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SCATTERING FROM IMPERFECTLY CONDUCTING SPHERES: THEORETICAL CONSIDERATIONS

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ABSTRACT

Scattering from imperfectly conducting spheres having both positive and negative dielectric constants is treated theoretically in a rigorous manner.

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SUMMARY

Exact formulas for the forward and backscattering cross sections of mono- and dual-region imperfectly conducting spheres are developed. Extensive numerical results based on the theory are presented in a companion paper.[1] The scattering body may consist of a plasma with collisions. Evidently the theory permits the electron densities and collision frequencies of the plasma comprising the shell and core regions to differ. The dielectric constant of the plasma may be positive or negative. Among other things it is proved that whenever $\sigma >> \omega|\epsilon|$ the scattering obstacle behaves essentially as though it were perfectly conducting, i.e., the back- and bistatic scattering cross sections are very insensitive to σ. This phenomenon was first observed in obtaining the approximate radar cross section of an imperfectly conducting scattering antenna when $\sigma >> \omega\epsilon$ (ϵ positive). The need for substantiating experimental evidence for either scattering body is thus obviated.

Laminar and turbulent scattering from over- and under-dense plasma spheres is discussed qualitatively. It is concluded that laminar scattering must be equal to or greater than turbulent scattering from such obstacles.

TABLE OF CONTENTS

SCATTERING FROM IMPERFECTLY CONDUCTING SPHERES: THEORETICAL CONSIDERATIONS

Introduction

In a recent investigation[2] it was discovered that the radar cross section of a thin cylindrical rod, for parallel incidence of the electric field, is very insensitive to the conductivity of the rod over the range $10 \leq \sigma \leq 10^7$ mhos/m. But it should be mentioned that the study was not exhaustive in that only several rod lengths and frequencies were considered. Moreover, certain approximations were made in the development of the theory. For instance, the internal impedance per unit length of the structure was specified. This severely limits the range of σ over which it was possible to obtain reliable results. To obviate the need for experimental evidence to substantiate the observed phenomenon it was considered important to develop a parallel theory for an imperfectly conducting scattering body amenable to exact analytical treatment--the sphere. For this obstacle geometry it is not necessary to define an internal impedance per unit length, so that the permitted range of σ and ϵ characterizing the material is arbitrary. This means that the scattering properties of plasmas of spherical shape possibly not physically realizable at the present time may be investigated.

Mathematical Representation of the Electromagnetic Fields

Figure 1 illustrates a homogeneous spherical plasma shell of outer radius a and inner radius b characterized by permeability μ_1, dielectric constant ϵ_1, and conductivity σ_1. It is embedded in an infinite homogeneous medium with

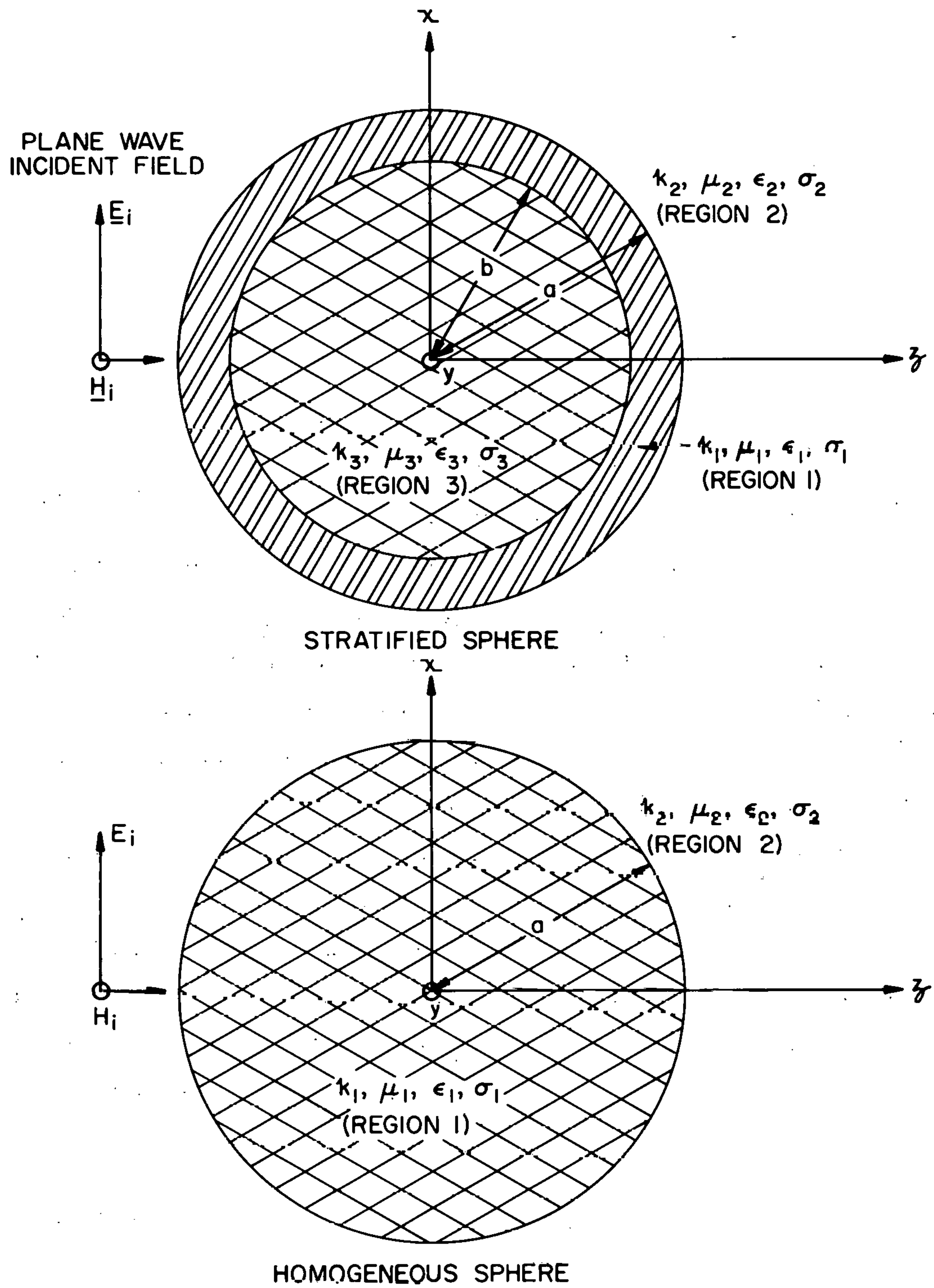

Figure 1. Dual- and Mono-Region Imperfectly Conducting Spheres

constitutive parameters, μ_2, ϵ_2, and σ_2. The core of the sphere is assumed to possess the electrical properties μ_3, ϵ_3, and σ_3. The center of the sphere is the origin of superimposed Cartesian and spherical coordinate systems. The unit vectors in these systems are $\hat{x}$, $\hat{y}$, and $\hat{z}$, and $\hat{\theta}$, $\hat{\Phi}$, and $\hat{R}$, respectively. θ is the angle between $\hat{z}$ and $\hat{R}$, Φ is the angle between $\hat{x}$ and the projection of $\hat{R}$ in the xy plane, and R is measured from the origin. The incident electric field is linearly polarized in the x-direction and propagates in the direction of the positive z-axis, so that it impinges on the sphere at the angle $\theta = \pi$. The scattered field of interest emanates from the sphere at $\theta = \pi$.

The expansions, in vector spherical wave functions, of the incident, scattered, shell and cavity fields may be written down by analogy with the work of Stratton.[3] The field expressions are

$$\underline{E}_i = \hat{x}E_o e^{-jk_2 z} = E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\underline{m}_{oln}^{(1)} + j\underline{n}_{eln}^{(1)}\right] \qquad R \geq a, \quad (1)$$

$$\underline{H}_i = \hat{y}H_o e^{-jk_2 z} = -\frac{k_2}{\omega\mu_2} E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\underline{m}_{eln}^{(1)} - j\underline{n}_{oln}^{(1)}\right] \qquad R \geq a, \quad (2)$$

$$\underline{E}_r = E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[a_n^r \underline{m}_{oln}^{(3)*} + jb_n^r \underline{n}_{eln}^{(3)*}\right] \qquad R \geq a, \quad (3)$$

$$\underline{H}_r = -\frac{k_2}{\omega\mu_2} E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[b_n^r \underline{m}_{eln}^{(3)*} - ja_n^r \underline{n}_{oln}^{(3)*}\right] \qquad R \geq a, \quad (4)$$

$$\underline{E}_s = E_o \left\{ \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[p_n \underline{m}_{oln}^{(3)*} + jq_n \underline{n}_{eln}^{(3)*} \right] \right.$$

$$\left. + \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[d_n \underline{m}_{oln}^{(3)} + jf_n \underline{n}_{eln}^{(3)} \right] \right\} \qquad b \leq R \leq a , \quad (5)$$

$$\underline{H}_s = -\frac{k_1}{\omega\mu_1} E_o \left\{ \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[q_n \underline{m}_{eln}^{(3)*} - jp_n \underline{n}_{oln}^{(3)*} \right] \right.$$

$$\left. + \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[f_n \underline{m}_{eln}^{(3)} - jd_n \underline{n}_{oln}^{(3)} \right] \right\} \qquad b \leq R \leq a , \quad (6)$$

$$\underline{E}_c = E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[a_n^c \underline{m}_{oln}^{(1)} + jb_n^c \underline{n}_{eln}^{(1)} \right] \qquad 0 \leq R \leq b , \quad (7)$$

$$\underline{H}_c = -\frac{k_3}{\omega\mu_3} E_o \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[b_n^c \underline{m}_{eln}^{(1)} - ja_n^c \underline{n}_{oln}^{(1)} \right] \qquad 0 \leq R \leq b . \quad (8)$$

The subscripts on the field vectors i, r, s, and c indicate incident, reflected, shell, and cavity, respectively. The time dependence assumed (and suppressed in writing Equations 1 through 8 is exp $(j\omega t)$.

The propagation constant of a given medium is given by

$$k = \omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = \beta - j\alpha , \qquad (9)$$

where the subscript corresponding to the region under consideration in Figure 1 must be used on k, μ, ϵ, and σ. In Equation 9 standard notation is employed.

E_o is the amplitude of the incident electric field.

a_n^r, b_n^r, p_n, q_n, d_n, f_n, a_n^c, and b_n^c are constants to be evaluated from the boundary equations.

$$\underline{m}^{(1)}_{\substack{o\\e}1n} = \pm \frac{j_n(kR)}{\sin\theta} P_n^1(\cos\theta) \begin{matrix}\cos\Phi\\ \sin\Phi\end{matrix} \hat{\theta} - j_n(kR) \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \begin{matrix}\sin\Phi\\ \cos\Phi\end{matrix} \hat{\Phi} , \tag{10}$$

$$\underline{n}^{(1)}_{\substack{o\\e}1n} = \frac{n(n+1)}{kR} j_n(kR) P_n^1(\cos\theta) \begin{matrix}\sin\Phi\\ \cos\Phi\end{matrix} \hat{R} + \frac{1}{kR}\left[kR j_n(kR)\right]' \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \begin{matrix}\sin\Phi\\ \cos\Phi\end{matrix} \hat{\theta}$$

$$\pm \frac{1}{kR\sin\theta}\left[kR j_n(kR)\right]' P_n^1(\cos\theta) \begin{matrix}\cos\Phi\\ \sin\Phi\end{matrix} \hat{\Phi} . \tag{11}$$

$\underline{m}^{(3)}_{\substack{o\\e}1n}$ is obtained from $\underline{m}^{(1)}_{\substack{o\\e}1n}$ by writing $h_n^{(1)}(kR)$ for $j_n(kR)$ throughout the expression. $\underline{n}^{(3)}_{\substack{o\\e}1n}$ is obtained from $\underline{n}^{(1)}_{\substack{o\\e}1n}$ in like manner.

In writing down the expressions for the magnetic field, the relations

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} , \tag{12}$$

and

$$\left.\begin{aligned} \nabla \times \underline{m} &= k\underline{n} \\ \nabla \times \underline{n} &= k\underline{m} \\ \nabla \times \underline{m}^* &= k\underline{n}^* \\ \nabla \times \underline{n}^* &= k\underline{m}^* \end{aligned}\right\} \tag{13}$$

are used. It is important to note that the notation $\underline{m}^{(3)*}_{\substack{o\\e}1n}$ and $\underline{n}^{(3)*}_{\substack{o\\e}1n}$ employed in this paper indicates that the complex conjugate of the function is to be taken. The argument of the function, even though complex, is to be left alone. Thus, $h_n^{(1)*}(kR) \rightarrow h_n^{(2)}(kR)$ and $h_n^{(2)*}(kR) \rightarrow h_n^{(1)}(kR)$, where k may be complex.

The Scattered Field from a Dual Region Sphere

The electric field scattered by the dual region sphere is given by Equation 3. At great distances from the sphere, when $\theta = \pi$, this relation may be simplified by use of the formulas

$$\left.\begin{aligned}
&\underset{z \to \infty}{\text{Limit}}\; h_n^{(2)}(z) \rightarrow \frac{j^{n+1}}{z}\, e^{-jz} \\
&\underset{\theta \to \pi}{\text{Limit}}\; \frac{P_n^1(\cos\theta)}{\sin\theta} \rightarrow (-1)^n \frac{n(n+1)}{2} \\
&P_n^1(\cos\theta)\Big|_{\theta=\pi} = 0 \\
&\underset{\theta \to \pi}{\text{Limit}}\; \frac{\partial}{\partial\theta} P_n^1(\cos\theta) \rightarrow -(-1)^n \frac{n(n+1)}{2} \\
&\underset{z \to \infty}{\text{Limit}}\; \frac{1}{z}\left[z h_n^{(2)}(z)\right]' \rightarrow j^n \frac{e^{-jz}}{z}
\end{aligned}\right\} . \qquad (14)$$

Substituting Equation 14 into Equation 3,

$$E_r = j \frac{E_o}{2} \frac{e^{-jk_2R}}{k_2R} \sum_{n=1}^{\infty} (-1)^n (2n+1)\left[a_n^r - b_n^r\right]\left[\cos\Phi\hat{\theta} + \sin\Phi\hat{\Phi}\right]. \tag{15}$$

But,

$$-\hat{x} = \cos\Phi\hat{\theta} + \sin\Phi\hat{\Phi} , \tag{16}$$

when $\theta = \pi$. Hence,

$$E_r = -j\hat{x} \frac{E_o}{2} \frac{e^{-jk_2R}}{k_2R} \sum_{n=1}^{\infty} (-1)^n (2n+1)\left(a_n^r - b_n^r\right). \tag{17}$$

This is the final expression for the backscattered field from the dual-region sphere. It is of interest to observe that the electric field propagating in the direction of the source (radar antenna) is a plane wave polarized parallel to the incident electric field.

The Boundary Equations

The boundary conditions that must be satisfied on the inner and outer surfaces of the region 1, Figure 1, are

$$\left.\begin{aligned}\left(\underline{E}_s\right)_\theta &= \left(\underline{E}_c\right)_\theta\\ \left(\underline{E}_s\right)_\Phi &= \left(\underline{E}_c\right)_\Phi\\ \left(\underline{H}_s\right)_\theta &= \left(\underline{H}_c\right)_\theta\\ \left(\underline{H}_s\right)_\Phi &= \left(\underline{H}_c\right)_\Phi\end{aligned}\right\} R = b\,, \tag{18}$$

$$\left.\begin{aligned}\left(\underline{E}_i + \underline{E}_r\right)_\theta &= \left(\underline{E}_s\right)_\theta\\ \left(\underline{E}_i + \underline{E}_r\right)_\Phi &= \left(\underline{E}_s\right)_\Phi\\ \left(\underline{H}_i + \underline{H}_r\right)_\theta &= \left(\underline{H}_s\right)_\theta\\ \left(\underline{H}_i + \underline{H}_r\right)_\Phi &= \left(\underline{H}_s\right)_\Phi\end{aligned}\right\} R = a\,. \tag{19}$$

Let the following notation be introduced:

$$\left.\begin{array}{lll} A_1 = j_n(k_2a) & A_2 = j_n(k_2b) & K_1 = j_n(k_3b) \\ B_1 = h_n^{(2)}(k_2a) & B_2 = h_n^{(2)}(k_2b) & K_2 = \left[k_3bj_n(k_3b)\right]' \\ C_1 = h_n^{(2)}(k_1a) & C_2 = h_n^{(2)}(k_1b) & \\ D_1 = h_n^{(1)}(k_1a) & D_2 = h_n^{(1)}(k_1b) & \\ E_1 = \left[k_2aj_n(k_2a)\right]' & E_2 = \left[k_2bj_n(k_2b)\right]' & \\ F_1 = \left[k_2ah_n^{(2)}(k_2a)\right]' & F_2 = \left[k_2bh_n^{(2)}(k_2b)\right]' & \\ G_1 = \left[k_1ah_n^{(2)}(k_1a)\right]' & G_2 = \left[k_1bh_n^{(2)}(k_1b)\right]' & \\ H_1 = \left[k_1ah_n^{(1)}(k_1a)\right]' & H_2 = \left[k_1bh_n^{(1)}(k_1b)\right]' & \end{array}\right\} \tag{20}$$

It can then be shown that the boundary equations take the following form:

$$A_1 + a_n^r B_1 = p_n C_1 + d_n D_1 \ , \tag{21}$$

$$\frac{1}{k_2}\left(E_1 + b_n^r F_1\right) = \frac{1}{k_1}(q_n G_1 + f_n H_1) \ , \tag{22}$$

$$\frac{k_2\mu_1}{k_1\mu_2}\left(A_1 + b_n^r B_1\right) = q_n C_1 + f_n D_1 \ , \tag{23}$$

$$\frac{\mu_1}{\mu_2}\left(E_1 + a_n^r F_1\right) = p_n G_1 + d_n H_1 \ , \tag{24}$$

$$p_n C_2 + d_n D_2 = a_n^c K_1 \ , \tag{25}$$

$$\frac{k_3}{k_1}\left(q_n G_2 + f_n H_2\right) = b_n^c K_2 \ , \tag{26}$$

$$\frac{k_1 \mu_3}{k_3 \mu_1}\left(q_n C_2 + f_n D_2\right) = b_n^c K_1 \ , \tag{27}$$

$$\frac{\mu_3}{\mu_1}\left(p_n G_2 + d_n H_2\right) = a_n^c K_2 \ . \tag{28}$$

In obtaining the simultaneous equations for the constants, one need only observe that $P_n^I(\cos\theta)$ and $\frac{\partial}{\partial\theta}P_n^I(\cos\theta)\sin\theta$ are linearly independent and the coefficients must be identically zero if any linear combination is to be zero for all θ in $(0, \pi)$. These equations yield

$$a_n^r = \frac{\left(A_1G_1 - C_1E_1\frac{\mu_1}{\mu_2}\right)\left(H_2K_1\frac{\mu_3}{\mu_1} - D_2K_2\right) + \left(A_1H_1 - D_1E_1\frac{\mu_1}{\mu_2}\right)\left(C_2K_2 - G_2K_1\frac{\mu_3}{\mu_1}\right)}{\left(C_1F_1\frac{\mu_1}{\mu_2} - B_1G_1\right)\left(H_2K_1\frac{\mu_3}{\mu_1} - D_2K_2\right) + \left(D_1F_1\frac{\mu_1}{\mu_2} - B_1H_1\right)\left(C_2K_2 - G_2K_1\frac{\mu_3}{\mu_1}\right)} \ , \tag{29}$$

$$b_n^r = \frac{\left(C_1E_1 - A_1G_1\frac{k_2^2}{k_1^2}\frac{\mu_1}{\mu_2}\right)\left(H_2K_1\frac{k_3^2}{k_1^2} - D_2K_2\frac{\mu_3}{\mu_1}\right) + \left(E_1D_1 - A_1H_1\frac{k_2^2}{k_1^2}\frac{\mu_1}{\mu_2}\right)\left(C_2K_2\frac{\mu_3}{\mu_1} - G_2K_1\frac{k_3^2}{k_1^2}\right)}{\left(B_1G_1\frac{k_2^2}{k_1^2}\frac{\mu_1}{\mu_2} - C_1F_1\right)\left(H_2K_1\frac{k_3^2}{k_1^2} - D_2K_2\frac{\mu_3}{\mu_1}\right) + \left(B_1H_1\frac{k_2^2}{k_1^2}\frac{\mu_1}{\mu_2} - F_1D_1\right)\left(C_2K_2\frac{\mu_3}{\mu_1} - G_2K_1\frac{k_3^2}{k_1^2}\right)} . \tag{30}$$

It is now only necessary to substitute the expressions for a_n^r and b_n^r given by Equations 29 and 30, respectively, into Equation 17. The latter expression is then summed, using a digital computer, to obtain the backscattered field from an imperfectly conducting spherical shell in terms of the amplitude of the incident electric field E_o.

Scattering from a Solid Homogeneous Imperfectly Conducting Sphere

The boundary equations for a solid homogeneous imperfectly conducting sphere are[3]

$$j_n(k_2a) + a_n^r h_n^{(2)}(k_2a) = a_n^c j_n(k_1a) , \tag{31}$$

$$\frac{k_1}{k_2}\left\{\left[k_2aj_n(k_2a)\right]' + b_n^r\left[k_2ah_n^{(2)}(k_2a)\right]'\right\} = b_n^c\left[k_1aj_n(k_1a)\right]' , \tag{32}$$

$$\frac{k_2\mu_1}{k_1\mu_2}\left\{j_n(k_2a) + b_n^r h_n^{(2)}(k_2a)\right\} = b_n^c j_n(k_1a) , \tag{33}$$

$$\frac{\mu_1}{\mu_2}\left\{\left[k_2aj_n(k_2a)\right]' + a_n^r\left[k_2ah_n^{(2)}(k_2a)\right]'\right\} = a_n^c\left[k_1aj_n(k_1a)\right]' . \tag{34}$$

Solving these equations simultaneously yields

$$a_n^r = - \frac{\mu_1 j_n(k_1 a)\left[k_2 a j_n(k_2 a)\right]' - \mu_2 j_n(k_2 a)\left[k_1 a j_n(k_1 a)\right]'}{\mu_1 j_n(k_1 a)\left[k_2 a h_n^{(2)}(k_2 a)\right]' - \mu_2 h_n^{(2)}(k_2 a)\left[k_1 a j_n(k_1 a)\right]'} , \tag{35}$$

$$b_n^r = - \frac{\mu_1 k_2^2 j_n(k_2 a)\left[k_1 a j_n(k_1 a)\right]' - \mu_2 k_1^2 j_n(k_1 a)\left[k_2 a j_n(k_2 a)\right]'}{\mu_1 k_2^2 h_n^{(2)}(k_2 a)\left[k_1 a j_n(k_1 a)\right]' - \mu_2 k_1^2 j_n(k_1 a)\left[k_2 a h_n^{(2)}(k_2 a)\right]'} . \tag{36}$$

a_n^r and b_n^r given above for a solid sphere parallel Equations 29 and 30 for the dual-region sphere.

Forward Scattering by a Sphere

It is of interest to calculate the scattered field in the forward direction of a sphere ($\theta = 0$) as well as in the backward direction ($\theta = \pi$). The limiting forms of the functions needed for $\theta = 0$ are

$$\left.\begin{aligned}
&\underset{\theta \to 0}{\text{Limit}} \frac{P_n^I(\cos\theta)}{\sin\theta} \to - \frac{n(n+1)}{2} \\
&P_n^I(\cos\theta) = 0 \\
&\qquad \theta = 0 \\
&\underset{\theta \to 0}{\text{Limit}} \frac{\partial}{\partial\theta} P_n^I(\cos\theta) \to - \frac{n(n+1)}{2} \\
&-\hat{x} = \cos\Phi\hat{\theta} - \sin\Phi\hat{\Phi} \\
&\qquad \theta = 0
\end{aligned}\right\} . \tag{37}$$

Substituting Equation 37 and two of the relations from Equation 14 into Equation 3 leads to the result

$$E_r = -j\hat{x}\,\frac{E_o}{2}\,\frac{e^{-jk_2R}}{k_2R}\sum_{n=1}^{\infty}(2n+1)\left(a_n^r + b_n^r\right). \tag{38}$$

This is the final expression for the field scattered in the forward direction. Expressions for a_n^r and b_n^r appear earlier in the paper.

The Scattering Cross Section of an Obstacle

According to King and Wu[4] the monostatic or backscattering cross section of a finite obstacle is defined to be the ratio of the total power $P^s_{isotropic}$ reradiated by a fictitious isotropic scatterer (that maintains the same field E^r in all directions as that maintained by the actual obstacle in the direction toward the source) to the real magnitude S^i of the Poynting vector of the incident plane wave at the obstacle. Thus

$$\sigma_s = \frac{P^s_{isotropic}}{S^i} = \lim_{R\to\infty} 4\pi R^2 \left|\frac{E_r}{E_o}\right|^2 . \tag{39}$$

Equation 17 is substituted into Equation 39 with a_n^r and b_n^r given by Equations 29 and 30 or by Equations 35 and 36, depending on whether a dual-region or mono-region sphere is under consideration, respectively. Formula 39 is also used to obtain the bistatic cross section of an obstacle. In such installations the receiver is located in an arbitrary direction with respect to the transmitter. Evidently, it is necessary to know the scattered field in the specified direction. In particular, the forward scattering cross section σ_f is obtained from Equation 39 using Equation 38.

Conductivity and Dielectric Constant of a Plasma with Collisions

The propagation constant k of a plasma for an assumed time dependence of $\exp(j\omega t)$ is $k = \omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)}$ where

$$\sigma = \frac{\epsilon_o \omega_{eff} \omega_p^2}{\omega^2 + \omega_{eff}^2}, \qquad (40)$$

$$\omega_p^2 = \frac{Nq^2}{m\epsilon_o}, \qquad (41)$$

and

$$\epsilon = \epsilon_o\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_{eff}^2}\right) = \epsilon_o \epsilon_r . \qquad (42)$$

Here

$\omega = 2\pi f$ where f is the radar frequency,

$\mu = \mu_o = 4\pi \times 10^{-7}$ henry/m is the fundamental magnetic constant of space,

ϵ = the effective dielectric constant of the plasma,

σ = the effective conductivity of the plasma,

$\epsilon_o = 8.85 \times 10^{-12}$ farads/m is the fundamental dielectric constant of space,

ω_{eff} = the collision frequency in collisions/sec,

ω_p = the plasma frequency in cycles/sec,

N = the electron density in electrons per cu/m,

q = the charge on an electron; $q = 1.602 \times 10^{-19}$ coulomb,

m = the mass of an electron; $m = 9.108 \times 10^{-31}$ kg.

Equations 40 through 42 apply when the motion of the ions can be neglected, i.e., ω is sufficiently high. Static fields that may be present are neglected.

Note also that σ and ϵ as defined here are real. $\sigma \geq 0$, but ϵ may take on positive or negative value.

Semiconductors Having the Permeability of Free Space

The complex propagation constant k, as delineated by Equation 9, may be written

$$\beta - j\alpha = k = \sqrt{\omega^2 \mu\epsilon - j\sigma\omega\mu} \ . \tag{43}$$

The writer believes this is the best form in which to express k to insure taking the indicated square root correctly, especially when the dielectric constant is negative. Harrison and Aronson[5] have prepared tables for obtaining $\sqrt{\pm a \pm jb}$. For checking computer programs the use of these tables is recommended. Let the assumption be made in the following discussion that the permeability of the semiconductor is the same as that of free space.

Case I: $\epsilon < 0$

In this instance

$$k = \sqrt{-\omega^2 \mu |\epsilon| - j\sigma\omega\mu} = k_o \sqrt{|\epsilon_r|}\,[g(p) - jf(p)] \ , \tag{44}$$

so that

and

$$\left.\begin{aligned} \beta &= k_o \sqrt{|\epsilon_r|}\, g(p) \\ \alpha &= k_o \sqrt{|\epsilon_r|}\, f(p) \end{aligned}\right\} . \tag{45}$$

Here

$$k_o = \frac{\omega}{c} , \tag{46}$$

$$c = \frac{1}{\sqrt{\epsilon_o \mu}} = 3 \times 10^8 \text{ m/sec} , \tag{47}$$

$$g(p) = \sinh\left(\frac{1}{2} \sinh^{-1} p\right) = p/2f(p) , \tag{48}$$

$$f(p) = \cosh\left(\frac{1}{2} \sinh^{-1} p\right) , \tag{49}$$

$$p = \frac{\sigma}{\omega|\epsilon|} = \frac{\sigma}{\omega\epsilon_o|\epsilon_r|} . \tag{50}$$

It is important to observe that g(p) is an odd function and that f(p) is an even function. If

$$\sigma \gg \omega|\epsilon| , \tag{51}$$

i.e., p becomes large,

$$f(p) \rightarrow g(p) \rightarrow \sqrt{p/2} . \tag{52}$$

It follows that Equation 44 may be written

$$k \simeq k_o \sqrt{|\epsilon_r|}\, f(p)(1 - j) \simeq (1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} \; . \qquad (53)$$

<u>Case II:</u> $\epsilon = 0$, i.e., $p \to \infty$

In this instance

$$k = \sqrt{-j\omega\mu\sigma} = (1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} \; . \qquad (54)$$

<u>Case III:</u> $\epsilon > 0$

In this instance it is evident that Equation 43 becomes

$$k = k_o \sqrt{\epsilon_r}\,[f(p) - jg(p)] \; , \qquad (55)$$

so that

$$\left.\begin{aligned} \beta &= k_o \sqrt{\epsilon_r}\, f(p) \\ \text{and} \qquad & \\ \alpha &= k_o \sqrt{\epsilon_r}\, g(p) \end{aligned}\right\} . \qquad (56)$$

Thus when Equation 51 applies, k is again given by Equation 53. Thus k correctly "joins up" as ϵ passes through zero. Consider now a plane-wave electric field propagating in the positive x direction and having the value E_o at $x = 0$. When the dielectric constant is positive,

$$E = E_o\, e^{-jkx} = E_o\, e^{-k_o x \sqrt{\epsilon_r}\, g(p)}\; e^{-jk_o x \sqrt{\epsilon_r}\, f(p)} \; , \qquad (57)$$

and when the dielectric constant is negative

$$E = E_o \, e^{-k_o x \sqrt{|\epsilon_r|} f(p)} \, e^{-jk_o x \sqrt{|\epsilon_r|} g(p)} \quad . \qquad (58)$$

It is understood that $p = \sigma/\omega|\epsilon|$ as before. Since $f(p) > g(p)$ (except for large p) it follows that the attenuation of a plane wave traversing a given distance is greater in a region having a negative as against positive dielectric constant.

An examination of Equations 57 and 58 show that correct roots of Equation 43 have been taken because the attenuation and phase-shift factors have the correct algebraic signs. The value of k must be the same for a sphere as for a plane, provided the wave equation in spherical and Cartesian coordinates is written in the same form. k must have the same form when $ka \ll 1$ as for $ka \gg 1$. As $a \rightarrow \infty$ the spherical surface approaches that of a plane. It also follows from Equations 57 and 58 that when it is possible to define a skin depth for a semiconductor it may be calculated from the relations

$$d_s = 1/\alpha = \frac{1}{k_o \sqrt{\epsilon_r}\, g(p)} \; , \qquad (59)$$

when $\epsilon > 0$ and

$$d_s = \frac{1}{k_o \sqrt{|\epsilon_r|}\, f(p)} \qquad (60)$$

when $\epsilon < 0$. d_s is the distance from $x = 0$ to the point where the field reaches the value $e^{-1} E_o$. Obviously whenever $\sigma \gg \omega|\epsilon|$,

$$d_s = \sqrt{\frac{2}{\omega \mu \sigma}} \; . \qquad (61)$$

In writing many of the equations in this section it is assumed that $\mu = \mu_o$.

Prognostication of Results

The author's remarks in this section will be confined to a foretoken of results anticipated in the case of scattering by a homogeneous imperfectly conducting sphere. Of course, the comments apply with equal force to the dual-region case. Now the backscattered field as given by Equation 17 is controlled by a_n^r and b_n^r. These constants are delineated by Equations 35 and 36, respectively. Examination of these expressions shows that a_n^r and b_n^r are determined by the values of k_1a and k_2a. Replacing k_2 by $k_o = \omega/c = 2\pi/\lambda_o$, $c = 3 \times 10^8$ m/sec, and k_1 by k (refer to Figure 1), it is clear that the backscattered field depends on the electrical size of the sphere in terms of the free space wavelength, and on the electrical size of the sphere in terms of the wavelength in the material. Evidently k_oa is not a function of the constitutive parameters σ and ϵ of the material. Hence one must study the behavior of ka. When the dielectric constant is positive $\beta a = k_oa\sqrt{\epsilon_r}f(p)$ and when it is negative, $\beta a = k_oa\sqrt{|\epsilon_r|}\,g(p)$. For large p, $f(p) \rightarrow \sqrt{p/2}$ and $g(p) \rightarrow \sqrt{p/2}$, as mentioned before. Since $p = \sigma/\omega|\epsilon|$ it follows that whenever σ, ω, or $|\epsilon|$ satisfy the inequality $\sigma >> \omega|\epsilon|$ the scattering obstacle must behave as though it were essentially perfectly conducting. Notice that moduli signs appear on ϵ. Hence the above statement applies whether ϵ is positive or negative. It is of interest to determine under what conditions a plasma with collisions will lead to large p. From Equations 40 and 42, $\sigma = \epsilon_o\omega_{eff}(1 - \epsilon_r)$. Hence $\sigma >> \omega\epsilon_o|\epsilon_r|$ when $\omega_{eff}(1 - \epsilon_r) >> \omega|\epsilon_r|$. If $\epsilon_r < 0$ and is large, the condition is simply $\omega_{eff}/\omega >> 1$.

Evidently what has been said relating to the insensitivity of the backscattering cross section of a sphere to σ and ϵ over certain ranges of these parameters applies with equal force in the case of bistatic cross sections. This includes, of course, the case of forward scatter. Before conducting this section, scaling for solid spheres should be mentioned. Since $ka = \omega a\sqrt{\mu\left(\epsilon - \frac{\sigma}{\omega}\right)}$ it becomes evident that if one increases σ and ω by a factor of 10, holds ϵ fixed, and lets ka = const., the same numbers for the scattering cross section must be obtained for $\sigma = 10^{-7}$,

$f = 10^8$ as for $\sigma = 10^{-6}$, $f = 10^9$. But notice that a has been divided by a factor of 10.

Turbulent Plasmas

It is common to model a turbulent plasma by a random distribution of homogeneous spheres having a diameter equal to the space scale of turbulence and possessing statistical constitutive parameters (ϵ, σ). Plane-wave scattering by a turbulent plasma depends on three parameters which are dimensionally distances. These are λ the wavelength of the incident field, λ_T the average radius of the spheres representing the turbulence, and a the radius of the spherical region within which the turbulence is contained. At distances greater than a, the region is unperturbed. Normally $a >> \lambda$ but $\lambda \gtrless (\lambda_T)_{average}$. The physical process involved in the scattering by a turbulent plasma region is different depending on whether the plasma is (1) overdense and (2) underdense.

An overdense plasma is defined as one in which $\omega < \omega_p$ if there are no collisional losses or the medium is only slightly lossy. In this case the field penetrating into the plasma is evanescent; hence, the scattering properties of an overdense turbulent plasma may be described in all essential respects by its surface properties. The surface of an overdense turbulent plasma sphere is rough so that the backscattered field is less than from a smooth sphere of the same radius. The roughness gives rise to diffuse scattering, that is, the scattered field will be necessarily highly depolarized and incoherent. Backscattering is usually defined in terms of plane waves. If a plane parallel to the incident phase fronts is tangential to the rough sphere at the point of contact on the illuminated spherical sector, then the backscattering cross section will be the same (approximately) as that of a smooth sphere of radius equal to the local radius of curvature of the rough sphere at the point of contact. Usually (statistically speaking), the plane parallel to the phase fronts is not tangential to the rough sphere at the point of contact. Thus the above mentioned situation does not ordinarily occur. Also, for a homogeneous sphere, the backscattering cross-section curve versus a/λ has peaks at certain values of a/λ. These are called geometrical resonances. In a rough sphere the radius of curvature changes along the periphery of the sphere. Hence one can expect the geometrical resonances characteristic of a uniform sphere to be destroyed in the case of a rough sphere. In view of the foregoing, the backscattering cross section for an overdense plasma sphere, in general, is smaller than for laminar or coherent backscattering that obtains from a homogeneous sphere of radius a, discussed in this paper.

In summary, one may conclude that for an overdense turbulent plasma sphere when $a \gg \lambda$

a. σ_s (laminar) $> \sigma_s$ (turbulent) for $\lambda < (\lambda_T)_{average}$,

b. σ_s (laminar) $\simeq \sigma_s$ (turbulent) for $\lambda > (\lambda_T)_{average}$.

From Figure 2(a) it is clear that when $\lambda < (\lambda_T)_{average}$ the surface of the region bounded by the sphere of radius a is rough. The surface roughness scatters power incoherently reducing the radar cross section. From Figure 2(b) it is seen that as λ becomes larger with respect to $(\lambda_T)_{average}$ the surface of the overdense turbulent plasma sphere becomes effectively more smooth, giving rise to more laminar or coherent scattering. Noncoherent scattering always reduces the effective received signal.

An underdense plasma is defined as one in which $\omega > \omega_p$. In this case the field propagates into the plasma, and one must solve a volume rather than a surface problem. As before the plasma is represented by a system of randomly positioned spheres of homogeneous electrical properties. In the theory of underdense turbulent scattering developed so far by workers in the field, the interactions between the spheres, i.e., coupling and multiple scattering is ignored. If the turbulent plasma sphere has a sufficiently large diameter (compared to the

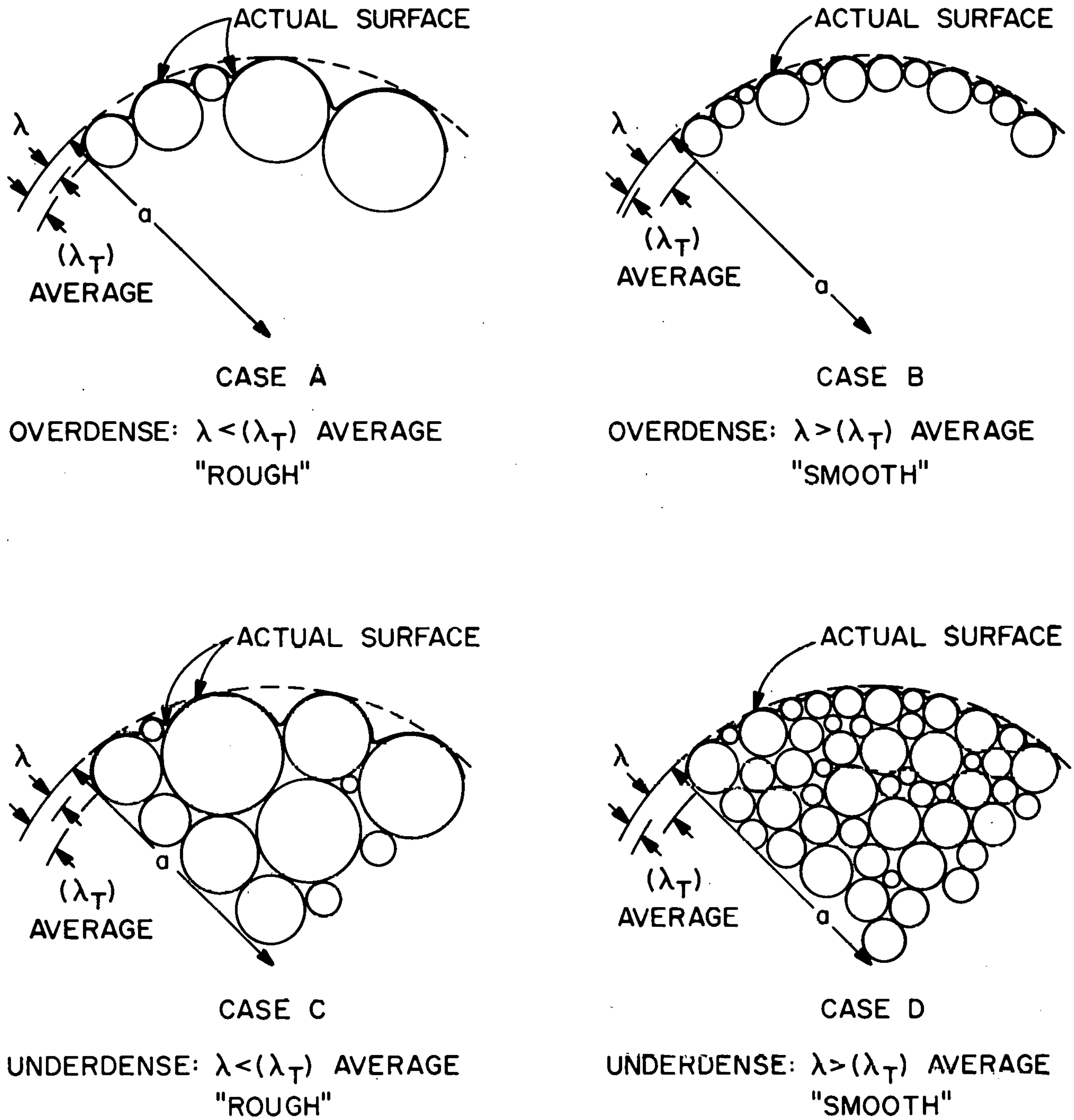

Figure 2. Turbulent Plasmas

space scale of turbulence) so that many scattering centers are contained in the scattering volume, the energy scattered from the sphere is incoherent. In this case the scattered signal is highly polarized, and is subject to random fluctuations induced by the turbulent fluctuations.

For an underdense turbulent plasma sphere when $a \gg \lambda$, one may conclude that:

c. σ_s (laminar) $> \sigma_s$ (turbulent) for $\lambda < (\lambda_T)_{average}$,

d. σ_s (laminar) $\geq \sigma_s$ (turbulent) for $\lambda > (\lambda_T)_{average}$.

One must remember that scattering from turbulent plasma is incoherent, and that as λ increases for fixed $(\lambda_T)_{average}$ one approaches more nearly a laminar scattering body (refer to Figures 2(c) and (d)).

From the foregoing remarks, it is clear that it is sufficient to compute the scattering cross section of an underdense or overdense homogeneous plasma sphere (which gives rise to laminar or coherent scattering) to obtain an upper bound for the scattering from a turbulent plasma sphere, provided ϵ and σ correspond to the constitutive parameters of the most effective scattering subsphere used in the collection of spheres employed in the turbulent plasma model. It can be shown that for the underdense case and $\lambda > (\lambda_T)_{average}$ a closer upper bound is obtained by the use of the mean value of ϵ and σ.

Evidently the above remark applies only to plasma bodies of finite size. There is no scattering in the direction of the source from a homogeneous plasma slab except for normal incidence of the electromagnetic field. On the other hand for arbitrary angles of incidence of the field on a turbulent plasma slab there will exist a scattered signal in the direction of the source. The problem of scattering from an overdense turbulent plasma slab is closely related to the problem of terrain radar return.

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