SAND--96-2367C

CONF-970146--3

Use of Quadratic Components for Buckling Calculations[1]

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Abstract

A buckling calculation procedure based on the method of quadratic components is presented. Recently developed for simulating the motion of rotating flexible structures, the method of quadratic components is shown to be applicable to buckling problems with either conservative or nonconservative loads. For conservative loads, stability follows from the positive definiteness of the system's stiffness matrix. For nonconservative loads, stability is determined by solving a nonsymmetric eigenvalue problem, which depends on both the stiffness and mass distribution of the system. Buckling calculations presented for a cantilevered beam are shown to compare favorably with classical results. Although the example problem is fairly simple and well-understood, the procedure can be used in conjunction with a general-purpose finite element code for buckling calculations of more complex systems.

1. Introduction

The elastic stability of structures has been a major focus area of mechanics for many years. There are several standard texts on buckling including, among others, those of Timoshenko [1], Bolotin [2], and Brush and Almroth [3]. These texts include classical results useful for determining the buckling loads of a variety of structures with simple geometries. General-purpose finite element codes such as MSC/NASTRAN [4] and ABAQUS [5] provide one with computational methods to solve certain classes of buckling problems not amenable to closed-form solution.

Recently, the method of quadratic components [6] was developed to simulate the motion of rotating flexible structures. As the name implies, the method expresses the deformation

[1]This work, performed at Sandia National Laboratories, was supported by the U.S. Department of Energy under contract DE–AC04–94AL85000.

of a structure as a quadratic function of a set of generalized coordinates. As an illustration, consider an inextensible cantilevered beam subjected to a transverse load at its tip. The linear part of the response consists of the transverse deformations while the quadratic part consists of the axial deformations. The second order axial deformations are required in order to satisfy the inextensibility constraint.

It turns out that second order terms in the quadratic component formulation appear as first order terms in the equations of motion if the structure undergoes significant rigid body motions. One can easily verify this fact for the cantilevered beam when its base rotates about an axis perpendicular to the transverse and axial directions. By including the second order axial deformations in the kinematics, one is able to correctly predict the spin stiffening effect. In a similar manner, the method of quadratic components can be used to model the effects of applied loads on the stiffness of a structure. In fact, the geometric stiffness matrix appears naturally through the use of quadratic components. Thus, the method provides a means to calculate buckling loads.

In the following section, the method of quadratic components is introduced and used as the basis for a buckling calculation procedure. In the third section, this procedure is applied to a cantilevered beam subjected to compressive loads of both the fixed-direction and follower types. For fixed-direction loads, the stiffness matrix of the beam is a linear function of the load and is symmetric. Thus, the buckling load can be determined by solving a symmetric eigenvalue problem. For the case of the follower-type load, it is shown that the stiffness matrix is a linear function of the load, but the matrix is nonsymmetric. In this case, the stability of the beam depends on the mass distribution as well as the stiffness of the beam. Buckling calculations obtained using the new procedure are compared with classical results.

2. Quadratic Components and Buckling

In this section, a procedure is developed for buckling calculations based on the method of quadratic components [6]. Under conditions of static equilibrium, the displacement field U of a structure subjected to an applied force field F can be expressed as

$$U = N(F) \tag{1}$$

where N is a nonlinear operator mapping F to U. With the method of quadratic components, the force field is expressed as a superposition of basis force fields:

$$F = s_i F^i \tag{2}$$

where each field F^i is time-independent and summation is performed over repeated indices. The index i is assumed to have values from 1 to n. Appropriate bases of force fields can be selected to reflect either static or modal-like responses. As is shown in the example problem, the basis force fields are not necessarily associated with the actual loading in the problem. The basis forces simply serve as generators for the nonlinear space of displacement configurations.

Expanding the nonlinear operator N as a Taylor series through quadratic terms and neglecting the higher-order terms yields

$$U(\{s_i\}) = s_i U^i + s_i s_j G^{ij} \tag{3}$$

where U^i and G^{ij} represent the linear and quadratic parts of the displacement field. Evaluating the displacement field given by Eq. (3) at the material point $\mathbf{x}$ and allowing the generalized coordinates $\{s_i\}$ to vary with time yields

$$\mathbf{u}(\mathbf{x}, t) = s_i(t)\mathbf{u}^i(\mathbf{x}) + s_i(t)s_j(t)\mathbf{g}^{ij}(\mathbf{x}) \tag{4}$$

One can show that the symmetry $\mathbf{g}^{ij}(\mathbf{x}) = \mathbf{g}^{ji}(\mathbf{x})$ holds.

For purposes of buckling calculations, the strain energy V and kinetic energy T of the system can be expressed in matrix notation as quadratic functions of the generalized coordinates:

$$V = \frac{1}{2}s^T K s \tag{5}$$

$$T = \frac{1}{2}\dot{s}^T M \dot{s} \tag{6}$$

where the overdot in Eq. (6) denotes the time derivative and

$$s = \begin{bmatrix} s_1(t) & \cdots & s_n(t) \end{bmatrix}^T \tag{7}$$

The elements in row i and column j of the matrices K and M are given by

$$k_{ij} = \int \mathbf{f}^i(\mathbf{x}) \cdot \mathbf{u}^j(\mathbf{x}) dV \tag{8}$$

$$m_{ij} = \int \rho(\mathbf{x})\mathbf{u}^i(\mathbf{x}) \cdot \mathbf{u}^j(\mathbf{x}) dV \tag{9}$$

where ρ is the mass density and $\mathbf{f}^i(\mathbf{x})$ is the basis force field F^i evaluated at $\mathbf{x}$. The integrals in Eqs. (8–9) are evaluated over the volume of the structure.

The load used for buckling calculations is assumed to be of the form

$$\mathbf{r}(\mathbf{x}, \mathbf{u}, p) = p\hat{\mathbf{r}}(\mathbf{x}, \mathbf{u}) \tag{10}$$

where p is a parameter used to scale the magnitude of the load applied to the structure. The dependence of $\mathbf{r}$ on the deformation field allows for the consideration of follower-type loads. Since $\mathbf{u}(\mathbf{x})$ is a function of the generalized coordinates, the right hand side of Eq. (10) can be expressed as a Taylor series in s as

$$\mathbf{r}(\mathbf{x}, s, p) = p[\hat{\mathbf{r}}_0(\mathbf{x}) + \mathbf{a}^i(\mathbf{x})s_i] + \mathcal{O}(s^2) \tag{11}$$

where $\mathcal{O}(s^2)$ denotes quadratic and higher order terms of the generalized coordinates.

The virtual work of the load is given by

$$\delta W = \int \mathbf{r}(\mathbf{x}, s, p) \cdot \delta \mathbf{u}(\mathbf{x}, t) dV \tag{12}$$

where δ is the variational symbol. Substituting Eqs. (4) and (11) into Eq. (12) and neglecting quadratic and higher order terms yields

$$\delta W = \delta s^T (b + pHs) \tag{13}$$

where the elements of the vector b and matrix H are given by

$$b_i = p \int \hat{\mathbf{r}}_0(\mathbf{x}) \cdot \mathbf{u}^i(\mathbf{x}) dV \tag{14}$$

$$h_{ij} = \int [2\hat{\mathbf{r}}_0(\mathbf{x}) \cdot \mathbf{g}^{ij}(\mathbf{x}) + \mathbf{u}^i(\mathbf{x}) \cdot \mathbf{a}^j(\mathbf{x})] dV \tag{15}$$

In the present study, damping mechanisms within the structure are neglected. Thus, Hamilton's principle is written as

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt = 0 \tag{16}$$

Substituting Eqs. (5–6) and (13) into Eq. (16), integrating by parts, and setting the coefficient of δs equal to zero yields the following equations of motion

$$M\ddot{s} + (K - pH)s = b \tag{17}$$

Neglecting the right hand side of Eq. (17) and assuming a solution of the form

$$s = e^{ct}\phi \tag{18}$$

yields the generalized eigenvalue problem

$$[(K - pH) - \mu M]\phi = 0 \tag{19}$$

where $\mu = -c^2$.

The system is stable provided that all the eigenvalues μ are real and greater than zero. Otherwise, there would exist a value of c with a positive real part. Thus, one can determine the stability of the system for any value of the load parameter p by examining the eigenvalues of Eq. (19).

For cases in which the matrix $K - pH$ is symmetric, the value of p associated with buckling can be determined by solving the generalized eigenvalue problem

$$(K - pH)\phi = 0 \tag{20}$$

where p is now considered as an eigenvalue. The lowest eigenvalue of Eq. (20) is the critical value of p for buckling. Symmetry of the matrix $K - pH$ is associated with conservative loads while nonsymmetry of this matrix connotes a nonconservative loading.

The buckling calculation procedure is summarized as follows:

1. Select a set of basis forces $\mathbf{f}^i(\mathbf{x})$.
2. Obtain expressions for the terms $\mathbf{u}^i(\mathbf{x})$ and $\mathbf{g}^{ij}(\mathbf{x})$ (see Eq. 4).
3. Obtain expressions for the matrices K, M and H (see Eqs. 8–9 and Eq. 15).
4. If the matrix $K - pH$ is symmetric, the critical value of the load parameter is the smallest eigenvalue of Eq. (20).
5. If the matrix $K - pH$ is nonsymmetric, the critical value of the load parameter is the smallest value of p for which at least one of the eigenvalues of Eq. (19) is not positive and real.

An appealing feature of this procedure is that it can be applied to a wide variety of structures by making use of existing finite element codes to aid in Steps 1 through 3. Guidelines on the use of such codes for this purpose are provided in Ref. [7]. While the same code used to aid in Steps 1 through 3 would likely have a buckling calculation capability, this capability may be limited to problems in which the H matrix is symmetric [5].

3. Example

As an example application of the buckling calculation procedure, consider the cantilevered beam shown in Figure 1. The beam is inextensible and has uniform mass and stiffness distribution along its length. Deformations are restricted to the $\mathbf{n}_1 - \mathbf{n}_2$ plane and the load is applied at the beam tip. The beam length, bending stiffness and mass per unit length are denoted by L, EI and $\hat{m}$, respectively.

Two load cases are considered. For Case 1, the direction of the load remains in the negative $\mathbf{n}_1$-direction. For Case 2, the load follows the rotation of the beam and remains parallel the neutral axis at the tip. The latter case is an example of a follower load and is nonconservative.

Step 1: The basis forces are chosen as those associated with the eigenmodes of the beam.

Step 2: The terms $\mathbf{u}^i(\mathbf{x})$ are the linear eigenmodes of the system and given by [8]

$$\mathbf{u}^i(x) = \left\{\cosh\frac{\lambda_i x}{L} - \cos\frac{\lambda_i x}{L} - \sigma_i\left[\sinh\frac{\lambda_i x}{L} - \sin\frac{\lambda_i x}{L}\right]\right\}\mathbf{n}_2 \tag{21}$$

where the coefficients λ_i and σ_i for $i = 1, \ldots, 5$ are given in Table 1. The terms $\mathbf{g}^{ij}(\mathbf{x})$ are determined by the constraint that the beam is inextensible. Thus,

$$\mathbf{g}^{ij}(x) = \left\{-\frac{1}{2}\int_0^x \frac{d\mathbf{u}^i(\tau)}{d\tau}\cdot\frac{d\mathbf{u}^j(\tau)}{d\tau}d\tau\right\}\mathbf{n}_1 = Lg^{ij}(x)\mathbf{n}_1 \tag{22}$$

Substituting Eq. (21) into Eq. (22), setting x equal to L, and using the integration tables of Ref. [8], one obtains

$$g^{ij}(L) = \begin{cases} -\frac{1}{2}[\sigma_i\lambda_i(2+\sigma_i\lambda_i)] & i = j \\ \frac{-2\lambda_i\lambda_j}{\lambda_i^4-\lambda_j^4}[(-1)^{i+j}(\sigma_j\lambda_i^3 - \sigma_i\lambda_j^3) - \lambda_i\lambda_j(\sigma_i\lambda_i - \sigma_j\lambda_j)] & i \neq j \end{cases} \tag{23}$$

Step 3: The elements of the stiffness and mass matrices are given by

$$k_{ij} = EI\int_0^L \left[\frac{d^2\mathbf{u}^i}{dx^2}\cdot\frac{d^2\mathbf{u}^j}{dx^2}\right]dx \tag{24}$$

$$m_{ij} = \hat{m}\int_0^L \left[\mathbf{u}^i\cdot\mathbf{u}^j\right]dx \tag{25}$$

Performing the integrations in Eqs. (24–25), one obtains

$$K = \frac{EI}{L}diag(\lambda_1^4, \ldots, \lambda_n^4) \tag{26}$$

$$M = \hat{m}L^3 I_n \tag{27}$$

where *diag* denotes a diagonal matrix and I_n is the identity matrix of dimension n.

Since the beam is a one-dimensional, Eq. (15) simplifies to

$$h_{ij} = \int_0^L [2\hat{\mathbf{r}}_0(x) \cdot \mathbf{g}^{ij}(x) + \mathbf{u}^i(x) \cdot \mathbf{a}^j(x)]dx \tag{28}$$

For both loading cases, the term $\hat{\mathbf{r}}_0$ is given by

$$\hat{\mathbf{r}}_0(x) = -\delta(x - L)\mathbf{n}_1 \tag{29}$$

where δ is the Dirac delta function. The term $\mathbf{a}^j(x)$ is equal to zero for Case 1. For Case 2,

$$\mathbf{a}^j(L) = -2\sigma_j\lambda_j(-1)^{j+1}\delta(x - L)\mathbf{n}_2 \tag{30}$$

The leading coefficient of $\delta(x - L)\mathbf{n}_2$ on the right hand side of Eq. (30) is the slope of the j'th eigenmode at the beam tip. Substituting Eqs. (21–22) and (29–30) into Eq. (28) yields

$$h_{ij} = -2L[g^{ij}(L) + 2\sigma_j\lambda_j(-1)^{i+j}] \tag{31}$$

Step 4: For Case 1 the matrix $K - pH$ is symmetric, therefore, the critical value of the load parameter p is the smallest eigenvalue of Eq. (20).

Step 5: For Case 2 the matrix $K - pH$ is nonsymmetric. Thus, the critical value of the load parameter is the smallest value of p for which at least one of the eigenvalues of Eq. (19) is not positive and real.

Results: The critical value of the load parameter p can be expressed in terms of the dimensionless variable α as

$$p = \frac{\alpha EI}{L^2} \tag{32}$$

Table 2 shows the values of α for $n = 1, \ldots, 5$ for both load cases. Recall that n is the number of generalized coordinates. The exact value of α for Case 1 is equal to $\pi^2/4 \approx 2.4674$ [1]. The exact value of α for Case 2 is equal to approximately 20.05 [2]. Notice in the table that α converges monotonically from above to a constant value for Case 1. For Case 2, α appears to be converging to a constant value, but the convergence is nonmonotonic.

4. Conclusions

The method of quadratic components was shown to be applicable to buckling problems with both conservative and nonconservative loads. The method was used to develop a buckling calculation procedure which was applied to an example problem. The results from the example problem were in excellent agreement with classical results. The procedure developed can be applied to a wide variety of different structures by making use of existing finite element software.

5. References

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Table 1: Beam coefficients in Eq. (21).

i	λ_i	σ_i
1	1.87510407	0.734095514
2	4.69409113	1.018467319
3	7.85475744	0.999224497
4	10.99554073	1.000033553
5	14.13716839	0.999998550

Table 2: Results for Cases 1 and 2.

	α	
n	Case 1	Case 2
1	2.6598	∞
2	2.4817	20.105
3	2.4740	20.113
4	2.4697	20.052
5	2.4688	20.061

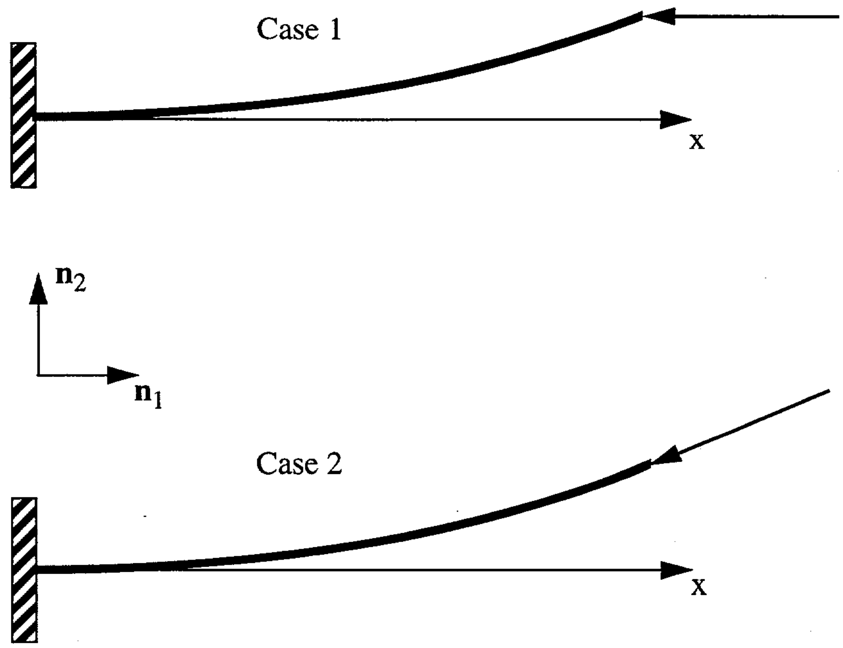

Figure 1: Load cases for cantilevered beam example.