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TIME OPTIMAL XENON SHUTDOWN ON THE XENON BOUNDARY

J. Lewins and R. D. Benham

May, 1966

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TIME OPTIMAL XENON SHUTDOWN ON THE XENON BOUNDARY

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ABSTRACT

The time optimal flux program to shut down a reactor without losing restart capability through xenon poisoning is reviewed. A more rigorous proof of the optimal nature of operating along the xenon boundary is offered, together with a discussion of situations which might not be optimal. It is concluded that for practical purposes operation along the boundary will be optimal. Part of the proof consists of analog computations, and a discussion is given of the suitability of the optimum xenon study to analog techniques.

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TIME OPTIMAL XENON SHUTDOWN ON THE XENON BOUNDARY

INTRODUCTION

When a reactor is shut down, the xenon concentration tends to increase and passes through a maximum or peak some 11 hr later.[1] Many reactors have insufficient excess reactivity to overcome this peak and are, therefore, unable to restart until the xenon has decayed sufficiently. To avoid this, we may institute a control period during which the flux is not shutdown to zero; instead some residual flux is used to burn out the accumulating xenon and keep it to an acceptable value. Since there may be many such control programs capable of meeting the stated objective, one can seek a program that is in some sense optimal,

Among many criteria of optimality, one of interest is to minimize the time spent in the control period before the flux is finally brought to zero, This problem has been studied by several authors[2,3] using the Pontryagin optimal theory.[4] Recently, the theory has been applied by Roberts and Smith.[5]

One particular aspect of the Roberts and Smith treatment is discussed in an attempt to strengthen the rigor of their results; also, a hypothesis of Dittman's[6] as to the boundary condition for an optimum trajectory is disproved. Parts of the proof were carried out by using an analog computer. Since these computers are available in many reactors for control purposes, there is some interest in the techniques for applying them to the determination of the optimum shutdown program. The report contains a short account of the use of analog computers for determining time optimal control programs. It also contains a discussion of the somewhat more elaborate procedure used here to prove optimality rigorously.

THEORY

Time Optimal Solutions

The basic time optimal problem is to secure restart capability at all times, t, after first shutdown. To do this, the xenon concentration, X(t), must be kept at or below some boundary value $\bar{X}$ determined by the available excess reactivity. We make the reasonable simplification that $\bar{X}$ is time independent. However, this would change with temperature, etc, but not enough to make a practical change in the following results. If the reactor can be permitted not to have restart capability over a selected time period, a more optimal trajectory may be possible.[9]

For the basic time optimal problem the control period can be terminated when the xenon, iodine or X, I phase plane trajectory reaches the target curve Ω. This curve, as shown in Figure 1, is the trajectory followed at zero flux whose peak just touches $\bar{X}$; once on the target curve the flux may be set to zero and control terminated. For a nontrivial case, the starting point will lie outside Ω (above or to the right).

Application of Pontryagin's method has suggested that one of three trajectory regimes may be optimum:

Single Pulse Control. Control will be terminated by a trajectory at the maximum attainable flux. In trivial cases the initial point may lie on this trajectory, but more generally the maximum flux will be preceded by a period of minimum (zero) flux. At an optimum switching point between minimum and maximum, the switching function vanishes. (Switching points are on ε in Figure 1).

Intermediate Control. A boundary, Γ, exists in the phase plane, limiting the region of optimum 'bang bang' control. Zero flux trajectories meeting this line may follow it downward while

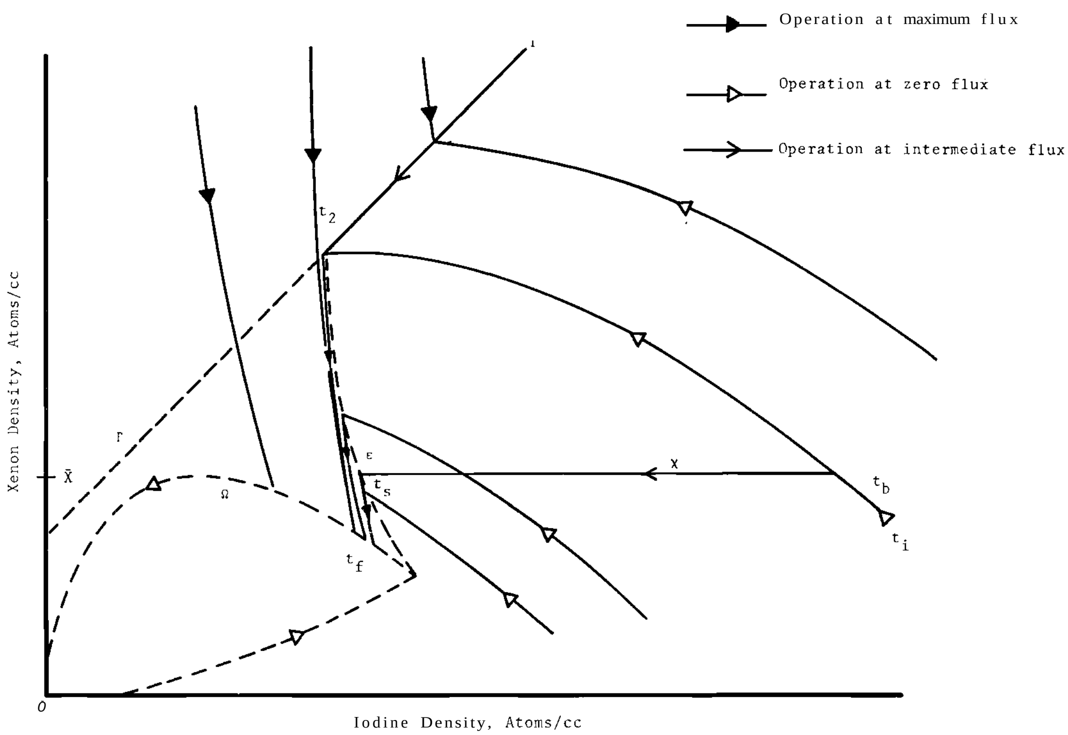

FIGURE 1. Sample Time Optimal Xenon Shutdown Trajectories

still satisfying a Pontryagin condition for the switching function, until meeting the unique single pulse switching point (t_2) leading (at maximum flux) to the target curve. Other trivial flux maximum trajectories lead to Γ and hence to the target curve.

Boundary Control. Both of the previous regimes may call for xenon values above the allowable $\bar{X}$, depending upon the location of the initial point in the phase plane. Since this is inadmissible, it is thought that the optimum trajectory will follow a zero flux path to the xenon boundary, $\bar{X}$, and therefore follow the boundary at constant xenon until meeting that unique switching point (t_s) that leads to Ω at maximum flux.

Our concern is the strict proof that Regime 3, when applicable, is optimal. We note at this point that there is a similarity between operation along Γ and operation along the various possible xenon boundaries. Indeed, without physical significance, one could generalize the boundary to be a regime of fixed (though not necessarily zero) slope in the phase plane. Gamma (Γ) is unique because the trajectory follows such a line without constraints.

The status on the rigorous proof of Regime 3 is given by Smith and Roberts[5] and by Woodcock[3] who take the following position:

- The flux program to hold the xenon on the boundary is uniquely specified in the phase plane. Therefore, if the optimum trajectory has to go along the boundary, the unique flux is the optimal flux.
- At t_s, a switching point to a flux maximum, the flux may indeed change while satisfying a Pontryagin condition. At t_b, the Hamiltonian can be made continuous by an allowable discontinuity of the adjoint functions.
- In the zero flux regime before t_b, the trajectory satisfies the conditions of the problem without xenon boundary.

Physically it is probably "clear" that such a trajectory is indeed optimal (if only on the grounds of what else can it do!). However, one could envisage situations (with other cost functionals perhaps) where the trajectory moves very slowly along the boundary. In such a case, the optimal path would avoid the boundary.

Specifically, there are two points that remain to be shown for a rigorous proof,

(1) Following the trajectory back from the target curve to the initial time, it may be that the trajectory starts along the boundary. We have not proved that it can continue along the boundary as far as desired.

(2) In the regime at zero flux, before t_b, it is true that the same equations are satisfied by the adjoint functions as in the unrestricted problem. But the boundary conditions are different. Thus, it cannot be said we have verified that the switching functicn does not have yet another switch until we have investigated the solution with the correct boundary conditions.

Fortunately, the simplification that makes the flux on the boundary unfque makes it rather easy to determine the adjoint functions explicitly in the boundary, This was not done by Smith and Roberts, Our proposal, therefore, is to find the full solution of the adjoint equations to the problem on the boundary and the effect these have on the switching function. We can then carry out the appropriate tests of the Pontryagin formulation, This will be done analytically on the boundary and during the zero flux period, as was previously done[5] by numerical solution over the range of interest. Due to the rapidity with which one can change the initial or property values of the problem, we found it convenient to generate these solutions with an analog computer, The accuracy is less than that offered by a digital computation but within the accuracy of the model and the uncertainty of the parameters.

From some of the previous applications of the Pontryagin method to this problem it appears that certain aspects of the optimum theorem are not well understood. Therefore, the optimum conditions are outlined in a way that differs somewhat from Pontryagin but accords more closely with classical calculus of variations.[7] The effect is to remove an arbitrary and negative normal-ization from the Pontryagin formulation with the following advantages:[8]

- The ad-joint functions in real time are positive.
- The adjoint functions may now be identified with the importance functions of conventional perturbation theory.

To minimize the control period, we are now to give an appropriate Hamiltonian its least value rather than its greatest value.

Optimum Control Conditions

We outline the necessary conditions to be satisfied for the unrestricted and restricted problems if a control program is to be optimum. The equations for xenon (X) and iodine (I) densities are assumed to be

$$\frac{dI}{dt} = -\lambda_i I + \gamma_i \Sigma_f \phi \tag{1}$$

$$\frac{dX}{dt} = \lambda_i I - \lambda_x X - \phi\sigma X + \gamma_x \Sigma_f \phi . \tag{2}$$

It is convenient to write this in vector and matrix form as

$$\frac{dN}{dt} = MN + S \tag{3}$$

where

$$N = [I, X],$$

$$S \quad [\gamma_i, \gamma_x]\Sigma_f \phi,$$

and M is a matrix of the coefficients of the homogeneous terms appearing in Equations (1) and (2).

Initial conditions for $N(t_i)$ are assumed specified. The flux, ϕ, is a control variable in this prompt jump approximation which may vary between a minimum, 0, and a maximum, $\bar{\phi}$. The xenon in the restricted problem is to lie on or below $\bar{X}$.

We define also an adjoint vector $N^+ = [I^*, X^+]$ and a full Hamiltonian as

$$\mathcal{H} = 1 + H = 1 + N^+[MN + S]. \tag{4}$$

Then in the usual way

$$\frac{\partial H}{\partial N^+} \quad \frac{dN}{dt} = — \qquad (5)$$

reproduces Equations (1) and (2).

Adjoint equations are defined through

$$\frac{\partial H}{\partial N} = - \frac{dN^+}{dt} \qquad (6)$$

which leads to the following equations:

$$- \frac{dI^+}{dt} = \frac{\partial H}{\partial I} = - \lambda_i [I^+ - X^+], \qquad (7)$$

$$- \frac{dX^+}{dt} = \frac{\partial H}{\partial X} = - [\lambda_x + \phi\sigma] X^+. \qquad (8)$$

It is easy to see that these equations can be written succinctly as

$$- \frac{dN^+}{dt} = M^T N^+ \qquad (9)$$

which involves the transpose of M, in this linear problem.

Control is terminated, at a time t_f say, when the target curve is reached. If alpha is the slope of the target curve at t_f

$$\alpha = \frac{dX}{dI} \Big|_\Omega \qquad (10)$$

then this termination or transversality condition leads to one adjoint boundary condition,

$$I^+(t_f) + \alpha X^+(t_f) = 0 \qquad (11)$$

The other boundary condition will be imposed by assigning a definite value to the Hamiltonian at t_f. We chose this value to be zero and will shortly demonstrate that for an optimal solution, $\mathcal{H}$ is zero at all previous times.*

The switching function $\frac{\partial\mathcal{H}}{\partial\phi}$, is

$$\frac{\partial\mathcal{H}}{\partial\phi} = I^{+}\gamma_{i}\Sigma_{f} + X^{+}\gamma_{x}\Sigma_{f} - X^{+}\sigma X \qquad (12)$$

It is seen that the adjoint equations, boundary conditions and switching function are homogeneous in N^{+} and do admit this arbitrary normalization of the Hamiltonian.

In the absence of the $\overline{X}$ restriction, the following necessary conditions are satisfied by an optimal trajectory that minimizes the control period:

(a) N^{+} is continuous and not identically zero at any time t

(b) $\mathcal{H}$= infinitum $\mathcal{H}(\phi)$

With our normalization of the adjoint functions, H is identically zero for an optimum and H identically - 1. The condition that $\mathcal{H}(\phi)$ is its infinitum for variations of ϕ with N^{+} and N constant, i.e. $\mathcal{H}$ takes its least value at every time, is equivalent to the three possibilities:

(a) if $\frac{}{\partial\phi} = 0$, ϕ may take any value

(b) if $\frac{\partial\mathcal{H}}{\partial\phi} > 0$, ϕ must take its maximum value, so that $\delta\phi \leq 0$

(c) if $\frac{\partial\mathcal{H}}{\partial\phi} < 0$, ϕ must take its minimum value, so that $\delta\phi \geq 0$

These conditions secure that the change in the control period due to a change in control variable is not negative around a minimum control period.

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* Pontryagin gives a proof that for free end time problems, as here, optimum $\mathcal{H}$ is indeed zero.

Optimum Conditions with Restricted Xenon

On the state boundary, the adjoint equations are modified. Let us first write the boundary equation in the form

$$g(N) = 0 \tag{13}$$

with the convention that $\frac{\partial g}{\partial N}$ is an outward pointing vector (i.e., in the forbidden direction).

In our problem

$$g = X - X; \qquad \frac{\partial g}{\partial N} = [0,1]. \tag{14}$$

The boundary condition is also satisfied if

$$\frac{dg}{dt} = \qquad = \frac{\partial g}{\partial N}[MN + S] \equiv p(N,\phi) = 0 \tag{15}$$

where p is a function of the state and control variables that, in our problem, is

$$p = \lambda_i I - \lambda_x X - \phi\sigma X + \gamma_x \Sigma_f \phi. \tag{16}$$

Then

$$\frac{\partial p}{\partial N} = [\lambda_i, - \lambda_x - \phi\sigma]. \tag{17}$$

A new Hamiltonian is defined

$$H^* = H + \lambda^* p. \tag{18}$$

where $\lambda^+(t)$ is a Lagrange multiplier for the additional (restricted xenon) equation to be satisfied by the state variables at every time during which the operation is an the boundary. Clearly, since P vanishes when the trajectory is on the boundary, the value of the Hamiltonian is unchanged. Furthermore, we have unchanged state equations

$$\frac{dN}{dt} = \frac{\partial H^*}{\partial N^+} = [MN + S] \qquad (19)$$

but new adjoint equations

$$-\frac{dN^+}{dt} = \frac{\partial H^*}{\partial N} = M^T N^+ + \lambda^+ \frac{\partial p}{\partial N}. \qquad (20)$$

Specifically, we have

$$\frac{dI^+}{dt} = \lambda_i I^+ - \lambda_i [X^+ + \lambda^+] \qquad (21)$$

$$\frac{dX^+}{dt} = [\lambda_x + \phi\sigma][X^+ + \lambda^+]. \qquad (22)$$

The optimum conditions for operations on the boundary now call for the following conditions:

(a) H^* takes its least value as a function of ϕ. This is essentially the previous theorem since H^* reduces to H on the boundary. Note that

$$\frac{\partial H^*}{\partial \phi} = I^+ \gamma_i \Sigma_f + [X^+ + \lambda^+][\gamma_x \Sigma_f - \sigma X]$$

(b) $N^+(t_o)$ is tangent to the boundary where t_o is some initial time. This is imposed to rule out trivial solutions for N^+. It is clear from the adjoint equations that we can add to a solution N^+ an additional constant of the form $\omega^+ \frac{\partial g}{\partial N}$ while subtracting ω^+ from λ^+. We may, therefore, always normalize N^+ at t_o such that the component colinear or parallel to $\frac{\partial g}{\partial N}$ vanishes. This has made $N^+(t_o)$ tangent to the boundary as long as it is not now identically zero, the trivial case to be avoided. It is not necessary to actually carry out this normalization; indeed t_o is not specified.

(c) $\frac{d\lambda^+}{dt} \leq 0$. This last condition is peculiar to Pontryagin's treatment of boundary control. It ensures that allowed perturbations off the boundary would not lead to a better solution.

Continuity Conditions. Pontryagin's jump conditions to connect a solution on the boundary with a solution off the boundary call for

$$N^+(\text{off}) = N^+(\text{on}) + \eta\frac{\partial g}{\partial N} \quad (23)$$

or

$$N^+(\text{off} + \eta\frac{\partial g}{\partial N} \equiv 0. \quad (24)$$

These two conditions are alternatives only, in view of condition (b) above, that $N^+(\text{on})$ is not trivial. In our problem, either condition requires the continuity of I^+ but not X^+ across the point of approaching or leaving the boundary. The number η is real; it is determined off the boundary from the value of the Hamiltonian. On the boundary, X^+ is not determined within an arbitrary constant since an equally acceptable solution of Equations 21 and 22 adds a value ω^+ say to X^+ and $-\omega^+$ to λ^+. However, the function $[X^+ + \lambda^+]$ is fully determined and (as we shall see) is continuous with X^+ off the boundary.

Constancy of Optimum Hamiltonian. Before turning to the determination of the adjoint solutions off and on the boundary, let us demonstrate that when the optimum flux program has been found, the Hamiltonian will be zero at all times. The rate of change of the Hamiltonian, in general, is given by

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial \mathcal{H}}{\partial N}\frac{dN}{dt} + \frac{\partial \mathcal{H}}{\partial N^+}\frac{dN^+}{dt} \quad (25)$$

For an autonomous system, as here, whose coefficients do not depend explicitly on time, the first partial derivative vanishes. The term in ϕ vanishes for an optimum distribution because either

$\frac{\partial \mathcal{H}}{\partial m}$ vanishes or the flux is a constant, maximum or minimum. Introducing the relations satisfied by the remaining terms yields

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial N}\frac{dN}{dt} + \frac{\partial \mathcal{H}}{\partial N^+}\frac{dN^+}{dt} = -\frac{dN^+}{dt}\frac{dN}{dt} + \frac{dN}{dt}\frac{dN^+}{dt} = 0. \quad (26)$$

Since we choose $\mathcal{H}(t_f)$ to be zero, $\mathcal{H}$ is identically zero for an * optimum distribution. Furthermore, the same result applies to H on the boundary since the term $\lambda^+ p$ then vanishes.

Adjoint Solutions

To secure the vanishing of $\mathcal{H}(t_f)$, we must select final values for X^+ and I^+, also satisfying the relation (11) of the form

$$I^+(t_f) = \frac{1}{\left.\frac{dI}{dt}\right|_{\bar{\phi}} \left[\frac{\beta}{\alpha} - 1\right]} \quad (27)$$

$$X^+(t_f) = \frac{1}{\left.\frac{dX}{dt}\right|_{\bar{\phi}} \left[\frac{\alpha}{\beta} - 1\right]} \quad (28)$$

where, as before, a is the slope of the target curve, Ω, while β is the slope of the flux maximum trajectory intercepting it. Clearly for interception $\beta \leq a$; we can see immediately that the equality is not acceptable or we would have infinite importances to give a zero Hamiltonian.

A little manipulation shows that

$$I^+(t_f) = -\alpha X^+(t_f), \quad (29)$$

$$\mathcal{H}(t_f) = 1 + I^+\frac{dI}{dt} + X^+\frac{dX}{dt} = 0 \quad (30)$$

as required, and that these final adjoint conditions are uniquely specified.

On the boundary, the solutions for the flux and resulting iodine that keep X constant at $\bar{X}$ are easily determined (Table I). Since $\frac{dX}{dt}$ vanishes, we obtain the explicit solution for I^+ on the boundary immediately:

$$\mathcal{H} = 1 + I^+ \left.\frac{dI}{dt}\right|_{\bar{X}} = 0 \tag{31}$$

$$I^+ = \frac{-1}{\gamma_i \Sigma_f \phi - \lambda_i I} \tag{32}$$

Table I. Analytical Solutions

1. Flux for operation on line Γ

$$\phi_\Gamma(t) = \frac{[2\lambda_i - \lambda_x] I(t) - \frac{\Sigma_f}{\sigma} \lambda_x \left[\gamma_i + \gamma_x - \gamma_x \frac{\lambda_x}{\lambda_i}\right]}{\sigma I(t) + \Sigma_f \left[2\gamma_i - \gamma_x \frac{\lambda_x}{\lambda_i}\right]}$$

2. Flux and iodine for operation on line χ

$$\phi_\chi(t) = \left[\frac{\lambda_i}{\sigma X - \gamma_x \Sigma_f}\right] \left\{ \left[I_o - \frac{\lambda_x}{\lambda_i} \frac{\gamma_i \Sigma_f X}{(\gamma_i + \gamma_x)\Sigma_f - \sigma X}\right] e^{\lambda_i [(\gamma_i + \gamma_x)\Sigma_f - \sigma X] t} + \frac{\lambda_x}{\lambda_i} \frac{(\gamma_i + \gamma_x)\Sigma_f - \sigma X}{\gamma[\sigma X - \gamma_x \Sigma_f] X} \right\}$$

$$I_\chi(t) = \left[I_o - \frac{\lambda_x}{\lambda_i} \frac{\gamma_i \Sigma_f X}{[(\gamma_i + \gamma_x)\Sigma_f - \sigma X]}\right] e^{\lambda_i [(\gamma_i + \gamma_x)\Sigma_f - \sigma X] t} + \frac{\lambda_x}{\lambda_i} \frac{\gamma_i \Sigma_f X}{[(\gamma_i + \gamma_x)\Sigma_f - \sigma X]}$$

3. Equations for boundary lines

$$\Gamma): \quad X = I + \left[\gamma_i + \gamma_x \frac{\lambda_i - \lambda_x}{\lambda_i}\right] \frac{\Sigma_f}{\sigma}$$

$$\chi): \quad = X = \bar{X}$$

4. Parametric Values Employed

$\gamma_i = 0.056$ $\lambda_i = 0.00174/\text{min}$ $\Sigma_f = 0.001$

$\gamma_x = 0.000$ $\lambda_x = 0.00126/\text{min}$ $\sigma = 3.5 \times 10^{-18}\ \text{cm}^2$

$\phi_\Omega = 2 \times 10^{13}\ \text{n/cm}^2\ \text{sec}$ $\bar{\phi} = 5 \times 10^{13}\ \text{n/cm}^2\ \text{sec}$

But we may differentiate this solution to compare it with the formal equation for $\frac{dI^+}{dt}$, Equation 21. We find that

$$\frac{dI^+}{dt} = \lambda_i I^+ - \lambda_i \frac{\gamma_i \Sigma_f}{[\sigma\overline{X} - \gamma_x \Sigma_f]} I^+ \qquad (33)$$

so that the comparison gives us

$$X^+ + \lambda^+ = \frac{\gamma_i \Sigma_f}{\sigma\overline{X} - \gamma_x \Sigma_f} I^+. \qquad (34)$$

But we also have Equation 22 for $\frac{dX^+}{dt}$, and can therefore establish

$$\frac{dX^+}{dt} = [\lambda_x + \phi\sigma][X^+ + \lambda^+] = \frac{\lambda_x + \phi\sigma}{\sigma\overline{X} - \gamma_x \Sigma_f} \gamma_i \Sigma_f I^+ \qquad (35)$$

from which we could determine X^+ to within the arbitrary constant to which it is defined on the boundary. Of more importance, a further differentiation establishes

$$\frac{d\lambda^+}{dt} = \frac{\gamma_i \Sigma_f I^+}{\sigma\overline{X} - \gamma_x \Sigma_f} \left[\lambda_i \quad \lambda_x \quad \phi\sigma \quad \lambda_i \frac{\gamma_i \Sigma_f}{\sigma\overline{X} - \gamma_x \Sigma_f}\right]$$

$$= \frac{\lambda_i \gamma_i \Sigma_f \sigma I^+}{[\sigma\overline{X} - \gamma_x \Sigma_f]^2} \left[\overline{X} - I - \left(\gamma_i + \gamma_x \frac{\lambda_i - \lambda_x}{\lambda_i}\right) \frac{\Sigma_f}{\sigma}\right] . \qquad (36)$$

The factor before the bracket is positive. If $\frac{d\lambda^+}{dt}$ is to be negative as required for an optimal solution, the bracketed term must be negative. Naturally, we are interested in determining the range over which this is possible. It will be seen that the vanishing of the term is a linear equation between $\overline{X}$ and I, defining a region of the phase plane where operation along the xenon boundary can be optimal.

It turns out that this linear equation is identical with the equation for the line Γ which divides the phase plane into a region where optimal switching is permissible. To see this, we recollect that Γ is determined by the requirement that the switching function, and its rate of change simultaneously vanish:

$$I^{+}\gamma_i\Sigma_f + X^{+}\gamma_x\Sigma_f - X^{+}\sigma X = 0 \tag{37}$$

$$\frac{dI^{+}}{dt}\gamma_i\Sigma_f + \frac{dX^{+}}{dt}\left[\gamma_x\Sigma_f - \sigma X\right] - X^{+}\sigma\frac{dX}{dt} = 0 \tag{38}$$

and substitution leads to the same linear equation inherent in Equation 36.

We can conclude that wherever the end point of the operation on the boundary can be satisfied then the operation on the boundary, itself, is-indeed optimal. Let us now show that these end conditions are the vanishing of the switching function $\frac{\partial H}{\partial \phi}$ and that, on the boundary, the switching function $\frac{\partial H^*}{\partial \phi}$ vanishes identically (thus allowing the Elux program that takes the trajectory along the boundary). Since I^{+} is continuous and known from its solution on the boundary, t-he continuity of the Hamiltonian at either end of the boundary leads to the continuity of

$$H = 1 + I^{+}\frac{dI}{dt} + X^{+}\frac{dX}{dt}. \tag{39}$$

But I^{+} is continuous and the discontinuity in $\frac{dI}{dt}$ and $\frac{dX}{dt}$ is caused by the discontinuity in the flux (I and X are continuous). Therefore, continuity of the Hamiltonian requires the partial derivative (with respect to the flux) to vanish at the end points; this is identical with the vanishing of the switching function at the end points. Furthermore, the relation between $X^{+} + \lambda^{+}$ and I^{+} on the boundary, Equation 34, is itself the condition for the vanishing of the switching function $\frac{\partial H^*}{\partial \phi}$ on the boundary as claimed. It follows that $X^+ + \lambda^+$ on the boundary is continuous with X^* off the boundary at either junction point.

Optimal Behavior Before the Boundary

We have shown analytically that if the boundary operation occurs in that section of the phase plane where the switching function at the end point can be made to vanish, then operation along the boundary is optimal for as long as required. Now it must be shown that the operation at zero flux before the boundary is reached is also optimal; that no additional switching points are found as the trajectory is traced backwards from the boundary.

As in Smith and Roberts treatment of the unrestricted problem, the restricted problem was studied numerically, with the trajectory followed over a range of examples. We chose to carry out by using an analog computer rather than a digital computer. The significant difference, however, is that we have introduced the correct boundary conditions, that the switching function vanishes at the junction with the boundary function; that is at t_b.

Where the computation was carried back until trajectory corresponded to a zero xenon value, no additional switching point was encountered (Figure 2). Perhaps it would be nice to have this shown analytically, but there are no physical or other grounds to anticipate any such additional switching point. Furthermore, since the logic to determine the switching point at t_s (the other end of the boundary) has to be set up, little additional computation is involved in verifying (case by case) that the period of operation at zero flux is optimal throughout the physically significant section of the trajectory.

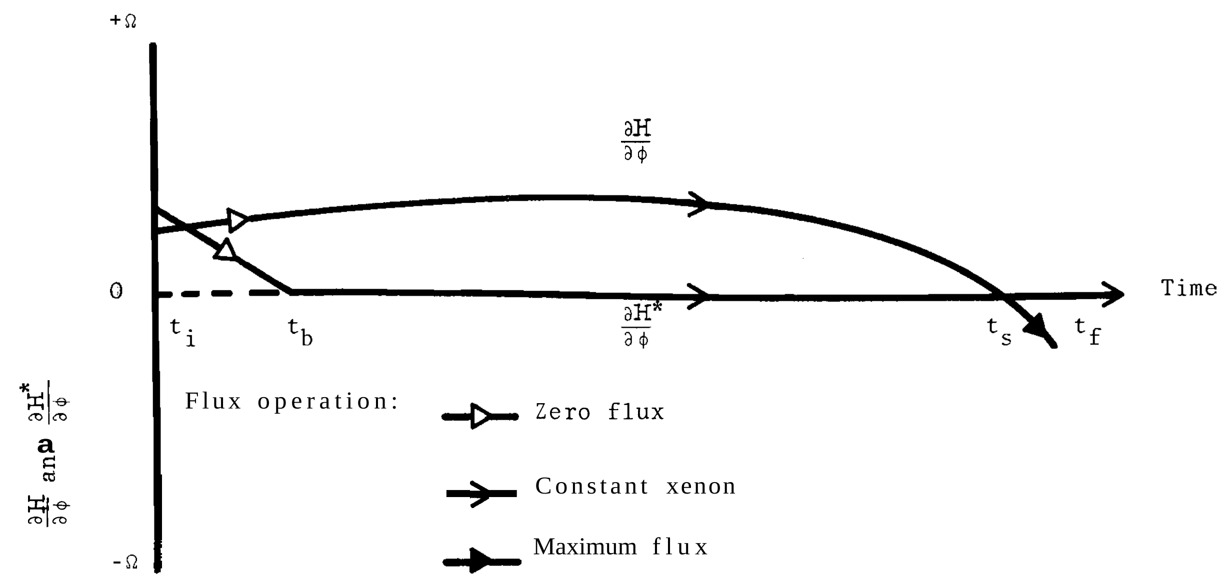

FIGURE 2. Switching Functions for Time-Optimal Trajectory on Boundary

The analog computer was used to illustrate some of the other analytical results by programming it to solve the density and adjoint equations in reverse time. Starting from arbitrary final points on the target curve adjusted until they fitted (a) the boundary value for $\bar{X}$ and, subsequently, (b) a switching to meet a given initial condition from the boundary. The switching function was computed to determine the switch at t_s as well as verify the subsequent optimal zero flux trajectory. Results shown include:

- The switching function, $\frac{\partial H^*}{\partial \phi}$ or $\frac{\partial H}{\partial \phi}$, with and without the boundary logic (Figure 2).
- The adjoint phase plane, X^+ or $X^+ + \lambda^+$ versus I^+ for operation on the boundary with and without the boundary logic (Figure 3)

typical flux-time and xenon-time trajectories (Figure 4).

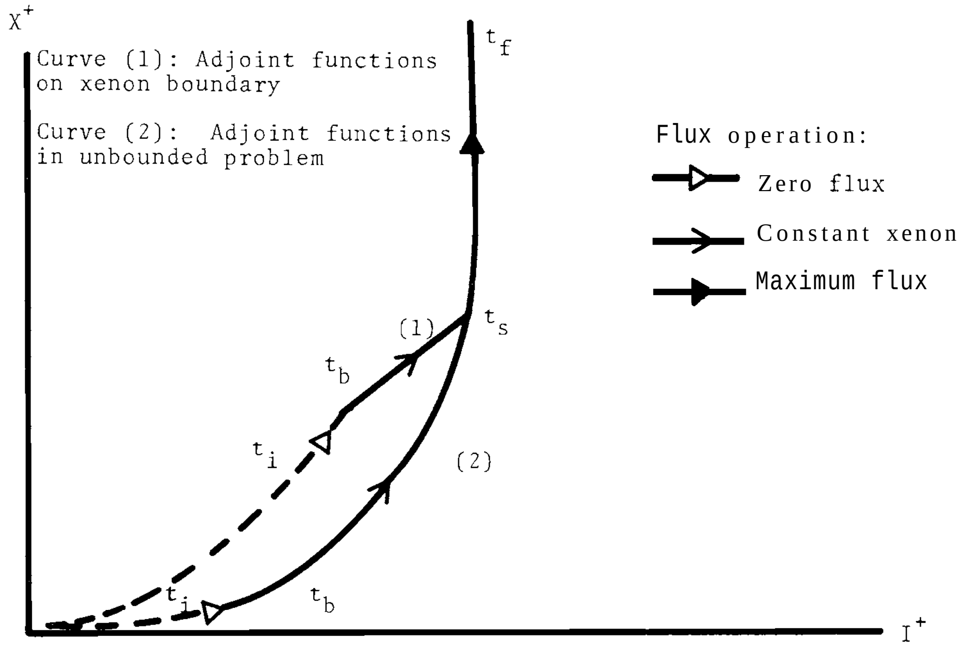

FIGURE 3. Adjoint Phase Plane for Time Optimal Trajectory on Boundary

Target Boundary Conditions

We have already disproved, after Equation 28, Dittman's hypothesis[6] that the time optimal trajectory is tangent to the target curve; this would lead to infinite adjoint functions to make $\mathcal{H}$ zero, or even more generally, to secure a finite value of $\mathcal{H}$ in Equation 4. Dittman's solution is optimal in a different sense in that given one pulse control, the tangent condition permits the minimum of the resulting maximum xenon.

A trivial exception allows Dittman's tangent condition in a limited sense for the case that the operation at maximum flux exists only for an infinitesimal time, and, therefore, the gradient of the flux maximum trajectory is not fully defined. This occurs for a switching at the steady operating point of the target curve;

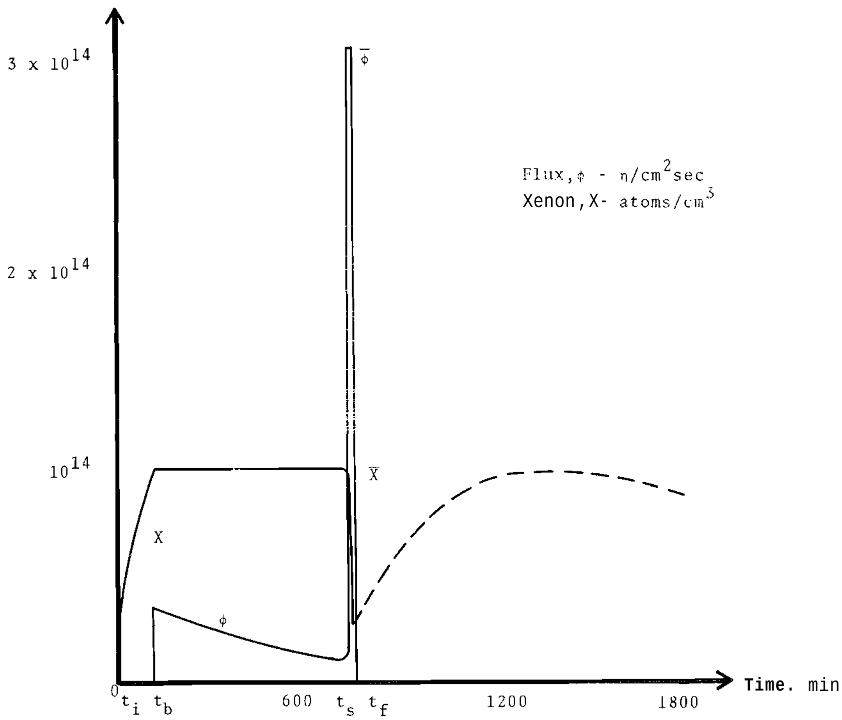

FIGURE 4. Time Optimal Flux and Resulting Xenon

initial points that lie on an extension of the target curve towards the iodine axis, clearly follow the target curve itself and no control is needed. On reaching the target curve, the switching function is momentarily zero, but no flux change is called for because the control period is terminated. To show this, recollect that the steady operating point, I_o, X_o, is given by

$$I_o = \frac{\gamma_i \Sigma_f}{\lambda_i} \phi_\Omega; \qquad X_o = \frac{\gamma_i + \gamma_x}{\lambda_x + \sigma \phi_\Omega} \Sigma_f \phi_\Omega \tag{40}$$

but

$$\alpha = \frac{dX}{dI} = \frac{\gamma_i \Sigma_f \phi_\Omega - \lambda_x X_o}{\gamma_i \Sigma_f \phi_\Omega} , \tag{41}$$

therefore,

$$\frac{\partial H(t_f)}{\partial \phi} = \alpha X^+ \gamma_i \Sigma_f + X^+ [\gamma_x \Sigma_f - \sigma X_o] = 0 \tag{42}$$

as supposed.

SUMMARY

We have formulated the optimum theorem in terms of a Hamiltonian function and its generalization for operation on a boundary of limited density or state variables. We see that the switching function for this generalized Hamiltonian vanishes identically on the boundary, which serves to connect the two regular switching points at either end of the boundary trajectory.

For this particular problem of a boundary at constant xenon, we have been able to prove analytically that wherever the end points of the boundary can be established optimally (ie to the right of the line Γ), then the boundary itself is optimal. In practice, this means that the boundary can extend as far to the right in the phase plane as desired. Of course, this is subject to the availability of a sufficiently large control flux carrying the trajectory to the right. We have also shown for a variety of reactor conditions that the operation at zero flux before the boundary is reached is optimal. These results now fully justify the general nature of the time optimal solutions discussed on page 2.

The fact that the region of allowed boundary control is identical with the region of allowed switching was anticipated. We envisaged a generalization of the restricted problem where the boundary line was no longer the horizontal line, $X = \overline{X}$, but some straight line of arbitrary slope, One such line is Γ itself. Since operation down Γ is already optimal within the framework of the unrestricted problem, it suffers no change on introducing this specific restriction. We could anticipate that any slope restriction to the right of Γ would lead to allowable boundary control. A further generalization would be to curved boundaries rather than straight boundaries. If the same result holds, we might be able to cover the practical case where the variation in the temperature, etc, of the reactor after shutdown leads to changes in the acceptable xenon maximum X.

We suggest, therefore, that a further point of study is to generalize the density boundary as indicated. For a linear boundary, this will be most easily done perhaps by seeking a linear transform of the density vector and equations into the direction of the new boundary, although it may not prove practical to generalize this to the problem of the curved boundary.

For very large excess reactivities, it can happen that the $\overline{X}$ boundary intersects the line Γ rather than reaching a direct switching point. In this case, the optimum bounded trajectory has an additional segment where it follows Γ to the highest switching point. This case lies outside the range of interest studied, but, if it arose, a simple modification could be made in the following analog program which would enable the line Γ to be followed. The equation for Γ and the flux on Γ are given in Table I

ANALOG COMPUTATION METHODS

Computing the Optimum Trajectory

An analog computer, which can be considered an electronic pilot plant, can be installed to monitor the on-line operation of a reactor. The computer is easily modified to undertake a speeded up calculation in an elapsed time of minutes for an optimum flux program. A significant advantage in having an on-line computer available for this task is that realistic values of xenon and iodine densities and admissible xenon maximum densities are available.

As an alternative, the optimum flux program can be solved readily 'off-line' and a tabulation for a range of operating conditions can be made available to the reactor operator. This section of the report describes how an EASE model 1132 analog computer was programmed for such an off-line computation. The program is readily adapted to other computers; in particular, there is an advantage in the use of an iterative analog computer since the task of determining a switching point can be undertaken simultaneously with the computation of the remainder of the flux program.

The representation of the reactor in the xenon-iodine phase plane, Figure 5, is useful in considering the system behavior. The value of the neutron flux determines the rate of change of iodine and xenon leading to a unique trajectory starting from an assumed initial point. Different flux programs or patterns lead to different trajectories from the initial point to the required terminal point on the target curve, Ω. A purely arbitrary program to keep the xenon concentration below a value $\bar{X}$ is shown in Figure 5. But according to the optimum theory, to secure a

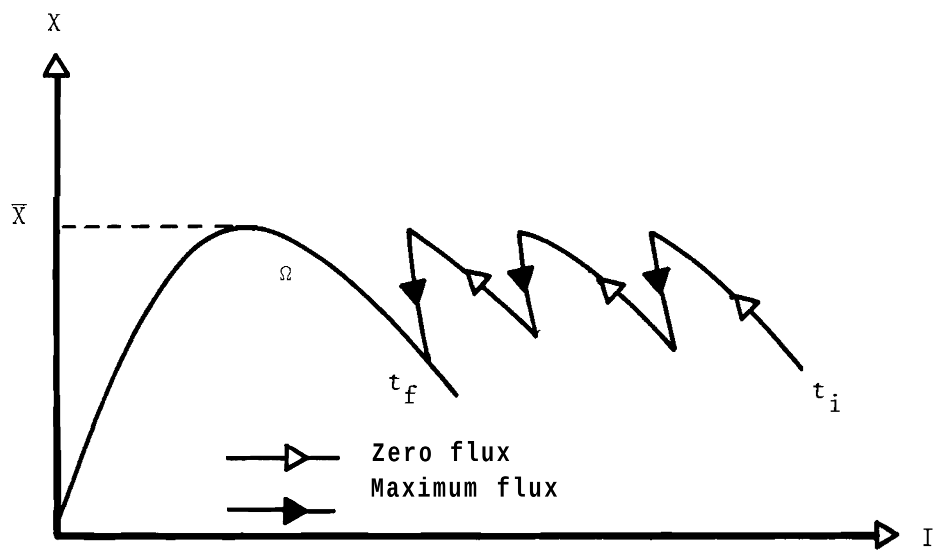

FIGURE 5. Arbitrary Trajectory and Flux Program

minimum control period or shutdown time, the trajectory should follow that shown in Figure 6.

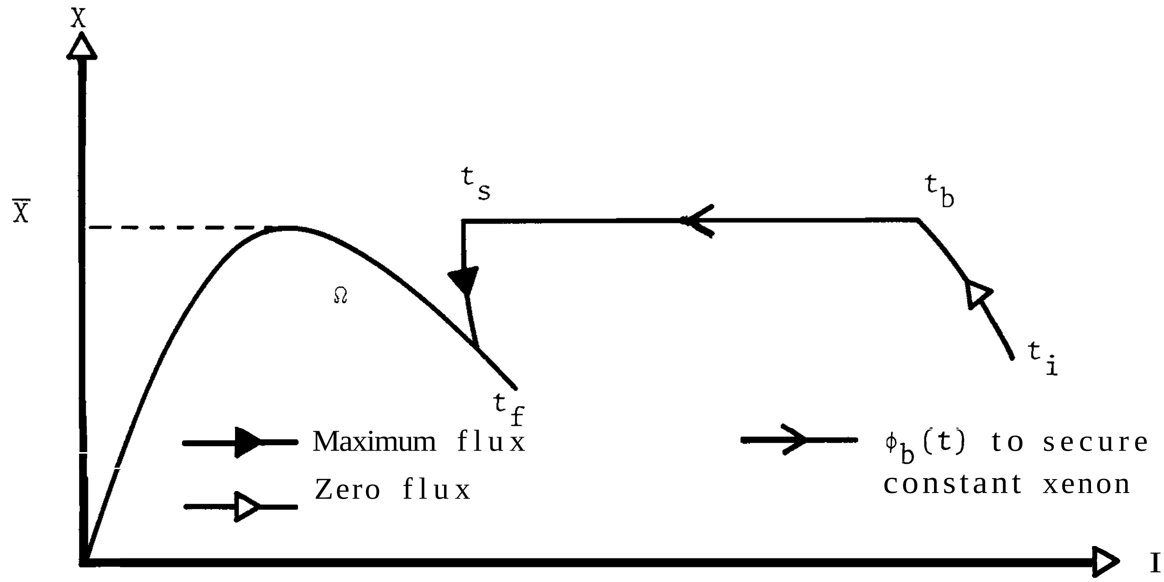

FIGURE 6. Optimum Trajectory and Flux Program

At t_i, the reactor flux is reduced to zero (first shutdown) until the maximum allowable xenon concentration $\overline{X}$ is reached at t_b. A variable flux is then required to control the xenon concentration which keeps it constant until the trajectory reaches a previously determined switching point at t_s. At the switching point, the maximum avialable control flux is applied to burn out xenon and to bring the trajectory to the target curve.

In the determination of the switching point, it is necessary to employ the adjoint equations, boundary conditions, and switching function of the theory already discussed. Figure 7 shows a family of curves for shutdown with no restriction on the xenon density. The switching point is the apex of the trajectory, which marks the change from a zero to a maximum flux. It is seen that for different end points on the target curve, there are different xenon values at the switching point. The task is to determine which end point will lead to a switching with the zenon value equal to $\overline{X}$ in the bounded problem.

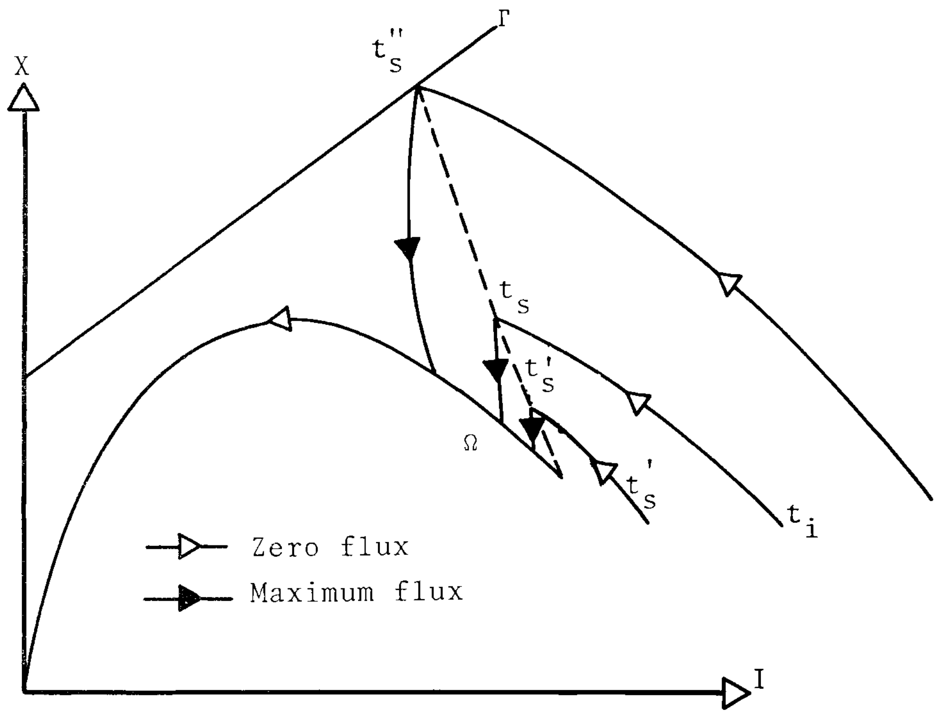

FIGURE 7. Unrestricted Switching Points

To follow the optimum trajectory, certain information is required by a reactor operator. He needs to know t_b, the time at which the flux is to be switched on after first shutdown, the flux trajectory to follow, the time t_s at which to switch. to a maximum flux, and the time t_f at which this flux is switched off and the control period finished. For given maxima flux and xenon densities, but for different initial conditions, this information can be determined and presented conveniently by measuring backwards from a fixed final time. (In this case, the switching time is also fixed, because for a given $\overline{X}$ and $\overline{\phi}$, the switching point is fixed.)

Figure 8 illustrates the combination of forward and reverse time followed by the analog in determining the target curve and

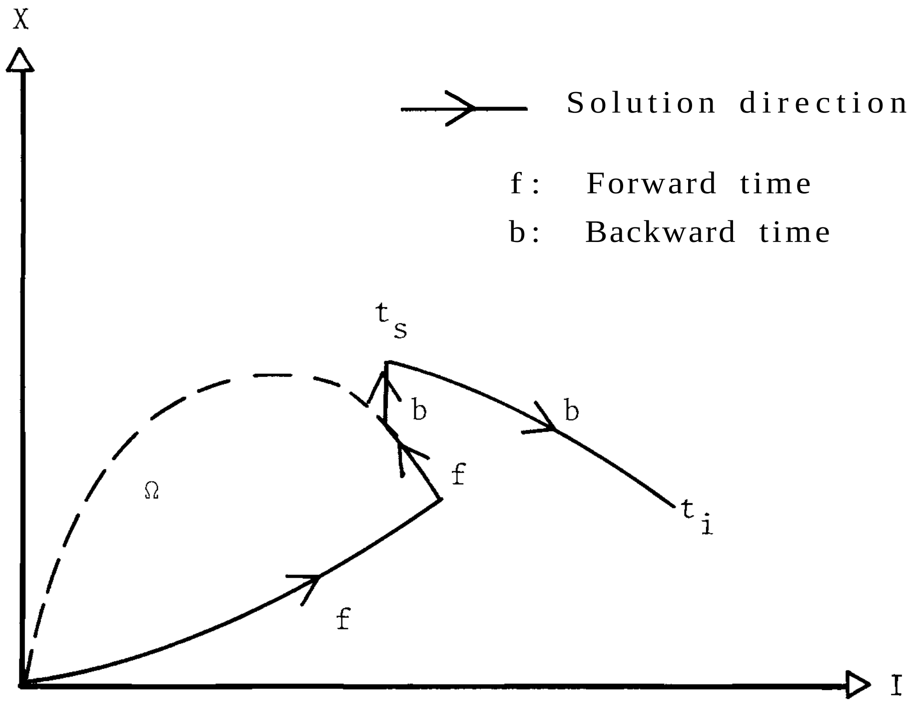

FIGURE 8. Solution Sequence to Find Switching Points

the switching point. First, the xenon-iodine target curve is established and a suitable point on the target curve is arbitrarily selected. The switching function is computed by solving the ndjoint and process equations in reverse time from the target curve with the computer 'holding' when the switching function vanishes. The xenon and iodine values are observed at this switching point, and the calculation is repeated, if necessary, from a different point on the target curve until switching occurs with the xenon density exactly equal to the required value $\overline{X}$.

Figure 9 presents the equations used in the different portions of the trajectory or in establishing the target curve (including equations for the formal proof of optimality, discussed in the next section). These equations have been categorized as:

- Establishing target curve - Initialization
- Switching point determination - Regime 1
- Operation on the boundary - Regime 2
- Returning to operating point - Regime 3.

The solid portion of the curves depicted with each set of equations shows where the equations apply.

Figure 10 shows the patching diagrams with special switches as used on the EASE 1132 analog computer. Integrators 2 and 3 simulate the iodine and xenon densities to satisfy the control flux (Amplifier 1). The adjoint equations are simulated with "special" Integrators 6 and 7. These integrators are special because their initial condition mode is controlled by Hand Switch 4 (SW4 in upper right hand corner of Figure 9). When SW4 is down, the integrators are in initial condition (I.C.)

Differential Relay 1 (K-1) provides the logic for altering the flux at the switching point and for automatically placing the computer in HOLD when SW1 is up. Differential Relay 2 and SW1 are used for automatically placing the computer in HOLD when the iodine value drops below the value by Hand Set Pot 1 (H1). Switch 7, together with Differential Relays K-3 and K-5, is used for changing from forward time to reverse time. Hand Switches 2, 3, 5 are used for placing multipliers in the divide mode. Figure 11 shows the sequence of operating all switches for satisfactory simulation of the problem.

Proof of Optimal Trajectory

To use the analog computer to repeat the proof of optimal shutdown in the zero flux section of the trajectory, it is necessary to make use of the additional equations shown in sections 3 and 4 of Figure 9. These are not required for just

INITIALIZATION

EQUATIONS FOR ESTABLISHING TARGET CURVES (IN FORWARD TIME)

$$\frac{dI}{dt} = -\lambda_i I + \gamma\Sigma_f\phi$$

$$\frac{dX}{dt} = \lambda_i I - \lambda_x X - \phi\sigma X$$

$$I_{(o)} = X_{(o)} = 0$$

$$\phi = \phi_{target} \text{ when } \frac{dI}{dt} > 0$$

$$= 0 \quad \frac{dI}{dt} \leq 0$$

REGIME 1

EQUATIONS FOR DETERMINING SWITCHING POINT (IN REVERSE TIME)

$$\frac{dI}{d\tau} = +\lambda_i I - \gamma\Sigma_f\phi$$

$$\frac{dX}{d\tau} = -\lambda_i I + \lambda_x X + \phi\sigma X$$

$I_{(tf)}$ = , $X_{(t)}$ = {Values that result in $\partial H/\partial\phi = 0$ at maximum allowable xenon density (determined by trial and error).}

$$\phi = \phi_{max} \quad \partial H/\partial\phi < 0$$

$$= 0 \quad \partial H/\partial\phi \geq 0$$

$$\frac{\partial H}{\partial\phi} = I^+\gamma\Sigma_f - X^+\sigma X$$

$$\frac{dI^+}{d\tau} = -\lambda_i[I^+ - X^+]$$

$$\frac{dX^+}{d\tau} = -\lambda_x X^+ - \phi\sigma X^+$$

$$X^+_{tf} = \frac{1}{dI/dt[\beta/\alpha - 1]}$$

$$I^+_{tf} = -\alpha X^+_{tf}$$

$$\beta = -dX/dI\big|_{tf} \text{ with } \phi_{max}$$

$$\alpha = -1 + \lambda_x X/\lambda_i I$$

REGIME 2

EQUATIONS FOR OPERATING ON THE BOUNDARY

Required Equations

$$\frac{dI}{d\tau} = +\lambda_i I - \gamma\Sigma_f\phi_t$$

$$\frac{dX}{d\tau} = -\lambda_i I + \lambda_x X + \phi_t\sigma X$$

$$\phi_\tau = \frac{\lambda_i I - \lambda_x X}{\sigma X}$$

$I_{(t_s)}$ = , $X_{(t_s)}$ = {Values that produce a zero switching function at maximum allowable xenon (determined by trial and error).}

Equations for Demonstrating Proof of Derivation

$$I^+ = \frac{1}{dI/dt}$$

$I^+_{(t_s)}$ is continuous

$$\frac{dX^+}{d\tau} = [\lambda_x + \phi_t\sigma]\frac{\gamma\Sigma_f}{\sigma X} I^+$$

$X^+_{(t_s)} = 0$ (Arbitrary initialization)

$$\lambda^+_t + X^+ = \frac{\gamma\Sigma_f I^+}{\sigma X}$$

REGIME 3

EQUATIONS FOR RETURNING TO OPERATING POINT (IN REVERSE TIME)

Required Equations

$$\frac{dI}{d\tau} = +\lambda_i I$$

$$\frac{dX}{d\tau} = -\lambda_i I + \lambda_x X$$

$I_{(t_b)}$, $X_{(t_b)}$ {Continuous

Equations for Demonstrating Proof of Derivation

$$\frac{\partial H}{\partial\phi} = I^+\gamma\Sigma_f - X^+\sigma X$$

$$\frac{dI^+}{d\tau} = -\lambda_i[I^+ - X^+]$$

$$\frac{dX^+}{d\tau} = -\lambda_x X^+$$

$I^+_{(t_b)}$ is continuous

$$X^+_{(t_b)} = \frac{\gamma\Sigma_f}{\sigma X} I^+_{(t_b)}$$

FIGURE 9. Summary of Equations for Solving the Time Optimal Boundary Limited Problem

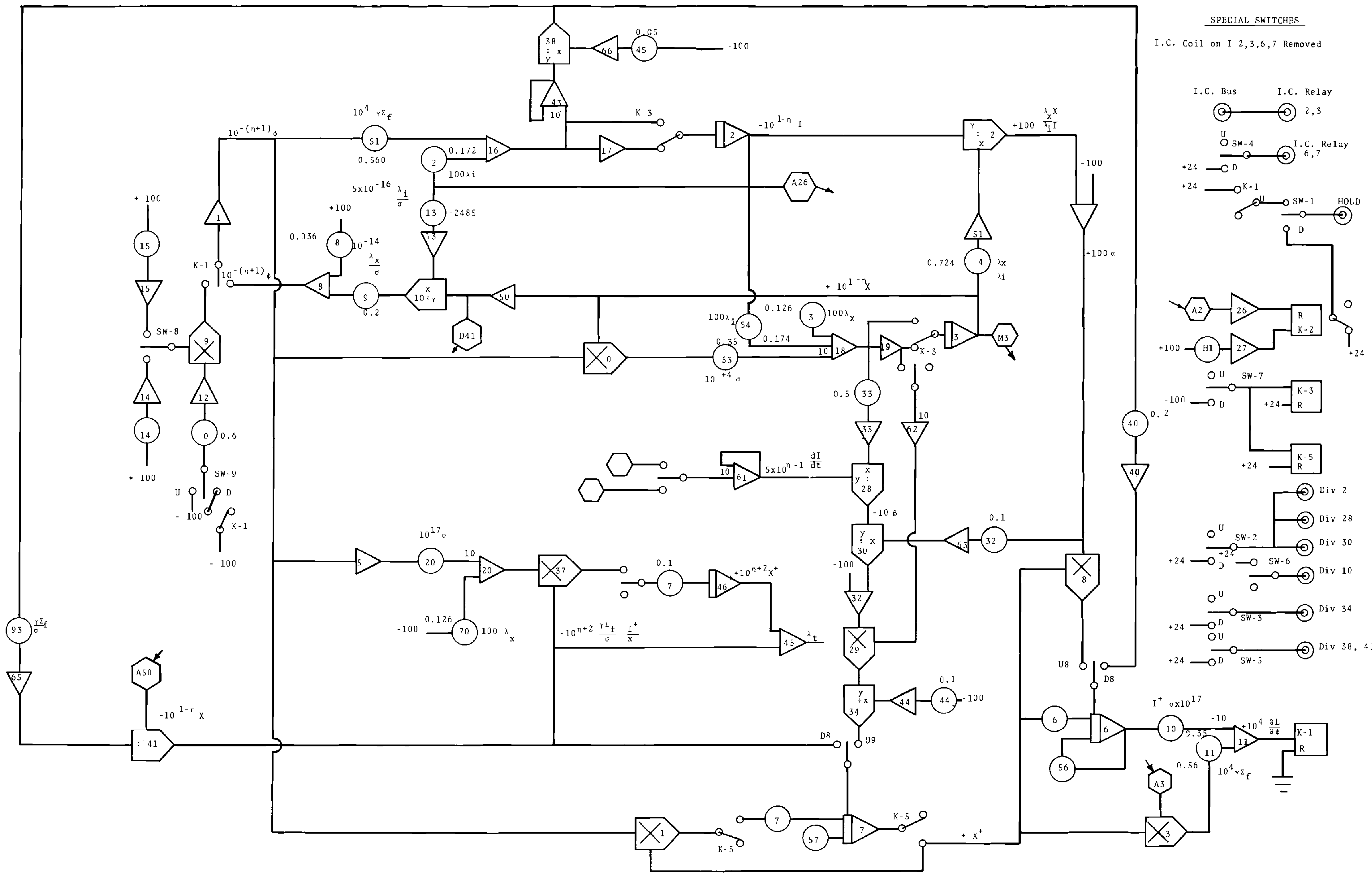

FIGURE 10. Xenon Time Optimal Shutdown 1132 Computer

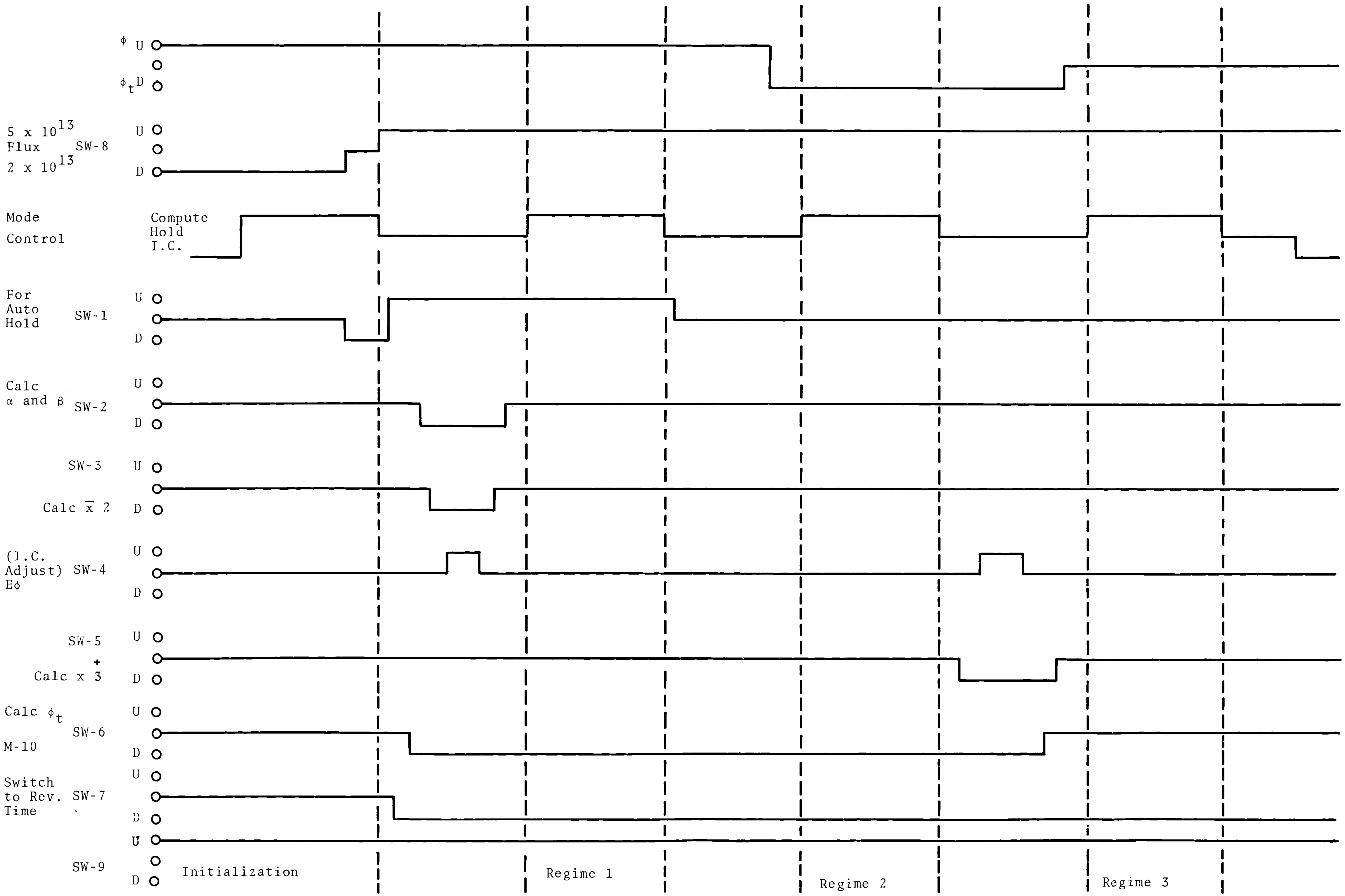

FIGURE 11. Switching Logic

determining the optimum trajectory. The adjoint equations in Regime 3 are the same as in Regime 1. Note that the initial conditions for the adjoint equations in Regime 3 are not the same as in Regime 1. Rather, the adjoint solution is generated in Regime 2 in order to provide the correct initial conditions in Regime 3. The necessary switching for this extension of the analog simulation is shown as part of Figure 11.

Presentation of Results

When the computation is done "off-line", the necessary information for standard operating procedures can be presented in the form of a chart of flux versus time. Such charts will vary with

- Initial condition of reactor - The most significant case where the reactor had been running at steady state before shutdown. It is easy to generate solutions for any initial conditions, but the operator must know what initial values of iodine and xenon he has, probably from an on-line computer.
- Maximum flux during shutdown - This will usually be the steady operating flux.
- Maximum admissible xenon density - This will vary with the available excess reactivity.

Dittman[6] discussed a nondimensional representation of the phase plane, which might be found useful in presenting optimal trajectories when these variables are altered.

Conclusion

The analog program and techniques here enable the optimal shutdown flux to be determined in minimum time while not exceeding restrictions on the maximum xenon poisoning permitted. The computer can find and present the information necessary for a reactor following such a program as in Figure 4.

Two observations should be made. The first concerns the sensitivity of the solution during the period of maximum flux, especially in the determination of the switching point. Although the method detailed here completes the optimum trajectory by following back from the switching point, there are advantages in using an alternative procedure (employing the same analog program, however). For example, the required switching point would be calculated separately, by use of a time scale that would enable a more accurate solution to be found.* With this switching point determined, it would be then elementary to follow the optimum trajectory at a higher computing speed for determining the time of switching relative to the first shutdown.

The second observation concerns an obvious generalization of the procedure. In the problem discussed so far, it was required that the reactor be available for restart at <u>all</u> times after t_f. In practice, it may be that a short outage would be accepted, with the advantage being that the control period can be shortened even further. In the phase plane, the trajectory would be terminated earlier on a target curve whose peak was greater than $\overline{X}$. There would be, therefore, a period where the xenon density rose about $\overline{X}$ and prevented restart. The analog computer is well suited for following this new target curve and determining the time at which the xenon density falls again to $\overline{X}$, allowing restart. It then becomes a matter of iteration to select the new target curve leading to the desired restart time. The technique for obtaining the optimum trajectory to the target curve is unchanged.

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* Note that the pulse of maximum flux may last for only tens of minutes in Regime 1 while Regimes 2 and 3 may last in all for 10 to 15 hr.

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