o, M_o\}$ contains a representative (system of one) particle, with particle parameters (P_o, M_o, m_o), velocity $U_o = M_o^{-1} P_o$, speed $u_o = |U_o|$, and trajectory (say) $R = U_o t$. As for any particle, we have the basic relationsof §1 for its parameters:

$$E_o = M_o c^2 \qquad e_o = m_o c^2 \qquad E_o = e_o + k_o \qquad \beta_o = u_o/c \tag{1}$$

$$M_o^2 u_o^2 = p_o^2 = c^2(M_o^2 - m_o^2)$$

or $$E_o^2 \beta_o^2 = (cp_o)^2 = E_o^2 - e_o^2$$

$$\gamma_o \equiv 1/(1-\beta_o^2)^{\frac{1}{2}} = M_o/m_o = E_o/e_o \qquad (m_o > 0,\ u_o < c)$$

$$\Psi_o = u_o^{-1} U_o = p_o^{-1} P_o \qquad (u_o > 0).$$

While all these quantities gain concreteness as the parameters of a particle, they are <u>a priori</u> functions of the <u>class</u> $\{P_o, M_o\}$, being determined solely from the values of P_o, M_o, which are in turn the total momentum and mass of each system of the class. We may therefore properly refer to U_o as the <u>class velocity</u>. Also, for reasons which will soon be apparent, m_o is called the <u>critical mass</u> of the class, and $e_o = m_o c^2$ its <u>critical energy</u>.

<u>Corollary 1</u>. The class velocity U_o is the CM velocity of every system of the class, and therefore the common velocity V_C of every coherent system of the class. Hence, for a coherent system, the total mo-

mentum P_o is distributed among its particles according to the relation

$$P_i = M_i U_o \equiv M_i(M_o^{-1}P_o) = (M_i/M_o)P_o.$$

Proof. From §7, we recall $\dot{R}_S = M_o^{-1}P_o \equiv U_o$ for every system, and $V_C = \dot{R}_S$ for every coherent system, of class $\{P_o, M_o\}$.

Since we have <u>defined</u> m_o by the relation $P_o^2 = c^2(M_o^2 - m_o^2)$ the principal result of Th. 7.1 may be restated as

<u>Theorem 2</u>. If m_S is the total ch. mass of a system S of class $\{P_o, M_o\}$, then necessarily

$$m_o \geqq m_S. \tag{T}$$

Equality holds if and only if S is coherent.

<u>Corollary 2</u>. The coherent systems of class $\{P_o, M_o\}$, in particular those like $S_o(m_o)$ consisting of I = 1 particle, possess the greatest total ch. energy, and hence the least kinetic energy, of all systems in their class.

Proof. Trivially $e_o \geqq e_S$, hence $k_S \equiv E_o - e_S \geqq E_o - e_o \equiv k_o$.

<u>Notes 8</u>.

1. The parameters E_o and cP_o are the indicated totals for all systems S of class $\{P_o, M_o\} \equiv \{cP_o, E_o\}$. Note however that $e_o \equiv m_o c^2$, and $k_o \equiv E_o - e_o$ (the ch. & kinetic energy of the representative particle) are <u>not</u> system totals e_S and k_S unless S is coherent.

2. To be precise, all systems S of a given class $\{P_o, M_o\}$ are de-

termined (except for trajectory origins R_i^o) by the solutions, for the number I, and the parameters (P_i, M_i, m_i), of the conditions

(C) 1. $\Sigma P_i = P_o$ 3. $P_i^2 = c^2(M_i^2 - m_i^2)$

$i = 1, \ldots, I;\ I \geqq 1$

2. $\Sigma M_i = M_o$ 4. $M_i > 0,\ m_i \geqq 0.$

3. Choice of the particular trajectory $R = U_o t$ for the representative particle is convenient (§10) but quite arbitrary. We do not of course imply, or require in the applications, the physical existence of a rest-mass m_o.

4. One may associate with each system $S(m_i)$ of a class an "equivalent" system (i.e., one in the same class) consisting of a single particle with the trajectory of its own CM, and particle parameters (P_o, M_o, m_o). Note: Here, m_o, not m_S, unless S is coherent.

9. The two kinds of classes. We here characterize completely the coherent systems, and in doing so, emphasize the fundamental distinction between a class of critical mass $m_o = 0$, for which the representative particle is immaterial, and one with $m_o > 0$, having a material representative particle, which can be brought to rest by a Lorentz transformation.

Theorem 1. (a) A system $S(m_i)$ belongs to a class $\{P_o, M_o\}$ of critical mass $m_o = 0$ $(u_o = c)$ if and only if it is coherent-immaterial.

(b) Every such system has parameters satisfying the relations

$$m_i \equiv 0, \quad M_i \equiv f_i M_o, \quad P_i \equiv f_i P_o, \quad I \geqq 1$$

where the f_i are positive, with sum $\Sigma f_i = 1$.

(c) If $\{P_o, M_o\}$ is a given class with $m_o = 0$, it contains a system S of an arbitrary number $I \geqq 1$ of ch. masses $m_i = 0$.

Proof. (a) follows from Th. 8.2, and (b) from Cor. 8.1, when f_i is defined as M_i/M_o. For (c), one need only choose <u>any</u> set of $f_i > 0$ with $\Sigma f_i = 1$, e.g., $f_i \equiv 1/I$, and define $M_i \equiv f_i M_o$, $P_i \equiv f_i P_o$, $m_i \equiv 0$. These obviously satisfy conditions (C) of Note 8.2.

<u>Theorem 2</u>. (a) A system $S(m_i)$ is coherent-material if and only if it belongs to a class $\{P_o, M_o\}$ with $m_o > 0$ ($u_o < c$), <u>and</u> has total ch. mass $m_S = m_o$.

(b) The parameters of such a system satisfy the conditions

$$m_i > 0, \quad \Sigma m_i = m_o, \quad M_i = m_i m_o^{-1} M_o \equiv m_i \gamma_o,$$

$$P_i = m_i m_o^{-1} P_o \equiv m_i \gamma_o U_o, \qquad I \geqq 1.$$

(c) If $\{P_o, M_o\}$ is a given class with $m_o > 0$, and $m_i > 0$ are any $I \geqq 1$ <u>given</u> rest-masses with sum m_o, then $\{P_o, M_o\}$ contains a system $S(m_1, \ldots, m_I)$.

Proof. (a) again follows from Th. 8.2. In (b), each particle has the speed u_o of the representative particle, hence $M_i/m_i = \gamma_o = M_o/m_o$ yields the stated M_i, and Cor. 8.1 the stated P_i. For (c), we <u>define</u> the M_i, P_i in terms of the given m_i as in (b), and easily verify (C) of Note 8.2. For (C3), we have from the definitions $P_i^2 = m_i^2 m_o^{-2} P_o^2$ $= m_i^2 m_o^{-2} c^2 (M_o^2 - m_o^2) = c^2 (M_i^2 - m_i^2)$.

2

Summarizing, we have identified the coherent immaterial systems as the entire contents of the classes with $m_o = 0$, and the material ones as those particular systems of the classes with $m_o > 0$ which have the maximal total ch. mass m_o. Every remaining system S is therefore a non-coherent member of a class with $m_o > 0$, and consists of at least 2 particles of total ch. mass $m_S < m_o$. We show in the following sections that every such class does indeed contain systems of any $I \geqq 2$ arbitrarily given ch. masses $m_i \geqq 0$, of sum $\Sigma m_i < m_o$.

10. The Σ'-frame of a class. Let $\{P_o, M_o\}$ be a particular class of systems $S(m_i)$ in Σ, with critical mass $m_o > 0$, and class velocity U_o of magnitude $u_o < c$. The frame Σ' moving with this velocity relative to Σ will be called the "Σ'-frame of the class." In it, the representative system $S_o(m_o)$ appears at rest, and at the origin $0'$ of the axes agreed upon in §2. (Cf. Fig. 2.1 for the standard configuration, and Fig. 2.3 for the general situation.) The transformations of §§2-6, based upon the class parameters u_o, β_o, γ_o defined in §8, relate all events in the frames Σ and Σ'.

Thus, each system $S(m_i)$ of $\{P_o, M_o\}$ appears in Σ' as a system $S'(m_i)$ in Σ', with the same ch. masses $m_i \geqq 0$, and valid parameters (P_i', M_i', m_i) related to those of $S(m_i)$ by the transformations of §4. Namely, for the standard axes $\mathcal{S}, \mathcal{S}'$,

$$p_{ix}' = \gamma_o(p_{ix} - u_o M_i) \qquad p_{iy}' = p_{iy} \qquad p_{iz}' = p_{iz}$$

$$M_i' = \gamma_o(-u_o c^{-2} p_{ix} + M_i).$$

Due to linearity, summing on $i = 1,\ldots,I$ yields the <u>same</u> <u>relation</u> between the system totals P_o', M_o' and P_o, M_o, namely

$$p_{ox}' = \gamma_o(p_{ox} - u_o M_o) \qquad p_{oy}' = p_{oy} \qquad p_{oz}' = p_{oz}$$

$$M_o' = \gamma_o(-u_o c^{-2} p_{ox} + M_o).$$

Since $P_o = M_o U_o$ has components $(M_o u_o, 0, 0)_g$ it follows that the system totals for $S'(m_i)$ in Σ' are

$$P_o' = 0 \qquad M_o' = M_o/\gamma_o \equiv m_o.$$

From this result, and a similar one based on the inverse transformations, we may derive

<u>Theorem 1</u>. For a Σ-class $\{P_o, M_o\}$ with $m_o > 0$ the Lorentz transformations based on the class velocity U_o induce a one-to-one correspondence $S \sim S'$ between all systems S of $\{P_o, M_o\}$ and <u>all</u> systems S′ of the class $\{0, m_o\}$ in the Σ'-frame of the class, m_o being the critical mass of both classes.

Every system S therefore appears in Σ' with total mass m_o, total energy e_o, and zero total momentum; its CM being at rest.

In particular, the coherent (material) systems of $\{P_o, M_o\}$ correspond to those of $\{0, m_o\}$ the latter being motionless. The representative system $S_o(m_o)$ appears as that of $\{0, m_o\}$, consisting of a particle of rest-mass m_o at rest at O′.

Notes 10.

1. The transformation (cPE) of §4 relates the energy-parameters of S and S′. In the present application we have $\beta_o = cp_o/E_o$ and $\gamma_o = E_o/e_o$, so (cPE) may be written with $cp_{ix} = (E_o cp'_{ix} + cp_o E'_i)/e_o$

$$E_i = (cp_o cp'_{ix} + E_o E'_i)/e_o.$$

We recall here again the simple device

$$cp_i = (E_i{}^2 - e_i{}^2)^{\frac{1}{2}}, \quad \cos\psi_i = cp_{ix}/cp_i.$$

2. Theorem 1 may be generalized and the proof simplified in the following way.

Let $\{P_o, M_o\}$ be an arbitrary Σ-class, of critical mass $m_o \geqq 0$, and Σ' a second frame with <u>any</u> constant velocity (of magnitude < c) relative to Σ. If $S(m_i)$ is a system in $\{P_o, M_o\}$, the linearity of the transformations insures that $(\Sigma P_i)^2 - c^2(\Sigma M_i)^2$ is invariant as well as the individual $P_i{}^2 - c^2 M_i{}^2$. It follows that the Σ-class appears in Σ' as a single class, with the same critical mass m_o, which is therefore an invariant also. Since $m_S = m_o$ is the criterion for a coherent system and both quantities are invariant, coherent systems are preserved. The latter is of course obvious from (3V) itself.

<u>11. Systems of zero total momentum</u>. In an <u>arbitrary</u> frame Σ', let $\{0, m_o\}$ be a class of systems $S'(m_i)$ with total momentum $P'_o = 0$ and total mass $M'_o = m_o > 0$, which is, <u>ipso facto</u>, the critical mass of the class. (Notation is chosen for the sake of the principal application). All

systems $S'(m_i)$ of the class are then determined (Note 8.2) from the solutions of

$$(C') \qquad 1.\ \Sigma P'_i = 0 \qquad 3.\ P'^2_i = c^2(M'^2_i - m_i^2)$$

$$2.\ \Sigma M'_i = m_o \qquad 4.\ M'_i > 0,\ \ m_i \geqq 0, \qquad I \geqq 1.$$

We know (Th. 9.2) that those with $m_S \equiv \Sigma m_i = m_o$ are the coherent-material ones, here motionless, with $M'_i \equiv m_i$, $P'_i \equiv 0$, while all others must have $m_S < m_o$ and $I \geqq 2$. The simplest of these are completely characterized in

<u>Theorem 1</u>. (a) A system $S'(m_1,m_2)$ of $\{0,m_o\}$ with $I = 2$ particles of total ch. mass $m_1 + m_2 < m_o$, has the unique masses

$$M'_1 = (m_o^2+m_1^2-m_2^2)/2m_o > m_1$$

$$M'_2 = (m_o^2+m_2^2-m_1^2)/2m_o > m_2$$

and oppositely directed momenta of equal magnitude, determined by (C′3).

(b) For <u>every</u> $m_1,m_2 \geqq 0$ with $m_1 + m_2 < m_o$, there exists a system $S'(m_1,m_2)$ of $\{0,m_o\}$ with these ch. masses, and an arbitrary direction for P'_1.

Proof. (a) From (C′1), $P'^2_1 = P'^2_2$. Hence, by (C′3),

$$M'^2_1 - M'^2_2 = m_1^2 - m_2^2.$$

From (C′2) $M'_1 + M'_2 = m_o > 0$.

Division of these equations and solution of the resulting linear system yields the stated values of the M'_i.

From $m_o > m_1 + m_2$ alone follows: $m_o^2 - 2m_o m_1 + m_1^2 > m_2^2$, and hence $(m_o^2+m_1^2-m_2^2)/2m_o > m_1$, with a similar result for the second particle.

(b) By the last remark, we know that the M_i as defined under (a) are positive, indeed $M'_i > m_i \geqq 0$, so (C′4) holds. Also, by definition, the M'_i have sum

$$M'_1 + M'_2 = m_o \qquad \text{(whence C}'2\text{)}$$

and difference $M'_1 - M'_2 = (m_1^2-m_2^2)/m_o$.

Multiplication of these equations leads to the result

$$c(M_1'^2-m_1^2)^{\frac{1}{2}} = c(M_2'^2-m_2^2)^{\frac{1}{2}} > 0.$$

Hence any two oppositely directed momenta P'_i with this common magnitude satisfy (C′1,3), which completes the proof.

Corollary 1. The system $S'(m_1,m_2)$ of Th. 1(a) has energies

$$E'_1 = (e_o^2+e_1^2-e_2^2)/2e_o > e_1$$

$$E'_2 = (e_o^2+e_2^2-e_1^2)/2e_o > e_2$$

and kinetic energies

$$k'_1 = \frac{e_2 + (k'_S/2)}{e_o} \cdot k'_S > 0$$

$$k'_2 = \frac{e_1 + (k'_S/2)}{e_o} \cdot k'_S > 0,$$

where $k'_S \equiv e_o - e_S$ is its total k.e. (strictly, $k'_{S'}$).

Proof. The above $E'_i \equiv M'_i c^2$ are obvious results of scaling. From the value of E'_1 we find, for example,

$$
\begin{aligned}
k'_1 \equiv E'_1 - e_1 &= \{(e_o-e_1)^2-e_2^2\}/2e_o \\
&= (e_o-e_1+e_2)(e_o-e_1-e_2)/2e_o \\
&= (e_o-e_S+2e_2)(e_o-e_S)/2e_o \\
&= (k'_S+2e_2)k'_S/2e_o
\end{aligned}
$$

which is the stated result.

Note that $m_1 > m_2$ implies $E'_1 > E'_2$ but $k'_1 < k'_2$, so m_1 has more than half the total energy e_o, but less than half the total k.e. k'_S.

For the existence theorem of the next section we require the following generalization of Th. 1(b).

<u>Theorem 2</u>. If $\{0,m_o\}$ is a given class, and $m_i \geqq 0$ are any $I \geqq 2$ ch. masses with sum $\Sigma m_i < m_o$, then $\{0,m_o\}$ contains a system $S'(m_1,\ldots,m_I)$.

Proof. Group the m_i in any way (there is at least one!) into non-empty disjoint subsets

$$\{m_k\} \qquad \{m_\ell\}$$

each individually with all $m_i > 0$ or all $m_i = 0$. Define

$$m_K = \Sigma m_k \qquad m_L = \Sigma m_\ell$$

where we know $m_K + m_L < m_o$.

Let $S'(m_K, m_L)$ be a two-particle system of the class $\{0, m_o\}$. Its particle parameters satisfy the conditions

1. $P'_K + P'_L = 0$ 3. $P'^2_K = c^2(M'^2_K - m^2_K)$, $P'^2_L = c^2(M'^2_L - m_L^2)$

2. $M'_K + M'_L = m_o$ 4. $M'_K > m_K \geqq 0$, $M'_L > m_L \geqq 0$.

By (1,2) it suffices to produce two systems:

$$S'(m_k) \text{ in } \{P'_K, M'_K\} \qquad \text{and} \qquad S'(m_\ell) \text{ in } \{P'_L, M'_L\}.$$

By (3,4) these classes are non-empty, with critical masses

$$m_K(=\Sigma m_k) \qquad \text{and} \qquad m_L(=\Sigma m_\ell).$$

The desired systems must therefore be coherent, and their existence is insured by parts (c) of Theorems 9.1, 9.2. For, the m_i of each subset are either all 0 (and we use Th. 9.1) or all positive (and we use Th. 9.2).

Note 11.

1. The formula for M'_1 in Th. 1(a) may also be inferred from the obvious relations

$$c^2(M_1'^2 - m_1^2) = P_1'^2 = P_2'^2 = c^2(M_2'^2 - m_2^2) = c^2\{(m_o - M_1')^2 - m_2^2\}.$$

We rely on this connection in §13.

12. The main existence theorem. The systems $S(m_i)$ which belong to a Σ-class $\{P_o,M_o\}$ of critical mass $m_o = 0$ have been determined as the coherent-immaterial ones of Th. 9.1.

In a given class $\{P_o,M_o\}$ with $m_o > 0$, all systems $S(m_i)$ must have $m_S \equiv \Sigma m_i \leqq m_o$ (Th. 8.2). Those with $m_S = m_o$ have been characterized as the coherent-material systems of Th. 9.2. Moreover, the correspondence of §10 provides a one-one mapping of the systems of this Σ-class on those of Σ'-class $\{0,m_o\}$, and we have seen (Th. 11.2) that, in addition to its coherent systems, the latter class contains (non-coherent) systems $S'(m_i)$ of any given number $I \geqq 2$ of arbitrarily specified $m_i \geqq 0$ of sum $\Sigma m_i < m_o$. These remarks establish the principal

Theorem 1. Given a Σ-class $\{P_o,M_o\}$ of critical mass $m_o > 0$, and $I \geqq 1$ specified ch. masses $m_i \geqq 0$ of sum $m_S \equiv \Sigma m_i$, then $\{P_o,M_o\}$ contains no system $S(m_1,\ldots,m_I)$ unless

$$m_o \geqq m_S. \qquad \text{(T)}$$

(a) If $m_o = m_S$, the class contains such a system if and only if all $m_i > 0$.

(b) If $m_o > m_S$, such a system belongs to the class if and only if $I \geqq 2$.

The systems $S(m_1,m_2)$ with $m_1+m_2 < m_o$, of class $\{P_o,M_o\}$ in Σ, all derive, via the transformations (4 PM), from the systems $S'(m_1,m_2)$ of class $\{0,m_o\}$ in the Σ'-frame of $\{P_o,M_o\}$. The latter have the unique masses M_i' and absolute momenta $p_1' = p_2'$ given in Th. 11.1(a). The values of P_i and

M_1 in Σ vary therefore only with the direction Ψ_1' of P_1' in Σ'. (For computation, see Note 10.1.) The geometric nature of this dependence is illustrated in Fig. 1, which is an obvious elaboration of Fig. 6.1, allowing

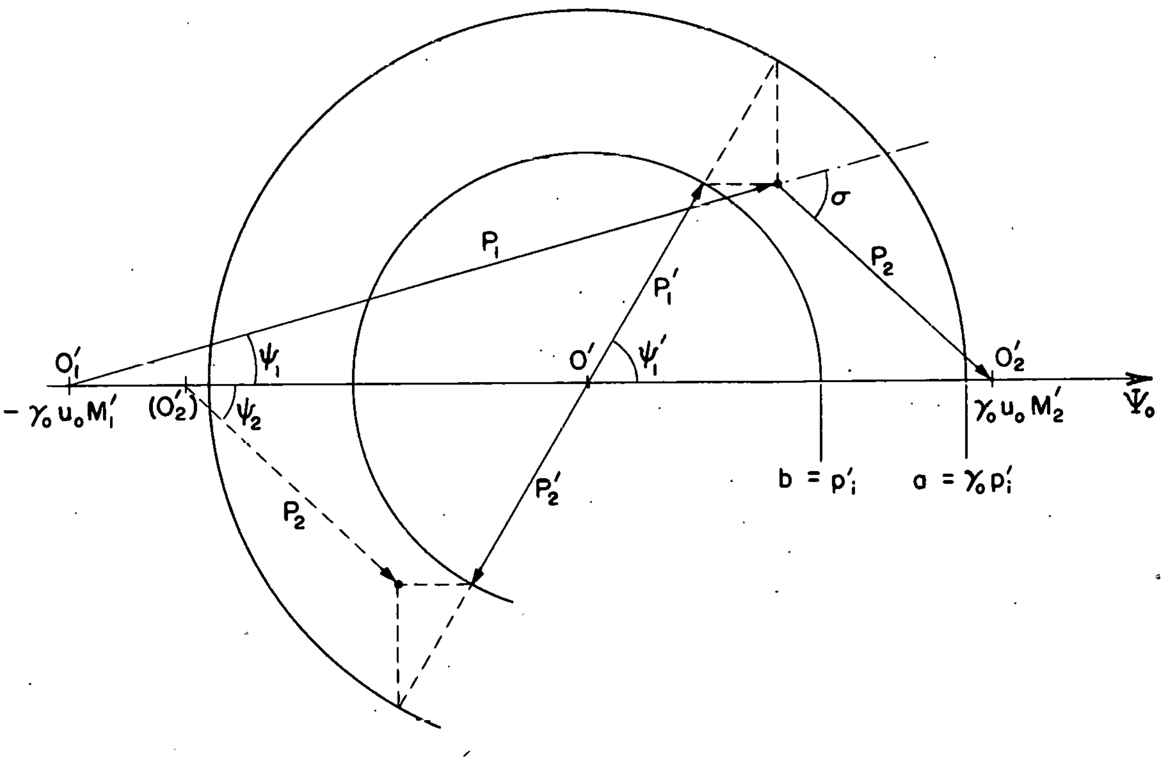


FIG. 12.1

the simultaneous construction of P_1 and P_2, together with their angle of separation $\sigma = \psi_1 + \psi_2$, $0 \leqq \sigma \leqq 180^{\circ}$.

Because of the importance of the 2-body case, the following remarks may be warranted.

1. The system $S'(m_1,m_2)$ is determined by Ψ_1' and e_o alone, being independent of β_o. Since $E_i' \geqq cp_1'$ is its only internal requirement (for <u>some</u> e_i) and $E_i' \lesseqgtr \beta_o^{-1} cp_1'$ determines the cases I, II, III of §6, where $\beta_o^{-1} cp_1' > cp_1'$, it is clear that all 9 combinations of these cases <u>may</u> occur for a two particle system in Σ.

2. Given the values of cp_i', $\gamma_o cp_i'$, and $\gamma_o \beta_o E_1'$ in the energy version of such a figure as Fig. 1, one may infer the values of γ_o, β_o, E_i', e_i, $e_o = E_1' + E_2'$, $E_o = e_o \gamma_o$, $cp_o = E_o \beta_o$. For the values so computed, the relations of Th. 11.1(a) and the identity $(cp_o)^2 = E_o^2 - e_o^2$ are automatic. Note in Fig. 1 the relation

$$p_{1x}' + p_{2x}' = O_1'O_2' = \gamma_o u_o M_1' + \gamma_o u_o M_2' = \gamma_o m_o u_o = M_o u_o = p_o.$$

3. Either from the vector relation $P_o = P_1 + P_2$, or the law of cosines in Fig. 1, we have for the angle σ, $p_o^2 = p_1^2 + p_2^2 + 2p_1p_2 \cos\sigma$, and therefore

$$\sin\sigma/2 = \left\{ \left(p_1+p_2\right)^2 - p_o^2 \right\}^{\frac{1}{2}} \Big/ 2\left(p_1p_2\right)^{\frac{1}{2}} \qquad (1)$$

which allows computation of σ, once the p_i are known.

<u>For systems with $m_1 = m_2$</u>, σ has some easily verified properties to which we refer later. For such systems, we know

$$O_1'O' = \gamma_o u_o M_1' = \gamma_o u_o M_2' = O'O_2' \equiv \Delta$$

and it is geometrically obvious that the usual three cases

$$\rho_i' = u_o/v_i' = \gamma_o u_o M_i'/\gamma_o p_i' \equiv \Delta/a \lesseqgtr 1$$

result in the following behavior:

Case I. $\rho_i' < 1$; σ has a minimum at $\psi_1' = 90^\circ$, and maxima ($= 180^\circ$) at $\psi_1' = 0^\circ, 180^\circ$.

Case II. $\rho_i' = 1$; σ has a minimum at $\psi_1' = 90^\circ$, and maxima ($= 90^\circ$) at $\psi_1' = 0^\circ, 180^\circ$.

Case III. $\rho_i' > 1$; σ has a <u>maximum</u> at $\psi_1' = 90^\circ$, and minima ($= 0^\circ$) at $\psi_1' = 0^\circ, 180^\circ$.

In all three cases, we have from (1), for the extremal angle $\bar{\sigma}$ occurring at $\psi_1' = 90^\circ$ ($p_1 = p_2$, $p_1{}^2 = \Delta^2 + b^2$)

$$\sin \bar{\sigma}/2 = (p_1{}^2 - \Delta^2)^{\frac{1}{2}}/p_1 = b/p_1 = 1/\{1+(\Delta/b)^2\}^{\frac{1}{2}}$$

or

$$\sin \bar{\sigma}/2 = 1/\{1+(\gamma_o \rho_i')^2\}^{\frac{1}{2}} \qquad (m_1 = m_2). \tag{2}$$

For, $\Delta/b = (\Delta/a)(a/b) = \rho_i' \gamma_o$.

Of geometric interest under Case I is the system of two photons ($\rho_i' = \beta_o < 1$). Since both origins are then at the foci, we have

$$\Delta = f = a\epsilon = a\beta_o$$

and it follows from the "string property" of the ellipse that there prevails a constant sum $p_1 + p_2 \equiv 2a$. Hence in (1),

$$(p_1+p_2)^2 - p_o^2 \equiv 4(a^2-\Delta^2) = 4a^2(1-\beta_o^2) = 4(a\gamma_o^{-1})^2$$

$$= (2p_1')^2 = c^{-2}(2E_1')^2 = c^{-2}e_o^2.$$

Therefore we obtain from (1) and (2), in energy form,

$$\sin\sigma/2 = e_o/2(E_1E_2)^{\frac{1}{2}} \geqq \sin\bar{\sigma}/2 = 1/\{1+(\gamma_o\beta_o)^2\}^{\frac{1}{2}} = 1/\gamma_o = e_o/E_o. \quad (3)$$

The underlying inequality $(E_1E_2)^{\frac{1}{2}} \leqq (E_1+E_2)/2$ is the simplest example of that in Note 1.5.

Another instance arises in the elastic scattering of a projectile on a target of equal rest-mass, at rest in Σ. For the Σ' frame of the class, the appropriate figure falls under Case II, with both O_i' on the ellipsoid. In such a case, since $v_i' = u_o$, $\rho_i' = 1$,

(2) reads $$\sin\bar{\sigma}/2 = 1/(1+\gamma_o^2)^{\frac{1}{2}} \quad (4)$$

or $$\cos\bar{\sigma} = (\gamma_o^2-1)/(\gamma_o^2+1) \quad (5)$$

giving the minimal angle of separation for the scattered projectile and recoil target directions in Σ.

Notes 12.

1. One may now deduce the following generalization of the validity condition (Th. 1.1) for particles:

A number M and vector P are possible values for the total mass and momentum of <u>some</u> system S of specified total ch. mass m_S if and only if

$$M > 0 \qquad \& \qquad P^2 \leqq c^2(M^2-m_S{}^2).$$

2. It should be noted that the Σ parameters of a two body system may be obtained, as functions of the angle ψ_1, directly from the basic relations

$$P_1 + P_2 = P_o \qquad \text{and} \qquad E_1 + E_2 = E_o \tag{6}$$

without introducing the frame Σ'. However, their dependence on the Σ' variable ψ_1' is essential for understanding their behavior under Cases I - III. For example, we have directly from (6)

$$(cp_2)^2 = (cp_o)^2 + (cp_1)^2 - 2(cp_o)(cp_1)\cos\psi_1$$

$$E_2{}^2 = E_o{}^2 + E_1{}^2 - 2E_oE_1.$$

Subtracting gives

$$-e_2{}^2 = -e_o{}^2 - e_1{}^2 + 2(E_oE_1-cp_ocp_1\cos\psi_1)$$

or $$2e_oE_1' = 2E_o(E_1-\beta_ocp_1\cos\psi_1)$$

where we have written $E_1' \equiv (e_o{}^2+e_1{}^2-e_2{}^2)/2e_o$ simply as an abbreviation. This yields $\gamma_o{}^{-1}E_1' = E_1 - \beta_ocp_1\cos\psi_1$

or $$\cos\psi_1 = (E_1-\gamma_o^{-1}E_1')/\beta_o(E_1{}^2-e_1{}^2)^{\frac{1}{2}}$$

E_1' being of course in reality the energy of m_1 in Σ'. (Cf. (4.5).)

13. Many particle systems. All systems $S'(m_i)$ with $I \geqq 2$ specified ch. masses $m_i \geqq 0$ of sum $m < m_o$, which belong to a given class $\{0,m_o\}$ of an arbitrary frame Σ', must be determined from the conditions

$$(C') \qquad 1.\ \Sigma P_i' = 0 \qquad\qquad 3.\ P_i'^2 = c^2(M_i'^2 - m_i^2)$$

$$2.\ \Sigma M_i' = m_o \qquad\qquad 4.\ M_i' > 0.$$

The existence of such a system, consisting of two coherent subsystems, was established in Th. 2 of §11, in just this context. Interpreting that result, with Σ' regarded as the Σ'-frame of a class, we obtained a corresponding existence theorem for an arbitrary class $\{P_o,M_o\}$ in §12, Theorem 1(b). We now exploit the latter result to clarify the nature of those systems $S'(m_i)$ of $\{0,m_o\}$ with $I \geqq 3$ particles. These lack the uniqueness properties of the 2-particle systems described in §11, and we shall determine completely the energy ranges permitted for their particles. This, in its turn, has immediate but complicated implications for an arbitrary class of the same critical mass, which will be mentioned only briefly in a later application (§23).

Suppose specified the class $\{0,m_o\}$, and $I \geqq 3$ ch. masses $m_i \geqq 0$ of sum $m \equiv \Sigma m_i < m_o$. We single out any one ch. mass m_1, and define

$$m_K = m_1 \qquad\qquad m_L = \Sigma_2^I m_i \qquad\qquad \text{where } m_K + m_L = m.$$

Let $M_K' \equiv (m_o^2+m_1^2-m_L^2)/2m_o > m_1$ and $M_L' \equiv (m_o^2+m_L^2-m_1^2)/2m_o > m_L$ be the unique masses of any system $S'(m_K,m_L)$ of $\{0,m_o\}$. Note that $M_K' < m_o$, since

$M'_K + M'_L = m_o$ and $M'_L > 0$. The possible range of values of M'_1 in systems $S'(m_i)$ of $\{0,m_o\}$ is given in the following two theorems.

<u>Theorem 1</u>. If $S'(m_i)$ is a system of $\{0,m_o\}$, with the $I \geqq 3$ specified ch. masses $m_i \geqq 0$ of sum $m \equiv \Sigma m_i < m_o$, and m_1 is any one of the m_i, then its mass M'_1 must satisfy

$$M'_1 \leqq M'_K.$$

Equality obtains if and only if the residual sub-system $S'_L \equiv S'(m_2,\ldots,m_I)$ is coherent.

Proof. Since the residual system has totals

$$\Sigma_2^I P'_i = - P'_1 \qquad \Sigma_2^I M'_i = m_o - M'_1 \qquad \Sigma_2^I m_i = m_L$$

it follows from Th. 7.1 that

$$c^2({M'_1}^2-m_1^2) \equiv (-P'_1)^2 \leqq c^2 \left\{(m_o-M'_1)^2 - m_L^2\right\}$$

or, equivalently $\quad M'_1 \leqq \left\{m_o^2 + m_1^2 - m_L^2\right\}\Big/2m_o$

with equality if and only if S'_L is coherent.

<u>Theorem 2</u>. Let $\{0,m_o\}$ be a class in Σ', and $m_i \geqq 0$ any $I \geqq 3$ stipulated ch. masses of sum $m \equiv \Sigma m_i < m_o$. Then, for every number M'_1 and vector P'_1 which are valid mass and momentum for m_1, there exists a system $S'(m_1,\ldots,m_I)$ of $\{0,m_o\}$ in which 1 has the stated parameters (P'_1,M'_1,m_1), provided only that $M'_1 < M'_K$. If the given $M'_1 = M'_K$, this is also true, provided all $m_2,\ldots,m_I$ are of the same kind (> 0 or $= 0$).

Proof. Clearly it suffices to produce a residual system $S'_L \equiv S'(m_2, \ldots, m_I)$ with $-P'_1$ and $m_o - M'_1$ as total momentum and mass. Since we are given in any case $M'_1 \leqq M'_K < m_o$, certainly $m_o - M'_1 > 0$. Moreover, the inequality $M'_1 \leqq M'_K$ is equivalent to

$$c^2\left(M_1'^2 - m_1^2\right) \leqq c^2\left\{\left(m_o - M'_1\right)^2 - m_L^2\right\}$$

which, in view of the given validity condition $P_1'^2 = c^2\left(M_1'^2 - m_1^2\right)$, is in turn equivalent to

$$(-P'_1)^2 \leqq c^2\left\{(m_o - M'_1)^2 - m_L^2\right\}.$$

This insures that the class $\{-P'_1, m_o - M'_1\}$ in which we seek S'_L is at least non-empty, and moreover has a critical mass $m^* \geqq m_L$ (Th. 8.1). Clearly, this last inequality is equivalent to the given one, $M'_1 \leqq M'_K$.

But m_L is the total ch. mass $\sum_2^I m_i$ of the desired system S'_L, and its existence follows from Th. 12.1 and Th. 9.1. In detail: if $M'_1 < M'_K$ is given, then $m^* > m_L$, the above class has critical mass $m^* > 0$, and contains a non-coherent S'_L by Th. 12.1(b), since the $m_2, \ldots, m_I$ are $I - 1 \geqq 2$ ch. masses; if $M'_1 = M'_K$ is given, then $m^* = m_L$, and S'_L exists as a coherent system, by Th. 9.1(c) if $m_L = 0$, or by Th. 12.1(a) if $m_L > 0$, since we have stipulated, in this limiting case, that the $m_i (i \geqq 2)$ are of a single kind.

Notes 13.

1. Under the conditions of Th. 1, the k.e. of m_1 must lie on the range

$$0 \leqq k'_1 \leqq k'_K \equiv \frac{e_L + (k'/2)}{e_o} \cdot k'$$

where $e_L \equiv \Sigma_2^I e_i$ and $k' \equiv e_o - \Sigma_1^I e_i$. The latter is of course the total k.e. of the two particle system $S'(m_1,m_L)$, and k'_K its k.e. for m_1. (Cor. 11.1.) Note that k' is also the total k.e. of <u>all</u> systems $S'(m_1,\ldots,m_I)$ of $\{0,m_o\}$.

2. Under the conditions of Th. 2, all values of k'_1 on the range $k'_1 \leqq k'_K$ are attainable, under the same provisos. The <u>lower</u> bound $k'_1 = 0$ is attainable if $m_1 > 0$, the residual system then being of class $\{0,m_o-m_1\}$.

CHAPTER III

TRANSMUTATIONS OF SYSTEMS

14. Transmutations. A transmutation in an event-space Σ is a localized, "black-box," physical process, of short duration, in which a set of free physical particles is converted into a second such set, with conservation of total momentum P_o and mass M_o.

We idealize such a process as an event (R,t), denoted by

$$A \to S \qquad (\dagger)$$

at which two systems A and S of the same class $\{P_o, M_o\}$, and concurrent at (R,t), interchange "reality," A becoming "virtual" as S becomes "real."

Thus the reverse process $S \to A$ does not here connote "time-reversal," but simply a reversed interpretation of "reality" for the same two "mathematical objects" A and S, as indicated in Fig. 1.

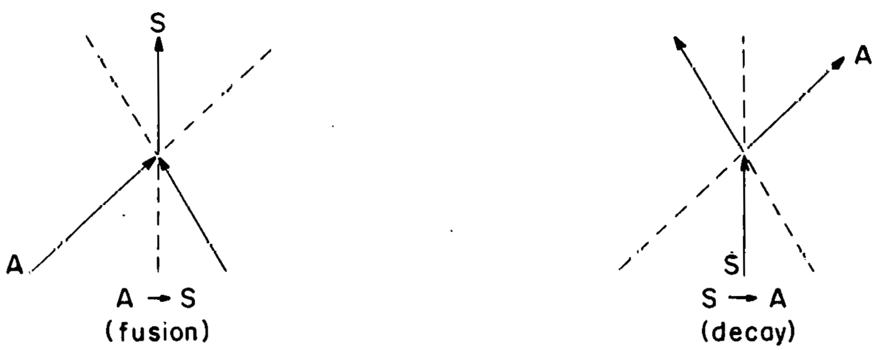


FIG. 14.1

While other conservation laws (for charge, spin, ...) may impose further restrictions on the physical process, we shall regard two arbitrary systems, as defined in §7, to be interconvertible if and only if (1) they are concurrent at an event, and (2) belong to the same class.

In particular, if S^* is a system of class $\{P_o, M_o\}$ consisting of a single particle, hence with parameters (P_o, M_o, m_o), and A and S are arbitrary systems of this class, concurrent with S^* at some event (R,t) of its trajectory, then the fusion $A \to S^*$, and the decay $S^* \to S$ are equally possible. Indeed, every transmutation $A \to S$ may be regarded, mathematically at least, as a composite process $A \to S^* \to S$, where S^* is of negligible duration.

The present chapter is, in the main, only an elaboration of Ch. II, which dealt with the systems belonging to a given class $\{P_o, M_o\}$, that is to say, having a given total momentum P_o and mass M_o. Now, we emphasize the dependence of P_o and M_o on the particle parameters of the initial system $A(m_h)$, and (although irrelevant for the dynamics) the necessary concurrence of A and S, a property not required before. We shall also consider in detail the problem of orientation on given spatial axes, and, to facilitate computation, a gradual transition to energy-parameters will be made.

We state below without proof the principal implications of Ch. II for transmutations.

Theorem 1. In any transmutation $A \to S$, S is coherent-immaterial if and only if A is. For such a system A, all possible resulting systems are "coalesced," with the single trajectory of A, and parameters determined to the extent indicated by Theorem 9.1.

Thus a free photon of energy $h\nu$ can only transmute into a system of immaterial particles of total energy $h\nu$, all superimposed on its own line of flight. It cannot produce, for example, an electron-positron pair, nor a divergent set of photons.

Theorem 2. If $A \to S$ is a transmutation between systems of class $\{P_0, M_0\}$, both A and S have the same CM velocity, namely the class velocity $U_0 = M_0^{-1} P_0$, and indeed, identical CM trajectories. If $|U_0| < c$ $(m_0 > 0)$, then $A \to S$ appears, in the Σ'-frame of their class, as a transmutation $A' \to S'$ between the corresponding systems of class $\{0, m_0\}$, occurring at their stationary CM. The latter systems, of zero total momentum, both have total mass m_0 and total energy e_0. Conservation of energy in Σ is expressed by the equation

$$e_A + k_A = E_0 = e_S + k_S$$

and in Σ' by

$$e_A + k_A' = e_0 = e_S + k_S'.$$

As a consequence of Th. 12.1, we have, in terms of energy parameters, the principal

Theorem 3. Let A be a system, of class $\{cP_0, E_0\}$ with $e_0 > 0$, and let $e_i \geq 0$ be any $I \geq 1$ specified ch. energies. Then, a transmutation of form

$$A \rightarrow S(e_i) \qquad (\dagger)$$

is impossible unless

$$e_o \geqq e_S \equiv \Sigma e_i. \qquad (T)$$

(a) If $e_o = e_S$, ($\dagger$) is possible if and only if all $e_i > 0$.

(b) If $e_o > e_S$, ($\dagger$) is possible if and only if $I \geqq 2$.

In case (a), S is a completely unique, coherent-material system, coalesced, with the single trajectory of the CM of A, and the parameters given in Th. 9.2. Fusion, with I = 1, is the only case of physical interest.

In case (b), details on the nature of S will be found in §§11-13.

Notes 14.

1. (Notation) In a transmutation $A(e_h) \rightarrow S(e_i)$, subscripts h and i designate the H particles of A, and the I particles of S, respectively. When $H \geqq 2$ and $I \geqq 2$, we adopt for simplicity the numbering convention $h = 1,\ldots,H;\ i = H+1,\ldots,H+I\}$.

2. To avoid constant repetition, we summarize here for reference purposes, and in broad outline, the main procedures involved in most of the problems occurring in the present chapter.

(a) For the initial system $A(e_h)$, given with respect to definite spatial axes $G = [X,Y,Z]$ in Σ, we find the totals

$$cP_o \equiv \Sigma cP_h, \qquad E_o \equiv \Sigma E_h, \qquad e_A \equiv \Sigma e_h, \qquad cp_o = |cP_o|.$$

(b) For its class $\{cP_o, E_o\}$, we obtain the class parameters (§8)

$$\beta_o = cp_o/E_o, \quad e_o = \{E_o{}^2-(cp_o)^2\}^{\frac{1}{2}}, \quad \gamma_o = E_o/e_o, \quad \Psi_o = cP_o/cp_o.$$

(c) The necessary condition $e_o \geqq \Sigma e_i \equiv e_S$ for formation of a proposed system $S(e_i)$ is tested. (Assuming the transformation to $S(e_i)$ possible, in accordance with Th. 3, its actual formation, rather than that of competing systems of other particles, rests on relative values of cross sections.)

(d) If indeed a non-coherent system $S(e_i)$ results, with $e_o > e_S$ and $I \geqq 2$ (the only non-trivial case), we require the Σ-parameters of its particles. From these, the trajectories are obvious and S is determined.

In general, it is necessary to consider for this purpose the corresponding transmutation $A' \to S'$ in a second inertial frame Σ', usually but not always the Σ'-frame of the class, and related to Σ via Lorentz transformations based upon their relative velocity.

Even in the simplest cases ($I = 2$), S' is not <u>completely</u> determined, and at this point we shall suppose physically <u>stipulated</u>, for the i-th particle of S', its energy E_i' (hence also cp_i'), and its direction Ψ_i' referred to the Σ' axes employed in the problem. In cases of "non-polarized emission" of i about a given <u>basic direction</u> Ψ' in Σ', we will show how the direction Ψ_i' may be chosen for Monte Carlo purposes. The following Note 3, applied in the frame Σ', should make this procedure clear.

(e) From the values of E_i' and $cP_i' = cp_i'\Psi_i'$ obtained in (d), we will indicate how to compute the Σ energy E_i, and the components of cP_i referred to the original axes $\mathfrak{a}$. For this, we shall require the appropriate (cPE) transformation, either in the simple form of §4, with auxiliary rotations from Appendix III if necessary, or in the vector form of Note 4.1

3. ("Standard device") Suppose a <u>basic</u> <u>direction</u> $\Psi = (a_x, a_y, a_z)_{\mathfrak{a}}$ is given relative to axes $\mathfrak{a}$ in an arbitrary frame Σ, and a second direction Ψ_i, in a "non-polarized distribution" about Ψ, is to be chosen by sampling. This means, in effect, that the "latitude" angle $\theta\,(0^\circ \leqq \theta \leqq 180^\circ)$ which Ψ_i makes with Ψ may be drawn from a given distribution, and that a second "longitudinal angle" φ, <u>uniformly</u> distributed on $0^\circ \leqq \varphi < 360^\circ$, and measured from <u>any</u> plane through Ψ, may be used to locate Ψ_i on the "cone" of angular opening θ with axis Ψ. Since the <u>auxiliary</u> <u>direction</u> (Fig. 2)

$$\Omega = (\cos\theta,\ \sin\theta\ \cos\varphi,\ \sin\theta\ \sin\varphi)_{\mathfrak{a}}$$

is distributed about X as Ψ_i should be about Ψ, it is clear that Ψ_i may be chosen as the point $\Psi_i = \delta\Omega$, where δ is any rotation which takes X into Ψ. The explicit rotation δ of Appendix III, Cor. 1, based on the given $\mathfrak{a}$-coordinates of Ψ, and having the matrix given there as D, is designed for this purpose, and the $\mathfrak{a}$-coordinates of Ψ_i are obtained from

$$(\Psi_i)_{\mathfrak{a}} = D(\Omega)_{\mathfrak{a}}$$

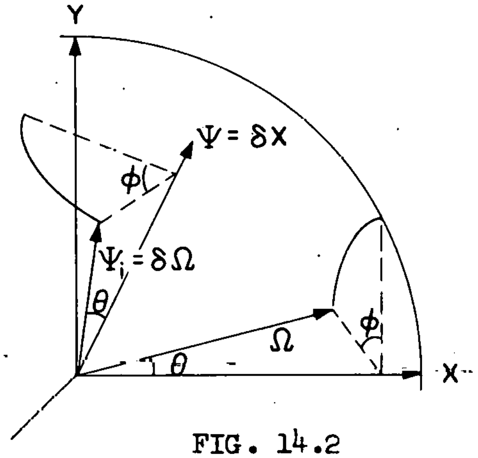


FIG. 14.2

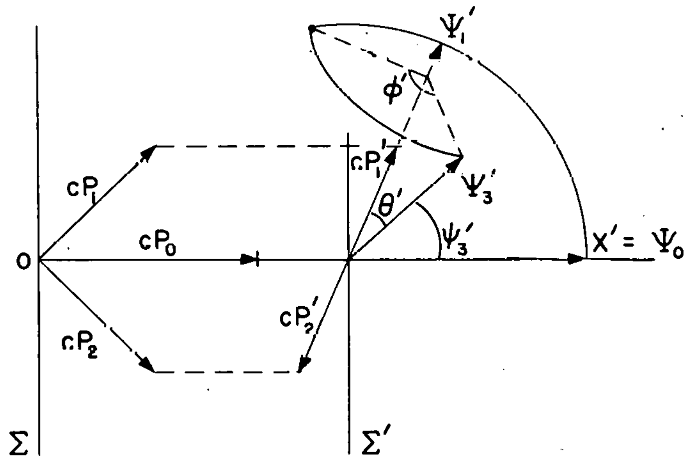


FIG. 14.3

in accordance with Th. I(b) of A III. If the basic direction Ψ is, or may be chosen to be, X itself, one simply sets $\Psi_i = \Omega$.

The choice $\Psi = X$ is always possible in case of an "isotropic" direction distribution, $\cos\theta$ being equi-distributed on $[-1,1]$.

4. The following problem illustrates many of these points. The energy unit is arbitrary, as usual.

PROBLEM. A photon (h = 1) of energy 4 strikes a particle (h = 2) of rest energy 3 and k.e. 2 at right angles, their directions being

$$\Psi_1 = (1/\sqrt{2},\ 1/\sqrt{2},\ 0)_G \qquad \Psi_2 = (1/\sqrt{2},\ -1/\sqrt{2},\ 0)_G$$

on given Σ axes G.

The collision results in two particles (i = 3,4) of rest energies 2,4 resp. The angles $\theta' = 45^\circ$ and $\varphi' = 135^\circ$ are chosen for location of Ψ_3' about the stipulated photon direction Ψ_1' in the Σ'-frame of the class (Fig. 3). Following Note 2, we have

(a) for the initial system $A(e_1,e_2)$

$$e_1 = 0,\ k_1 = 4,\ E_1 = 4,\ cp_1 = 4,\quad \Psi_1 = (1/\sqrt{2},\ 1/\sqrt{2},\ 0)_G,\quad cP_1 = cp_1\Psi_1$$

$$e_2 = 3,\ k_2 = 2,\ E_2 = 5,\ cp_2 = 4,\quad \Psi_2 = (1/\sqrt{2},\ -1/\sqrt{2},\ 0)_G,\ cP_2 = cp_2\Psi_2$$

with totals $cP_o = (4\sqrt{2},\ 0,\ 0)_G$, $E_o = 9$, $e_A = 3$, $cp_o = 4\sqrt{2}$.

(b) For the class of A,

$$\beta_o = 4\sqrt{2}/9, \qquad e_o = 7, \qquad \gamma_o = 9/7, \qquad \Psi_o = (1,\ 0,\ 0)_G.$$

Since $\Psi_o = X$, the given axes G are standard ones, and we define parallel axes G' in Σ' in the standard configuration of Fig. 2.1.

(c) For the proposed $S(e_3,e_4)$, we have $e_3 = 2$, $e_4 = 4$, $e_S = 6 < e_o = 7$, $I = 2$, and the process is possible with a non-coherent result.

(d) Since $I = 2$, the <u>scalar</u> parameters of $S'(e_3,e_4)$ are unique, as in §11. In one of many ways, we find its parameters from $k_S' = e_o - e_S = 1$ to be

$$k_3' = \{e_4 + (k_S'/2)\} k_S'/e_o = 9/14 \qquad k_4' = k_S' - k_3' = 5/14$$

$$E_3' = e_3 + k_3' = 37/14 \qquad E_4' = e_4 + k_4' = 61/14$$

$$cp_i' = \{k_3'(E_3'+e_3)\}^{\frac{1}{2}} = 3\sqrt{65}/14; \; i = 3,4.$$

Fig. 3 makes plain why the latitude θ' of Ψ_3' about Ψ_1' does not alone determine the angle ψ_3' upon which E_3 depends. Although we return to this type of problem in §24, we indicate here the remaining steps. To find the basic direction Ψ_1' we compute from the inverse of (cPE), Note 10.1,

$$cp_{1x}' = (E_o \, cp_{1x} - cp_o \, E_1)/e_o = 2\sqrt{2}/7$$

$$E_1' = (- cp_o \, cp_{1x} + E_o \, E_1)/e_o = 20/7$$

$$cp_1' = (E_1'^2 - e_1^2)^{\frac{1}{2}} = 20/7$$

$$\cos \psi_1' = cp_{1x}'/cp_1' = \sqrt{2}/10.$$

Therefore $\Psi_1' = (\sqrt{2}/10,\ 7\sqrt{2}/10,\ 0)_{G'}$. Applying the device of Note 3, with the given θ', φ', we have $\Omega' = (1/\sqrt{2},\ -1/2,\ 1/2)_{G'}$ for the auxiliary direction, and

$$(\Psi_3')_{G'} = D(\Omega')_{G'} = (.1 + .7/\sqrt{2},\ .7 - .1/\sqrt{2},\ +.5)_{G'}$$

where
$$D = \begin{vmatrix} \sqrt{2}/10 & -7\sqrt{2}/10 & 0 \\ 7\sqrt{2}/10 & \sqrt{2}/10 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

is the matrix of A III, Cor. 1, based on Ψ_1'. (All vectors are understood to be <u>column</u> vectors, despite appearances.)

(e) Since cp_3' and E_3' are known from (d), we have only to apply (cPE) of Note 10.1 to $(cP_3')_{G'} = cp_3'(\Psi_3')_{G'}$ and E_3' to obtain $(cP_3)_G$ and E_3.

N.B. The <u>scalar</u> parameters of $A'(e_1, e_2)$ may be obtained, if desired, just as were those of $S'(e_3, e_4)$ in (d). Thus $k_A' = e_o - e_A = 4$

$$k_1' = \{e_2 + (k_A'/2)\}k_A'/e_o = 20/7 \qquad k_2' = k_A' - k_1' = 8/7$$

$$E_1' = 20/7 \qquad E_2' = 29/7$$

$$cp_h' = 20/7;\ h = 1,2.$$

For the <u>direction</u> of Ψ_1', we must use the transformation (cPE) itself at least for p_{1x}', as in (d). (It is a good exercise to sketch Fig. 12.1 for both A and S.) If the PROBLEM is carried through completely, the equations

$$E_1 + E_2 = E_3 + E_4$$

$$cP_1 + cP_2 = cP_3 + cP_4$$

should provide a final check. Alternatively, they may of course be used to obtain cP_4, E_4 by default.

15. The Q-value. The Q-value of a (proposed) transmutation

$$A(e_h) \to S(e_i) \qquad (\dagger)$$

is defined as the intrinsic difference

$$Q = e_A - e_S \qquad (1)$$

in the total ch. energies of the two systems. The required equation

$$e_A + k_A = E_o = e_S + k_S \qquad (2)$$

for energy conservation in Σ is thus expressible in the form

$$Q = k_S - k_A \qquad (3)$$

emphasizing that, in the conversion of A into S, the "loss" in ch. energy must balance the "gain" in kinetic energy. This also makes obvious the invariance of the kinetic energy difference for two systems of the same class, under arbitrary Lorentz transformations.

If A and S are in a class with $e_o > 0$, then, for the corresponding transmutation $A' \to S'$ in the Σ'-frame of the class, the required energy conservation is expressed in Σ' by

$$e_A + k'_A = e_o = e_S + k'_S. \qquad (4)$$

It may be noted that this version makes obvious the necessity of the

condition $$e_o \geqq e_S \tag{T}$$

and at the same time shows its equivalence with the condition

$$k'_A \geqq e_S - e_A \equiv -Q \tag{T$'$}$$

in the Σ'-frame, signifying that the k.e. of A′ must suffice to make up the ch. energy excess of S over A.

The transmutation (†) is said to be elastic in case the total kinetic energy is conserved, as well as E_o and P_o. Such a change is therefore one for which we have the additional stipulation that

$$k_A = k_S \tag{5}$$

or, equivalently, $$Q = 0. \tag{6}$$

Clearly a transmutation $A \rightarrow S$ appearing elastic in Σ must so appear in all inertial frames.

Note 15.

1. The (invariantly expressed) condition (T) is equivalent to the

inequality $$k_A \geqq (-Q) + (\gamma_o - 1)e_S$$

in an arbitrary frame Σ, and reduces to $k_A \geqq -Q$ when $U_o = 0$ ($\Sigma \equiv \Sigma'$), as it must. It is tempting, but misleading (cf. §16) to assert that (T) requires A to have k.e. k_A sufficient to supply $-Q$ plus the k.e. of a particle of ch. energy e_S riding at its CM.

16. Decay. By a decay we understand here any transmutation of form

$$A(e_0) \to S(e_1,\ldots,e_I) \qquad \text{(D)}$$

in which a single material particle is converted into a system S of an arbitrary number $I \geqq 2$ of particles. If the decaying particle has momentum P_0 and mass M_0, then its rest mass is of course the critical mass $m_0 > 0$ of its class $\{P_0, M_0\}$, and indeed all class parameters cP_0, E_0, β_0, e_0, γ_0, and Ψ_0 are simply the parameters of the particle itself.

Moreover, the Σ'-frame of the class is the rest frame of the particle, in which it appears stationary, with energy e_0, and intrinsic life time τ_0'. From §2, we recall that its apparent life time in Σ is $\tau_0 = \gamma_0\tau_0'$, during which it travels a distance $\delta_0 = u_0\tau_0 = \beta_0\gamma_0 c\tau_0'$.

The Q-value of (D) is

$$Q = e_0 - e_S$$

so that the necessary condition $e_0 \geqq e_S$ is here simply

$$Q \geqq 0. \qquad (T_D)$$

From Th. 14.3, we conclude that the decay is prohibited if $Q < 0$ ($e_0 < e_S$), regardless of the k.e. of the particle. If $Q = 0$, the decay is possible if and only if all $e_i > 0$, with a trivial coherent result. If $Q > 0$ (the only case of interest), then (D) is always possible (since we have stipulated $I \geqq 2$). Such a process appears in Σ' as an "explosion" of a particle at rest, with a conversion of characteristic to kinetic energy indicated by the equation $$Q = e_o - e_S = k'_S .$$

Notes 16.

1. We list some examples of physical decays forbidden by the laws of energy and momentum conservation.

(a) For a stable nucleus $\left({}^A_Z X\right)$ of charge Zq "containing" A nucleons (Z protons p^+, $N \equiv A - Z$ neutrons n^o) the decay

$$\left({}^A_Z X\right) \longrightarrow Z(p^+) + N(n^o) \qquad (D_1)$$

is impossible, since

$$e_o \equiv e\left({}^A_Z X\right) < Ze(p^+) + Ne(n^o) \equiv e_S$$

and so $Q = e_o - e_S < 0$. Here $^-Q > 0$ is called the "binding energy" of the stable nucleus.

Note: If $\left[{}^A_Z X\right]$ denotes the neutral atom, with Z electrons (ϵ^-) in ground-state about the bare nucleus, the process

$$\left[{}^A_Z X\right] \longrightarrow \left({}^A_Z X\right) + Z(\epsilon^-)$$

has negative Q-value, ^-Q being the "binding energy of the electrons." In nuclear processes this is neglected. Thus in (D_1) one takes

$$Q \cong e\left[{}^A_Z X\right] - Ze\left[{}^1_1 H\right] - Ne(n^o).$$

Table III gives the values of $e\left[{}^A_Z X\right]$ for a few neutral atoms in "atomic mass units." For example, the binding energy (2.225 Mev) of the deuteron $\left({}^2_1H\right)$ may be found in this way.

(b) The decay $p^+ \rightarrow n^0 + \epsilon^+ + \nu_\epsilon$, where ϵ^+ is a positron and ν_ϵ a neutrino, has Q = - 1.8 Mev (Table II). While therefore forbidden to the free proton, this is nevertheless the underlying process in positron emission from unstable nuclei.

(c) The electron decay $\epsilon^- \rightarrow \epsilon^- + \gamma$ has Q = 0 but is forbidden by $m(\gamma) = 0$. Hence an electron can neither "emit" nor "absorb" a photon.

Remark. A transmutation $\epsilon^- + \gamma \rightarrow \epsilon^-$ is obviously impossible, as the reverse of (c). It is all the more curious that the impossibility appears to lie deeper where $A(\epsilon^-,\gamma)$ is regarded as the initial system.

(d) The process $p^+ \rightarrow \epsilon^+$ has Q > 0 but is impossible since I = 1 (Th. 14.3). Consider the reverse here!

2. We indicate two methods of dealing with a "Monte Carlo" type decay problem in which the <u>given</u> axes $G = [X,Y,Z]$ of Σ are (here for the first time) <u>not</u> in standard configuration. The first is based on an auxiliary rotation of axes, provided for in A III, Cor. 1; the second on the "vector form" of the (cPE) transformation in Note 4.1. The generalities of Notes 14.2,3 should be consulted as required. All energies are in (say) Mev.

<u>PROBLEM</u>. A particle of rest-energy $e_o = 3$, k.e. $k_o = 2$, and direction $\Psi_o \equiv (a_{ox}, a_{oy}, a_{oz})_G \equiv (2/3, 2/3, 1/3)_G$ on given Σ-axes $G = [X,Y,Z]$, decays in flight into two particles of equal rest-energy $e_1 = e_2 = 1$. If, as we shall assume, the decay product 1 is emitted isotropically in the

rest-frame Σ' of the decaying particle, the "basic direction of emission" Ψ' is ours to choose, and we shall do so in different ways in the two methods. We shall suppose chosen the coordinates (2/7,6/7,3/7) for the "auxiliary direction" Ω' in either case, i.e., $\cos\theta' = 2/7$, etc.

(a) The totals for the initial system $A(e_o)$ are $e_o = 3$, $E_o = 5$, $cp_o = (E_o{}^2 - e_o{}^2)^{\frac{1}{2}} = 4$.

(b) The class parameters are those of the decaying particle: $\beta_o = cp_o/E_o = 4/5$, $e_o = 3$, $\gamma_o = E_o/e_o = 5/3$, $\Psi_o = (2/3,2/3,1/3)_{\bar{G}}$.

(c) The proposed system $S(e_1,e_2)$ has $e_1 = e_2 = 1$, $e_S = 2 < e_o = 3$, and $I = 2$, so non-coherent decay into S is possible.

(d) Since $e_1 = e_2$, S' in Σ' obviously has $E_i' = e_o/2 = 3/2$, $cp_i' = \left(E_i'^2 - e_i{}^2\right)^{\frac{1}{2}} = \sqrt{5}/2$; $i = 1,2$. Now:

Method I. Suppose spatial axes $\bar{G} = [\bar{X},\bar{Y},\bar{Z}] = [\delta X,\delta Y,\delta Z]$ determined in Σ by the rotation δ of A III, Cor. 1, with $\delta X = \Psi_o$, the direction of the class velocity (Fig. 1). The associated matrix D, based on Ψ_o, is found to be

$$D = \begin{vmatrix} 2/3 & -2/3 & -1/3 \\ 2/3 & 11/15 & -2/15 \\ 1/3 & -2/15 & 14/15 \end{vmatrix}.$$

Defining Σ' axes $\bar{G}' = [\bar{X}',\bar{Y}',\bar{Z}']$ in standard configuration (§2) with $\bar{G}$, we select $\Psi_o = \bar{X}'$ as the basic direction for emission of 1, and hence define Ψ_1' immediately as the auxiliary direction itself:

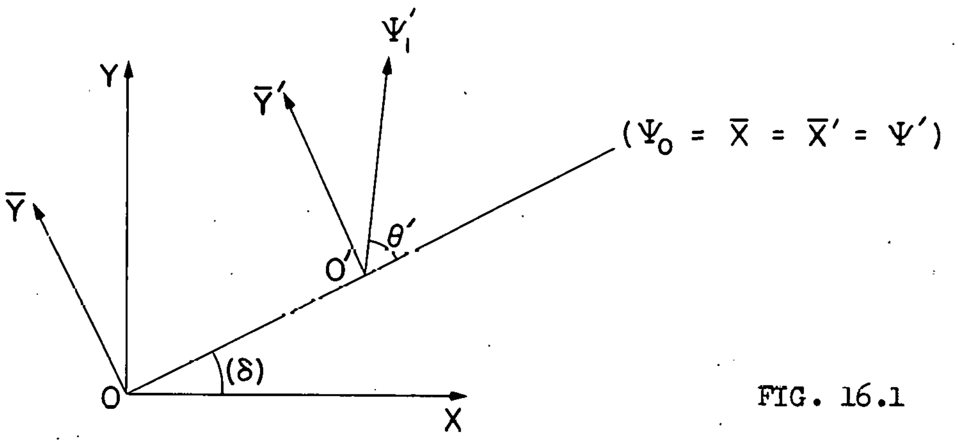


FIG. 16.1

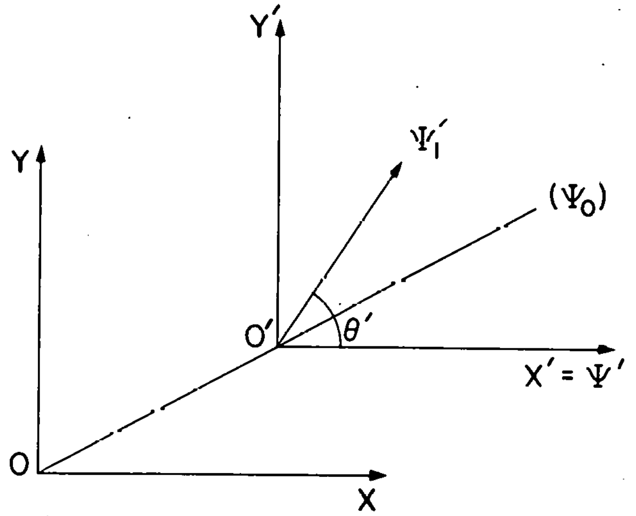


FIG. 16.2

$$\Psi_1' = (2/7, 6/7, 3/7)_{\bar{G}'}.$$

Thus $E_1' = 3/2$, and $cP_1' = (\sqrt{5}/2)\Psi_1'$ on standard Σ' axes, are determined.

(e) To these the simple (cPE) of Note 10.1 applies, yielding $E_1 = 4\sqrt{5}/21 + 5/2$, $cP_1 = (5\sqrt{5}/21 + 2,\ 3\sqrt{5}/7,\ 3\sqrt{5}/14)_{\bar{G}}$ on the standard axes $\bar{G}$ of Σ. (Note the check: $E_1{}^2 - (cP_1)^2 = 1 = e_1{}^2$.)

The components of cP_1 relative to the original axes G are then obtained from $(cP_1)_G = D(cP_1)_{\bar{G}}$, as in Th. 1(a) of A III.

Method II. We may equally well choose axes $G' = [X', Y', Z']$ in Σ' parallel to the given Σ axes $G = [X, Y, Z]$, and now select X' as the basic direction Ψ'. We now take the Σ' direction of 1 as

$$\Psi_1' = (2/7, 6/7, 3/7)_{G'}$$

(it is of course not the same absolute direction as in Method I) and so have $E_1' = 3/2$, and $cP_1' = (\sqrt{5}/2)\Psi_1'$ on the axes G'.

(e) These may be substituted directly into the vector form (cPE) of Note 4.1, based on the class parameters in (b). The inner product required is $cP_1' \cdot \Psi_o = 19\sqrt{5}/42$. The energy is found to be $E_1 = 38\sqrt{65}/63 + 5/2$, while the vector equation

$$(cP_1)_G = (cP_1')_{G'} + \{19\sqrt{65}/63 + 2\}(\Psi_o)_G$$

indicates how the components of cP_1 on Σ axes G are to be computed.

17. Decay into two particles. In a decay

$$A(e_o) \rightarrow S(e_1, e_2)$$

where $$Q = e_o - (e_1+e_2) = k'_S > 0$$

the products emerge in opposite directions in the Σ'-frame, with the unique energies E'_i and k'_i of Cor. 11.1. If e_o is at rest in Σ, the frames Σ and Σ' coincide, and all parameters of S are of course those of S′.

I. In the simplest cases, $e_1 = e_2$, so that $Q = e_o - 2e_1$

and $k'_i = Q/2, \quad E'_i = e_o/2, \quad cp'_i = \left(e_o^2 - 4e_1^2\right)^{\frac{1}{2}}/2; \quad i = 1,2.$

Thus, in the kaon decay (TABLE II)

$$K_1^o \rightarrow \pi^- + \pi^+$$

$Q = 218.8$ Mev, and each pion has k.e. 109.4 Mev in Σ'.

In particular, when $e_1 = e_2 = 0$, the decay involves a total conversion of the rest-energy e_o into kinetic energy

$$Q = e_o = k'_S$$

with $k'_i = E'_i = cp'_i = e_o/2$. This is the case for which we have the simple result of §12,

$$\sin \sigma/2 = e_o/2\,(E_1E_2)^{\frac{1}{2}} \geq e_o/E_o$$

for the angle σ between the two lines of flight in Σ.

For example, the decay

$$\pi^{o} \longrightarrow \gamma + \gamma$$

of a 135 Mev (k.e.) neutral pion yields photons with a minimum angle of separation of 60^{o} in Σ. In this case, each has energy $h\nu_i$ = 135 Mev in Σ.

A second instance is provided by the decay of "parapositroneum": $\{\epsilon^{+}\uparrow, \epsilon^{-}\downarrow\} \longrightarrow \gamma\uparrow + \gamma\downarrow$. Neglecting its binding energy, each photon in Σ' has energy $h\nu_i' = e_\epsilon$ = .511006 Mev and wavelength λ_i' (by definition, Note 1.4) the Compton wavelength of the electron.

Note. Although a free positron ϵ^{+} is stable, it comes to rest locally when liberated (as in pair production and positron emission) in the presence of matter, and may then combine with an electron ϵ^{-} to form a very unstable "double star" complex $\{\epsilon^{+}, \epsilon^{-}\}$ called positroneum. When the component spins are opposite (the usual case) the result is parapositroneum, with the decay mode above, the photon spins also being opposite. Spin conservation is indicated by the equation $\frac{1}{2} - \frac{1}{2} = 0 = 1 - 1$. The alternative result is "orthopositroneum", with the decay $\{\epsilon^{+}\uparrow, \epsilon^{-}\uparrow\} \longrightarrow \gamma\uparrow + \gamma\uparrow + \gamma\downarrow$, and the spin conservation $\frac{1}{2} + \frac{1}{2} = 1 = 1 + 1 - 1$.

II. In another important case, one has $e_1 > 0$ and $e_2 = 0$, with $Q = e_o - e_1 = k_S' > 0$. Here, the formula (§11) for the Σ' k.e. of e_1 becomes

$$k_1' = Q\rho \qquad \text{where} \qquad \rho \equiv Q/2e_o$$

and the relations

$$cp_i' = E_2' = k_2' = Q(1-\rho), \qquad E_1' = k_1' + e_1 \equiv e_o - E_2'$$

indicate an easy computational scheme for the remaining parameters of S'.

Since $Q = e_o - e_1 < e_o$, it is clear that $\rho < \frac{1}{2}$, and consequently

$$k_1' < k_S'/2 < k_2'.$$

On the other hand, a glance at the energy formulas (Cor. 11.1)

$$E_1' = (e_o/2)\{1+(e_1/e_o)^2\}$$

$$E_2' = (e_o/2)\{1-(e_1/e_o)^2\}$$

shows that

$$E_1' > e_o/2 > E_2'.$$

(These inequalities are true whenever $e_1 > e_2$, as noted in §11.)

The last equation correlates the rest-energies e_o, e_1 of the two material particles with the k.e. $k_2' - E_2' = h\nu_2'$ of the immaterial one. For example, the implied relation

$$e_o = k_2' + (k_2'^2 + e_1^2)^{\frac{1}{2}}$$

may be used to determine the rest energy of the decaying particle from e_1 and $k_2' = h\nu_2'$ in cases of γ-emission.

The decay modes (TABLE II) of some of the "fundamental" particles fall under Case II, e.g.

$$\Sigma^o \rightarrow \Lambda^o + \gamma.$$

In this decay, $Q = 77$ Mev, $\rho = .0323$, $k_1' = 2.49$ Mev, and $k_2' = h\nu_2' =$ 74.51 Mev.

In nuclear decay of this type, $Q = e_o - e_1$ may be thought of as a difference in energy levels of the "same" nucleus, having rest-energies $e_o > e_1$ in the two corresponding states. In such a photon emission, it is interesting to compare the photon wavelength $\lambda_2' \equiv c/\nu_2'$ with the "normal" wavelength $\lambda_o \equiv c/\nu_o$, where by definition

$$h\nu_o \equiv Q$$

is the difference in energy levels. Dividing this equation by

$$h\nu_2' = k_2' = Q(1-\rho)$$

yields

$$\lambda_2'/\lambda_o = \nu_o/\nu_2' = 1/(1-\rho)$$

and therefore

$$z' = (\lambda_2'-\lambda_o)/\lambda_o = \rho/(1-\rho) = k_1'/k_2' = Q/(Q+2e_1) = (e_o-e_1)/(e_o+e_1)$$

is the "red shift due to recoil" (in Σ').

For example, in the γ-emission

$$\left({}^2_1H\right)^* \longrightarrow \left({}^2_1H\right) + \gamma$$

with $Q = 2.225$ Mev, one finds $z' \cong 6 \times 10^{-4}$.

III. When $e_1 > e_2 > 0$, the general formulas of §11 are required. The decay modes $\Xi^- \longrightarrow \Lambda^o + \pi^-$ and $\Sigma^- \longrightarrow n^o + \pi^-$ of TABLE II, and the classical nuclear emission of α particles $\left({}^4_2He\right)$, are of this kind.

18. Decay into three particles. In a decay

$$A(e_o) \longrightarrow S(e_1, e_2, e_3)$$

where $$Q = e_o - (e_1+e_2+e_3) = k'_S > 0$$

the Σ' energies of S' are not unique, and as shown in §13, any one of its particles (i = 1) may have for its k.e. values on the range

$$0 < k'_1 < k'_K \equiv \frac{e_L + (k'/2)}{e_o} \cdot k'$$

where $e_L \equiv e_2 + e_3$, and $k' \equiv e_o - (e_1+e_2+e_3)$, which is here (as in all decays) Q itself. We recall that k'_K is the unique k.e. of e_1 in a 2 particle system $S'(e_1, e_L)$ of class $\{0, e_o\}$. (The technicalities involved in attainment of the bounds are given in §13, but are of no physical interest.)

For example, the Σ' energy range of any one of the three photons produced in orthopositroneum decay (§17) is $0 < h\nu'_1 \leqq .511$ Mev.

Perhaps the most famous instance is the decay

$$n^o \longrightarrow \epsilon^- + p^+ + \bar{\nu}_\epsilon \qquad (Q = .783 \text{ Mev})$$

of the free neutron, which is unstable, with mean lifetime 1013 sec. The electron should have a k.e. range

$$0 \leqq k'_1 < k'_K = (938.648/939.550)(.783) = .782 \text{ Mev},$$

which indeed is observed experimentally.

Note. A decay of form $n^{o} \rightarrow \epsilon^{-} + p^{+}$ has the same positive Q-value, and is also mechanically allowed, but would result in a unique k.e. $k_{1}' = k_{K}'$ for ϵ^{-}, in conflict with experiment, and would violate conservation of spin, since: $\pm \frac{1}{2} \neq \pm \frac{1}{2} \pm \frac{1}{2}$ for any choice of signs.

Neutron decay is the basic process involved in electron-emission from unstable nuclei:

$$\left({}^{A}_{Z}X\right) \rightarrow \epsilon^{-} + \left({}^{A}_{Z+1}Y\right) + \bar{\nu}_{\epsilon}$$

e.g., in the decay of the "triton"

$$\left({}^{3}_{1}H\right) \rightarrow \epsilon^{-} + \left({}^{3}_{2}He\right) + \bar{\nu}_{\epsilon} \qquad (Q = .0182 \text{ Mev}).$$

The analogous nuclear positron emission

$$\left({}^{A}_{Z}X\right) \rightarrow \epsilon^{+} + \left({}^{A}_{Z-1}Y\right) + \nu_{\epsilon}$$

is observed, although the process for the free proton is forbidden. (Note 16.1.) An example is

$$\left({}^{11}_{6}C\right) \rightarrow \epsilon^{+} + \left({}^{11}_{5}B\right) + \nu_{\epsilon} \qquad (Q = .96 \text{ Mev}).$$

Note here that, for such decays, "adding" Z electrons to each side results in the neutral atom "reaction"

$$\left[{}^{Z}_{A}X\right] \rightarrow \epsilon^{+} + \epsilon^{-} + \left[{}^{A}_{Z-1}Y\right] + \nu_{\epsilon}$$

so that $Q \cong e\left[{}^{Z}_{A}X\right] - 2e_{\epsilon} - e\left[{}^{A}_{Z-1}Y\right]$ where $e_{\epsilon} = .511$ Mev. (Cf. Note 16.1.)

19. Collisions with target at rest. Every transmutation $A(e_h) \rightarrow S(e_i)$ in which A consists of a single particle (H=1) may be regarded as a "decay." All others, with $H \geqq 2$, are called "collisions," and we shall consider only those of the form

$$A(e_1,e_2) \rightarrow S(e_i) \qquad \text{(c)}$$

where particle 1 will be called the "projectile" and 2 the "target."

We study first (§§19-23) the important special case in which the target is a (necessarily) material particle at rest in its own rest frame Σ, with the projectile moving toward it on collision course. As a common basis for the sections referred to, we consider given an initial system $A(e_1,e_2)$, with particle parameters

$$k_1 > 0 \qquad E_1 > e_1 \geqq 0 \qquad cp_1 = \left(E_1^{\,2}-e_1^{\,2}\right)^{\frac{1}{2}} > 0 \qquad cP_1 = cp_1\Psi_1$$

$$k_2 = 0 \qquad E_2 = e_2 > 0 \qquad cp_2 = 0 \qquad cP_2 = 0.$$

Following Note 14.2, we see first that

(a) the totals of the system are

$$cP_o = cP_1 \text{ with } cp_o = \left(E_1^{\,2}-e_1^{\,2}\right)^{\frac{1}{2}}; \qquad E_o = E_1 + e_2; \qquad e_A = e_1 + e_2.$$

(b) Hence its class $\{cP_o,E_o\}$ has parameters

$$\beta_o = \left(E_1^{\,2}-e_1^{\,2}\right)^{\frac{1}{2}}/(E_1+e_2), \ e_o = \left(e_1^{\,2}+e_2^{\,2}+2e_2E_1\right)^{\frac{1}{2}} \equiv \left(e_A^{\,2}+2e_2k_1\right)^{\frac{1}{2}},$$

$\gamma_o = (E_1+e_2)/e_o$; and the direction of the class velocity is $\Psi_o = \Psi_1$. It is convenient to know $\left(\gamma_o^{\,2}-1\right)^{\frac{1}{2}} = \gamma_o\beta_o = cp_o/e_o = \left(E_1^{\,2}-e_1^{\,2}\right)^{\frac{1}{2}}/e_o$.

(c) The ch. energies e_i, of sum $e_S \equiv \Sigma e_i$, being stipulated for the proposed system $S(e_i)$, it appears from (b) that the (invariantly expressed) necessary condition

$$e_o \geqq e_S \qquad (T)$$

assumes, in the rest frame of the target, the significant form

$$k_A \equiv k_1 \geqq \frac{e_A+e_S}{2e_2}\,(^-Q) \equiv k_T \qquad (T_{CR})$$

where $\qquad Q \equiv e_A - e_S$

is the Q-value of the reaction (C). The energy k_T so defined is called the "kinetic energy threshold" for the process, and one speaks accordingly of k_1 as below, on, or above threshold in the cases $k_1 \lesseqgtr k_T$. Note the analogy between the form of condition (T) as it appears in the target rest frame, and its form $k'_A \geqq (^-Q)$ in the zero-momentum frame Σ' (§15).

Since the inequality $e_o \geqq e_S$ is here strictly equivalent to $k_1 \geqq k_T$, we may interpret the results of Th. 14.3 in the following convenient form.

I. When $Q < 0$ ($e_A<e_S$), then k_T is <u>positive</u>, and the necessary condition $k_1 \geqq k_T$ "has teeth," to wit:

(i) if $k_1 < k_T$ ($e_o < e_S$), (C) is forbidden.

(ii) if $k_1 = k_T$ ($e_o = e_S$), (C) is possible iff all $e_i > 0$. (Fusion, with I = 1, is the case of interest, and must occur exactly "on threshold.")

(iii) if $k_1 > k_T$, (C) may occur iff $I \geqq 2$.

It should be noted that, when $Q < 0$ ($e_A < e_S$), one has

$k_T \equiv \frac{e_A+e_S}{2e_2}\ (^-Q) > (2e_A/2e_2)\ (^-Q) \geqq (^-Q)$, so necessarily $k_T > (^-Q)$.

II. When $Q \geqq 0$ ($e_A \geqq e_S$), then $k_T \leqq 0$ and its value is irrelevant. For, collision then occurs above threshold, with $k_1 > 0 \geqq k_T$ ($e_o > e_S$), and (C) is possible iff $I \geqq 2$, just as in (iii) above.

Only in the case $Q = 0$ ($e_A = e_S$) under (II) is the collision elastic, with $k_1 = k_S$. In general, the energy equation reads

$$e_A + k_1 = E_o = e_S + k_S$$

or

$$k_1 + Q = k_S$$

so that

$$k_S \lesseqgtr k_1 \quad \text{as} \quad Q \lesseqgtr 0.$$

(d) If the collision is to be studied in the Σ'-frame of the class, the following information may be required, in addition to the class parameters in (b), governing the Lorentz transformations between Σ and Σ'.

In Σ', the collision $A'(e_1,e_2) \to S'(e_1)$ involves two systems of class $\{0,e_o\}$, the <u>colliding</u> system A', with $e_A < e_o = (e_A{}^2+2e_2k_1)^{\frac{1}{2}}$, necessarily (§11) having the unique energies $E_h' = (e_h{}^2+e_2E_1)/e_o$, $h = 1,2$, or, more simply,

$$E_2' = e_2E_o/e_o = \gamma_o e_2 \qquad\qquad E_1' = e_o - \gamma_o e_2$$

and oppositely directed vectors cP_h' of equal magnitude:

$$cp_h' = \left(E_2'-e_2{}^2\right)^{\frac{1}{2}} = \gamma_o\beta_o e_2 = (e_2/e_o)cp_1; \quad h = 1,2.$$

The above form $E_2' = e_2\gamma_o$ reflects the physically obvious fact that

2 in Σ' has the speed $v_2' = u_o$ of the CM of A in Σ. This accounts for the position of O_2' on the ellipsoid $\mathcal{E}$ in Fig. 1, showing the relation between the initial systems A and A′. (Cf. §§6,12.)

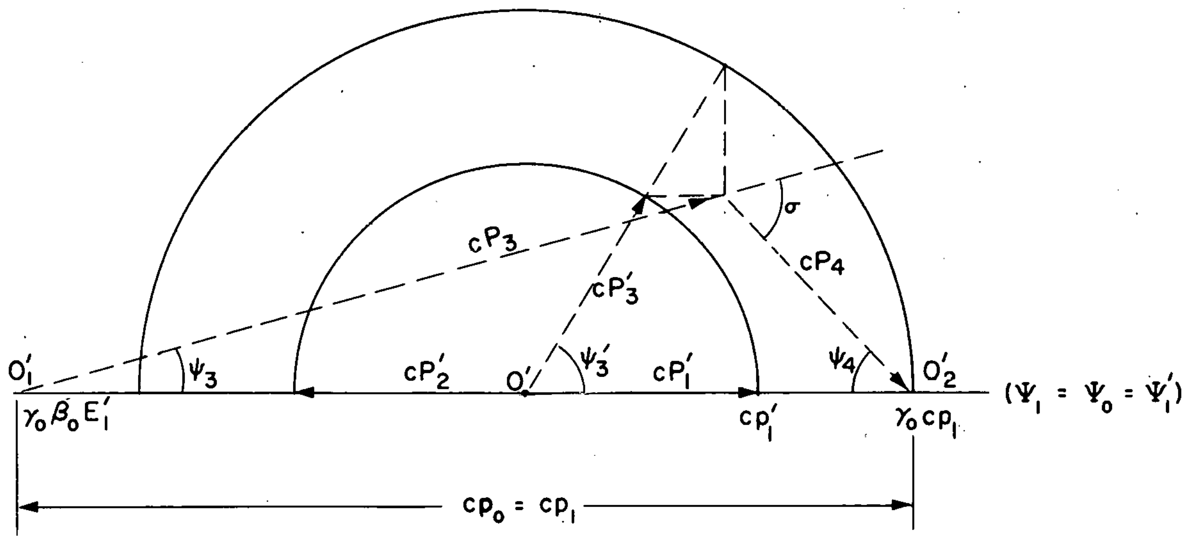


FIG. 19.1 $(e_1 > e_2)$

In the figure, the sphere radii are $cp_1' = \gamma_o\beta_o e_2$, $\gamma_o cp_1' = \gamma_o^2\beta_o e_2$, and the origins O_i' are at distances $O_1'O' = \gamma_o\beta_o E_1'$, $O'O_2' = \gamma_o\beta_o E_2' = \gamma_o^2\beta_o e_2$ from O′.

These details are of importance in the elastic case $S = S(e_1,e_2)$, since the resulting system $S'(e_1,e_2)$ will then differ from $A'(e_1,e_2)$ only in direction. (Such a result is indicated by dashed lines in the figure.) With this in mind, we will further note here that O_1' falls

(I) inside, (II) on, or (III) outside $\mathcal{E}$ according as

$$e_1 \lesseqgtr e_2$$

i.e., in the cases of a projectile of ch. mass "lighter" than, equal to, or "heavier" than that of the target. To see this, we need only note that, in the present instance,

$$O_1'O_1 \equiv \gamma_o\beta_o E_1' \lesseqgtr \gamma_o(cp_1') = O'O_2' \equiv \gamma_o\beta_o E_2'$$

as

$$E_1' \lesseqgtr E_2'$$

and the remark follows from the equation

$$E_1'^2 - e_1^2 = cp_1'^2 = cp_2'^2 = E_2'^2 - e_2^2.$$

<u>Notes 19.</u>

1. There are good reasons for considering first, in some detail, collisions in the rest frame of the target, aside from their greater simplicity.

(a) In many physical collisions, the target may be assumed essentially at rest in the laboratory frame Σ.

(b) If the target is a material particle moving with velocity $V_2 \neq 0$ in Σ, a preliminary Lorentz transformation based on V_2 will carry the colliding system into a frame in which the target <u>is</u> at rest, and to which the simpler theory applies.

(c) The "general" methods presented later (§24) really <u>require</u> the target to be moving (with a well-defined direction), specializing to the rest case only in a limiting sense, and it would be witless to treat the simpler case in such a way.

2. In Note 16.2 we gave two methods of dealing with a general decay problem. Strictly analogous methods may be used for collisions with target at rest in Σ, as indicated in the following

<u>PROBLEM</u>. A projectile of rest energy $e_1 = 5$, k.e. $k_1 = 8$, and direction $\Psi_1 = (2/3,2/3,1/3)_G$ on given Σ-axes $G = [X,Y,Z]$ strikes a particle of rest-energy $e_2 = 7$ which is at rest in Σ. The result is an elastic scattering $A(e_1,e_2) \rightarrow S(e_1,e_2)$, which is to be treated in the Σ'-frame of the class, with the direction Ψ_1' of the projectile as the basic direction Ψ' for non-polarized scattering of E_1. The coordinates $(2/7,6/7,3/7)$ will be used for the auxiliary direction Ω'.

For $A(e_1,e_2)$ in Σ, we are given

$$e_1 = 5,\ k_1 = 8,\ E_1 = 13,\ cp_1 = \left(E_1{}^2 - e_1{}^2\right)^{\frac{1}{2}} = 12,\ \Psi_1 = (2/3,2/3,1/3)_G,\ cP_1 = 12\Psi_1$$

$$e_2 = 7,\ k_2 = 0,\ E_2 = 7,\quad cp_2 = 0,\ cP_2 = 0.$$

(a) The totals of A are $cP_o = cP_1$ with $cp_o = 12$, $E_o = 20$, $e_A = 12$.

(b) Its class therefore has parameters

$$\beta_o = cp_o/E_o = 3/5,\ e_o = \left\{E_o{}^2 - (cp_o)^2\right\}^{\frac{1}{2}} = 16,\ \gamma_o = E_o/e_o = 5/4,\ \Psi_o = \Psi_1.$$

(c) Elastic collision of this type is trivially possible; obviously we may have $A \equiv S$. Note that, in general, we need only verify $e_o \geqq e_S$. Computation of k_T is a "luxury."

(d) For $A'(e_1,e_2)$, $k'_A = e_o - e_A = 4$, $k'_1 = \{e_2+(k'_A/2)\}k'_A/e_o = 9/4$, $k'_2 = k'_A - k'_1 = 7/4$. Hence

$E'_1 = e_1 + k'_1 = 29/4$, $E'_2 = e_2 + k'_2 = 35/4$, $cp'_h = \left(E_2{}^2-e_2{}^2\right)^{\frac{1}{2}} = 21/4$.

For $S'(e_1,e_2)$ therefore

$E'_3 = E'_1$, $E'_4 = E'_2$, $cp'_i = cp'_h$; $i,h = 1,2$.

We have now the two methods: (Fig. 2,3):

Method I. As in Note 16.2, standard axes $\bar{\mathfrak{a}},\bar{\mathfrak{a}}'$ are chosen, using the same δ and matrix D. Since $\bar{X} = \delta X = \Psi'_1$ is the basic direction for scattering of e_1, we take

$$\Psi'_3 = \Omega' = (2/7,6/7,3/7)_{\bar{\mathfrak{a}}'}.$$

Then from (d), $E'_3 = 29/4$, $cP'_3 = (21/4)\Psi'_3$.

(e) Applying (cPE) of Note 10.1 to E'_3 and $(cP'_3)_{\bar{\mathfrak{a}}'}$ yields E_3 and $(cP_3)_{\bar{\mathfrak{a}}}$. Finally $(cP_3)_{\mathfrak{a}} = D(cP_3)_{\bar{\mathfrak{a}}}$, on the original axes $\mathfrak{a}$.

Method II. The vector method of Note 16.2 also applies, and could be used just as before if we were free to choose X' as the basic direction Ψ' (as for example in isotropic scattering in Σ'). However, since the stipulated $\Psi' = \Psi_o \neq X'$, we use the device of Note 14.3. From the auxiliary direction $\Omega' = (2/7,6/7,3/7)_{\mathfrak{a}'}$ and the same matrix D as in Method I (the rotation δ being the same, although used here for a different purpose) one finds

$$(\Psi'_3)_{\mathfrak{a}'} = D(\Omega')_{\mathfrak{a}'},\ (cP'_3)_{\mathfrak{a}'} = (21/4)(\Psi'_3)_{\mathfrak{a}'},\ E'_3 = 29/4.$$

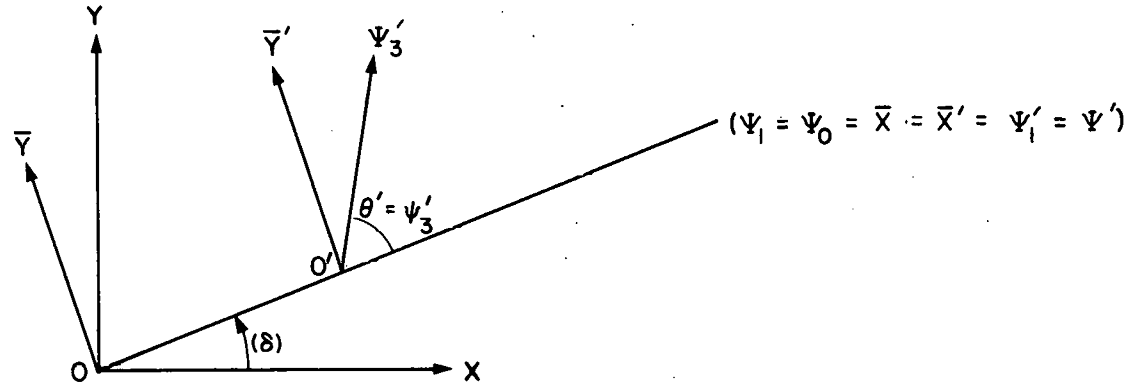


FIG. 19.2

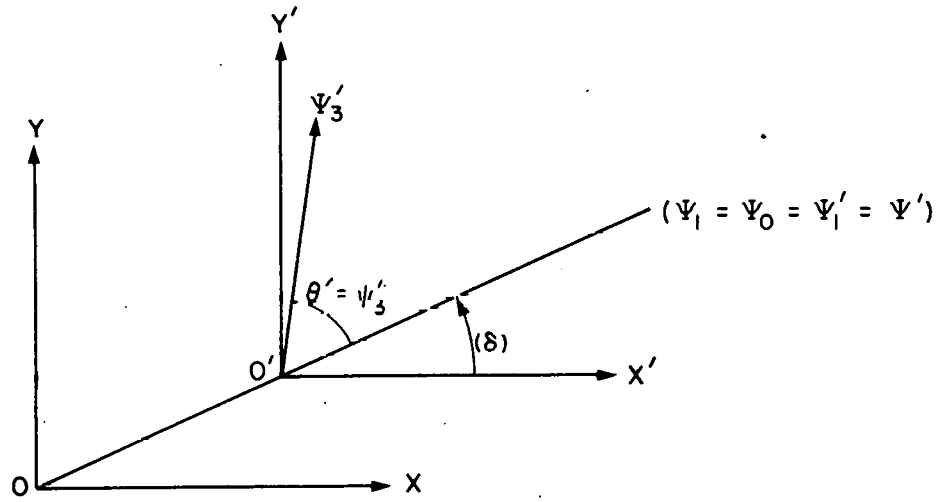


FIG. 19.3

Hence, in step (e), we substitute the latter vector and energy directly into (cPE) of Note 4.1 to obtain $(cP_3)_G$ and E_3 in Σ. The required values of $\beta_o, \gamma_o, (\Psi_o)_G$ are those in (b) above.

Note that we have purposely used only the basic principles. The dismaying plethora of special "formulas" in the current sections are helpful in understanding the nature of the collision process, but may usually be bypassed in computation if desired.

3. The following collisions (with the second particle at rest in the laboratory frame Σ) are historical landmarks in artificial transmutation. The Mev values of Q and k_T may be verified from TABLES II and III.

		Q	k_T	
A	${}^4_2He + {}^{14}_7N \rightarrow {}^{17}_8O + {}^1_1H$	- 1.19	1.53	Rutherford, 1914
B	${}^4_2He + {}^9_4Be \rightarrow {}^{12}_6C + n^o$	5.7	----	Chadwick, 1932
C	$n^o + {}^{14}_7N \rightarrow {}^{14}_6C + {}^1_1H$	.63	----	Feather, 1932
D	${}^1_1H + {}^7_3Li \rightarrow 2\left({}^4_2He\right)$	17.3	----	Cockcroft, Walton, 1932
E	${}^4_2He + {}^{27}_{13}Al \rightarrow {}^{30}_{15}P + n^o$	- 2.65	3.05	Joliots, 1934
F	$\gamma + {}^2_1H \rightarrow {}^1_1H + n^o$	- 2.225	$(-Q)^+$	Chadwick, Goldhaber, 1935
G	$p^+ + p^+ \rightarrow 3p^+ + p^-$	- 1877.	5630	Segré, Chamberlain, 1954
H	$\bar{\nu}_\epsilon + p^+ \rightarrow n^o + \epsilon^+$	- 1.805	1.807	Reines, Cowan, 1956
I	$\bar{\nu}_\mu + p^+ \rightarrow n^o + \mu^+$	- 107	113.	(Brookhaven) 1962

20. Fusion. For an arbitrarily given colliding system $A(e_h)$, of class $\{cP_o, E_o\}$ with critical mass $e_o \equiv \left(E_o^2 - (cp_o)^2\right)^{\frac{1}{2}} > 0$, a fusion is always possible, provided only that the single particle resulting has a rest energy precisely equal to e_o. While mathematically the reverse of a decay process, there is here the physical implication that the fused particle incorporates into its own rest energy the given critical energy e_o, which can hardly be regarded as an intrinsic property of unique species of particle.

In a collision with target at rest, therefore, we may consider the fusion

$$A(e_1, e_2) \longrightarrow S(e_o) \tag{F}$$

where by definition

$$e_o \equiv \left(e_A^2 + 2e_2k_1\right)^{\frac{1}{2}} > e_A$$

is the rest energy of the resulting particle.

Technically, the Q-value of (F) is then the negative number

$$Q = e_A - e_o < 0$$

and the "threshold" energy k_T is k_1 tautologically, the fusion (F) necessarily occurring precisely "on threshold." (§19)

The fused particle rides at the CM of A, with the class velocity U_o, and energy E_o; indeed with all its parameters those of the class $\{cP_o, E_o\}$ of A.

The energy conservation equation

$$k_1 + Q = k_S < k_1$$

indicates a conversion of kinetic energy to rest energy. (We recall that a coherent system has the least k.e. and greatest ch. energy of all systems in its class.)

In the Σ'-frame of the class, the two particles of A′ fuse into a motionless one, with a total conversion of kinetic energy into rest-energy:

$$k'_A \equiv k'_1 + k'_2 = (^-Q).$$

A fusion sometimes results in a particle which possesses a (more or less) stable "ground state" of minimal rest energy e_g. In such a case, one has necessarily

$$e_o \geqq e_g$$

and the fused particle is said to be formed with an "energy of excitation"

$$e^* \equiv e_o - e_g \geqq 0.$$

If the fusion occurs at vanishingly small incident energies k_1, as it does for example in the neutron-capture

$$n^o + (^{235}U) \rightarrow (^{236}U)^*$$

then, since $e_o \equiv (e_A{}^2 + 2e_2k_1)^{\frac{1}{2}} \rightarrow e_A$ as $k_1 \rightarrow 0$, necessarily

$$e_A \geqq e_g$$

and the (intrinsic) energy

$$e_g^* \equiv e_A - e_g \geqq 0$$

is the "minimum energy of excitation" with which the particle can be formed. In the case cited, one finds from TABLE III

$$e_g^* = e(^{235}U) + e(n^o) - e(^{236}U) = 6.4 \text{ Mev.}$$

This is more than (^{236}U) can stand, and results in fission.

In "radiative capture" (= neutron capture followed by γ emission, a less drastic result), the excited nucleus formed in the fusion

$$n^o + \left({}^A_Z X\right) \rightarrow \left({}^{A+1}_Z X\right)^* \qquad \text{(F)}$$

has a ground state to which it <u>may</u> drop by emitting a photon:

$$\left({}^{A+1}_Z X\right)^* \rightarrow \left({}^{A+1}_Z X\right) + \gamma. \qquad \text{(D)}$$

The Q-value of the latter "decay" is then precisely the energy of excitation with which $\left({}^{A+1}_Z X\right)$ was formed in the fusion (F).

Nuclear γ absorption

$$\gamma + \left({}^A_Z X\right) \rightarrow \left({}^A_Z X\right)^*$$

affords a further example. Here, the ground state rest energy e_g is e_2 itself, and the energy of excitation has the simple form

$$e^* \equiv e_o - e_g = \left(e_2^{\,2} + 2e_2k_1\right)^{\frac{1}{2}} - e_2$$

$$= e_2 \left\{- 1 + \sqrt{1 + (2k_1/e_2)}\right\} \geq e_g^* \equiv 0.$$

<u>Notes 20</u>.

1. If the collision of Note 19.2 results in fusion, the particle formed has $E_o = 20$, $e_o = 16$, $k_o = 4$ and $cP_o = (8,8,4)_G$.

21. Elastic collision. We consider in this section in general, and in the next, with $e_1 = 0$, the important case of elastic collision on target at rest, of the simple form

$$\Lambda(e_1, e_2) \longrightarrow S(e_3, e_4)$$

where $$e_1 \equiv e_3 \geqq 0 \qquad e_2 \equiv e_4 > 0$$

and hence $Q = 0$, $k_1 = k_3 + k_4$.

In the corresponding collision $A' \longrightarrow S'$ in the Σ'-frame of the class, even the individual kinetic energies are unchanged:

$$k_1' = k_3' \qquad k_2' = k_4'$$

and indeed, as we know, A' and S', as systems of class $\{0, e_o\}$ with $e_1 + e_2 < e_o$, can differ only in direction. From §19 we may therefore write immediately

(d) $$E_1' = E_3' = e_o - \gamma_o e_2 \qquad E_2' = E_4' = \gamma_o e_2$$

and $$cp_1' = cp_3' = \gamma_o \beta_o e_2 = cp_2' = cp_4'.$$

Once the direction Ψ_3' is specified in Σ', the systems S' and S are fully determined, and a complete computational procedure has already been outlined in the PROBLEM of Note 19.2. To understand the physical nature of such collisions, it is important to study them further as they appear on the standard axes involved in Fig. 19.1, and here denoted by $\mathcal{S}, \mathcal{S}'$. In particular, we will consider the dependence of the final system on the physically meaningful angle $\psi_3' (0 \leqq \psi_3' \leqq 180^o)$ at which the projectile scatters from its Σ' line of flight. (The initial direction Ψ_1' of the latter, under the present "target at rest" assumption, is of course $\Psi_o = \Psi_1$.) The mathematical limiting case $\psi_3' = 0$ and $\psi_3' = 180^\circ$ will be referred to for obvious reasons as a "miss", and a "direct hit", respectively.

Since $cp_{4x}' = - cp_{3x}' = - cp_3' \cos \psi_3' = - \gamma_o\beta_o e_2 \cos \psi_3'$, and $E_4' = e_2\gamma_o$ we find first (as easier) the Σ <u>target</u> parameters

$$cp_{4x} = \gamma_o(cp_{4x}'+\beta_o E_4') = \gamma_o^2\beta_o e_2(1-\cos \psi_3') = 2\gamma_o^2\beta_o e_2\sin^2(\psi_3'/2) \quad (1)$$

$$E_4 = \gamma_o(\beta_o cp_{4x}'+E_4') = \gamma_o^2 e_2(1-\beta_o^2\cos \psi_3')$$

$$= e_2 \left\{1 + (\gamma_o^2-1)\ (1-\cos \psi_3')\right\} . \quad (2)$$

The "recoil k.e." of the target is therefore

$$k_4 = E_4 - e_2 = e_2(\gamma_o^2-1)\ (1-\cos \psi_3'). \quad (3)$$

From these, the energies

$$k_3 = k_1 - k_4 \quad (4)$$

$$E_3 = e_1 + k_3 = E_o - E_4 \quad (5)$$

of the scattered projectile in Σ may be inferred, as they depend on ψ_3'. From E_3 and E_4 we may also obtain formulas for the magnitudes

$$cp_3 = \left(E_3^2-e_1^2\right)^{\frac{1}{2}} \quad \text{and} \quad cp_4 = \left(E_4^2-e_2^2\right)^{\frac{1}{2}} \quad (6)$$

of the corresponding vectors cP_3, cP_4.

For the scattered projectile in Σ, we find directly from conservation of momentum

$$cp_{3x} = cp_o - cp_{4x} = E_o\beta_o - \gamma_o^2\beta_o e_2(1-\cos\psi_3'). \tag{7}$$

If desired, the results of (1), (6), and (7) may be used to obtain

$$\cos\psi_3 = cp_{3x}/cp_3 \quad \text{and} \quad \cos\psi_4 = cp_{4x}/cp_4 \tag{8}$$

for the Σ angle of deflection of the projectile, and the recoil angle of the target, as they depend on ψ_3'. In amplification of (8), one should note the following remarks.

1. The position of O_2' <u>on</u> the ellipsoid $\mathcal{E}$ of Fig. 19.1 indicates that Case II (§6) always obtains for the <u>target</u>, which therefore scatters <u>forward</u> in Σ, its angle ψ_4 ranging from (a limiting) 90° to 0° as the projectile angle ψ_3' ranges from 0° (miss) to 180° (direct hit) in Σ'. In (8), the exceptional case $cp_4 = 0$ ($E_4 = e_2$, $k_4 = 0$) occurs only in the event of a miss, with the trivial result $S \equiv A$.

2. As we know from §§6, 19, the dependence of the <u>projectile</u> angle ψ_3 on ψ_3' is more complex, the range of ψ_3 depending on the cases

I. $e_1 < e_2$ (O_1' inside $\mathcal{E}$) with $0 \leqq \psi_3 \leqq 180^\circ$

II. $e_1 = e_2$ (O_1' on $\mathcal{E}$) with $0 \leqq \psi_3 < 90^\circ$ (see Note 2)

III. $e_1 > e_2$ (O_1' outside $\mathcal{E}$) with $0 \leqq \psi_3 \leqq \bar{\psi}_3 < 90^\circ$

as explained in §6. (The value of $\bar{\psi}_3$ is given in Note 3.) In (8) the exceptional case $cp_3 = 0$ ($E_3 = e_1$, $k_3 = 0$) occurs only in the event of a direct hit on a target of equal rest mass ($\psi_3' = 180^\circ$, $e_1 = e_2$). In this case, we see from (4) that $k_4 = k_1$, showing that the projectile is stopped "dead in its tracks," while the target recoils with its entire k.e. (See

Note 1.)

3. While the values of $\cos\psi_i$ are perhaps most easily computed from (8), we may recall that (3Y1) gives the explicit formulas

$$\cos\psi_i = (\cos\psi_i' + \rho_i')/D_i' \qquad i = 3,4 \tag{9}$$

where $D_i' = \left\{(\cos\psi_i' + \rho_i')^2 + \gamma_o^{-2}(1-\cos^2\psi_i')\right\}^{\frac{1}{2}}$

and here $\rho_i' \equiv u_o/v_i' = \beta_o E_i'/cp_i' = E_i'/\gamma_o e_2$.

Now $\rho_4' = 1$, and $\cos\psi_4' = -\cos\psi_3'$, so that, for $i = 4$, (9) reads

$$\cos\psi_4 = (1-\cos\psi_3')\Big/\left\{(1-\cos\psi_3')^2 + \gamma_o^{-2}(1-\cos^2\psi_3')\right\}^{\frac{1}{2}}. \tag{10}$$

On the other hand, $\rho_3' = E_1'/\gamma_o e_2 = (e_o - \gamma_o e_2)/\gamma_o e_2$, which may be used in (9) to obtain $\cos\psi_3$. We know from §3 that, in the case $e_1 = 0$ (immaterial projectile, $\rho_3' = \beta_o$) the latter result reduces to the simpler form

$$\cos\psi_3 = (\cos\psi_3' + \beta_o)/(\beta_o\cos\psi_3' + 1). \tag{10c}$$

4. Finally, we recall from §4 the formula (4.5)

$$\cos\psi_i = (E_i - \gamma_o^{-1}E_i')\big/\beta_o cp_i \qquad i = 3,4$$

which gives $\cos\psi_i$ directly in terms of E_i and the fixed E_i', by eliminating the Σ' parameter ψ_i'. Substituting the present values

$$E_3' = e_o - \gamma_o e_2, \quad E_4' = \gamma_o e_2$$

and remembering that

$\gamma_o = E_o/e_o$, $\beta_o = cp_1/E_o$, $E_o = E_1 + e_2$, $e_o^2 = e_1^2 + e_2^2 + 2e_2E_1$, we obtain from (4.5)

$$\cos \psi_3 = (E_oE_3 - e_1^2 - e_2E_1)/cp_1cp_3 \qquad (11)$$

$$\cos \psi_4 = E_ok_4/cp_1cp_4.$$

These may also be used to obtain the $\cos \psi_i$, once the E_i are known. Again in the special case $e_1 = 0$, to which we turn in the next section, (11) reduces to

$$\cos \psi_3 = 1 + (e_2/E_1) - (e_2/E_3) \qquad (11c)$$

$$\cos \psi_4 = \{1+(e_2/E_1)\}k_4/cp_4.$$

The first of these is the polar equation of the ellipse (Note 6.2), with O_1' at the left focus.

Notes 21.

1. In the exceptional case $cp_3 = 0$ in (8), we have from (3,4)

$k_1 = k_4 = e_2(\gamma_o^2-1)(1-\cos \psi_3') \leqq 2e_2(\gamma_o^2-1) = 2e_2(E_1^2-e_1^2)/e_o^2 = 2e_2k_1(E_1+e_1)/e_o^2$ and so also, $e_o^2 \equiv e_1^2 + e_2^2 + 2e_2E_1 \leqq 2e_2E_1 + 2e_2e_1$, or $(e_1-e_2)^2 \leqq 0$. Hence $e_1 = e_2$ and $\cos \psi_3' = -1$. The result is of course geometrically obvious.

2. In Case II, $e_1 = e_2 > 0$, the minimum angle $\bar{\sigma}$ of separation occurs when $\psi_3' = 90^\circ$, with $\cos \bar{\sigma} = (\gamma_o^2-1)/(\gamma_o^2+1) = k_1/(k_1+4e_1)$. Cf. (12.5).

3. In Case III, the value of the maximum angle $\bar{\psi}_3$ of deflection is given (Note 6.3) by $\tan \bar{\psi}_3 = 1/\gamma_o(\rho_3'^2-1)^{\frac{1}{2}}$, where $\rho_3' = (e_o-\gamma_oe_2)/\gamma_oe_2$, as shown under (10). Hence

$$\tan \bar{\psi}_3 = 1\Big/\left\{(e_1/e_2)^2 - 1\right\}^{\frac{1}{2}}, \qquad e_1 > e_2.$$

22. Compton scattering. It was first noted by Compton and Debye that the laws of elastic collision govern the scattering of x-rays by free electrons, provided the "rays" are regarded as particles of mass $M = h\nu/c^2$ and absolute momentum $Mc = h\nu/c$.

For the present section, we define "Compton scattering" as any elastic collision of form $A(0,e_2) \longrightarrow S(0,e_2)$ in which an immaterial particle scatters from a material target, and consider, under the target at rest assumption, an initial system $A(0,e_2)$ with parameters

$$h\nu_1 = k_1 = E_1 > e_1 = 0, \quad cp_1 = k_1, \quad cP_1 = k_1\Psi_1$$

$$k_2 = 0, \quad E_2 = e_2 > 0, \quad cp_2 = 0, \quad cP_2 = 0$$

the basic scalars being k_1 and e_2.

Specializing §19 to the case $e_1 = 0$, we have

(a) A totals: $cP_o = cP_1, \quad cp_o = k_1, \quad E_o = k_1 + e_2, \quad e_A = e_2.$

(b) Class parameters: $\beta_o = k_1/(k_1+e_2), \quad e_o = \left(e_2^2+2e_2k_1\right)^{\frac{1}{2}},$
$\gamma_o = (k_1+e_2)/e_o, \quad \left(\gamma_o^2-1\right)^{\frac{1}{2}} = \beta_o\gamma_o = k_1/e_o.$

(c) $e_o > e_S$, or $k_1 > k_T = Q = 0$ indicates the possibility of elastic collision for $I = 2$.

(d) Σ'-parameters of A',S' (§21)

$$k_1' = E_1' = E_3' = k_3' = cp_h' = cp_1' = \gamma_o\beta_o e_2 = (k_1/e_o)e_2; \qquad E_2' = E_4' = \gamma_o e_2.$$

In the ellipsoid construction, one has

$O_1'O' = \gamma_o\beta_o E_1' = (\gamma_o\beta_o)^2 e_2 = (k_1/e_o)^2 e_2$, with O_1' at the left focus.

$O'O_2' = \gamma_o cp_1' = \gamma_o{}^2/\beta_o e_2 = (k_1/e_o)\gamma_o e_2$, $\quad cp_1' = (k_1/e_o)e_2$. Since the figure falls under Case I, we know that, as the deflection angle ψ_3' of the projectile in Σ' ranges from 0° to 180°, so does its deflection angle ψ_3 in Σ, while the recoil angle of the target ranges from (a limiting) 90° to 0°.

(e) The formulas (3,4,10,10c,11c) of §21 yield the following Σ parameters of the resulting system $S(0,e_2)$, as they depend, in effect, on ψ_3'.

The recoil k.e. of the target is

$$k_4 = (k_1/e_o)^2 e_2(1-\cos\psi_3') \tag{1}$$

with a range

$$0 \leqq k_4 \leqq 2(k_1/e_o)^2 e_2 = k_1/\{1+(e_2/2k_1)\}. \tag{2}$$

The energy of the scattered projectile is therefore

$$k_3 = E_3 = k_1 - k_4 \tag{3}$$

with a corresponding range

$$k_1 \geqq k_3 \geqq k_1/\{1+(2k_1/e_2)\}. \tag{4}$$

The deflection angles ψ_i as they depend explicitly on ψ_3' may be obtained from

$$\cos\psi_3 = (\cos\psi_3'+\beta_o)/(1+\beta_o\cos\psi_3') \tag{5}$$

$$\cos\psi_4 = (1-\cos\psi_3')/\{(1-\cos\psi_3')^2 + \gamma_o^{-2}(1-\cos^2\psi_3')\}^{\frac{1}{2}} \tag{6}$$

while their dependence on k_3 and k_4 is indicated by

$$\cos \psi_3 = 1 + (e_2/k_1) - (e_2/k_3) \tag{7}$$

$$\cos \psi_4 = k_4\{1+(e_2/k_1)\}\Big/\left(E_4{}^2-e_2{}^2\right)^{\frac{1}{2}}$$

$$= \{1+(e_2/k_1)\}/\{1+(2e_2/k_4)\}^{\frac{1}{2}} \tag{8}$$

Note here that $1 + e_2/k_1 = \beta_o^{-1}$, and therefore (7) may be written as $cp_3 = k_3 = e_2\beta_o/\{1-\beta_o\cos\psi_3\} = \gamma_o^{-1}E_3'/\{1-\beta_o\cos\psi_3\}$. This is of course the polar equation of the ellipsoid, as shown in Note 6.2. Equation (7) may also be expressed in the form

$$k_3 = k_1/\{1+(k_1/e_2)\ (1-\cos\psi_3)\}. \tag{7a}$$

The angles ψ_3, ψ_4 are correlated by the equation

$$\tan\psi_4 = \left(\frac{e_2}{e_2+k_1}\right)\cot\psi_3/2 \qquad 0 < \psi_3 < 180^\circ. \tag{9}$$

This may be obtained by division of the self-evident "momentum" equations ($cp_3 = k_3$, $cp_o = k_1$)

$$cp_4 \sin\psi_4 = k_3 \sin\psi_3$$

$$cp_4 \cos\psi_4 = k_1 - k_3\cos\psi_3$$

and substitution of k_1/k_3 from (7a) into the result.

Notes 22.

1. Since u_o is the target speed in Σ', and $(\gamma_o\beta_o E_1') \lesseqgtr (cp_1') = E_1'$ as $\beta_o \lesseqgtr 1/\sqrt{2}$, it follows that, in the ellipsoid figure, the focus $0_1'$ falls inside, on, or outside the smaller sphere as the Compton wavelength

of the target compares with the common wavelength of both particles in Σ', i.e., as

$$hc/e_2 \equiv \lambda_{2c} \lesseqgtr \lambda_i' = h/p_i'.$$

(Cf. Note 1.4.)

2. Setting $k_1 = h(c/\lambda_1)$ and $k_3 = h(c/\lambda_3)$ in (7), one obtains $\Delta\lambda \equiv \lambda_3 - \lambda_1 = \lambda_{2c}(1-\cos\psi_3) = 2\lambda_{2c}\sin^2\psi_3/2$, where λ_{2c} is the Compton wavelength of the target. For $\psi_3 = 90^\circ$, $\Delta\lambda = \lambda_{2c}$.

3. We have indicated general methods in Note 19.2 for dealing with target at rest problems when treated in the Σ'-frame of the class. We now show how such collisions are handled if we need not leave the target rest frame. The method is then quite simple, since the Lorentz transformation is not invoked, and will be sufficiently obvious from the following "Compton collision" example, which neglects polarization effects.

N.B. Since the "Klein-Nishina" differential cross section (Note 4) for photon scattering on free electrons is <u>given</u> in the electron rest frame, it is natural to deal with such elastic collisions in this way. Moreover, since the <u>energy</u> distribution has the simpler form algebraically, one customarily samples the energy k_3 on its range (4), obtaining $\cos\psi_3$ <u>a fortiori</u> from (7). Finally, use of the energy <u>unit</u> e_ϵ = .511006 Mev allows the formulas of this section to be read with $e_2 = 1$.

<u>PROBLEM</u>. A photon of energy $k_1 = 4$ (i.e., 2.044024 Mev) and direction $\Psi_1 = (2/3,2/3,1/3)_G$ on given Σ axes G, collides elastically with a free motionless electron. The energy $k_3 = 4/5$ is chosen on the range

$4 \geq k_3 \geq 4/9$ by sampling the K - N energy distribution for $k_1 = 4$. By (7), the corresponding deflection angle is $\psi_3 = 90^\circ$. The direction ψ_1 being basic for scattering of the photon, ψ_3 is itself the latitude angle θ for location of Ψ_3 and Ψ_1. If the longitude φ, here assumed uniformly distributed, is chosen as 300°, then

$$\Omega = (0,1/2,-\sqrt{3}/2)_G$$

is the auxiliary direction about X, and

$$(\Psi_3)_G = D(\Omega)_G$$

locates Ψ_3 about Ψ_1, where D is the matrix of Note 16.2.

The final photon momentum is therefore given by

$$(cP_3)_G = \frac{4}{5}\,(\Psi_3)_G$$

on the Σ axis G.

If desired, one may obtain $k_4 = k_1 - k_3 = 16/5$, $E_4 = 1 + k_4 = 21/5$ $cp_4 = (E_4{}^2-1)^{\frac{1}{2}} = 4\sqrt{26}/5$, and $(cP_4)_G = (cP_1)_G - (cP_3)_G$, where $(cP_1)_G = 4\cdot(\Psi_1)_G$.

All energies may be converted to Mev on multiplication by $e_\epsilon = .511006$ Mev.

4. Neglecting polarization effects, one obtains from the "Klein-Nishina" formula (A IV) the differential cross section

$$\sigma(k_1;a)da = \pi r^2(k/k_1)^2\{k_1/k + k/k_1 - (1-a^2)\}da \quad cm^2$$

for the Compton scattering of a photon of energy $k_1 = h\nu_1/e_\epsilon$ from a motionless free electron, where $k \equiv k_3$, $a \equiv \cos \psi_3$. (For $r \equiv r_\epsilon$, see TABLE I.) A more convenient cross section, defined by

$$\tilde{\sigma}(k_1;k)dk \equiv \sigma(k_1;a)da$$

is obtained by using the relation (7), $a = 1 + 1/k_1 - 1/k$, with $da/dk = 1/k^2$, namely

$$\tilde{\sigma}(k_1;k) = (\pi r_\epsilon^{\,2}/k_1^{\,2})\{k^{-2}-(2k_1^{-1}+2-k_1)k^{-1} + (k_1^{-2}+2k_1^{-1}) + k_1^{-1}k\}.$$

Integration on the range (4), $k_1/(1+2k_1) \leqq k \leqq k_1$, yields the (total) cross section

$$\sigma(k_1) = 2\pi r_\epsilon^{\,2}\{2k_1^{-2} + (1+k_1)(1+2k_1)^{-2} - (k_1^{-3}+k_1^{-2}-2^{-1}k_1^{-1})\ell n(1+2k_1)\}.$$

Norming gives the probability density function $p(k_1;k)= \sigma(k_1;k)/\sigma(k_1)$, and the equation

$$r = \int_k^{k_1} p(k_1;k)dk$$

indicates the dependence of k on the random number r in Monte Carlo practice.

The inverse function $k = F(k_1;r)$ has been fitted by B. Carlson ($k_1 \leq 4$) and E. D. Cashwell ($k_1 \leq 24$) as follows

$k_1 \leqq 4$	$k = F_1$
$4 < k_1 \leqq 8.5$	$k = F_1 + F_2$
$8.5 < k_1 \leqq 15$	$k = F_1 + F_2 + F_3$
$15 < k \leqq 24$	$k = F_1 + F_2 + F_3 + F_4$

where

$$F_1 = k_1/\{1 + r[S + (2k_1-S)r^2]\}, \qquad S \equiv k_1/(1+.5625\, k_1)$$

$$F_2 = \tfrac{1}{2}(k_1-4)r^2(1-r)^2$$

$$F_3 = -6(k_1-8)r(\tfrac{1}{4}-r)(1-r)^4$$

$$F_4 = \begin{cases} f_4 \equiv k_1 r^2(1-r)(.4-r)(.85-r), & r \le .85 \\ f_4 + 6(1-r)(.85-r), & r > .85. \end{cases}$$

For higher energies (> 12 Mev), scattering is extremely forward, and a rejection technique employed on two subranges of $p(k_1,k)$ seems indicated. The cross sections for energies ≤ 500 Mev are graphed in N.B.S. Circular 542.

In Appendix IV, we consider the Compton collision of plane polarized photons. In this more general case, the cross section $\tilde{\sigma}(k_1;k)$ of Note 4 is used as just indicated to obtain the scattered photon energy and defection angle ψ in Σ.

5. For tables of the integral $\int_k^{k_1} p(k_1;k)dk$, $k \le 25$, see H. Mayer et al, LAMS 1199.

23. <u>Pair production</u>. As a final example of a collision with target at rest, we consider the case of "pair production," in which a sufficiently energetic photon interacts with a charged particle, the transmutation, of form $A(0,e_2) \longrightarrow S(e_2,e_\epsilon,e_\epsilon)$,

resulting in the recoil unexcited target, of rest energy $e_3 = e_2$, together with an electron-positron pair, with rest energies $e_4 = e_5 = e_\epsilon = .511006$ Mev.

The reaction is seen to have a Q-value

$$Q = -2e_\epsilon = -1.022 \text{ Mev}$$

and consequently an energy threshold condition

$$h\nu_1 = E_1 = k_1 \geqq k_T = \frac{e_2+e_\epsilon}{e_2}(2e_\epsilon) = (2e_\epsilon)\{1+(e_\epsilon/e_2)\}$$

which is here sufficient as well as necessary (§19(c)). Note that $k_T \cong 2e_\epsilon$ for a nuclear target (the usual case), whereas $k_T = 4e_\epsilon$ for a target electron. We shall assume that $k_1 > k_T$, with S non-coherent.

The initial system being identical with that of §22, all particle parameters of A, and of A′ in the Σ'-frame of its class, as well as the parameters of the class itself, are already given there.

The system $S'(e_2,e_\epsilon,e_\epsilon)$, of class $\{0,e_o\}$ in Σ', has a total k.e.

$$k_S' = e_o - (e_2+2e_\epsilon) > 0, \quad \text{where} \quad e_o = \left(e_2{}^2+2e_2k_1\right)^{\frac{1}{2}}.$$

The sharing of this among the three particles is of course not unique. The target, for example, may recoil in Σ' with any k.e. on the range

$$0 \leqq k_3' \leqq k_K' = \frac{2e_\epsilon + (k_S'/2)}{e_o} k_S' .$$

Technically, both end points are attainable. The limiting case $k_3 = k_K'$ would demand the coalesced emission of the charged pair in the direction opposite to that of the target recoil, which is physically absurd. However, their nearly parallel emission in Σ is indeed observed. If the target were left motionless in Σ', with $k_3' = 0$, the pair would then form a two particle system of class $\{0, e_o - e_2\}$, having the unique energies

$$E_i' = (e_o - e_2)/2, \quad k_i' = k_S'/2, \quad i = 4,5$$

and oppositely directed momenta of equal magnitude, with

$$cp_i' = \left(E_i'^2 - e_\epsilon^2\right)^{\frac{1}{2}}.$$

24. <u>Collision with target in motion</u>. Finally, we turn to collisions $A(e_1, e_2) \longrightarrow S(e_i)$ in which both particles of a non-coherent system A are in motion in Σ, hence with parameters

$$k_h > 0,\ E_h > e_h \geqq 0,\ cp_h = (E_h^2 - e_h^2)^{\frac{1}{2}} > 0,\ cP_h = cp_h \Psi_h;\ h = 1,2. \qquad (1)$$

(a) The totals of A are then

$$cP_o = cP_1 + cP_2, \text{ with } cp_o \equiv |cP_o|,\ E_o = E_1 + E_2,\ e_A = e_1 + e_2. \qquad (2)$$

While computable in the usual way, it is sometimes convenient to express cp_o and quantities depending upon it, in terms of the physically meaningful angle σ $(0 \leqq \sigma \leqq 180^\circ)$ between cP_1 and cP_2, which may either be given, or easily obtained from

$$\cos\sigma = \Psi_1 \cdot \Psi_2. \qquad (3)$$

Thus, $(cp_o)^2 = |cP_1 + cP_2|^2 = (cp_1)^2 + (cp_2)^2 + 2cP_1 \cdot cP_2$

$$= E_1^2 - e_1^2 + E_2^2 - e_2^2 + 2cp_1cp_2\cos\sigma$$

$$= (E_1 + E_2)^2 - e_1^2 - e_2^2 - 2(E_1E_2 - cp_1cp_2\cos\sigma)$$

so $$cp_o = (E_o^2 - e_1^2 - e_2^2 - 2E_\sigma^2)^{\frac{1}{2}} \qquad (4)$$

where by definition

$$E_\sigma^{\ 2} \equiv E_1E_2 - cp_1cp_2 \cos\sigma > e_1e_2 \geqq 0 \quad \text{(Note 1)}. \tag{5}$$

(b) We write therefore for the parameters of the class $\{cP_o, E_o\}$ of A,

$$\beta_o = cp_o/E_o,\ e_o = \left(e_1^{\ 2}+e_2^{\ 2}+2E_\sigma^{\ 2}\right)^{\frac{1}{2}} = \left\{e_A^{\ 2}+2\left(E_\sigma^{\ 2}-e_1e_2\right)\right\}^{\frac{1}{2}} > e_A,\ \gamma_o = E_o/e_o. \tag{6}$$

The direction of the class velocity is $\Psi_o = cP_o/cp_o$, as always.

(c) The necessary condition $e_o \geqq e_S$ for the proposed transmutation here takes the form

$$E_\sigma^{\ 2} \geqq e_1e_2 + \frac{e_S+e_A}{2}\,(-Q) \tag{T_C}$$

where $Q = e_A - e_S$ is the Q-value of the reaction. We shall suppose $e_o > e_S$, and $I \geqq 2$, so that a non-coherent result $S(e_i)$ is possible (Th. 14.3).

(d) If the collision $A'(e_1,e_2) \longrightarrow S'(e_i)$ is to be studied in the Σ'-frame of the class, we may require the (unique) parameters

$$E_h' = \left(e_h^{\ 2}+E_\sigma^{\ 2}\right)\Big/e_o,\ cp_h' = \left(E_\sigma^{\ 4}-e_1^{\ 2}e_2^{\ 2}\right)^{\frac{1}{2}}\Big/e_o;\ h = 1,2 \tag{7}$$

of A′, given by §11, and the angles ψ_1, ψ_1' which cP_1 and cP_1' make with the direction Ψ_o of the class velocity.

The first of these is obtained from

$$\cos\psi_1 = (cP_o \cdot cP_1)/cp_ocp_1 = \{(cP_1+cP_2)\cdot(cP_1)\}/cp_ocp_1$$

$$= \{(cp_1)^2 + cp_1cp_2\cos\sigma\}/cp_ocp_1. \tag{8}$$

Ignoring §3, we will obtain ψ_1' directly from the Lorentz transforma-

3

tion for cp'_{1x} on standard axes (Note 10.1).

$$cp'_{1x} = (E_o cp_{1x} - cp_o E_1)/e_o = (E_o cp_1 \cos \psi_1 - cp_o E_1)/e_o.$$

Hence, using (8), we find

$$cp_o(e_o cp'_{1x}) = E_o\{(cp_1)^2 + cp_1 cp_2 \cos \sigma\} - (cp_o)^2 E_1.$$

Substituting $E_o = E_1 + E_2$, $(cp_1)^2 = E_1{}^2 - e_1{}^2$, $cp_1 cp_2 \cos \sigma = E_1 E_2 - E_\sigma{}^2$, and $(cp_o)^2 = E_o{}^2 - e_1{}^2 - e_2{}^2 - 2E_\sigma{}^2$, we obtain upon simplification

$$cp_o e_o cp'_{1x} = e_2{}^2 E_1 - e_1{}^2 E_2 + E_\sigma{}^2(E_1 - E_2).$$

But $cp'_{1x} = cp'_1 \cos \psi'_1$, so we find from this and (7) the result

$$\cos \psi'_1 = \left\{e_2{}^2 E_1 - e_1{}^2 E_2 + E_\sigma{}^2(E_1 - E_2)\right\} \Big/ cp_o\left(E_\sigma{}^4 - e_1{}^2 e_2{}^2\right)^{\frac{1}{2}}. \qquad (9)$$

Note that formulas (4-9) involve only the given scalars E_h, e_h and $\cos \sigma$.

With the general objective and plan of Notes 14.2, 14.3, and reliance on basic principles as far as possible, some procedures are given below for various kinds of collision problems with target in motion in Σ.

Method I. Assumptions: cP_h given on Σ axes G; collision treated in Σ'-frame of class; Ψ'_1 specified as basic direction for non-polarized emission in Σ'; θ', φ' chosen for location of Ψ'_3 about Ψ'_1. (Fig. 1.)

1. $(cP_o)_G = (cP_1)_G + (cP_2)_G$ and $cp_o = |cP_o|$ yield $(\Psi_o)_G = (cP_o)_G/cp_o$ for direction of class velocity.

2. $(cP_1)_{\bar{G}} = D^\tau (cP_1)_G$ gives components of cP_1 on standard Σ-axes $\bar{G} \equiv [\bar{X}, \bar{Y}, \bar{Z}] \equiv [\delta X, \delta Y, \delta Z]$, where D is the matrix of rotation δ, based on $(\Psi_o)_G$, as in AIII, Cor. 1.

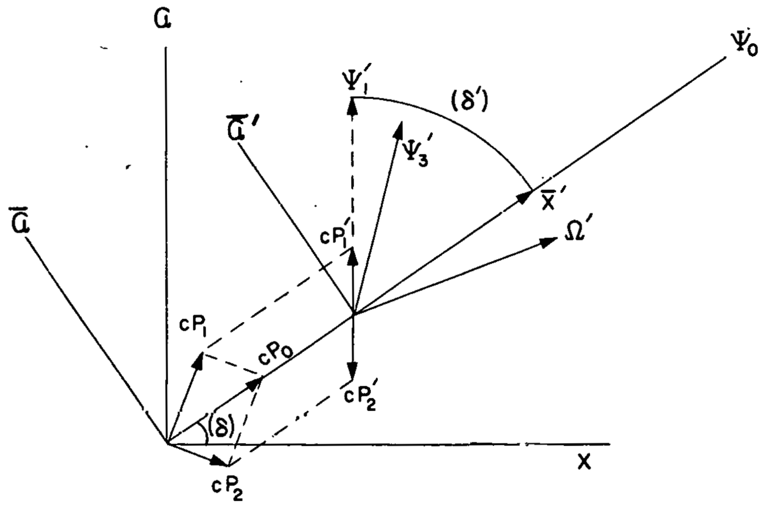


FIG. 24.1

3. For standard Σ' axes $\bar{G}' = [\bar{X}', \bar{Y}', \bar{Z}']$ parallel to $\bar{G}$,

$$(cP_1')_{\bar{G}'} = ((E_0 cp_{1\bar{x}} - cp_0 E_1)/e_0,\ cp_{1\bar{y}},\ cp_{1\bar{z}})$$

then gives the $\bar{G}'$-components of cP_1' in Σ'. (Note 10.1.)

Here one computes $E_0 = E_1 + E_2$, $e_0 = \left(E_0{}^2 - (cp_0)^2\right)^{\frac{1}{2}}$, where cp_0 is known from step (1). The necessary condition $e_0 \geqq e_S$ may be tested at this point.

4. $(\Psi_1')_{\bar{G}'} = (cP_1')_{\bar{G}'}/cp_1'$ is obtained for the basic (projectile) direction in Σ', where $cp_1' = |(cP_1')_{\bar{G}'}|$.

5. If collision is elastic, we set $cp_3' = cp_1'$, $E_3' = E_1'$, where $E_1' = (-cp_0 cp_{1\bar{x}} + E_0 E_1)/e_0$. In any case, we must suppose E_3', cp_3', θ', φ' determined at this point.

The matrix D' of the rotation δ' (AIII, Cor. 1) taking $\overline{X}'$ into $(\Psi_1')_{\overline{G}'}$, and based on the latter unit vector, is applied to the auxiliary direction $\Omega' \equiv (\cos\theta', \sin\theta' \cos\varphi', \sin\theta' \sin\varphi')_{\overline{G}'}$, to obtain the direction of emission

$$(\Psi_3')_{\overline{G}'} = D'(\Omega')_{\overline{G}'}$$

and the vector $(cP_3')_{\overline{G}'} = cp_3'(\Psi_3')_{\overline{G}'}$, on Σ' axes $\overline{G}'$.

6. The transformation of Note 10.1 (with i = 3 and barred x,y,z) applied to E_3' and $(cP_3')_{\overline{G}'}$ yields E_3, and $(cP_3)_{\overline{G}}$ on standard Σ axes.

7. Finally, $(cP_3)_G = D(cP_3)_{\overline{G}}$ gives the components of cP_3 on the original Σ axes G, where D is the matrix of step (2).

Except for the complication of non-standard axes, this is the scheme used in Note 14.4.

Method II. This is a modification of (I) which simplifies the work in Σ' at the expense of a more complicated rotation in Σ.

1. From cP_o and cP_1 we obtain both directions $(\Psi_o)_G$, $(\Psi_1)_G$.

2. The rotation δ_1, with matrix D_1 based on these unit vectors, (AIII, Th. 3) defines axes $\overline{G}$ in Σ such that $\overline{X} \equiv \delta_1 X = \Psi_o$ as before, and with Ψ_1 lying in the upper half of the $\overline{X},\overline{Y}$-plane, which now contains the parallelogram of Fig. 1. The matrix D_1 is not used until the final step.

3. We now compute $\cos \psi_1'$ from (9), and, if required for elastic collision, also E_1' and cp_1' from (7). Here one may prefer to use the principles from which these formulas were derived, namely

$$\cos \psi_1 = \Psi_o \cdot \Psi_1$$

$$cp_{1x} = cp_1 \cos \psi_1$$

$$cp_{1x}' = (E_o cp_{1x} - cp_o E_1)/e_o$$

$$E_1' = (-cp_o cp_{1x} + E_o E_1)/e_o$$

$$cp_1' = (E_1'^2 - e_1^2)^{\frac{1}{2}}$$

$$\cos \psi_1' = cp_{1x}'/cp_1'.$$

4. The basic direction $\Psi_1' = (\cos \psi_1', \sin \psi_1', 0)_{\bar{G}'}$, on Σ' axes $\bar{G}'$ parallel to $\bar{G}$, is now known. Note that $\sin \psi_1' = + (1-\cos^2 \psi_1')^{\frac{1}{2}}$.

5. The matrix D' of the rotation δ' taking $\bar{X}'$ into Ψ_1' is here simply

$$D' = \begin{vmatrix} \cos \psi_1' & -\sin \psi_1' & 0 \\ \sin \psi_1' & \cos \psi_1' & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

and just as before $(\Psi_3')_{\bar{G}'} = D'(\Omega')_{\bar{G}'}$ yields the Σ'-direction of 3.

The final steps (6,7) are those of (I), except that the matrix D_1 is used in place of D in (7).

<u>Method III</u>. In one type of problem, the parameters $(cP_1)_G$, E_1, e_1 of the projectile are given, but the nature of the target is subject to chance. Suppose cp_2, E_2, e_2 suitably chosen, and that the direction Ψ_2

is in a non-polarized distribution about Ψ_1. Finally, suppose the (latitude) angle of separation σ, and a longitude φ chosen for location of Ψ_2 about the "basic direction" Ψ_1.

We may of course proceed by (I) or (II) if we first specify $(\Psi_2)_G$, which may be done by using the device of Note 14.3, with $\Omega =$ $(\cos\sigma, \sin\sigma\cos\varphi, \sin\sigma\sin\varphi)_G$, and D the matrix of AIII, Cor. 1, based on Ψ_1.

However, the problem admits a simpler strategy outlined in IAMS 2360 (Metropolis, Turkevich, et al.) and slightly modified here.

1. We first define Σ axes

$$G_1 = [\delta_1 X, \delta_1 Y, \delta_1 Z] = [X, Y, Z] D_1$$

where $\delta_1 X = \Psi_1$, and D_1 is the matrix based on $(\Psi_1)_G$, as in AIII Cor. 1. (See Fig. 2.)

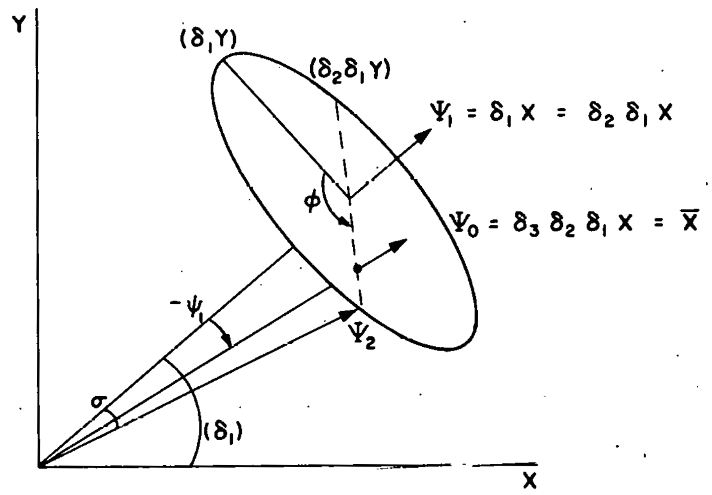


FIG. 24.2

2. We <u>now</u> specify $\left(\Psi_2\right)_{G_1} = (\cos\sigma, \sin\sigma\cos\varphi, \sin\sigma\sin\varphi)_{G_1}$ on the Σ axes G_1, where $0^\circ \leqq \sigma \leqq 180^\circ$, $0 \leq \varphi < 360^\circ$.

3. The rotation δ_2, about Ψ_1 through $\varphi + 180^\circ$ is defined, relative to the G_1 axes, by

$$G_2 \equiv \delta_2[\delta_1 X, \delta_1 Y, \delta_1 Z] = [\delta_1 X, \delta_1 Y, \delta_1 Z]D_2$$

where

$$D_2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi+180^\circ) & -\sin(\varphi+180^\circ) \\ 0 & \sin(\varphi+180^\circ) & \cos(\varphi+180^\circ) \end{vmatrix} .$$

The directions Ψ_1 and Ψ_2 appear in the $\delta_2\delta_1 X, \delta_2\delta_1 Y$-plane, with $\Psi_1 = \delta_2\delta_1 X$, and Ψ_2 in its <u>lower</u> half.

4. The angle ψ_1 between cP_o and cP_1 is next computed from (8), and used to obtain the matrix

$$D_3 = \begin{vmatrix} \cos(-\psi_1) & -\sin(-\psi_1) & 0 \\ \sin(-\psi_1) & \cos(-\psi_1) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

of the rotation δ_3, about $\delta_2\delta_1 Z$ through $-\psi_1$, which determines final Σ axes:

$$[\overline{X}, \overline{Y}, \overline{Z}] \equiv \overline{G} = \delta_3(\delta_2\delta_1 X, \delta_2\delta_1 Y, \delta_2\delta_1 Z) = (\delta_2\delta_1 X, \delta_2\delta_1 Y, \delta_2\delta_1 Z)D_3 .$$

The situation is now precisely that of Method II, with $\overline{X} = \Psi_o$, and cP_1 in the <u>upper</u> half of the $\overline{X}, \overline{Y}$-plane, although no computing has been done aside from evaluation of the three matrices D_i.

We now follow steps (4,5,6) of (II), relying on the formulas (4-9) for required parameters, and so obtaining E_3 and $(cP_3)_{\bar{G}}$.

7. It only remains to compute the components

$$(cP_3)_G = D_4(cP_3)_{\bar{G}}$$

on the original Σ axes G, using the matrix D_4 defined by

$$\delta_4[X,Y,Z] = [X,Y,Z]D_4$$

where δ_4 is the composite rotation $\delta_3\delta_2\delta_1$. Retracing our steps, we see that

$$\delta_4 G = \delta_3(\delta_2\delta_1 G) \equiv \delta_3 G_2 = G_2 D_3$$
$$= \delta_2 G_1 D_3 = G_1 D_2 D_3$$
$$= \delta_1 G D_2 D_3 = G D_1 D_2 D_3$$

so that we must use $D_4 = D_1 D_2 D_3$ in the final step.

Method IV. With the data given in Method I, the vector form of cPE in Note 4.1 may be used, with the parameters $\beta_o, \gamma_o, \Psi_o$ of the class computed from first principles.

1. Its inverse, applied to $(cP_1)_G$ gives $(cP'_1)_{G'}$ in Σ' on axes G' parallel to G, with direction $(\Psi'_1)_{G'} = (cP'_1)_{G'} / cP'_1$.

2. The rotation δ, with matrix D based on the latter unit vector (AIII, Cor. 1), takes the auxiliary direction Ω' into an emission direction Ψ'_3 with Σ' coordinates

$$(\Psi'_3)_{G'} = D(\Omega')_{G'}.$$

3. From the stipulated E_3', and $cP_3' = cp_3'(\Psi_3')_{G'}$, the direct (cPE) of Note 4.1 yields E_3 and $(cP_3)_G$. This method requires a minimum of "formulas" and may well surpass the others in speed.

Method V. Assumptions: cP_h given on Σ-axes G, collision treated in rest frame Σ' of target (with $e_2 > 0$); Ψ_1' specified as basic direction for non-polarized emission in Σ'; θ', φ' chosen for location of Ψ_3' about Ψ_1'. The method is indicated when differential cross sections are given for the target at rest, as in Compton scattering.

The appropriate Lorentz transformations between Σ and Σ' are those of §4 or Note 4.1, the parameters $\beta_o, \gamma_o, \Psi_o$ being, in the present case, the target parameters

$$\beta_2 = cp_2/E_2, \quad \gamma_2 = E_2/e_2, \quad \Psi_2 = cP_2/cp_2.$$

It was shown in Note 4.4 that

$$E_1' = (E_1E_2 - cP_1 \cdot cP_2)/e_2$$

is the projectile energy in the target rest frame Σ'. The geometric procedure is that of Method I or Method IV, with the target rest frame playing the rôle the Σ'-frame of the class. The emission parameters $\Sigma_3', cp_3', \theta', \varphi'$ must of course be stipulated in Σ'. The procedure for Compton collision has been indicated in Notes 22.3, 22.4.

Notes 24.

1. The "colliding" system A is understood to be non-coherent. It is easy to verify for the quantities defined,

(a) $E_\sigma^{\,2} \geq e_1e_2$, with equality if and only if $e_2E_1 = e_1E_2$ and $\sigma = 0^\circ$, which is a necessary and sufficient condition for coherence of $A(e_1,e_2)$.

(b) $E_\sigma^{\,2} = 0$ if and only if A is coherent-immaterial.

2. We have assumed $cP_o \neq 0$, with a well-defined direction Ψ_o. This indeed fails in the single case $\sigma = 180^\circ$, $cp_1 = cp_2$. The frame Σ is then identical with the Σ'-frame of the class, and Methods I, IV by-pass the Lorentz transformation.

3. We have assumed $cP_2 \neq 0$, with a well-defined direction Ψ_2. If σ is fixed, and $cp_2 \rightarrow 0$, the general formulas reduce in the limit to the "target at rest" relations (§19).

4. Figure 1 is only schematic. The true relations between the systems A,A′ may be seen as usual from a suitable ellipsoid figure, based on sphere radii cp_1',$\gamma_o(cp_1')$ and distances $|O_h'O'| = \gamma_o\beta_oE_h'$. The initial projectile angle ψ_1' with Ψ_o is given by (9), from which the rest of the figure may be drawn. Here of course we are <u>given</u> the cP_i and ψ_i, a priori.

In such a figure, one can show that

$$\gamma_o\beta_oE_1' \lesseqgtr \gamma_o(cp_1')$$

as
$$cp_1cp_2\cos\sigma \lesseqgtr (E_2-e_1)k_1.$$

(The relation may be derived from the equivalent condition $e_1\gamma_o \lesseqgtr E_1'$ of §6, using $\gamma_o = E_o/e_o$, E_1' from (7), and $E_\sigma^{\,2}$ from (5).)

As usual, this governs the position of O_1' relative to the ellipsoid, and hence the behavior of ψ_3 under Cases I, II, III (§6) in case of elastic collision $A(e_1,e_2) \rightarrow S(e_1,e_2)$.

5. (The colliding beam problem.) For a collision

$$A(e_1,e_1) \rightarrow S(e_i) \tag{C}$$

between two particles of equal rest energy $e_1 = e_2$, and fixed energies E_1,E_2, in the lab frame Σ, the total energy of the system in the Σ'-frame of the class is given by (6) as

$$e_o = (2e_1^2+2E_1E_2-2cp_1cp_2 \cos \sigma)^{\frac{1}{2}}.$$

This energy, which is critical for the production of new particles, as witnessed by the necessary condition

$$e_o \geqq e_S \tag{T}$$

naturally reaches its maximum value

$$e_o(E_1,E_2) = (2e_1^2+2E_1E_2+2cp_1cp_2)$$

in the case of head-on collision, with $\sigma = 180^\circ$.

Now suppose $E_1 > e_1$ is fixed, say at the greatest energy to which such a particle can be accelerated in Σ by present methods. Then, if the <u>target</u> energy varies from $E_2 = e_1$ (limit case, target at rest) to $E_2 = E_1$ (as in two optimal colliding beams), this Σ' energy rises from

$$e_o(E_1,e_1) = \left(2e_1^2+2e_1E_1\right)^{\frac{1}{2}}$$

to its maximal value

$$e_o(E_1,E_1) = \left(2e_1^2+2E_1^2+2cp_1^2\right)^{\frac{1}{2}}$$

$$= \left\{2e_1^2+2E_1^2+2\left(E_1^2-e_1^2\right)\right\}^{\frac{1}{2}} = 2E_1.$$

In the latter case, the oppositely directed momenta are of equal magnitude, the frames Σ and Σ' coincide, and of course the total Σ' energy is $e_o = E_1 + E_1$.

Thus a factor $2E_1 \Big/ \left(2e_1^{\,2}+2e_1E_1\right)^{\frac{1}{2}} > 1$

in the critical energy e_o is attained. To appreciate this factor, one must ask what Σ projectile energy E_1^* would be required for collision on target at rest to achieve the same energy $2E_1$ in Σ'. The answer is obviously provided by the equation

$$\left(2e_1^{\,2}+2e_1E_1^*\right)^{\frac{1}{2}} = 2E_1$$

namely,

$$E_1^* = \left(2E_1^{\,2}-e_1^{\,2}\right)\Big/e_1 .$$

It is interesting to evaluate these quantities (in Bev) for energies E_1 in the range of present design for proton beam-proton target systems ($e_1 = e_2 \cong 1$ Bev).

E_1	$e_o(E_1, e_1)$	$2E_1$	E_1^*
10	4.7	20	199
25	7.2	50	1249
100	14.2	200	19,999

CHAPTER IV

CROSS SECTIONS

25. Mean free path in a gas. If, in traversing a distance $d\delta$ in Σ, through a medium of n_i identical particles 2 per cm^3, a projectile 1 of k.e. $k_1 > 0$ has probability $n_i \sigma_i d\delta$ of collision, we call $\sigma_i = \sigma_i(k_1)$ the cross section of the second particles for the first. More generally, for a medium of total density n, composed of $I \geqq 1$ such submedia i, presumed independent, with densities $n_i = f_i n$ and cross sections σ_i, the corresponding collision probability is the sum

$$\sum n_i \sigma_i d\delta \equiv nsd\delta$$

where $s = s(k_1) = \sum f_i \sigma_i$ is the "effective cross section."

The assumption of an "infinitesimal" collision probability $nsd\delta$ is equivalent to the law $dT = - T(nsd\delta)$ for the probability $T(\delta)$ of transmission (without collision) through a finite distance δ, i.e., $T(\delta) = e^{-ns\delta}$. In this situation

$$P(\delta) = 1 - e^{ns\delta} = 1 - T(\delta)$$

is the appropriate "probability distribution function" for (first) collision distance $\leqq \delta$. Accordingly, in Monte Carlo practice, for a random number r uniformly distributed on (0,1), the equation $r = P(\delta)$ determines the distance $\delta = - (1/ns)\ln(1-r)$ of particle 1 to collision. The length $L \equiv 1/ns$, naturally scaling the above formulas, is called the mean free path, since for the "density function"

$$p(\delta) \equiv P'(\delta)$$

for collision on $(\delta,\delta+d\delta)$, the average collision distance is seen to be

$$\bar{\delta} \equiv \int_0^\infty \delta p(\delta)d\delta = 1/ns \equiv L.$$

For media at rest in Σ, the cross sections $\sigma_r(k_1)$ so defined are those ordinarily listed, and used as indicated in problems warranting the rest assumption. If the medium consists of a "gas" of particles in a known k.e. distribution, the "cross section" required by Monte Carlo procedure is an "effective" one determining a transmission probability $T(\delta)$. An attempt is made below to derive such a cross section for a pure material gas, in terms of its k.e. distribution and its rest cross sections σ_r.

As a preliminary step, consider a projectile 1 of k.e. $k_1 > 0$, direction Ψ_1, traversing distance $d\delta$ through a medium of n_1 identical particles 2 per cm^3, all with energy E_2 and direction Ψ_2. In the common rest frame of the targets, the projectile has energy and k.e. (Note 4.4)

$$E_1' = (E_1E_2 - cp_1cp_2\cos\sigma)/e_2 \equiv E_\sigma^2/e_2; \quad k_1' = E_1' - e_1. \tag{1}$$

Here, E_σ^2 is the abbreviation (5) of §24, with $\cos\sigma = \Psi_1\cdot\Psi_2$. The corresponding momentum magnitude in Σ' is therefore given by

$$cp_1' = \left(E_1'^2 - e_1^2\right)^{\frac{1}{2}} = \left(E_\sigma^4 - e_1^2e_2^2\right)^{\frac{1}{2}}/e_2. \tag{2}$$

The projectile undergoes a corresponding displacement (Note 6.6)

$$d\delta' = d\delta \; cp_1'/cp_1 \tag{3}$$

through a medium at rest in Σ', with density (Note 2.6)

$$n_i' = n_i/\gamma_2 \quad \text{where} \quad \gamma_2 = E_2/e_2. \tag{4}$$

We might therefore expect a probability of collision

$$n_i'\sigma_r(k_1')d\delta' \tag{5}$$

on $d\delta'$ in Σ', and hence on $d\delta$ in Σ. Substitution of (2), (3), (4) in (5) yields

$$n_i \left\{\sigma_r(k_1') \left(E_\sigma{}^4 - e_1{}^2 e_2{}^2\right)^{\frac{1}{2}} \Big/ E_2 cp_1\right\} d\delta \tag{6}$$

where we shall regard

$$\sigma_i = \sigma_i(k_1;k_2,\sigma) = \left\{\sigma_r(k_1') \left(E_\sigma{}^4 - e_1{}^2 e_2{}^2\right)^{\frac{1}{2}} \Big/ E_2 cp_1\right\} \tag{7}$$

as the cross section of the medium particles for the projectile, in Σ.

Now consider the traversal of the same projectile 1 through distance $d\delta$ of an isotropic gas of n particles 2 per cm^3, in a k.e. "distribution" $f(k_2)dk_2$. There are then a fraction

$$f_i = f(k_2)dk_2 \; d\Psi_2/4\pi$$

on (k_2, k_2+dk_2), $(\Psi_2, \Psi_2+d\Psi_2)$, which we regard as a submedium contributing the cross section σ_i of (7). The argument at the outset would then lead, in the limit, to an "effective cross section"

$$s = s(k_1) = \iint \sigma_i(k_1;k_2,\sigma)f(k_2)dk_2 d\Psi_2/4\pi$$

for the gas, and hence a free path $L = 1/ns$ determining first collision distance $\delta = - L \ln(1-r)$ for the projectile.

Adopting spherical coordinates σ, φ for location of Ψ_2 about the projectile direction Ψ_1, and setting $a = \cos\sigma$, we see that

$$s = s(k_1) = (2cp_1)^{-1}\int_0^\infty \int_{-1}^1 \sigma_r(k_1')\left(E_\sigma{}^4 - e_1{}^2 e_2{}^2\right)^{\frac{1}{2}} E_2^{-1} f(k_2)\,dk_2\,da. \qquad (8)$$

The involved dependence of the integrand on the variables of integration is provided by the relations

$$k_1' = E_1' - e_1, \quad E_1' = E_\sigma{}^2/e_2, \quad E_\sigma{}^2 = E_1E_2 - cp_1cp_2a, \quad cp_2 = \left(E_2{}^2 - e_2{}^2\right)^{\frac{1}{2}},$$

$$E_2 = k_2 + e_2,$$

where $E_1 = e_1 + k_1$ and $cp_1 = \left(E_1{}^2 - e_1{}^2\right)^{\frac{1}{2}}$ are constants of the projectile.

Notes 25.

1. A non-relativistic analogue of (8), which is "well-known," reads

$$s(k_1) = v_1^{-1}\int \sigma_r(k_1')|V_1 - V_2|F_2(V_2)dV_2 = (2v_1)^{-1}\int_0^\infty \int_{-1}^1 \sigma_r(k_1')v_1' f(v_2)dv_2 da,$$

where $v_1' = \left(v_1{}^2 + v_2{}^2 - 2v_1v_2a\right)^{\frac{1}{2}}$. In case $\sigma_r(k_1')$ is constant, one can show that $s(k_1) > \sigma_r$ for arbitrary distribution $f(v_2)$. This answers the question (C. Mark) whether it is easier to cross Times Square with traffic in motion or at rest. For a Maxwell distribution,

$f(v_2) = \left(4\beta_2{}^3/\sqrt{\pi}\right)v_2{}^2 \exp\left(-\beta_2{}^2 v_2{}^2\right)$, with $\beta_2 = (m_2/2k_BT)^{\frac{1}{2}}$, one finds

$$s(k_1) = \sigma_r\{(1/w\sqrt{\pi})\exp(-w^2) + [1 + (1/2w^2)]\,\mathrm{Erf}(w)\}$$

where $w \equiv \beta_2 v_1$, $\mathrm{Erf}(w) \equiv (2/\sqrt{\pi}) \int_0^w \exp(-x^2)dx$. The "mean free path" of kinetic theory, with particles of types 1 and 2 both in Maxwell distributions involves the surprisingly more tractable integral

$$\bar{\bar{v}}_1' = \iint |V_1 - V_2| F_1(V_1) F_2(V_2) dV_1 dV_2 = (8k_B T/\pi\mu)^{\frac{1}{2}}, \text{ where}$$

$\mu = m_1 m_2/(m_1+m_2)$.

2. The cross section (8) for the case of a photon beam reduces to

$$s(k_1) = (1/2) \int_0^\infty \int_{-1}^1 \sigma_r(k_1') \{1 - (cp_2/E_2)a\} f(k_2) dk_2 da, \text{ where}$$

$k_1' = k_1(E_2 - cp_2 a)/e_2$. Here, in the physically uninteresting case σ_r constant, one sees that $s(k_1) \equiv \sigma_r$ regardless of $f(k_2)$. Hence, for a photon crossing Times Square ...

3. Unfortunately no adequate reference for (8) has been found, and some manifest subtleties may vitiate the result, which is offered tentatively. It should be emphasized in any case that the "cross section" here considered is not Lorentz invariant, and is only a means to a free path. In the following section we revert to standard practice, regarding (non-differential) cross sections as intrinsic properties of the target particle, as measured in its rest frame.

26. Transformation of differential cross sections. Let $\sigma_T = \sigma_T(k_1)$ denote the (total) cross section for collision of a particle 1, having k.e. k_1, with particles of a single species 2, at rest in Σ. Various types of transmutation may result from such a collision, the probability p_K of each defining the partial cross section $\sigma_K = p_K \sigma_T$ for its occurrence.

We now focus attention on any one such process

$$A(e_1,e_2) \longrightarrow S(e_1) \qquad (K)$$

of cross section $\sigma \equiv \sigma_K$, the resulting system S consisting of $I \geqq 3$ particles, of which $\mu = \mu_j$ are of the same species j. Then the probability of emission $f(E,\Psi)dEd\Psi$ of a j-particle, with E,Ψ on the indicated ranges in Σ, has the operational meaning that, in a large number N of K-processes, one expects to find $\mu N f dEd\Psi$ such particles of species j. The corresponding differential cross section is then given by

$$\sigma(E,\Psi)dEd\Psi = \mu\sigma f dEd\Psi \text{ cm}^2$$

with the integral

$$\iint \sigma(E,\Psi)dEd\Psi = \mu\sigma \qquad (\sigma = \sigma_K).$$

If $f'(E',\Psi')dE'd\Psi'$ denotes the corresponding probability of emission for the K-process $A' \longrightarrow S'$ as it appears in a second frame Σ' moving with constant velocity $U_o = u_o\Psi_o$ relative to Σ, then the equation

$$f'(E',\Psi')dE'd\Psi' = f(E,\Psi)dEd\Psi \qquad (1)$$

is dictated by the invariance of j-particle counts. Regarding $\sigma = \sigma_K$ as invariant, the same relation is seen to govern the corresponding differential cross sections σ' and σ. For the standard axes of Fig. 2.1, and polar coordinates (ψ,φ), (ψ',φ') for location of Ψ,Ψ' about $\Psi_o = X = X'$, we may write (1) in the form

$$f'(E',\psi',\varphi')dE' \sin\psi' d\psi' d\varphi' = f(E,\Psi,\varphi)dE \sin\psi d\psi d\varphi.$$

Setting $a' = \cos\psi'$, $a = \cos\psi$, and noting that $\varphi' = \varphi$ for standard axes, this becomes

$$f'(E',a',\varphi)dE'da'd\varphi = f(E,a,\varphi)dEdad\varphi. \qquad (2)$$

It follows that

$$f'(E',a'\varphi)dE'da' = f(E,a,\varphi)|\partial(E,a)/\partial(E',a')|dE'da' \qquad (3)$$

where the factor denotes the absolute value of the Jacobian

$$J = \det\begin{bmatrix}\partial E/\partial E' & \partial a/\partial E' \\ \partial E/\partial a' & \partial a/\partial a'\end{bmatrix} \qquad (4)$$

of the transformation $E = E(E',a')$, $a = a(E',a')$ from Σ' to Σ. The latter is concealed implicitly in the (cPE) transformation, namely

$$\begin{aligned} cp\cdot a &= \gamma_o(cp'\cdot a'+\beta_o E') \\ E &= \gamma_o(\beta_o cp'\cdot a'+E') \end{aligned} \qquad (5)$$

where $(cp)^2 = E^2 - e^2$, $(cp')^2 = E'^2 - e^2$.

Since $dcp/dE = E/cp$ we obtain formally

$$\begin{aligned} \partial(cp\cdot a)/\partial E' &= (E/cp)(\partial E/\partial E')a + cp(\partial a/\partial E') \\ \partial(cp\cdot a)\partial a' &= (E/cp)(\partial E/\partial a')a + cp(\partial a/\partial a'). \end{aligned} \qquad (6)$$

Hence, multiplying the second column in (4) by cp, and adding to the result the multiple (E/cp)a of the first, we see that

$$cp\ J = \det \begin{bmatrix} \partial E/\partial E' & \partial(cp\cdot a)/\partial E' \\ \partial E/\partial a' & \partial(cp\cdot a)/\partial a' \end{bmatrix}.$$

These partials are readily found from (5) to be, respectively,

$$\begin{matrix} \gamma_o(\beta_o E'a'/cp'+1) & \gamma_o(E'a'/cp'+\beta_o) \\ \gamma_o\beta_o cp' & \gamma_o cp' \end{matrix}$$

so that $cpJ = cp'$. Thus (3) reduces to the symmetric relation

$$f'(E',a',\varphi)dE'da'/cp' \equiv f(E,a,\varphi)dEda/cp. \qquad (7)$$

Analogous formulas obtain for other variables. Thus one may prove in similar fashion

$$f'(cp',a',\varphi)E'd(cp')da'/(cp')^2 = f(cp,a,\varphi)Ed(cp)da/(cp)^2.$$

The condition $I \geqq 3$ imposed above on the system $S(e_i)$ was necessary for the independence of the variables E',a'. For a two particle system S, E' is uniquely determined by the initial system A, and one speaks of a probability of emission

$$f'(a',\varphi)da'd\varphi = f(a,\varphi)dad\varphi$$

related by

$$f'(a',\varphi)da = f(a,\varphi)(da/da')da'.$$

We have derived in §3 the required formulas

$$a = (a'+\rho')/D' \qquad da/da' = \gamma_o^{-2}(1+\rho'a')/D'^3$$

where $D = \left\{(a'+\rho')^2 + \gamma_o^{-2}(1-a'^2)\right\}^{\frac{1}{2}}$; $\rho' = \beta_o E'/cp'$ (constant)

with the simpler version

$$a = (a'+\beta_o)/d' \qquad da/da' = \gamma_o^{-2}/d'^2$$

$$d' = \beta_o a' + 1$$

for the case of an immaterial j-particle.

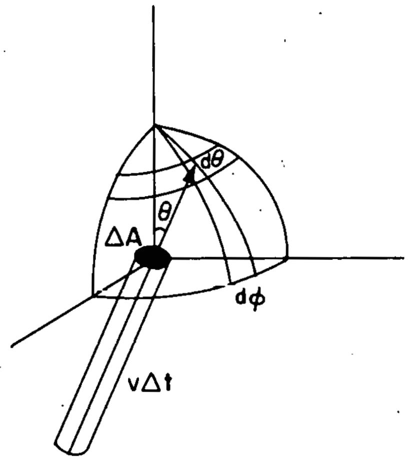
Note 26.

1. For a more complete discussion, including singularities, see K. G. Dedrick, Rev. Mod. Phys. 34, 1962, 429-442.

APPENDIX I

A RELATIVISTIC GAS

Consider a homogeneous, isotropic gas of n particles per cm^3, each of ch. mass m, of which the fraction $f(k)dk$ have kinetic energy on



$(k, k+dk)$, $0 < k < \infty$. The numerical flux $N(k,\theta,\varphi)dk d\theta d\varphi/cm^2$ sec of particles, in the indicated ranges of k and (direction spherical-coordinates) θ, φ is seen from the figure to be $n(\Delta A \cdot v\Delta t \cos\theta)f(k)dk(\sin\theta d\theta d\varphi/4\pi)/\Delta A \Delta t = (n/4\pi)vf(k)dk \sin\theta\cos\theta d\theta d\varphi$, where $v = v(k)$ is the speed. Successive integrations, on $0 \leqq \varphi < 2\pi$, $0 \leqq \theta \leqq \pi/2$, and $0 < k < \infty$ show the various resulting (one-way) numerical fluxes to be

1. $N(k,\theta)dk d\theta = (n/2)vf(k)dk \sin\theta\cos\theta d\theta/cm^2$ sec
2. $N(k)dk = (n/4)vf(k)dk$ $/cm^2$ sec
3. $N = (n/4)\overline{v}$ $/cm^2$ sec

while for the kinetic energy flux,

4. $\phi = \int_0^\infty kN(k)dk = (n/4)\overline{kv}$ erg/cm^2 sec.

Regarding pressure $\mathbb{P}$ at a "wall" as the total change of normal component of momentum per sec, per cm^2, we find from (1)

$$5.\quad \mathbb{P} = \int_0^\infty \int_0^{\pi/2} (2p\cos\theta)N(k,\theta)dkd\theta = (n/3)\overline{pv} = (n/3)\overline{Mv^2} \qquad erg/cm^3$$

where $p = p(k)$ is the absolute momentum.

The kinetic energy per unit volume, on the interval $(k,k+dk)$ is

$$6.\quad K(k)dk = k\cdot nf(k)dk \qquad erg/cm^3$$

the total being

$$7.\quad K \equiv \int_0^\infty K(k)dk = n\bar{k} \qquad erg/cm^3.$$

For a gas of photons ($m = 0$, $v = c$, $k = E = h\nu$) at "temperature" $\theta \equiv k_B T$ ergs, the function $K(k)$ in (6) is known as the Planck distribution, namely

$$8.\quad K(k)dk = 8\pi(hc)^{-3}k^3(e^{k/\theta}-1)^{-1}dk \qquad erg/cm^3.$$

From this as a starting point, we infer from (6) that

$$9.\quad nf(k)dk = 8\pi(hc)^{-3}k^2(e^{k/\theta}-1)^{-1}dk \qquad /cm^3$$

and upon integration, we find that (Note 1)

$$10.\quad n = 16\pi\zeta(3)(hc)^{-3}\theta^3 \qquad photons/cm^3$$

is the (temperature dependent!) numerical density.

From (9), the probability of k on $(k,k+dk)$ is therefore

$$11.\quad f(k)dk = (2\zeta(3)\theta^3)^{-1}k^2(e^{k/\theta}-1)^{-1}dk.$$

Direct integration of (8) shows the total energy density to be (Note 1)

12. $K = (8/15)\pi^5(hc)^{-3}\theta^4 \equiv n\bar{k}$ erg/cm^3

so that, from (12) and 10), the average photon energy is

13. $\bar{k} = K/n = \pi^4\theta/30\,\zeta(3)$ erg.

Evaluation of N, ∅ and ℙ from (3), (4), and (5) is quite trivial. Thus, the total numerical flux is

14. $N = (c/4)n = 4\pi\zeta(3)c(hc)^{-3}\theta^3$ photons/cm^2 sec

carrying an energy

15. $\emptyset = (c/4)n\bar{k} = (c/4)K = (2/15)\pi^5 c(hc)^{-3}\theta^4 \equiv \sigma T^4$ erg/cm^2 sec

where σ is the "Stefan-Boltzmann constant."

Finally, for the radiation pressure,

16. $\mathbb{P} = (n/3)\overline{Mc^2} = (1/3)n\bar{k} = K/3$ erg/cm^3.

Notes I.

1. The values of n and K for a photon gas may be verified from the formula

$$(*) \qquad P(s) \equiv \int_0^\infty x^{s-1}(e^x-1)^{-1}dx = \Gamma(s)\zeta(s), \qquad \text{real } s > 1$$

where $\Gamma(s) \equiv \int_0^\infty x^{s-1}e^{-x}dx$ $(s > 0)$ is the "Γ-function" with values

$\Gamma(s) = (s-1)!$ for integral $s = 1,2,3,\ldots$ $(0! \equiv 0)$, and

$\zeta(s) \equiv \sum_{m=1}^\infty m^{-s}$ $(s > 1)$ is the "Riemann ζ-function."

One knows $\zeta(3) \cong 1.2021$, $\zeta(4) = \pi^4/90$.

The formula (*) can be obtained by termwise integration using the geometric series

$$e^{-x}(1-e^{-x})^{-1} = \sum_{m=1}^{\infty} e^{-mx} \qquad (x > 0)$$

and the obvious relation

$$\int_0^{\infty} x^{s-1}e^{-mx}dx = m^{-s}\Gamma(s) \qquad (s > 0,\ m > 0).$$

2. For a gas of <u>material</u> particles, the pressure formula $\mathbb{P} = (n/3)\overline{Mv^2}$ is not expressible in terms of K, although in non-relativistic approximation $\mathbb{P} \cong (2/3)n\overline{k} = (2/3)$ K (compare (16)). Strictly, $Mv^2 = pv = h(v/\lambda)$ so we might write $\mathbb{P} = (n/3)\overline{hf}$ for a "frequency" f such that $\lambda f \equiv v$. We recall for $\nu \equiv E/h$ that $\lambda\nu = c^2/v$. (Cf. §1.)

APPENDIX II

THE GENERAL LORENTZ TRANSFORMATION

Let $c > 0$ be a specified constant; $\Sigma' = \mathcal{R}^{n+1} (n \geq 2)$ an "event space" of vectors $\xi' = \begin{vmatrix} X' \\ t' \end{vmatrix}$, where X' is a "position" in Euclidean $\mathcal{R}^n$, and $t' \equiv x'_{n+1} \in \mathcal{R}^1$. A Lorentz transformation (L.T.) here means any non-singular (n.s.) linear transformation (l.t), (n+1 order matrix),

$$T = \begin{vmatrix} A & B \\ C^{\tau} & d \end{vmatrix} \qquad \text{(A being } n \times n\text{)}$$

of Σ', with the property

(L) For every $\xi' = \begin{vmatrix} X' \\ t' \end{vmatrix}$ with $|X'| = c|t'|$, its image $\begin{vmatrix} X \\ t \end{vmatrix} \equiv \xi = T\xi'$ $= \begin{vmatrix} AX' + Bt' \\ C^{\tau}X' + dt' \end{vmatrix}$ also satisfies $|X| = c|t|$.

Defining the symmetric matrices $Q = \begin{vmatrix} I_n & 0 \\ 0^{\tau} & -c^2 \end{vmatrix}$ and $P \equiv T^{\tau}QT$, (L) may be expressed in the form:

$$\text{(L}'\text{)} \qquad \xi'^{\tau}Q\xi' \equiv \Sigma_1^n x_j'^2 - c^2t'^2 = 0 \qquad (1)$$

$$\text{implies} \qquad \xi'^{\tau}T^{\tau}QT\xi' \equiv \xi^{\tau}Q\xi \equiv \Sigma_1^n x_j^2 - c^2t^2 = 0$$

$$\text{i.e.,} \qquad \xi'^{\tau}P\xi' \equiv \Sigma_1^{n+1} x_i'p_{ij}x_j' = 0. \qquad (2)$$

Theorem 1. If T is a L.T., then

$$P \equiv T^{\tau}QT = qQ \tag{3}$$

where
$$q = d^2 - c^{-2}|B|^2 \neq 0, \tag{4}$$

hence the identity
$$\textstyle\sum_1^n x_j^2 - c^2t^2 = q(\sum_1^n x_j'^2 - c^2t'^2). \tag{5}$$

Conversely, if T is a matrix such that $T^{\tau}QT = qQ$ with $q \neq 0$, then T is a L.T.

Proof. Let δ_i be position vector with i-th component 1, all others zero. Since $\xi' = \pm\,\delta_i$, $t' = c^{-1}$, and $\xi' = 3\delta_i + 4\delta_j$, $t' = 5c^{-1}$ all satisfy (1), it follows from (2) that $p_{ii} = -\,p_{n+1n+1}c^{-2}$, $p_{in+1} = 0$, $i = 1,\ldots,n$, and $p_{ij} = 0$, $i \neq j$, $i,j = 1,\ldots,n$. Hence $T^{\tau}QT = P$ $= -\,p_{n+1n+1}c^{-2}Q \equiv qQ$. Here $q \neq 0$ since T is n.s. Finally, (5) is obvious, and implies the value of q in (4), upon setting $X' = 0$, $t' = 1$ and the corresponding $X = B$, $t = d$.

The converse statement is trivial.

Corollary 1. The set of all L.T.'s is a group.

Proof. This follows formally from the N. & S. condition: $T^{\tau}QT = qQ$, $q \neq 0$.

Corollary 2. The matrix T is a L.T. iff

(6) $A^{\tau}A - c^2CC^{\tau} = qI_n$ and (7) $A^{\tau}B = c^2dC$, where

(8) $q \equiv d^2 - c^{-2}|B|^2$ is non-zero.

Proof. By block multiplication,

$$T^{\tau}QT = \begin{vmatrix} A^{\tau}A - c^2CC^{\tau} & A^{\tau}B - c^2dC \\ B^{\tau}A - c^2dC^{\tau} & B^{\tau}B - c^2d^2 \end{vmatrix}.$$

The result follows at once from Th. 1.

<u>Corollary 3</u>. If T is a L.T., then $d \neq 0$, and for

$$T = dT_1 = d \begin{vmatrix} A_1 & B_1 \\ C_1^{\tau} & 1 \end{vmatrix}$$

we must have

(9) $A_1^{\tau}A_1 - c^2C_1C_1^{\tau} = q_1I_n$ and (10) $A_1^{\tau}B_1 = c^2C_1$, where

(11) $q_1 = q/d^2 = 1 - c^{-2}|B_1|^2 \neq 0.$

Proof. Suppose $d = 0$. Let X' be a non-zero vector (existence obvious) such that $C^{\tau}X' = 0$, and define $t' = c^{-1}|X'|$. Then $|X'| = c|t'|$, and we must also have

$$|AX'+Bt'| \equiv |X| = c|t| = c|C^{\tau}X'+dt'| = 0.$$

But a n.s. T cannot take $\xi' \neq 0$ into $\xi = 0$. The rest is clear from Cor. 2.

<u>Corollary 4</u>. If $T = dT_1$ is a L.T., as in Cor. 3, then $B_1 = 0$ iff $C_1 = 0$. In such a case, $q_1 = 1$, $q = d^2 > 0$, and A_1 is a rotation of $\mathfrak{R}^n$.

Proof. These statements follow at once from (9), (10), (11).

Note: A "rotation" means here any $n \times n$ matrix R such that $R^{\tau}R = I_n$. A "space rotation" is a l.t. of Σ' of form

$$S = \begin{vmatrix} R & 0 \\ 0^\tau & 1 \end{vmatrix}, \quad R \text{ a rotation.}$$

The set S^* of all such is a group, and for $S \in S^*$, $S^{-1} = S^\tau$ and $\det S = \pm 1$.

Corollary 5. The set of all matrices of form dS, $d \neq 0$, $S \in S^*$ is a group, consisting of precisely those L.T.'s with $B = C = 0$.

Proof. This is clear from Cor. 2 and Cor. 4.

Now suppose T is a L.T. as in Cor. 3, with $|B_1| \equiv b_o > 0$, $|C_1| \equiv c_o > 0$, and consider the equation

$$S_1T_1S'^\tau = \begin{vmatrix} R_1 & 0 \\ 0^\tau & 1 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ C_1^\tau & 1 \end{vmatrix} \begin{vmatrix} R'^\tau & 0 \\ 0^\tau & 1 \end{vmatrix} = \begin{vmatrix} R_1A_1R'^\tau & R_1B_1 \\ C_1^\tau R'^\tau & 1 \end{vmatrix} \equiv T_1'$$

where S_1, S' are space rotations.

Letting R_1, R' be rotations such that $R_1B_1 = b_o\delta_1$ and $R'C_1 = c_o\delta_1$ (Note 1), we obtain a L.T.

$$T_1' = \begin{vmatrix} A_2 & b_o\delta_1 \\ c_o\delta_1^\tau & 1 \end{vmatrix}$$

which, by Cor. 3, satisfies

(12) $A_2^\tau A_2 - c^2c_o^2\delta_1\delta_1^\tau = q_1I_n$ and (13) $A_2^\tau\delta_1 = c^2b_o^{-1}c_o\delta_1$ where

(14) $q_1 = 1 - c^{-2}b_o^2 = q/d^2 \neq 0$

is the same as for T_1, since $b_o \equiv |B_1|$.

By (13), A_2 has the form

$$A_2 = \begin{vmatrix} c^2 b_o^{-1} c_o & 0^\tau \\ U & A_3 \end{vmatrix} .$$

From (12) and (14) we then conclude that in A_2,

(15) $c^4 b_o^{-2} c_o{}^2 + U^\tau U - c^2 c_o{}^2 = q_1 = 1 - c^{-2} b_o{}^2 \neq 0$

(16) $A_3{}^\tau U = 0^\tau$

(17 $A_3{}^\tau A_3 = q_1 I_{n-1}$.

Since $q_1 \neq 0$, we see from (17) that $q_1 \equiv 1 - c^{-2} b_o{}^2 > 0$ (hence $q = q_1 d^2 > 0$ in Th. 1), and so $b_o < c$. Defining $\gamma_o = q_1^{-\frac{1}{2}} = (1 - c^{-2} b_o{}^2)^{-\frac{1}{2}} > 0$, we have from (17) that $\gamma_o A_3$ is a rotation R_{n-1} of $\mathcal{R}^{n-1}$; from (16) that $U = 0$; and from (15) that $c_o = c^{-2} b_o$.

Collecting these results, it appears that

$$T_1' = S_1 T_1 S'^\tau = \begin{vmatrix} A_2 & b_o \delta_1 \\ c^{-2} b_o \delta_1{}^\tau & 1 \end{vmatrix} \quad \text{where } A_2 = \begin{vmatrix} 1 & 0^\tau \\ 0 & \gamma_o^{-1} R_{n-1} \end{vmatrix} .$$

Finally, defining the space rotations

$$S_2 = \begin{vmatrix} R_2 & 0 \\ 0^\tau & 1 \end{vmatrix} , \quad \text{where } R_2 = \begin{vmatrix} 1 & 0^\tau \\ 0 & R_{n-1}^\tau \end{vmatrix}$$

and $S = S_2 S_1$, a straightforward computation for $S T_1 S'^\tau = S_2 (S_1 T_1 S'^\tau) = S_2 T_1'$ yields the final

Theorem 2. Every Lorentz transformation T is expressible in the form

$$T = d\gamma_o^{-1} S^\tau L S'$$

where S and S′ are space rotations, and

$$L = \begin{vmatrix} \gamma_o & 0^\tau & \gamma_o b_o \\ 0 & I_{n-1} & 0 \\ \dfrac{\gamma_o b_o}{c^2} & 0^\tau & \gamma_o \end{vmatrix}$$

with $\gamma_o = (1-b_o{}^2c^{-2})^{-\frac{1}{2}}$, $0 \leqq b_o = |B_1| < c$, $\det L = 1$.

Moreover, in Th. 1, $q = d^2\gamma_o^{-2} > 0$.

Corollary 6. Given a Lorentz transformation $\xi = T\xi'$ relative to the coordinates ξ, ξ', the transformation relative to the "standard" coordinates $\bar{\xi} \equiv S\xi$, $\overline{\xi'} \equiv S'\xi'$ has the form

$$\bar{\xi} = d\gamma_o^{-1} L \overline{\xi'}.$$

Note II.

1. The "Gram-Schmidt" process affords a construction of a "rotation" R which takes δ_1 into a given unit (column) vector Ψ_1. Since $\Psi_1 \neq 0$, suppose its i-th component $a_i \neq 0$. Then the set of n vectors: Ψ_1, and all δ_j, $j \neq i$, is linearly independent. The orthonormalization algorithm produces from these an orthonormal set

$$R = [\Psi_1, \Psi_2, \ldots, \Psi_n]$$

of column vectors. Regarded as a matrix R, we have

$$R\delta_1 = \Psi_1.$$

APPENDIX III

COORDINATES AND ROTATIONS

Let $\mathfrak{R}^3$ denote a Euclidean 3-space of position vectors R. A "set of axes" $\mathfrak{G} = [X,Y,Z]$ means any right-handed set of mutually perpendicular unit vectors X,Y,Z. The relation

$$R = [X,Y,Z] \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathfrak{G}} \equiv Xx + Yy + Zz \tag{1}$$

then determines x,y,z as the <u>$\mathfrak{G}$-coordinates</u> of R.

A <u>rotation</u> δ of $\mathfrak{R}^3$ is (intuitively) a "rigid motion" about the origin, and is completely defined by its action on any set of axes $\mathfrak{G}$, as indicated by an equation of the form

$$\delta\mathfrak{G} \equiv \delta[X,Y,Z] \equiv [\delta X,\delta Y,\delta Z] = [X,Y,Z]D \equiv \mathfrak{G}D$$

where $D = [d_{ij}]$ is a 3×3 matrix with $D^{-1} = D^{\tau} \equiv [d_{ji}]$ and $\det D = +1$. The vectors $\delta X,\delta Y,\delta Z$ then form a set of axes also, with $\mathfrak{G}$-coordinates given by the column vectors of D.

<u>Theorem 1</u>. Let $\mathfrak{G} = [X,Y,Z]$ be a set of axes, and δ a rotation, defined by $\delta\mathfrak{G} = \mathfrak{G}D$. Then

(a) a point R with $\mathfrak{G}$-coordinates $(x,y,z)_{\mathfrak{G}}$ has $\delta\mathfrak{G}$-coordinates

$$\begin{vmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{vmatrix}_{\delta\mathfrak{G}} = D^{\tau} \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathfrak{G}} \qquad \text{whereas}$$

(b) the point δR has $\mathcal{G}$-coordinates

$$\begin{vmatrix} x' \\ y' \\ z' \end{vmatrix}_{\mathcal{G}} = D \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathcal{G}} .$$

Proof. From $\delta\mathcal{G} = \mathcal{G}D$ and the definitions, we have

(a) $R = \mathcal{G}\begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathcal{G}} = \delta\mathcal{G}\cdot D^{\tau} \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathcal{G}} \equiv \delta\mathcal{G}\begin{vmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{vmatrix}_{\delta\mathcal{G}}$ and

(b) $\delta R = \delta\mathcal{G}\begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathcal{G}} = \mathcal{G}\cdot D \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{\mathcal{G}} \equiv \mathcal{G}\begin{vmatrix} x' \\ y' \\ z' \end{vmatrix}_{\mathcal{G}}$.

Every rotation may be achieved by a right-handed rotation through an angle θ ($0^{\circ} \leqq \theta \leqq 180^{\circ}$) about some fixed unit vector Ψ_{o}, and defined explicitly as in

Theorem 2. If a unit vector Ψ_{o} has $\mathcal{G}$-coordinates $(b_x,b_y,b_z)_{\mathcal{G}}$, and θ is given on $[0^{\circ},180^{\circ}]$, with $C \equiv \cos\theta$, $C' \equiv 1 - \cos\theta$, $S \equiv \sin\theta \geqq 0$, then the right-handed rotation δ about Ψ_{o} through θ is defined by $\delta\mathcal{G} = \mathcal{G}D$, where

$$D = \begin{vmatrix} C + b_x^{2}C' & -b_zS + b_yb_xC' & b_yS + b_zb_xC' \\ b_zS + b_xb_yC' & C + b_y^{2}C' & -b_xS + b_zb_yC' \\ -b_yS + b_xb_zC' & b_xS + b_yb_zC' & C + b_z^{2}C' \end{vmatrix} .$$

Proof. One has only to verify that the above column vectors are the $\mathcal{G}$-coordinates of $\delta X,\delta Y,\delta Z$ for the δ defined. Let $X_{\parallel} = (X\cdot\Psi_{o})\Psi_{o} =$

$= (b_x{}^2, b_x b_y, b_x b_z)_G$, $X_\perp = X - X_\parallel = (1-b_x{}^2, -b_x b_y, -b_x b_z)_G$, where $|X_\perp| = (1-b_x{}^2)^{\frac{1}{2}}$. Then the desired δX is

$$\delta X = \delta(X_\parallel + X_\perp) = X_\parallel + \delta X_\perp \quad \text{where} \quad |\delta X_\perp| = |X_\perp| .$$

It is clear from Fig. III.1 that $\delta X_\perp$ must satisfy the (dependent) conditions

(1) $\Psi_o \cdot \delta X_\perp = 0$ (2) $X_\perp \cdot \delta X_\perp / |X_\perp|^2 = C$

and (3) $X_\perp \times \delta X_\perp / |X_\perp|^2 S = \Psi_o$

when $X_\perp \neq 0$, $S \neq 0$. (If $X_\perp = 0$, $b_x{}^2 = 1$, $b_y = b_z = 0$, and the first column correctly reads (1,0,0). The column is also correct if $S = 0$, $C = \pm 1$. This we leave for the reader.)

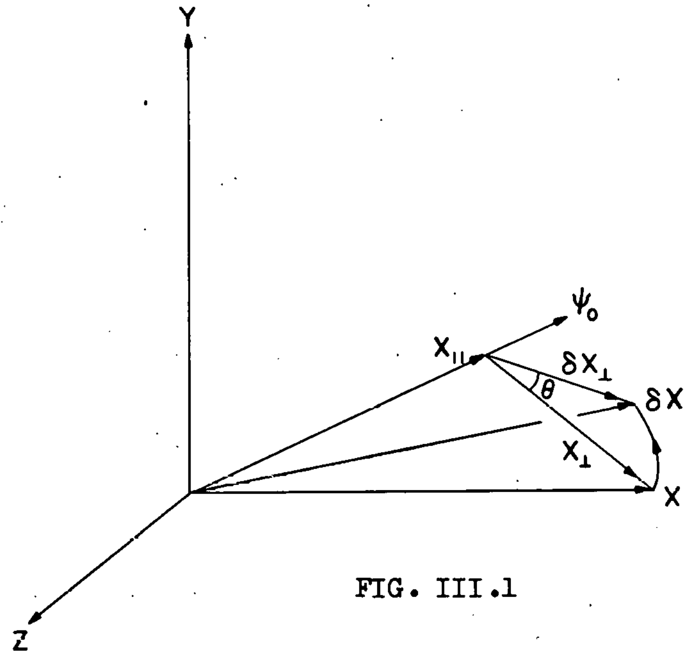


FIG. III.1

Letting the unknown $\delta X_\perp = (p_x, p_y, p_z)_G$, these conditions yield

(4) $b_x p_x + b_y p_y + b_z p_z = 0$

(5) $(1-b_x^2)p_x - b_x b_y p_y - b_x b_z p_z = C(1-b_x^2)$

(6) $b_x b_z p_y - b_x b_y p_z = S(1-b_x^2)b_x$ (first component).

From (4) and (5) we have at once $p_x = C(1-b_x^2)$, so that δX has X coordinate $b_x^2 + C(1-b_x^2) = C + b_x^2 C'$ as claimed. If $b_x \neq 0$ and $b_x^2 \neq 1$ as assumed above, (4) and (6) then yield

$$p_y = b_z S - b_x b_y C \qquad \text{and} \qquad p_z = - b_y S - b_x b_z C$$

and the remaining coordinates of δX are seen to be those in D. Finally, if $b_x = 0$, then $p_x = C$, and condition (3) gives trivially $p_y = b_z S$, $p_z = - b_y S$, as required in D. The verification for δY, δZ is immediate by cyclic permutation.

<u>Corollary 1</u>. Let $(a_x, a_y, a_z)_G$ be the G-coordinates of a given unit vector Ψ. Then δ, defined by $\delta G = GD$, where

$$\text{(a)} \quad D = \begin{vmatrix} a_x & -a_y & -a_z \\ a_y & 1 - (a_y^2/\Delta) & -a_y a_z/\Delta \\ a_z & -a_y a_z/\Delta & 1 - (a_z^2/\Delta) \end{vmatrix} \qquad \Delta \equiv 1 + a_x \neq 0$$

$$\text{or} \quad \text{(b)} \quad D = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{when } \Delta = 0$$

is a rotation which takes X into $\delta X = \Psi$.

Proof. If $a_x = 1$, then $a_y = a_z = 0$, and $D = I$ in (a). If $a_x = -1$, the D in (b) defines a rotation of 180° about Z. Otherwise, $\Psi \neq \pm X$, so that X and Ψ lie in a well-defined plane, and a right-handed rotation through an angle θ on $[0^\circ, 180^\circ]$ with $C \equiv \cos\theta = X \cdot \Psi = a_x$, about its normal $\Psi_o = (X \times \Psi)/S = (0, -a_z, a_y)_G /S$, $S \equiv \sin\theta = (1-a_x^2)^{\frac{1}{2}}$, will obviously serve (Fig. III.2). The above matrix D results from that of Th. 2 upon making these substitutions.

It is sometimes convenient to make a rotation which not only takes X into a specified $\Psi = \delta X$, but has the property that a second given vector Ψ_1 lies in the new $\delta X, \delta Y$ plane. For this we have

<u>Theorem 3</u>. Let $(a_x, a_y, a_z)_G$, $(a_{1x}, a_{1y}, a_{1z})_G$ be the G coordinates of two unit vectors Ψ, Ψ_1. Define $C = \cos\psi_1 = \Psi \cdot \Psi_1$, $0 \leq \psi_1 \leq 180^\circ$, and set $S = \sin\psi_1 \geq 0$. Then we obtain a rotation δ_1, defined by $\delta_1 G = GD_1$, such

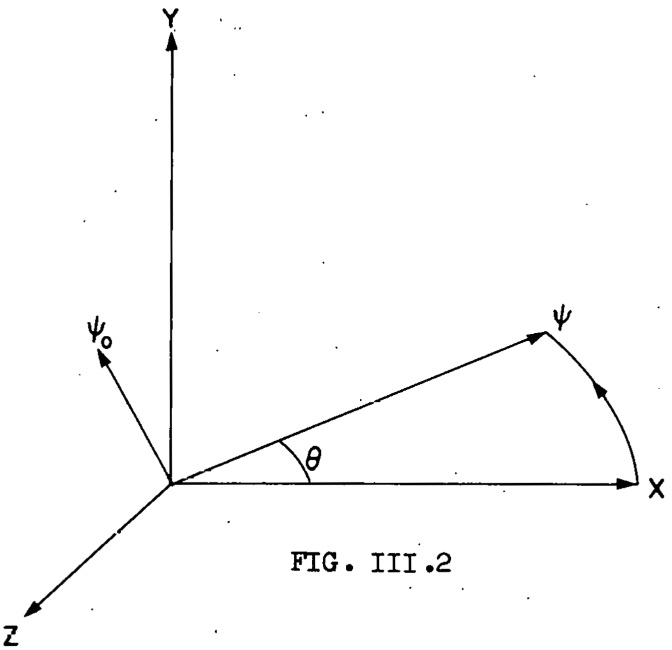


FIG. III.2

that (a) $\delta_1 X = \Psi$ and (b) Ψ_1 lies in the "upper half" of the $\delta_1 X, \delta_1 Y$ plane, with $\delta_1 G$-coordinates $(\cos\psi_1, \sin\psi_1, 0)_{\delta_1 G}$, provided we take D_1 as the D of Cor. 1 in the trivial case $\Psi_1 = \pm\Psi$, and otherwise set

$$D_1 = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}$$

where $c_x = (a_y a_{1z} - a_z a_{1y})/S$, $c_y = (a_z a_{1x} - a_x a_{1z})/S$, $c_z = (a_x a_{1y} - a_y a_{1x})/S$

and $b_x = c_y a_z - c_z a_y = (a_{1x} - a_x C)/S$

$b_y = c_z a_x - c_x a_z = (a_{1y} - a_y C)/S$

$b_z = c_x a_y - c_y a_x = (a_{1z} - a_z C)/S$.

Proof. The matrix D_1 is uniquely determined by the conditions $\delta_1 X = \Psi$, $\delta_1 Z = \Psi \times \Psi_1/S$, $\delta_1 Y = \delta_1 Z \times \delta_1 X$ (Fig. III.3).

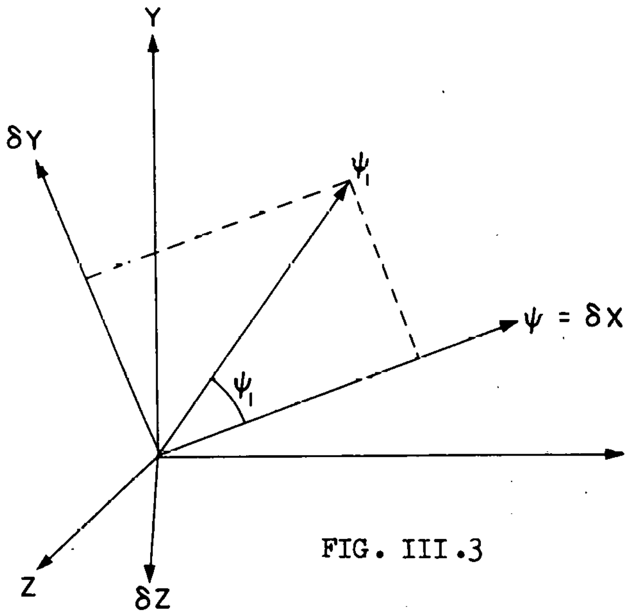


FIG. III.3

Corollary 2. Let $(a_x, a_y, a_z)_G$, $(a_{1x}, a_{1y}, a_{1z})_G$ be the G-coordinates of two orthogonal unit vectors Ψ, Ψ_1. Then δ_1, defined by $\delta_1 G = GD_1$, where

$$D_1 = \begin{vmatrix} a_x & a_{1x} & a_y a_{1z} - a_z a_{1y} \\ a_y & a_{1y} & a_z a_{1x} - a_x a_{1z} \\ a_z & a_{1z} & a_x a_{1y} - a_y a_{1x} \end{vmatrix} = [\Psi, \Psi_1, \Psi \times \Psi_1]_G$$

is a rotation such that $\delta_1 X = \Psi$ and $\delta_1 Y = \Psi_1$.

Proof. Set C = 0, S = 1 in Th. 3.

This is the basic rotation used in polarized Compton scattering (A IV).

Notes III.

1. The matrix $D^* = \begin{vmatrix} a_x & a_y & a_z \\ a_y & (a_y^2/\Delta)-1 & a_y a_z/\Delta \\ a_z & a_y a_z/\Delta & (a_z^2/\Delta) - 1 \end{vmatrix}$, $\Delta = 1 + a_x \neq 0$

obtained from D in Cor. 1 by changing signs of its last two columns is symmetric, and defines a rotated set of axes $\delta^* G$ (with $\delta^* X = \Psi$) upon which X appears with the same coordinates a_x, a_y, a_z as has Ψ on the original axes G.

2. Another alternative for Cor. 1 is the rotation $\tilde{\delta}$ with matrix

$$\tilde{D} = \begin{vmatrix} a_x & -S & 0 \\ a_y & a_x a_y/S & -a_z/S \\ a_z & a_x a_z/S & a_y/S \end{vmatrix} \qquad S \equiv (1-a_x^2)^{\frac{1}{2}} > 0$$

which achieves the result $\tilde{\delta}X = \Psi$ by successive rotations about Z and X. The square root is a computational disadvantage.

APPENDIX IV

COMPTON SCATTERING OF PLANE POLARIZED PHOTONS

1. Klein-Nishina cross section. A plane polarized photon may be characterized in a frame Σ by its energy $k_1 = h\nu_1/m_\epsilon c^2$, direction Ψ_1, and an electric unit vector[(1)] e_1 in the plane π_1 orthogonal to Ψ_1 (Fig. 1). Its Klein-Nishina differential cross section for scattering (on a free unpolarized electron at rest in Σ) into a direction within $d\Psi_2$ of Ψ_2, with an e-vector e_2 (in the plane π_2 orthogonal to Ψ_2) at angle Θ with e_1 (Fig. 2), i.e., with

$$e_2 \cdot e_1 = \cos \Theta$$

is given by

$$\sigma(\Psi_2, e_2)d\Psi_2 = (r^2/4)(k_2/k_1)^2\{K_2 - 2 + 4\cos^2\Theta\}d\Psi_2 \quad cm^2 \tag{1}$$

where $K_2 \equiv k_2/k_1 + k_1/k_2$, $k_2 = k_1/\{1+k_1(1-\cos\psi_2)\}$, and $\cos\psi_2 \equiv \Psi_2 \cdot \Psi_1$.

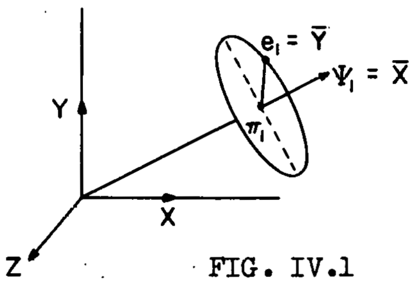


FIG. IV.1

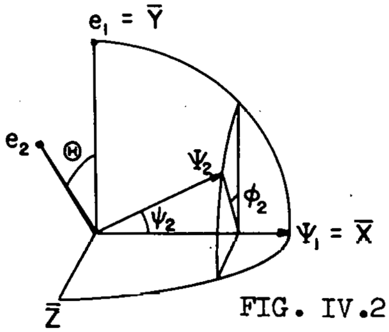


FIG. IV.2

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(1) For computation, we suppose $(\Psi_1)_G$, $(e_1)_G$ given on Σ axes $G = [X,Y,Z]$.

For a given scatter direction Ψ_2, one defines two basic directions in the plane π_2 (Fig. 3)

$$e_1^{\perp} = \Psi_2 \times e_1 / |\Psi_2 \times e_1| \quad \text{and} \quad e_1^{\parallel} = e_1^{\perp} \times \Psi_2 . \tag{2}$$

Note that $[\Psi_2, e_1^{\parallel}, e_1^{\perp}]$ form a right-handed set of axes, with $e_1^{\perp}$ orthogonal to Ψ_2, e_1, and $e_1^{\parallel}$, which are therefore coplanar. These vectors are also shown in Fig. 4, where the "plane of scattering" Ψ_1, Ψ_2 appears horizontally.

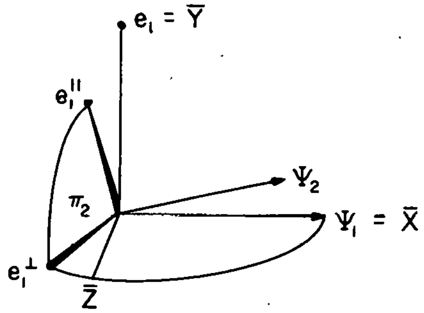

FIG. IV.3

Then, for an arbitrary e_2 in π_2, we may write $e_2 = A + B$, where $A = (e_2 \cdot e_1^{\parallel}) e_1^{\parallel}$ and $B = (e_2, e_1^{\perp}) e_1^{\perp}$ are its components on $e_1^{\parallel}, e_1^{\perp}$ resp. Thus, with η, η' as in Fig. 4, we find[(1)] $\cos^2 \Theta \equiv (e_2 \cdot e_1)^2 = ((A+B) \cdot e_1)^2$ $= (A \cdot e_1)^2 = (e_2 \cdot e_1^{\parallel})^2 (e_1^{\parallel} \cdot e_1)^2 = (e_2 \cdot e_1^{\parallel})^2 \cos^2 \eta' = (e_2 \cdot e_1^{\parallel})^2 \sin^2 \eta$ $= (e_2 \cdot e_1^{\parallel})^2 (1 - \cos^2 \eta)$, so that

$$\cos^2 \Theta = (e_2 \cdot e_1^{\parallel})^2 [1 - (\Psi_2 \cdot e_1)^2] .$$

---

(1) The η, η' relation $(e_1^{\parallel} \cdot e_1)^2 = 1 - (\Psi_2 \cdot e_1)^2$ may be verified vectorially, using the identity $A \cdot (B \times C) = C \cdot (A \times B) = B \cdot (C \times A)$.

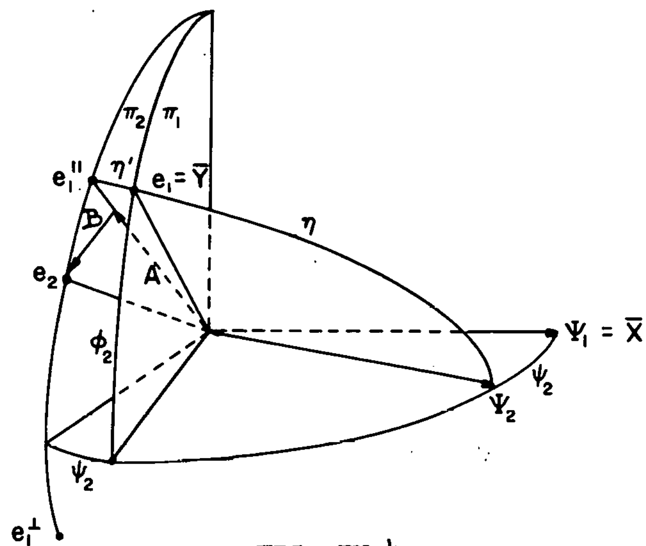

FIG. IV.4

In these terms, (1) becomes

$$\sigma(\Psi_2,e_2)d\Psi_2 = (r^2/4)(k_2/k_1)^2\{K_2-2+4(e_2\cdot e_1^{\parallel})^2[1-(\Psi_2\cdot e_1)^2]\}d\Psi_2. \quad (3)$$

We assume that, for each Ψ_2, e_2 must be exactly one of the two vectors $e_1^{\parallel}$ or $e_1^{\perp}$. From (3), the cross sections for the two corresponding events are

$$\sigma(\Psi_2,e_1^{\parallel})d\Psi_2 = (r^2/4)(k_2/k_1)^2\{K_2+2-4(\Psi_2\cdot e_1)^2\}d\Psi_2$$

$$\geq \sigma(\Psi_2,e_1^{\perp})d\Psi_2 = (r^2/4)(k_2/k_1)^2\{K_2-2\}d\Psi_2. \quad (4)$$

Their sum is the cross section for Ψ_2 scatter,

$$\sigma(\Psi_2)d\Psi_2 = (r^2/2)(k_2/k_1)^2\{K_2-2(\Psi_2\cdot e_1)^2\}d\Psi_2. \quad (5)$$

Introducing spherical coordinates ψ_2,φ_2 relative to the axes $\bar{G}$, with $\bar{X} = \Psi_1$, $\bar{Y} = e_1$, we have (Fig. 2)

$$\Psi_2 = (\cos\psi_2, \sin\psi_2\cos\varphi_2, \sin\psi_2\sin\varphi_2)_{\bar{G}}; \; e_1 = (0,1,0)_{\bar{G}} \quad (6)$$

so that $\Psi_2 \cdot e_1 = \sin \psi_2 \cos \varphi_2.$

For these coordinates, (5) reads

$$\sigma(\Psi_2)d\Psi_2 \equiv \sigma(\psi_2,\varphi_2)d(\cos \psi_2)d\varphi_2$$

$$= (r^2/2)(k_2/k_1)^2\{K_2-2 \sin^2 \psi_2 \cos^2 \varphi_2\}d(\cos \psi_2)d\varphi_2 \qquad (7)$$

where we note the non-uniformity on the "ψ_2-cone."

Integration on $0 \le \varphi_2 < 2\pi$ then gives

$$\sigma(\psi_2)d(\cos \psi_2) = \pi r^2(k_2/k_1)^2\{K_2-\sin^2\psi_2\}d(\cos \psi_2)$$

as the cross section for scattering within $d(\cos \psi_2)$ of the ψ_2-cone about Ψ_1. (Fig. 2.) This is identical with the cross section of Note 22.4, from which one may obtain the "energy" cross section $\tilde{\sigma}(k_2)dk_2$ and the total cross section σ, just as indicated there.[(1)]

2. <u>Simpleminded Monte Carlo</u>. One may follow a single (k_1,Ψ_1,e_1)-photon through a Compton collision as follows:

a. From k_1, one obtains k_2 and $\cos \psi_2$ as in Note 22.4.

b. $\cos \varphi_2$ is then obtained with good efficiency from (7) by the standard rejection technique, applied to the rectangle enclosing the curve

$$f(\varphi_2) = K_2 - 2 \sin^2 \psi_2 \cos^2 \varphi_2, \quad 0 \le \varphi_2 < 2\pi.$$

(See Notes 1-4 for details.)

[(1)]For simplicity, we here use subscript 2 in place of 3.

c. $(\Psi_2)_{\bar{G}}$ is next obtained from (6).

d. By (4) the relative probability of $e_2 = e_1^{\perp}$ is $(K_2-2)/2\{K_2-2 \sin^2 \psi_2 \cos^2 \varphi_2\}$, by which the alternatives $e_2 = e_1^{\perp}$ or $e_2 = e_1^{\parallel}$ are sampled.

e. The $\bar{G}$ coordinates $(e_2)_{\bar{G}}$ of the e_2 selected are obtained from (6) and (2).

f. If $(\Psi_1)_G$ and $(e_1)_G$ are the incident vectors on given Σ axes G, then the rotation δ_1 of AIII, Cor. 2, with matrix

$$D_1 = \left[(\Psi_1)_G, (e_1)_G, (\Psi_1)_G \times (e_1)_G\right]$$

takes the axes G into the axes $\bar{G}$ on which Ψ_2 and e_2 are known from (c), (e). Hence, $(\Psi_2)_G = D_1(\Psi_2)_{\bar{G}}$, $(e_2)_G = D_1(e_2)_{\bar{G}}$ give the direction and e-vector of the scattered photon on the original axes G (AIII, Th. 1(a)).

3. Stokes parameters. If one is not concerned with the e-vectors as such, but only in the successive changes of direction Ψ of a photon beam (upon which its energy depends) a much more ingenious method is available.[1]

Under our basic assumption, those photons of an initial "pure" (k_1, Ψ_1, e_1)-beam which scatter in a common direction Ψ_2 will be of two

[1] The method, described in part (5) below, is apparently due to L. V. Spencer, C. Wolff (Physical Review, 90, 1953, 510-514); the version here is due to G. I. Bell, who discovered it independently.

kinds, having e-vectors $e_1^{\|}$ or $e_1^{\perp}$, in the expected ratios defined by (4). Always following that portion of the beam which scatters in a common direction, we should expect in general (cf. Note 5) after n collisions a composite residual beam with 2^n e-vectors, all orthogonal to the final direction. It seems sufficient therefore to consider monoenergetic (k_1,Ψ_1)-beams with a finite number of discrete e-vectors e_1^i.

The "Stokes parameters" Q_1,U_1 appear here as two numbers, dependent upon the e-vector composition of such a beam, which (a) suffice to determine its probability $P(\Psi_2)d\Psi_2$ of scattering direction, and (b) serve to determine, for that part of the beam which scatters at Ψ_2, the accompanying Stokes parameters Q_2,U_2 required for its next collision. The following argument is intended to make plausible this point of view.

Consider then a composite beam of (k_1,Ψ_1)-photons, S_1^i denoting the fraction $(\Sigma S_1^i = 1)$ having their e_1 vector $e_1^i (i = 1,\ldots,N)$ at angle η_1^i from a specified direction e_1^o ("Stokes vector") in the plane π_1 orthogonal to Ψ_1 (Fig. 5). We now fix attention on a particular direction Ψ_2 of scatter

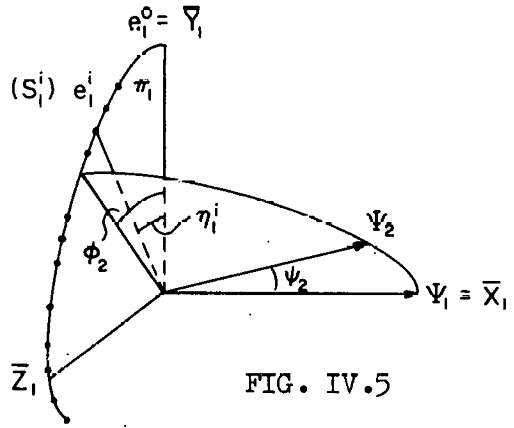

FIG. IV.5

with coordinates

$$\Psi_2 = (\cos\psi_2, \sin\psi_2 \cos\varphi_2, \sin\psi_2 \sin\varphi_2)_{\bar{G}_1} \tag{8}$$

on the axes $\bar{G}_1$ with $\bar{X}_1 \equiv \Psi_1$, and $\bar{Y}_1 = e_1{}^o$, <u>the given Stokes vector</u>. On these axes we have also

$$e_1{}^i = (0, \cos\eta_1{}^i, \sin\eta_1{}^i)_{\bar{G}_1} \tag{8a}$$

whence $$\Psi_2 \cdot e_1{}^i = \sin\psi_2 \cos(\varphi_2 - \eta_1{}^i). \tag{9}$$

For the fixed Ψ_2, and each $e_1{}^i$, there are then two possible resulting e-vectors

$$e_1{}^{i\perp} = \Psi_2 \times e_1{}^i / |\Psi_2 \times e_1{}^i| \quad \text{and} \quad e_1{}^{i\parallel} = e_1{}^{i\perp} \times \Psi_2 \tag{10}$$

for the scattered photon (Fig. 6). Their associated probability density functions are seen from (4) to be

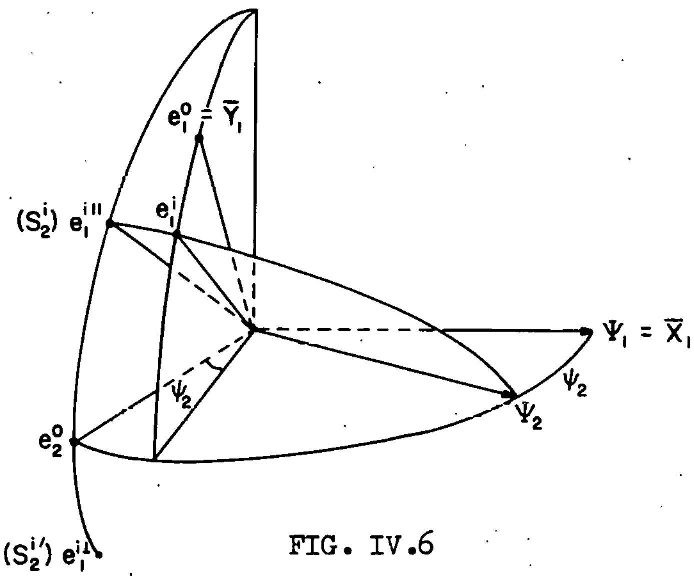

FIG. IV.6

$$p(\Psi_2, e_1^{i\parallel})d\Psi_2 = \{\tfrac{1}{2}K_2+1-2(\Psi_2\cdot e_1^{i})^2\}d\Psi_2/\vartheta_2$$
$$p(\Psi_2, e_1^{i\perp})d\Psi_2 = \{\tfrac{1}{2}K_2-1\}d\Psi_2/\vartheta_2 \qquad (11)$$

where $\vartheta_2 = \sigma/(r^2/2)(k_2/k_1)^2$ is a function of ψ_2.

Hence, the probability of a beam photon having $e_1 = e_1^{i}$, scattering to Ψ_2, and having $e_2 = e_1^{i\parallel}$, or $e_2 = e_1^{i\perp}$, are given by

$$S_1^{i}p(\Psi_2, e_1^{i\parallel})d\Psi_2 \quad \text{and} \quad S_1^{i}p(\Psi_2, e_1^{i\perp})d\Psi_2. \qquad (12)$$

The sum of the two is the probability of a beam photon having $e_1 = e_1^{i}$ and scattering to Ψ_2 (cf. (9)), namely

$$S_1^{i}\{K_2-2(\Psi_2\cdot e_1^{i})^2\}d\Psi_2/\vartheta_2 = S_1^{i}\{K_2-2\sin^2\psi_2\cos^2(\varphi_2-\eta_1^{i})\}d\Psi_2/\vartheta_2$$
$$= S_1^{i}\{K_2-\sin^2\psi_2-\sin^2\psi_2\cos 2(\varphi_2-\eta_1^{i})\}d\Psi_2/\vartheta_2.$$

Summing on i yields the probability of a <u>beam photon</u> scattering to Ψ_2

$$P(\Psi_2)d\Psi_2 = \{K_2-2\sum S_1^{i}(\Psi_2\cdot e_1^{i})^2\}d\Psi_2/\vartheta_2$$
$$= \{K_2-\sin^2\psi_2-\sin^2\psi_2\sum S_1^{i}\cos 2(\varphi_2-\eta_1^{i})\}d\Psi_2/\vartheta_2. \qquad (13)$$

Therefore we may write

$$P(\Psi_2)d\Psi_2 = \rho(\Psi_2)d\Psi_2/\vartheta_2 \qquad (14)$$

where $\rho(\Psi_2) = K_2 - \sin^2\psi_2 - \sin^2\psi_2 Q(\varphi_2)$ (15)

$$Q(\varphi_2) = Q_1\cos 2\varphi_2 + U_1\sin 2\varphi_2 \qquad (16)$$

and $Q_1 = \sum S_1^{i}\cos 2\eta_1^{i}, \quad U_1 = \sum S_1^{i}\sin 2\eta_1^{i}$ (17)

are the "Stokes parameters" serving alone to determine $P(\Psi_2)d\Psi_2$ for the composite beam. For later use, we also define here

$$U(\varphi_2) = -Q_1 \sin 2\varphi_2 + U_1 \cos 2\varphi_2. \tag{18}$$

4. Stokes parameters of the scattered beam. We now regard the $(\Psi_2, \Psi_2+d\Psi_2)$ scattered beam as a new source. Of these photons, the fractions having $e_2 = e_1^{i\parallel}$ and $e_2 = e_1^{i\perp}$ are, by (11), (12), and (14)

$$\begin{aligned} S_2^{\,i} &= (S_1^{\,i}/\rho(\Psi_2))\{\tfrac{1}{2}K_2+1-2(\Psi_2\cdot e_1^{\,i})^2\} \\ S_2^{\,i'} &= (S_1^{\,i}/\rho(\Psi_2))\{\tfrac{1}{2}K_2-1\} \end{aligned} \tag{19}$$

where $\Sigma(S_2^{\,i}+S_2^{\,i'}) = 1$.

Now the argument establishing (14-17) was quite general, and we may interpret it for the next scattering once we have referred the vectors e_2 to a basic Stokes vector $e_2^{\,o}$ in their plane. This we take to be the vector (Fig. 6)

$$\begin{aligned} e_2^{\,o} &= (\cos(\psi_2+90^o),\ \sin(\psi_2+90^o)\cos\varphi_2,\ \sin(\psi_2+90^o)\sin\varphi_2)_{\bar{G}_1} \\ &= (-\sin\psi_2,\ \cos\psi_2\cos\varphi_2,\ \cos\psi_2\sin\varphi_2)_{\bar{G}_1} \end{aligned} \tag{20}$$

on the axes $\bar{G}_1$!

Accordingly, we define $\eta_2^{\,i}$ and $\eta_2^{\,i'} = \eta_2^{\,i} + 90^o$ as the angles from $e_2^{\,o}$ to the vectors $e_1^{i\parallel}$ and $e_1^{i\perp}$ resp. The new source then appears as in Fig. 7, which is the exact analogue of Fig. 5.

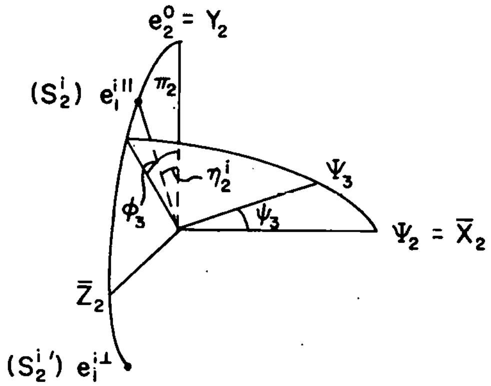

FIG. IV.7

It is clear that we may write at once the probability for Ψ_3 scatter of the new source in the forms of (14-17):

$$\tilde{P}(\Psi_3)d\Psi_3 = \tilde{\rho}(\Psi_3)d\Psi_3/\vartheta_3 \tag{21}$$

with $$\tilde{\rho}(\Psi_3) = K_3 - \sin^2\psi_3 - \sin^2\psi_3\tilde{Q}(\varphi_3) \tag{22}$$

$$\tilde{Q}(\varphi_3) = \tilde{Q}_2 \cos 2\varphi_3 + \tilde{U}_2 \sin 2\varphi_3 \tag{23}$$

where now $\tilde{Q}_2 = \Sigma S_2^{\,i} \cos 2\eta_2^{\,i} + \Sigma S_2^{\,i'} \cos 2\eta_2^{\,i'} = \Sigma(S_2^{\,i'} - S_2^{\,i})\cos 2\eta_2^{\,i'}$ (24)

and similarly $\tilde{U}_2 = \Sigma(S_2^{\,i'} - S_2^{\,i})\sin 2\eta_2^{\,i'}$.

These are the Stokes parameters for the new beam, determining the probability $\tilde{P}(\Psi_3)$ for the next scattering. From (19) we see that

$$S_2^{\,i'} - S_2^{\,i} = -\left(2S_1^{\,i}/\rho(\Psi_2)\right)\left[1-\left(\Psi_2\cdot e_1^{\,i}\right)^2\right] \equiv -2S_1^{\,i}R_2^{\,i}/\rho(\Psi_2)$$

where we have set $R_2^{\,i} = 1 - (\Psi_2\cdot e_1^{\,i})^2 \equiv 1 - \sin^2\psi_2 \cos^2(\varphi_2 - \eta_1^{\,i})$. (25)

Thus from (24) we have, for the new Stokes parameters,

$$\tilde{Q}_2 = -(2/\rho(\Psi_2))\Sigma S_1^{\,i}R_2^{\,i} \cos 2\eta_2^{\,i'},\ \tilde{U}_2 = -(2/\rho(\Psi_2))\Sigma S_1^{\,i}R_2^{\,i}\sin 2\eta_2^{\,i'}. \tag{26}$$

It remains to verify their relation to the previous ones Q_1, U_1, namely,

$$\tilde{Q}_2 = \{-\sin^2\psi_2 + (1+\cos^2\psi_2)Q(\varphi_2)\}/\rho(\Psi_2)$$
$$\tilde{U}_2 = 2\cos\psi_2 U(\varphi_2)/\rho(\Psi_2) \tag{27}$$

where the functions indicated are those in (15), (16), and (18), depending only on ψ_2,φ_2 and Q_1,U_1.

To do this we must relate $\cos\eta_2^{i'} \equiv e_2^{o}\cdot e_1^{i\perp}$ to $\cos\eta_1^{i} \equiv e_1^{o}\cdot e_1^{i}$. Now $e_1^{i\perp} \equiv \Psi_2 x e_1^{i}/|\Psi_2 x e_1^{i}|$, where $|\Psi_2 x e_1^{i}| = \left(1-(\Psi_2\cdot e_1^{i})^2\right)^{\frac{1}{2}} = (R_2^{i})^{\frac{1}{2}}$ as in (25). Hence[1] $(R_2^{i})^{\frac{1}{2}}\cos\eta_2^{i'} = e_2^{o}\cdot(\Psi_2 x e_1^{i}) \equiv e_1^{i}\cdot(e_2^{o} x \Psi_2)$, and it is easily verified from (8) and (20) that $e_2^{o} \times \Psi_2 = (0,\sin\varphi_2,-\cos\varphi_2)_{\overline{G}_1}$. From (8a) therefore we have $e_1^{i}\cdot(e_2^{o} x\Psi_2) = \sin(\varphi_2-\eta_1^{i})$. The required relation is then

$$\cos\eta_2^{i'} = \sin(\varphi_2-\eta_1^{i})/(R_2^{i})^{\frac{1}{2}} \tag{28}$$

whence $\cos 2\eta_2^{i'} = 2\cos^2\eta_2^{i'} - 1 = (R_2^{i})^{-1}\{2\sin^2(\varphi_2-\eta_1^{i}) - 1 + \sin^2\psi_2\cos^2(\varphi_2-\eta_1^{i})\} = (R_2^{i})^{-1}\{1-(1+\cos^2\psi_2)\cos^2(\varphi_2-\eta_1^{i})\}$.

From (26) then, $\rho(\Psi_2)\tilde{Q}_2 = \Sigma S_1^{i}\{-2+(1+\cos^2\psi_2)[1+\cos 2(\varphi_2-\eta_1^{i})]\}$

$= -\sin^2\psi_2 + (1+\cos^2\psi_2)[(\Sigma S_1^{i}\cos 2\eta_1^{i})\cos 2\varphi_2 + (\Sigma S_1^{i}\sin 2\eta_1^{i})\sin 2\varphi_2]$

$= -\sin^2\psi_2 + (1+\cos^2\psi_2)[Q_1\cos 2\varphi_2 + U_1\sin 2\varphi_2] = -\sin^2\psi_2 + (1+\cos^2\psi_2)Q(\varphi_2)$

as claimed in (27).

---

(1) $A\cdot(B x C) \equiv C\cdot(A x B)$ is vector identity.

For $\tilde{U}_2$ we shall require

$$\sin \eta_2^{i'} = \sin(\eta_2^{i}+90^{o}) = \cos \eta_2^{i} \equiv e_2^{o}\cdot e_1^{i\parallel} = e_2^{o}\cdot(e_1^{i\perp}\times\Psi_2)$$

$$= \Psi_2\cdot(e_2^{o}\times e_1^{i\perp}) = e_1^{i\perp}\cdot(\Psi_2\times e_2^{o}) = (\Psi_2\times e_1^{i})\cdot(\Psi_2\times e_2^{o})/(R_2^{i})^{\frac{1}{2}}$$

$$= (e_1^{i}\times\Psi_2)\cdot(e_2^{o}\times\Psi_2)/(R_2^{i})^{\frac{1}{2}}.$$

We have just seen that $e_2^{o}\times\Psi_2 = (0, \sin\varphi_2, -\cos\varphi_2)_{\bar{\alpha}_1}$, while reference to (8) and (8a) shows that $\Psi_2\times e_1^{i} = (*, \cos\psi_2 \sin\eta_1^{i}, -\cos\psi_2 \cos\eta_1^{i})$.

Hence

$$\sin \eta_2^{i'} = \cos\psi_2 \cos(\varphi_2-\eta_1^{i})/(R_2^{i})^{\frac{1}{2}}. \qquad (29)$$

From (28), (29) then

$$\sin 2\eta_2^{i'} = 2 \sin \eta_2^{i'} \cos \eta_2^{i'} = 2(R_2^{i})^{-1} \cos\psi_2 \cos(\varphi_2-\eta_1^{i})\sin(\varphi_2-\eta_1^{i})$$

$$= (R_2^{i})^{-1} \cos\psi_2 \sin 2(\varphi_2-\eta_1^{i}).$$

Turning to (26), we have finally

$$\rho(\Psi_2)\tilde{U}_2 = -2\sum S_1^{i}R_2^{i} \sin 2\eta_2^{i'} = -2\cos\psi_2 \sum S_1^{i} \sin 2(\varphi_2-\eta_1^{i})$$

$$= 2\cos\psi_2\{-(\sum S_1^{i}\cos 2\eta_1^{i})\sin 2\varphi_2 + (\sum S_1^{i}\sin 2\eta_1^{i})\cos 2\varphi_2\}$$

$$= 2\cos\psi_2\{-Q_1\sin 2\varphi_2+U_1\cos 2\varphi_2\} \equiv 2\cos\psi_2 U(\varphi_2)$$

as in (27).

5. Stokes method. One follows a beam of (k_1,Ψ_1)-photons through a Compton collision as follows:

Initially, one must assume known its e-vector composition, and compute Q_1,U_1 by (17). For a beam initially "pure" (i=1) one may take $Q_1 = 1$, $U_1 = 0$. At a given collision we suppose known $k_1,(\Psi_1)_G$, the Stokes vector $(e_1^{\,o})_G$, and the current Stokes parameters Q_1,U_1 of the beam.

(a) One obtains k_2, $\cos\psi_2$ as in (a) of part (2) above.

(b) φ_2 is then obtained from (15) by rejection technique, applied to the rectangle enclosing the curve $\rho(\varphi_2) = K_2 - \sin^2\psi_2 - \sin^2\psi_2 Q(\varphi_2)$

where $\qquad Q(\varphi_2) = Q_1\cos 2\varphi_2 + U_1\sin 2\varphi_2 \qquad$ (Notes 1-4).

(c) $(\Psi_2)_{\bar{G}_1}$ and $(e_2^{\,o})_{\bar{G}_1}$ are obtained from (8) and (20), on axes $\bar{G}_1$!

(d) New Stokes parameters are computed from (27).

(e) The new direction and Stokes vector on the original axes G are $(\Psi_2)_G = D_1(\Psi_2)_{\bar{G}_1}$, $(e_2^{\,o})_G = D_1(e_2^{\,o})_{\bar{G}_1}$, where

$$D_1 = \left[(\Psi_1)_G,(e_1^{\,o})_G,(\Psi_1)_G\times(e_1^{\,o})_G\right].$$

Notes IV.

1. "Rejection technique." Let M be the maximum of a probability density function for x on its domain (a,b), with d = b - a. Then a correctly distributed x is obtained by successively "throwing" pairs of

random numbers r, r' on $(0,1)$ and accepting the first $x = a+rd$ for which $p(x) \geqq Mr'$. The "efficiency" of the method is $e = \int_a^b p(x)dx/Md$. If m is the minimum of $p(x)$, it is trivial that $e \geqq \int_a^b (p(x)-m)dx/(M-m)d$. These remarks apply equally well to any function $f(x) = kp(x)$, $k > 0$.

2. For the function $f(\varphi_2) = K_2 - 2\sin^2\psi_2 \cos^2\varphi_2$ on $(0,2\pi)$, we have $M = K_2$, $m = K_2 - 2\sin^2\psi_2$, $d = 2\pi$, and an efficiency $e \geqq \int_0^{2\pi}(1-\cos^2\varphi_2)d\varphi_2/2\pi = \int_0^{2\pi}\sin^2\varphi_2 d\varphi_2/2\pi = 1/2$, with the above method.

3. For the function $\rho(\varphi_2) = K_2 - \sin^2\psi_2 - \sin^2\psi_2(Q_1\cos 2\varphi_2 + U_1\sin 2\varphi_2)$ on $(0,2\pi)$ we may write $\rho(\varphi_2) = A - B\cos 2(\varphi_2 - \varphi_0)$, where $A = K_2 - \sin^2\psi_2$ and $B = \sin^2\psi_2(Q_1^2 + U_1^2)^{\frac{1}{2}}$, the meaning of φ_0 being obvious. We see from this form that $M = A+B$, $m = A-B$, the rejection technique again having $e \geqq 1/2$.

4. In applying the method of Note 1 to the functions $f(\varphi_2), \rho(\varphi_2)$ in steps (b) of parts (2), (5) above, it may be noted that, instead of "throwing" $\varphi_2 = 2\pi r$ and then computing the required functions $\cos^2\varphi_2$ or $\cos 2\varphi_2$, $\sin 2\varphi_2$ one may use von Neumann's device for "throwing" directly for the $\cos\varphi_2$, $\sin\varphi_2$ of a φ_2 uniformly distributed on $(0,2\pi)$, and then computing $\cos^2\varphi_2$ or $\cos 2\varphi_2 (= \cos^2\varphi_2 - \sin^2\varphi_2)$, $\sin 2\varphi_2 (= 2\sin\varphi_2\cos\varphi_2)$. The device referred to (itself a rejection method) goes as follows: Of a sequence of random number pairs (r, r') one accepts the first for which $s \equiv r^2 + (r')^2 \leqq 1$, and defines $\cos\varphi_2 = (r^2 - (r')^2)/s$, and $\sin\varphi_2 = \pm 2rr'/s$, each sign having probability 1/2. The "efficiency" here is exactly $\pi/4$.

If the device is used in conjunction with the methods of Notes 2, 3, the overall efficiency is of course reduced by this factor.

5. The argument of parts (3,4) does not require that the e-vectors be distinct, e.g., in one 90° scattering, all e-vectors present collapse to two.

TABLE I

Some physical constants

c	2.997925	x 10^{10}	cm sec^{-1}	speed limit
h	6.62554	x 10^{-27}	erg sec	Planck's Constant
q	4.80296	x 10^{-10}	esu	charge quantum
m_ϵ	9.10904	x 10^{-28}	gm	electron rest mass
e_ϵ	.5110058		Mev	electron rest energy
G	6.670	x 10^{-8}	gm^{-1} cm^3 sec^{-2}	gravitation constant
H	3.24	x 10^{-18}	sec^{-1} = 100 Km sec^{-1}/Mpc	Hubble's constant
k_B	1.38053	x 10^{-16}	erg/°K	Boltzmann constant
F	2.892616	x 10^{14}	esu = 96487.27 C	Faraday (^{12}C)
N_o	6.02257	x 10^{23}		Avogadro's number = F/q (^{12}C)
R	8.3143	x 10^7	erg/°K	Gas constant = $N_o k_B$ (^{12}C)
$\hbar$	1.05449	x 10^{-27}	erg sec	Angular momentum unit = $h/2\pi$
a_o	.5291659	x 10^{-8}	cm	1st Bohr radius = $\hbar^2/m_\epsilon q^2$
r_ϵ	2.81776	x 10^{-13}	cm	electron "radius" = $q^2/m_\epsilon c^2$
λ_ϵ	.02426206	x 10^{-8}	cm	Compton ε-wave-length = $h/m_\epsilon c$
μ_B	9.27314	x 10^{-21}	erg/gauss	Bohr magneton = $q\hbar/2m_\epsilon c$

Units

f	10^{-13} cm (fermi)
Å	10^{-8} cm (Ångstrom)
AU	1.49598 x 10^{13} cm (Astron. unit)
LY	9.460 x 10^{17} cm (light year)
pc	3.0856 x 10^{18} cm = 3.262 LY (parsec)
Mpc	10^6 pc (Megaparsec)
Y	3.16 x 10^7 sec (year)
C	c/10 esu (coulomb)
V	10^8/c esu volt (Volt)
Mev	q x 10^{14}/c = 1.602095 x 10^{-6} erg
amu	931.476 Mev (^{12}C atomic mass unit)

TABLE II

A few "elementary" particles

		e (Mev)	approx. m/m_ϵ			Mean life (sec)	Main mode
Baryons $J = \frac{1}{2}$	Hyperons	1320.8	2585	Ξ^-	Ξ^+	1.3×10^{-10}	$\Xi^- \to \Lambda^0 + \pi^-$
		1314.3	2572	Ξ^0	$\overline{\Xi^0}$	10^{-10}	$\Xi^0 \to \Lambda^0 + \pi^0$
		1197.08	2343	Σ^-	$\overline{\Sigma^-}$	1.6×10^{-10}	$\Sigma^- \to n^0 + \pi^-$
		1192.4	2333	Σ^0	$\overline{\Sigma^0}$	$< 10^{-14}$	$\Sigma^0 \to \Lambda^0 + \gamma$
		1189.41	2328	Σ^+	$\overline{\Sigma^+}$	$.79 \times 10^{-10}$	$\Sigma^+ \to n^0 + \pi^+$
		1115.40	2183	Λ^0	$\overline{\Lambda^0}$	2.6×10^{-10}	$\Lambda^0 \to p^+ + \pi^-$
	Nucleons	939.550	1839	n^0	$\overline{n^0}$	1013	$n^0 \to p^+ + \epsilon^- + \bar{\nu}_\epsilon$
		938.256	1836	p^+	p^-	∞	
Mesons $J = 0$		498.0	975	K_1^0	$\overline{K_1^0}$	$.91 \times 10^{-10}$	$K_1^0 \to \pi^- + \pi^+$
		493.8	966	K^+	K^-	1.2×10^{-8}	$K^+ \to \mu^+ + \nu_\mu$
		139.60	273	π^-	π^+	2.6×10^{-8}	$\pi^- \to \mu^- + \bar{\nu}_\mu$
		135.01	264	π^0		1.8×10^{-16}	$\pi^0 \to \gamma + \gamma$
Leptons $J = \frac{1}{2}$	Muon family	105.659	207	μ^-	μ^+	2.2×10^{-6}	$\mu^- \to \epsilon^- + \bar{\nu}_\epsilon + \nu_\mu$
		0	0	ν_μ	$\bar{\nu}_\mu$	∞	
	Electron family	.511006	1	ϵ^-	ϵ^+	∞	
		0	0	ν_ϵ	$\bar{\nu}_\epsilon$	∞	
Photon $J = 1$		0	0	γ		∞	

Note. The numbers are of course obsolete, but serve well enough for various examples.

TABLE III

Some neutral atom rest masses

	amu	half life
n^{o}	1.008 6654	12.8 m (ϵ^-)
$\left[{}^{1}_{1}H\right]$	1.007 82522	
$\left[{}^{2}_{1}H\right]$	2.014 1022	
$\left[{}^{3}_{1}H\right]$	3.016 0494	12.26 Y (ϵ^-)
$\left[{}^{3}_{2}He\right]$	3.016 0299	
$\left[{}^{4}_{2}He\right]$	4.002 6036	
$\left[{}^{7}_{3}Li\right]$	7.016 005	
$\left[{}^{7}_{4}Be\right]$	7.016 931	53.6 d (ϵ^- cap)
$\left[{}^{9}_{4}Be\right]$	9.012 186	
$\left[{}^{10}_{4}Be\right]$	10.013 535	2.5 x 10^6 Y (ϵ^-)
$\left[{}^{11}_{5}B\right]$	11.009 3051	
$\left[{}^{11}_{6}C\right]$	11.011 433	20.4 m (ϵ^+)
$\left[{}^{12}_{6}C\right]$	12*	
$\left[{}^{13}_{6}C\right]$	13.003 354	
$\left[{}^{14}_{6}C\right]$	14.003 2419	5720 Y (ϵ^-)
$\left[{}^{14}_{7}N\right]$	14.003 0744	
$\left[{}^{16}_{8}O\right]$	15.994 9149	
$\left[{}^{17}_{8}O\right]$	16.999 133	
$\left[{}^{18}_{9}F\right]$	18.000 950	110 m (ϵ^+)
$\left[{}^{27}_{13}Al\right]$	26.981 535	
$\left[{}^{30}_{15}P\right]$	29.978 32	2.5 m (ϵ^+)
$\left[{}^{235}_{92}U\right]$	235.043 93	7.13 x 10^8 Y (α)
$\left[{}^{236}_{92}U\right]$	236.045 73	2.4 x 10^7 Y (α)

*${}^{12}_{6}C$ scale: 1 amu = 931.476 Mev

TABLE IV

Vector forms of the transformations

$$\delta_o \equiv \gamma_o - 1$$

(L)	$R = R' + \{\delta_o R'\cdot\Psi_o + \gamma_o u_o t'\}\Psi_o$	$t = \gamma_o\{u_o c^{-2} R'\cdot\Psi_o + t'\}$	N2.2
(V)	$V = (\gamma_o d')^{-1}[V' + \{\delta_o V'\cdot\Psi_o + \gamma_o u_o\}\Psi_o]$	$d' = \{u_o c^{-2} V'\cdot\Psi_o + 1\}$	N3.1
(Ψ)	$\Psi = (\gamma_o D')^{-1}[\Psi' + \{\delta_o \Psi'\cdot\Psi_o + \gamma_o \rho'\}\Psi_o]$	$D' = \{(\Psi'\cdot\Psi_o + \rho')^2 + \gamma_o^{-2}(1 - (\Psi'\cdot\Psi_o)^2)\}^{\frac{1}{2}}$	N3.2
		$\rho' = u_o/v'$	
(PM)	$P = P' + \{\delta_o P'\cdot\Psi_o + \gamma_o u_o M'\}\Psi_o$	$M = \gamma_o\{u_o c^{-2} P'\cdot\Psi_o + M'\}$	N4.1
(cPE)	$cP = cP' + \{\delta_o cP'\cdot\Psi_o + \gamma_o \beta_o E'\}\Psi_o$	$E = \gamma_o\{\beta_o cP'\cdot\Psi_o + E'\}$	N4.1
(F)	$F = (\gamma_o d')^{-1}[F' + \{\delta_o F'\cdot\Psi_o + \gamma_o \beta_o c^{-1} F'\cdot V'\}\Psi_o]$		N4.2
(M)	$M = M'\gamma_o d'$	(E) $E = E'\gamma_o d'$	§4
(d)	$(\gamma_o d)(\gamma_o d') = 1$		§3