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LECTURES ON REGGE POLES AND THEIR PHENOMENOLOGICAL APPLICATION IN HIGH-ENERGY PHYSICS

by

R. C. Arnold

Printed in USA. Price $3.00. Available from the Clearinghouse for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Springfield, Virginia 22151

ANL-7173
Physics (TID-4500)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois
60439

LECTURES ON REGGE POLES AND THEIR PHENOMENOLOGICAL APPLICATION IN HIGH-ENERGY PHYSICS

by

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February 1966

Operated by The University of Chicago
under
Contract W-31-109-eng-38
with the
U. S. Atomic Energy Commission

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LECTURES ON REGGE POLES AND THEIR PHENOMENOLOGICAL APPLICATION IN HIGH-ENERGY PHYSICS

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R. C. Arnold

PREFACE

These lectures will be divided into two parts. In Part One, we will discuss bound states and scattering from a nonrelativistic, potential-theory viewpoint. Regge poles will be introduced as generalizations of bound states and scattering resonances. Regge trajectories are discussed as aids in classifying spectra of complex two-body systems with many bound or resonance states. A classification of presently known, strongly interacting particles and resonances is exhibited to the extent that data on the spectrum are presently available.

In Part Two, crossing relations and the Sommerfeld-Watson transform are employed to discuss the influence of Regge poles on low-momentum-transfer, high-energy reactions in crossed channels. Phenomenological analyses of data from selected two-body reactions are discussed, and successes and failures of simple Regge-pole models are explicated. Peripheral models for inelastic reactions are treated as special cases. Finally, a (presently) semiphenomenological optical-model framework, including as a special case the absorptive correction method, is briefly described as an example of a more general approach to high-energy reactions in which the Regge poles appear as an approximation valid in the empirically successful cases described above.

PART ONE

BOUND STATES, RESONANCES, AND REGGE POLES IN NONRELATIVISTIC POTENTIAL THEORY

I. Simple Model Illustrating Regge Trajectories and Signatures

Consider a diatomic molecule, e.g., H_2, with two identical, spin-1/2 nuclei, bound such that a rigid rotator model is a good approximation. Consider nuclei only, treating electrons as a self-consistent phenomenological potential. Assume no nuclear-spin dynamical coupling. (See Landau and Lifshitz.[1]) There are two nuclear-spin wave functions, S = 1 (symmetric) and S = 0 (antisymmetric). Because of the Pauli principle, only odd ℓ rotational states occur in the first case (ℓ = 1, 3, 5, ...) and even in the second (ℓ = 0, 2, 4, ...). The rotational spectrum of this molecule then consists of two disjoint sequences of energy levels (para and ortho),

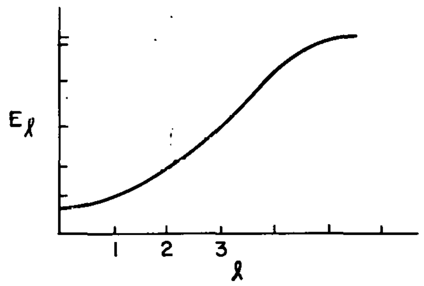
$$E_\ell^{(1)} = E_0 + \hbar\omega_0 \ell(\ell + 1), \quad \ell = 1, 3, 5, \ldots;$$

and

$$E_\ell^{(2)} = E_0 + \hbar\omega_0 \ell(\ell + 1), \quad \ell = 0, 2, 4, \ldots .$$

Note ω_0^2 is inversely proportional to the moment of inertia, hence inversely proportional to (nuclear separation)^2. If a small nuclear-spin coupling is introduced, then the two sequences will no longer linearly interpolate each other, but will be displaced a small amount, e.g., $E_0 \to E_0{}^{\pm}$.

We can plot E_ℓ vs ℓ; this appears as shown in the following sketch, for small ℓ. The large ℓ part of the curve is the breakdown of rigid-rotator model. Now observe:



a. There are obvious, simple, analytic functions $E^{\pm}(\ell) = E_0^{\pm} + \ell(\ell + 1)\hbar\omega_0$, which interpolate between the bound states at integer ℓ. Such a function can be inverted to give $\ell(E)$, called a trajectory.

b. Other trajectories usually exist, with different radial (i.e., vibrational) quantum number, which have different ω_0.

c. Trajectories will turn over eventually since rotation will pull the molecule apart if rotation is too energetic.

II. Integral Equation for Scattering Amplitude and Analyticity in ℓ with General Potentials in NR Schrödinger Equation

When bound states are present, we can show that there are such interpolating trajectories, which represent bound states when $\ell(E)$ passes upward through an integer. In general, there will be (+) and (−) trajectories due to the existence of exchange forces, which provide different potentials in even and odd ℓ states.

Not only bound states, but also scattering resonances may be connected by a trajectory. To show this, we must make a close connection between scattering and bound-state solutions of the Schrödinger equation (SE), and exhibit smooth behavior of SE solutions as ℓ varies.

We will replace SE by an integral equation for the scattering amplitude; these unified properties will be easy to see, and an approximate solution (for noninteger ℓ as well as integer) can be exhibited.

Let us consider a two-particle system, bound (E<0) or scattering (E>0). Put $k^2 = E$; then k is real for scattering, and imaginary for the bound state. Angular and radial-wave functions

$$\psi_k(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{U_\ell(k,r)}{r} P_\ell(\cos\theta)$$

are separated whenever potential depends only on $|\vec{r}|$. The Schrödinger equation

$$(\vec{\nabla}^2 + k^2)\,\psi(\vec{r}) = V(r)\,\psi(\vec{r})$$

then separates, giving radial equations for each ℓ.

For either scattering or bound-state boundary conditions, these radial-wave functions for angular momentum ℓ satisfy

$$\frac{d^2U_\ell(k,r)}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] U_\ell(k,r) = V(r)\,U_\ell(k,r). \tag{1}$$

The boundary condition (BC) appropriate for scattering (k real) is

$$\psi_k(r) \underset{r\to\infty}{\to} e^{i\vec{k}\cdot\vec{r}} + \frac{f(\theta)}{r} e^{ikr},$$

where $\vec{k}$ is a vector along the incoming beam direction with magnitude k, and $f(\theta)$ is called the scattering amplitude.

For bound states, the BC is (with k imaginary)

$$N_k \equiv \int d^3\vec{r}\,|\psi_k(r)|^2 < \infty.$$

Note that these are mutally exclusive conditions.

The BC's in terms of radial wave functions may be written

$$\int_0^\infty dr|U_\ell(k,r)|^2<\infty$$

for a bound state of angular momentum ℓ, and

$$\frac{U_\ell(k,r)}{r} - j_\ell(kr) \underset{r\to\infty}{\to} A_\ell(k) \frac{e^{i(kr - \ell\pi/2)}}{r}.$$

for scattering in a state of angular momentum ℓ. The j_ℓ term represents the plane-wave part (see below). A complete set of solutions of the homogeneous version (V = 0) of (1) are spherical Bessel function $j_\ell(kr)$, $y_\ell(kr)$, which have asymptotic behavior as $r \to \infty$, as follows:

$$kr\ j_\ell(kr) \to \sin\ (kr - \ell\pi/2)$$

and

$$kr\ y_\ell(kr) \to \cos\ (kr - \ell\pi/2).$$

A plane wave, which satisfies the scattering BC with V = 0, hence f = 0, is represented as follows with j_ℓ functions only;

$$\psi_k^{(0)}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} e^{i\pi\ell/2}\ j_\ell(kr)\ P_\ell(\cos\ \theta).$$

A purely outgoing wave solution can also be constructed of the form

$$h_\ell^{(1)}(kr) = y_\ell(kr) + ij_\ell(kr) \to \frac{e^{i(kr - \ell\pi/2)}}{kr}.$$

Knowing a complete set of solutions for the homogeneous version of (1) enables us to construct Green's functions for (1) and to convert (1) to an integral equation. (An alternative to our approach is presented in Ref. 2.)

For any BC, we first find a $G_\ell(k;r,r')$ such that

$$\frac{d^2G_\ell(k;r,r')}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] G_\ell(k;r,r') = \delta(r - r'). \qquad (2)$$

For scattering BC, we then construct G such that

$$G_\ell(k;r,r') \underset{\substack{r\to\infty,\\ r'\ \text{finite}}}{\to} B_\ell(k) \frac{e^{i(kr - \ell\pi/2)}}{r}.$$

Given such a G, we now show that (1) with scattering BC can be replaced by the integral equation

$$U_\ell(k,r) = j_\ell(k,r) + \int_0^\infty dr'\ G_\ell(k;r,r')\ V(r')\ U_\ell(k,r'), \tag{3}$$

provided V drops off rapidly enough with increasing r.

To show that a solution of (3) satisfies scattering BC, it is sufficient to examine the integral over G, which is

$$\lim_{r\to\infty}\left[\frac{U_\ell(k,r)}{r} - j_\ell(kr)\right].$$

This integral just depends on the BC for G, which was specified correctly above.

Explicitly, we now assert that G can be represented as

$$G_\ell(k;r,r') = kj_\ell(kr_<)\ h_\ell^{(1)}(kr_>). \tag{4}$$

where

$$r_< = \min(r,r') \text{ and } r_> = \max(r,r').$$

The BC is obvious. The fact that G satisfies (2) is not so obvious; it is however, easy to see that the left-hand side of (2) vanishes for $r \neq r'$.

The representation (4) may be replaced, by using a product formula for Bessel functions, by the integral representation

$$G_\ell(k;r,r') = \int_0^\infty \frac{dq}{q^2 - k^2 - i\varepsilon}\ j_\ell(qr)\ j_\ell(qr'). \tag{5}$$

Now the scattering amplitude $f_\ell(k)$ is defined by

$$\frac{e^{i(kr - \ell\pi/2)}}{r}\ f_\ell(k) = \lim_{r\to\infty}\left[\frac{U_\ell(k,r)}{2} - j_\ell(kr)\right]$$

$$= \lim_{r\to\infty} \int_0^\infty dr'\ G_\ell(k;r,r')\ V(r')\ U_\ell(k,r')$$

$$= \lim_{r\to\infty} kh_\ell^{(1)}(kr) \int_0^\infty dr' j_\ell(kr')\ V(r')\ U_\ell(k,r')$$

$$= \frac{e^{i(kr - \ell\pi/2)}}{r} \int_0^\infty dr'\ j_\ell(kr')\ V(r')\ U_\ell(k,r').$$

Thus,

$$f_\ell(k) = \int_0^\infty dr\ j_\ell(kr)\ V(r)\ U_\ell(k,r). \tag{6}$$

This requires the prior computation of U_ℓ. Now we will obtain an integral equation which yields f_ℓ directly without first computing U_ℓ. Define $f_\ell(k,k')$ by the formula

$$f_\ell(k,k') = \int_0^\infty dr\ j_\ell(kr)\ V(r)\ U_\ell(k',r), \tag{7}$$

where U_ℓ is the solution of (3). Note then that $f_\ell(k) = f_\ell(k,k)$ is the physical scattering amplitude. We now derive an integral equation for $f_\ell(k,k')$, which we will denote as the off-energy-shell scattering amplitude

Substituting (3) into (7), and using (5), we have

$$f_\ell(k,k') = \tilde{V}_\ell(k,k') + \int_0^\infty dr\ j_\ell(kr)\ V(r) \int_0^\infty dr' \int_0^\infty \frac{dq}{q^2 - k^2 - i\varepsilon}.$$

$$j_\ell(qr)\ j_\ell(qr')\ V(r')\ U_\ell(k',r'),$$

where

$$\tilde{V}_\ell(k,k') = \int_0^\infty dr\ j_\ell(kr)\ V(r)\ j_\ell(k'r).$$

Now the r integral can be carried out in terms of V_ℓ by inverting the orders of integration; we have then

$$f_\ell(k,k') = \tilde{V}_\ell(k,k') + \int_0^\infty \frac{dq}{q^2 - k'^2 - i\varepsilon}\ \tilde{V}_\ell(k,q) \int_0^\infty dr'\ j_\ell(qr')\ V(r')\ U_\ell(k',r').$$

The r' integral now yields $f_\ell(q,k')$, and we have, finally,

$$f_\ell(k,k') = \tilde{V}_\ell(k,k') + \int_0^\infty \frac{dq}{q^2 - k'^2 - i\varepsilon}\ \tilde{V}_\ell(k,q)\ f_\ell(q,k'). \tag{8}$$

Notes: 1. In this equation, k' is fixed and is treated as a parameter when the equation is being solved. When a solution is obtained, we set $k' = k$ to obtain the physical partial-wave scattering amplitude $f_\ell(k)$.

2. For bound states, $k'^2 < 0$, so the $i\varepsilon$ is unimportant; there is no singularity in the kernel then.

For a simple Yukawa potential $V(r) = (ge^{-\mu r})/r$, we obtain

$$\tilde{V}_\ell(k,q) = \frac{g}{2k^2}\ Q_\ell\left(\frac{k^2 + q^2 + \mu^2}{2kq}\right).$$

For a superposition of Yukawas, then,

$$V(r) = \int_\mu^\infty d\lambda \; g(\lambda) \frac{e^{-\lambda r}}{r},$$

we obtain

$$\tilde{V}_\ell(k,q) = \frac{1}{2k^2} \int_\mu^\infty d\lambda \; g(\lambda) \; Q_\ell \left(\frac{k^2 + q^2 + \lambda^2}{2kq} \right).$$

Now $Q_\ell(Z)$ is an analytic function of ℓ, for $\ell \neq -1, -2, \ldots$, which (see Appendix A) can be represented as a hypergeometric function. Thus (8) yields (in general) solutions $f_\ell(k)$ for (almost all) complex ℓ values, which coincide with physical scattering amplitudes when $\ell = 0, 1, 2, \ldots$.

III. Formal Solution of Integral Equation for Scattering Amplitude; Bound States; First-order Determinantal Approximation

In the previous section, we have gone as far as one can go with scattering states without obtaining explicit solutions for (8). We will now exhibit a method of solution, show that the bound states are also obtained from the solutions of (8), and derive a rough approximation for the bound-state locations. This will be used in a spirit similar to the Born approximation, which is usually taken as a rough guide to scattering. The approximation will retain analyticity in ℓ and thus exhibit Regge trajectories for any potential that can be represented as a superposition of Yukawas.

Take $\varepsilon \neq 0$ for the present. Consider approximating the integral in (8) by a sum over N discrete q values $\{q_n\}$, with weights $\{\omega_n\}$ for integration. Then, evaluating k also at these values, we obtain (for fixed k') the approximate equations

$$f_\ell(k_n,k') = \tilde{V}_\ell(k_n,k') + \sum_{m=1}^{N} \left\{ \omega_m \frac{\tilde{V}_\ell(k_n,q_m)}{q_m^2 - k'^2 - i\varepsilon} \right\} f_\ell(q_m,k') \qquad (9)$$

for n = 1, 2, ..., N. Let

$$\vec{f}_\ell = \begin{pmatrix} f_\ell(k_1,k') \\ f_\ell(k_2,k') \\ \cdot \\ \cdot \\ \cdot \\ f_\ell(k_N,k') \end{pmatrix}, \qquad \vec{V}_\ell = \begin{pmatrix} \tilde{V}_\ell(k_1,k') \\ \tilde{V}_\ell(k_2,k') \\ \cdot \\ \cdot \\ \cdot \\ \tilde{V}_\ell(k_N,k') \end{pmatrix},$$

and K_ℓ = matrix whose $(n,m)^{th}$ entry is the expression in braces { } in Eq. (9). Note that K_ℓ (as well as $\vec{V}_\ell$) is analytic in ℓ, in the sense that all terms in K (and $\vec{V}$) are analytic functions of ℓ. Then (9) can be written

$$\vec{f}_\ell = \vec{V}_\ell + K_\ell \vec{f}_\ell, \quad \text{or} \quad (I - K_\ell)\vec{f}_\ell = \vec{V}_\ell, \tag{9a}$$

where $(I)_{mn} \equiv \delta_{mn}$ and the matrices I and K are N x N. If $(I - K_\ell)$ is a non-singular matrix, we may invert it and obtain the following solution to (9):

$$\vec{f}_\ell = (I - K_\ell)^{-1}\vec{V}_\ell \tag{10}$$

We can then approximate the solution of (8) arbitrarily well by taking $N \to \infty$.

Since the inverse M^{-1} of a matrix M depends analytically on the elements M_{ij}, $\vec{f}_\ell$ will be analytic in ℓ except for isolated values of ℓ such that $\det(I - K_\ell) = 0$. Let $D_\ell(k') = \det(I - K_\ell)$. (Recall that k' is a parameter in K_ℓ.) Now we prove the following: when $D_\ell = 0$, and $k^2 < 0$, a bound state exists for these values of ℓ and k. The term "bound state" will be applied at the moment for integer (physical) ℓ values.

To prove this assertion, consider the matrix equation (9a) (for arbitarily large N). If $D \neq 0$, there is a unique solution to (9a); but if $D = 0$, (9a) has a solution if and only if the inhomogeneous term $\vec{V}_\ell$ is zero. (This is a well-known matrix-theory theorem.) Thus, if $D = 0$, $\vec{f}$ satisfies the homogeneous equation

$$\vec{f}_\ell^{(0)} = K_\ell \vec{f}_\ell^{(0)}. \tag{11}$$

Passing back $(N \to \infty)$ to the continuous functions, we obtain the following homogeneous equation analogous to (8):

$$f_\ell^{(0)}(k,k') = \int_0^\infty \frac{dq}{q^2 - k'^2 - i\varepsilon} \tilde{V}_\ell(k,q)\, f_\ell^{(0)}(q,k').$$

Retracing the steps leading to (8) and replacing $f_\ell^{(0)}$ by its expression in terms of wave functions, we find that the associated radial-wave function $U_\ell^{(0)}(k,r)$ must satisfy the homogeneous equation corresponding to (3), or

$$U_\ell^{(0)}(k,r) = \int_0^\infty dr'\, G_\ell(k;r,r')\, V(r')\, U_\ell^{(0)}(k,r'). \tag{12}$$

Now this wave function is normalizable; using the BC for G_ℓ for $r \to \infty$, we obtain

$$U_\ell^{(0)}(k,r) \to C_\ell(k) \frac{e^{i(kr - \ell\pi/2)}}{r}.$$

This shows the integrability of $|U|^2$ at the upper limit if $k^2 < 0$. G_ℓ is integrable at $r \to 0$. Thus when $D_\ell(k) = 0$, there exists $U_\ell^{(0)}(k,r)$ such that

$$\int_0^\infty dr \ |U_\ell^{(0)}(k,r)|^2 < \infty,$$

and hence there is a bound state at this k value.

This does not show what happens for $k^2 > 0$; but since K is complex in such a case it is at least plausible that $\det(I - K) \neq 0$ if $k^2 > 0$. However, we can have $\text{Re}\, D_\ell(k) = 0$ for $k^2 > 0$, and in this case we obtain in general a scattering resonance at that k value in the ℓ^{th} partial-wave amplitude, as we show later.

Now we show how to develop D_ℓ in a convergent power series in $\tilde{V}_\ell$; we will then obtain a rough approximation for the solution of (8) by retaining only the first-order term in $\tilde{V}_\ell$ in D_ℓ (analog of Born approximation).

For explicitly solving (8), we need to give an explicit formula for $(I - K_\ell)^{-1} \vec{V}_\ell$, which can be interpreted in terms of continuous functions, as $N \to \infty$. For this purpose, we use the following expansion of the inverse of a matrix in determinants of submatrices (Cramer's rule):

$$(M^{-1})_{ij} = (-1)^{i+j} \det[M^{(ij)}]/\det[M], \tag{13}$$

where $M^{(ij)}$ is the $(N - 1) \times (N - 1)$ matrix obtained from M by deleting the i^{th} row and the j^{th} column.

We can characterize $\Delta^{(ij)} \equiv \det(M^{(ij)}) \times (-1)^{i+j}$ as a partial derivative of det M with respect to the $(i,j)^{th}$ element of M, as follows:

$$\Delta^{(ij)} = \frac{\partial}{\partial M_{ij}} (\det M). \tag{14}$$

[This can be verified simply in the 2 x 2 case,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det M = (ad - bc).$$

The definition of $M^{(ij)}$ yields explicitly

$$\det[M^{(11)}] = \det M_{22} = M_{22} = d,$$

$$\det[M^{(12)}] = \det M_{21} = M_{21} = c,$$

(The determinant of a 1 x 1 "matrix" α is the value of α.)

$$\det[M^{(21)}] = b,$$

$$\det[M^{(22)}] = a.$$

On the other hand, the formula (14) yields

$$\Delta^{(11)} = d, \quad \Delta^{(12)} = -c, \quad \Delta^{(21)} = -b, \text{ and } \quad \Delta^{(22)} = a$$

in agreement with the explicit results above.]

For continuous functions, the discrete formula (14) may be generalized in a functional derivative notation,

$$\Delta(k,q) = \frac{\delta}{\delta M(k,q)} \, [\det M], \tag{15}$$

where [det M] is the limit of det M as $N \to \infty$, expressed as a functional of $M(k,q)$ [for example, a power series involving integrals over $M(k,q)$]. (We will not need to use this form explicitly in what follows.)

Then $f_\ell(k)$ can be written as $f_\ell(k,k)$, where

$$f_\ell(k,k') = N_\ell(k,k')/D_\ell(k'),$$

or

$$f_\ell(k) = N_\ell(k)/D_\ell(k); \tag{16}$$

where

$$N_\ell(k,k') = \int_0^\infty dq \; \Delta(k,q) \; \tilde{V}_\ell(q,k'); \; N_\ell(k) \equiv N_\ell(k,k) = \int_0^\infty dq \; \Delta(k,q) \; \tilde{V}_\ell(q,k). \tag{17}$$

Thus the solution to (8) may be written down, provided we can express D_ℓ as a functional of K_ℓ. The bound-state positions (i.e., spectrum), however, may be obtained (from zeros of D_ℓ) without employing any functional differentiation.

We will express the N^{th}-order determinant D as a series in $\tilde{V}$, such that the first few terms can be explicitly evaluated in the limit $N \to \infty$. For this purpose, the formula

$$\det M = \exp\{\text{trace } (\log M)\} \tag{18}$$

is employed. The matrix (log M) is defined by the power-series expansion of the function

$$\log(1 + Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \ldots \; .$$

Thus, if Z is a matrix and I the identity matrix, we define

$$\log(I + Z) = Z - ZZ/2 + ZZZ/3 - \ldots \; .$$

The trace of a matrix M is $\sum_j M_{ij}$.

The formula (18) may be proved simply if we diagonalize M; in this case,

$$\det M = \prod_j \lambda_j$$

where $\{\lambda_j\}$ are the eigenvalues of M; and on the other hand

$$\log M = \begin{pmatrix} \log \lambda_1 & & \\ & \ddots & \\ & & \log \lambda_N \end{pmatrix}$$

so

$$\text{trace } (\log M) = \sum_j \log \lambda_j ,$$

whence

$$\exp[\text{trace } (\log M)] = \prod_j \lambda_j = \det M ,$$

as required.

Returning to the case M = I - K, we obtain

$$D_\ell = \exp\{\text{trace}[\log (I - K)]\}$$

$$= \exp\left(- \text{trace } K + \frac{\text{trace } K^2}{2} - \frac{\text{trace } K^3}{3} + \ldots \right) .$$

Expanding the exponential in power series (in the strength of the potential), we obtain then

$$D_\ell = 1 - \text{trace } K + \frac{1}{2} [(\text{trace } K)^2 - \text{trace } K^2] - \ldots . \tag{19}$$

If we keep only the lowest order in $\tilde{V}$, this becomes

$$D_\ell \cong 1 - \text{trace } K$$

$$= 1 - \sum_{j=1}^{N} \left\{ \omega_j \frac{\tilde{V}(q_j , q_j)}{q_j^2 - k'^2 - i\varepsilon} \right\} .$$

Returning now to continuous functions ($N \to \infty$), and setting $k' = k$, we obtain

$$D_\ell(k) \cong 1 - \int_0^\infty \frac{dq}{q^2 - k^2 - i\varepsilon} \tilde{V}_\ell(q,q). \qquad (20)$$

This is called the FIRST-ORDER DETERMINANTAL APPROXIMATION (for D), abbreviated FODA.

If we put $k^2 = -E \leq 0$, then the bound-state energies E are such that (in the FODA)

$$\int_0^\infty \frac{dq}{q^2 + E} \tilde{V}_\ell(q,q) = 1. \qquad (20a)$$

If we define $E(\ell)$ such that this equation is satisfied whenever $E = E(\ell)$, then this defines a Regge trajectory (in FODA).

Remarks

1. The solution of (8) obtained by the formula (16) together with the series (19), is known as the Fredholm solution for (8); the use of D as given by (19) is known in potential theory as the determinantal method.[3,4] This was developed by Brown *et al.*[5] for complex ℓ.

2. For Yukawa potentials, (20) cannot be evaluated in terms of simple tabulated functions. However, some general properties of bound states can be immediately deduced from (20), as we will see later.

3. The coulomb scattering amplitude can be obtained by considering the limit $\mu \to 0$. However, this is a delicate limit, and although the coupling strength may be small, the series (19) does not converge rapidly for small binding energy; thus the FODA can, at best, yield a crude result for the most deeply bound states.

The corresponding approximation for f_ℓ now is obtained by using

$$\Delta^{(ij)} \equiv \Delta_\ell(k_i,k_j') = -\frac{\partial}{\partial K_{ij}} \det(I - K)$$

and passing to the continuous limit, where $\det(I - K) \equiv D$ is evaluated in FODA; we have then

$$D = \det(I - K) \cong 1 - \text{trace}\, K = 1 - \sum_m K_{mm},$$

so that

$$\Delta^{(ij)} = -\frac{\partial}{\partial K_{ij}}\left(1 - \sum_m K_{mm}\right) = \begin{Bmatrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{Bmatrix} = \delta_{ij},$$

yielding, from (17),

$$N_{ij} \equiv N_\ell(k_i, k_j') \overset{\sim}{=} \sum_m \delta_{im} \tilde{V}_\ell(k_m, k_j') = \tilde{V}_\ell(k_i, k_j'),$$

or, passing to the continuous limit,

$$N_\ell(k,k') \overset{\sim}{=} \tilde{V}_\ell(k,k'). \tag{21}$$

Thus the FODA result for $f_\ell(k)$, using (20) and (21), can be written

$$f_\ell(k) = \frac{B_\ell(k)}{1 - \int_0^\infty dq \, \frac{B_\ell(q)}{q^2 - k^2 - i\varepsilon}}, \tag{22}$$

where $B_\ell(k) \equiv \tilde{V}_\ell(k,k)$ is the Born approximation. Recall that this function is analytic in ℓ and provides an interpolation between bound states ($k^2 < 0$) and scattering amplitudes ($k^2 > 0$).

Further manipulations with the expressions (14) and (15) yield the result that the exact f_ℓ can be written in similar form,

$$f_\ell(k) = \frac{N_\ell(k)}{D_\ell(k)}, \tag{23a}$$

with D given by

$$D_\ell(k) = 1 - \int_0^\infty \frac{dq \, N_\ell(q)}{q^2 - k^2 - i\varepsilon}, \tag{23b}$$

where $N_\ell(k)$ is a real function for $k^2 > 0$, analytic in ℓ.

Accepting this, or relying on (22) as a guide, we can now show that a zero of Re $D_\ell(k)$ for $k^2 > 0$ and integer ℓ corresponds to a scattering resonance in the ℓ^{th} partial wave.

The phase-shift representation of f_ℓ is

$$f_\ell(k) = \frac{e^{i\delta_\ell(k)} \sin \delta_\ell(k)}{k} = (e^{2i\delta_\ell(k)} - 1)/2ik. \tag{24}$$

Thus $\delta_\ell(k)$ is the phase of f_ℓ. When $\delta_\ell = \pi/2$, we obtain a resonance; this means Re $f_\ell = 0$. But Re $f_\ell = 0$ means Re $f_\ell^{-1} = 0$; and since N_ℓ is real, for $k^2 > 0$, we find the condition for a resonance is

$$\text{Re } D_\ell(k) = 0, \; k^2 > 0. \tag{25}$$

For Regge-pole discussion for small $|B_\ell|$, see Appendix B.

IV. Regge Poles and Trajectories; Signature

Now we can exhibit the poles of $f_\lambda(k)$, in the complex λ variable. These are called Regge poles; their location varies with k (or E), and their path in the complex λ plane is called a Regge trajectory.

Assuming for the moment that some bound states occur, consider the most tightly bound one, with energy E_1 and angular momentum ℓ_1 (e.g., $\ell_1 = 0$ if the lowest bound state is S wave).

Then, since D_λ is simultaneously analytic in λ and E, an analytic function $\lambda_1(E)$ exists such that

(1) $D_{\lambda_1}(E) \equiv 0$,

(2) $\lambda_1(E_1) = \ell_1$.

This λ_1 defines a Regge trajectory, say (1), which may yield other bound states.

To find other trajectories, see if there is more than one angular momentum state, bound or resonant, at a given energy. If so, then there must be other interesting trajectories, which pass through the other bound state. (See also Ahmadzadeh, et al.[6]) In the Coulomb case, there are indefinitely many ℓ values bound at small binding energy (close to the continuum); hence there are infinitely many trajectories that produce physical bound states.[7]

A single Regge pole in f_λ, say $\lambda_1(E)$, can be represented in the complex λ plane by

$$[f_\lambda(E)]_1 = \frac{\beta_1(E)}{\lambda - \lambda_1(E)}, \tag{26}$$

where β_1 is the residue of f_λ at $\lambda \to \lambda_1$. We can express β in terms of N_λ and D_λ:

$$\beta_1(E) = N_{\lambda_1(E)}(k)\Big/\left[\frac{\partial}{\partial\lambda} D_\lambda(k)\right]\Bigg|_{\lambda = \lambda_1(E)}, \tag{27}$$

where $k^2 = E$.

For later reference, we note that [at least if $\lambda_1(E) > 0$] if $\partial D_\lambda/\partial\lambda$ is nonsingular (at $\lambda = \lambda_1$) as $k^2 \to 0^+$, then the threshold ($k^2 \to 0^+$) behavior of $\beta(E)$ will be the same as the threshold behavior of $N_{\lambda_1(0)}(k)$; and in FODA,

$$N_\lambda(k) = B_\lambda(k) = \frac{g}{2k^2} Q_\lambda(1 + \mu^2/2k^2) \to \sim \left(\frac{2k^2}{\mu^2}\right)^\lambda \text{ as } k^2 \to 0^+$$

for a simple Yukawa potential $V(r) = ge^{-\mu r}/r$.

Putting $R = \mu^{-1}$, we obtain

$$N_\lambda(k) \xrightarrow[(k^2 \to 0^+)]{} \sim (kR)^{2\lambda}.$$

Thus the residues β_n have the behavior

$$\beta_n(E) \xrightarrow[(E \to 0^+)]{} \tilde{\beta}_n(0) \cdot (kR)^{2\lambda_n(0)} ,$$

where $\tilde{\beta}_n(0)$ is finite.

By similar reasoning, we can show that the slope of $\lambda(E)$ is related to R^{-2} (but weighted in a more complicated way), as would be expected by simple rigid-rotator models, where the moment of inertia is proportional to R^2. An alternative derivation of these facts is one of the concerns in the appendix of Ref. 8. These facts are relevant to a qualitative understanding of the the order of magnitudes involved.

If we take all λ_n linear in E (or E^2 in the relativistic case), we can attempt to put resonances and bound states empirically on trajectories.

The potentials $\tilde{V}_\ell$ will be, in general, different functions in even and odd ℓ states, due to exchange contributions (see Section I above). As a consequence, there will be two sets of trajectories; one set will contribute to bound states and resonances with ℓ an even integer, the other to bound states and resonances with ℓ an odd integer. We can express this formally by the word signature; we say $\lambda_n(E)$ represents an even signature trajectory if, when $\lambda_n(E)$ = even integer, $\beta_n(E) \neq 0$, but when $\lambda_n(E)$ = odd integer, $\beta_n(E) = 0$.

Conversely, we say λ_n represents an odd signature trajectory if $\beta_n(E) = 0$ (Comment: the definitions of β_n will be different later when signature is exhibited in terms of even and odd Regge representations.) when $\lambda_n(E)$ = even integer.

Here we assumed there were sufficiently many bound states so we can unambiguously classify their trajectories. If there are only one or two bound states, it is necessary to give a more formal definition. However, we can always think of increasing the potential strength g to get sufficiently many bound states, then classifying trajectories, and then continuing back to its physical value, thereby retaining the trajectory labelling.

V. Regge Poles in Relativistic Theories; Qualitative Remarks

The general properties (i.e., poles, analytic in ℓ) of the scattering amplitude $f_\ell(E)$ we have found are retained if a relativistic (e.g., Bethe-Salpeter) equation is used to obtain bound states, provided the relativistic potentials are analytic functions of ℓ, and the relativistic kinematics are used. This is adequate to show that Regge poles are intrinsically connected with the idea of a bound state (at least of two particles), if produced by a potential with sufficient analytic properties in ℓ of its spherical Bessel transform, not a nonrelativistic phenomenon. Analyticity in ℓ of the potential $\tilde{V}_\ell$ is connected with the limiting form of V at small distances (provided we exclude Coulomb potentials, i.e., bad long-range behavior). In particular, scalar meson exchange, being the relativistic form of a Yukawa interaction, has the same smooth behavior as in the nonrelativistic case with a simple Yukawa potential.

The FODA can be extended to relativistic problems, and the formula (22) remains valid when $B_\ell(q)$ is the relativistic Born approximation for

$$f_\ell(k) = \frac{e^{i\delta_\ell} \sin \delta_\ell}{k}.$$

With mass μ scalar meson exchange, with (Lagrangian field theory) coupling constant g,

$$B_\ell(k) = \frac{g^2}{16\pi} \cdot \frac{W}{2k^2} \cdot Q_\ell(1 + \mu^2/2k^2), \tag{28}$$

where $W = (k^2 + M^2)^{1/2}$, M being the mass of the particles undergoing scattering or binding. The complete story on the Bethe-Salpeter equation with scalar meson-exchange kernel is contained in Ref. 9.

We will henceforth assume qualitative properties for relativistic bound-state problems that are contained in (22) with (28) (or a superposition with different μ's) for the Born approximation, except when otherwise noted.

Remarks

1. In a more realistic field-theory model, the potential terms ("Born approx.") will be complex above the threshold for three-body, inelastic (production) processes. Associated with these thresholds, N_ℓ will have an imaginary part also, so the phase of D_ℓ will not coincide with the scattering phase shift.

2. Our observations on relativistic theories apply specifically to properties of bound states and resonances and their associated trajectories, but not necessarily to the complete scattering amplitude in a field-theory model.

VI. Multichannel Scattering and Factorization of Residues

In practice, problems in high-energy physics almost never are concerned simply with single-channel reactions. Even in two-body decay processes, more than one channel is available for the heavier resonances. More generally, a bound state or resonance must be considered as a composite of (at least) all the two-body states that can exist with the quantum numbers of the given state. Thus the ρ mesonic state appears experimentally only through its 2π decay mode, yet it can in principle be considered as a bound state of $K\bar{K}$ and/or $N\bar{N}$ which decays by coupling to the $\pi\pi$ channel. We will see in

Part Two that such multichannel considerations for a given bound state or resonance are important for obtaining a predictive element in the Regge-pole concept.

Multichannel scattering theory (when only two-body, nonrelativistic channels are involved) can be formulated by using a matrix (in channel indices) generalization of the formalism presented in Sections II and III above. We now sketch this generalization and discuss implications of the final results.

Consider N coupled two-body channels. We can describe the scattering and transition amplitudes by N coupled Schrödinger equations for the N two-body, scattering-wave functions $\psi_j(\vec{r})$, j = 1, 2, ..., N; using the appropriate BC's, we can define a scattering matrix $F_{ij}(\vec{k}_i,\vec{k}_j)$, which is a generalization of the scattering amplitude $f(\theta)$, and partial-wave scattering matrices $(f_\ell)_{ij}$, which are generalizations of the partial-wave scattering amplitudes f_ℓ. The calculation of these amplitudes requires the specification of a generalized potential matrix $V_{ij}(r)$ whose off-diagonal elements describe transitions between channels. The Schrödinger equation then is written

$$\frac{d^2}{dr^2}(U_\ell)_i + \left[k_i^2 - \frac{\ell(\ell+1)}{r^2}\right](U_\ell)_i = \sum_j V_{ij}(r)[U_\ell(k_j,r)]_j. \qquad (29)$$

We can construct a matrix Green's function G_ℓ for this system of equations, and obtain after some manipulation an integral equation analogous to (8) involving matrices in channel indices; if, for simplicity, we take all channels to have equal masses, we obtain

$$f_\ell(k,k') = \tilde{V}_\ell(k,k') + \int_0^\infty \frac{dq}{q^2 - k'^2 - i\varepsilon}\tilde{V}_\ell(k,q)\, f_\ell(q,k') \qquad (30)$$

where $\tilde{V}_\ell$ is the spherical Bessel transform of the potential matrix $V(r)$. Now a determinantal method (Fredholm solution) is applicable to (30), and by analogy with the one-channel case, we can obtain

$$f_\ell(k) = N_\ell(k)[D_\ell(k)^{-1}], \qquad (31)$$

where N_ℓ and D_ℓ have convergent expansions in powers of V_ℓ.

The poles of f_ℓ represent bound states and are obtained by solving the equation

$$\det D_\ell(k) = 0. \qquad (32)$$

(Here recall that the matrix indices refer to discrete channels.)

The FODA for (31) can be written

$$N_\ell(k) \cong \tilde{V}_\ell(k); \qquad D_\ell(k) \cong I - \int_0^\infty \frac{dq}{q^2 - k^2 - i\varepsilon}\tilde{V}_\ell(q). \qquad (33)$$

Assuming V(r) is short range and not too singular at the origin, we can conclude that $\tilde{V}_\ell$ is analytic in ℓ, and hence that f_λ is analytic except for poles where det $D_\lambda(k) = 0$ for some complex λ which depends on k. We can extend the Regge-pole concept then to the multichannel case without difficulty. A pole [say $\lambda_n(E)$] in the complex λ plane will give a contribution to f(k) of the form

$$[f_\lambda(E)]_{ij} = \frac{\beta_{ij}^{(n)}(E)}{\lambda - \lambda_n(E)}. \tag{34}$$

(This form is independent of kinematics, e.g., whether masses are equal or not.)

Now we obtain a new result, known as factorization of residues, which states that for a given pole (at given E) the channel-index dependence of β_{ij} can be factorized as follows:

$$\frac{\beta_{ii}}{\beta_{ij}} = \frac{\beta_{ij}}{\beta_{jj}}, \tag{35}$$

provided another trajectory does not intercept the given one at the given energy.

The essential observation is the existence of a simple (i.e., linear) zero of det D, implied by the assumption that only one eigenvalue of

$$(D_\lambda N_\lambda)^{-1} = f_\lambda^{-1}$$

passes through zero at $\lambda = \lambda_1$. [Note that det N_λ^{-1} must be well behaved, or else we would obtain a second-order pole at λ; and

$$\det(D_\lambda N_\lambda^{-1}) = (\det D_\lambda) \times (\det N_\lambda^{-1})].$$

If we represent f_λ in terms of its eigenvalues α_n and eigenvectors $\vec{\xi}^{(n)}$, we can write

$$(f_\lambda^{-1})_{ij} = \sum_{n=1}^{N} \alpha_n \, \xi_i^{(n)*} \, \xi_j^{(n)}.$$

The corresponding representation of f_λ will be

$$(f_\lambda)_{ij} = \sum_{n=1}^{N} \frac{1}{\alpha_n} \xi_i^{(n)*} \, \xi_j^{(n)}.$$

Now at $\lambda \to \lambda_1$, we have seen that exactly one α, say α_1, vanishes with a simple zero; thus we can write, near λ_1,

$\alpha_1 = c(\lambda - \lambda_1)$, other α's nonzero.

This shows that as $\lambda \to \lambda_1$, we obtain

$$(f_\lambda)_{ij} = \frac{c^{-1}\xi_i^{(1)*}\xi_j^{(1)}}{\lambda - \lambda_1} \quad + \text{ terms nonsingular in } \lambda \text{ at } \lambda_1.$$

Identifying $c^{-1}\ \xi_i^{(1)*}\ \xi_j^{(1)}$, with $\beta_{ij}^{(1)}$, we see that we can write

$$\beta_{ij}^{(1)} = \gamma_i^{(1)}\ \gamma_j^{(1)}, \tag{36}$$

which proves our assertion.

This kind of factorization is intuitively expected in nuclear-reaction resonance interpretation, where the resonance has probability amplitude γ_i of formation, and γ_j for decay.

VII. Known Resonances and Possible Regge Trajectory Classification

To estimate the location of recurrences, it will be assumed that (a) the λ's are linear in $\underline{S}$ (this is indicated by relativistic models), and (b) the slope λ is of order 1 (Bev)$^{-2}$. The latter is a reasonable value if the characteristic bound-state radius is of order $(2\mu_\pi)^{-1}$.

A. Meson States

The lightest states (π,K) are 0^-; these would require even-signature trajectories (with negative intrinsic parity). The first recurrences would be expected to be 2^-. No such mesons have been established to date; one would expect them around 1.5 BeV.

The next lightest states are $1^-(\rho,\omega)$ and would suggest odd-signature trajectories (with positive intrinsic parity). Their recurrences would be 3^- mesons around 2 BeV, none of which have been seen to date.

There are some established 2^+ meson (resonant) states between 1.2 and 1.5 BeV (f°, A2, K^*_{1400}, $f^{\circ\prime}$) which will require even-signature trajectories with even intrinsic parity. They could presumably recur as 4^+ meson states near 3.5 BeV. It is also true that if the trajectories are extrapolated downward, one might expect 0^+ mesons at negative S (imaginary mass). Such stable states would, however, violate general requirements of quantum mechanics (e.g., unitarity) and must not exist. Thus either (a) trajectories bend up and never cross integers for $S < 0$, or (b) the residues vanish at crossing points. (In model calculations, both phenomena have been observed.)

It is conceivable (and actual in some models) that exchange forces are not very important in the mesonic bound states; in such a case, trajectories of even and odd signature would be degenerate. This would allow us to place 1^- and 2^+ states on the same (degenerate pair) trajectory. Empirically, such a hypothesis _is_ consistent with the slope estimated above and the mass differences between the 1^- octet (or nonet) and the 2^+ octet (or nonet). A more detailed discussion is presented in Ref. 10.

The 1^- and 2^+ trajectories' intercepts at $\underline{S} = 0$ have physical significance in terms of high-energy forward scattering in the crossed channel, as we will see in Part Two of these lectures.

B. Baryon States

(Here the trajectories refer to orbital angular momentum; there are distinct trajectories for even and odd parity states, as well as two signatures.)

We consider first the nonstrange, Y = +1 states.

The lowest mass state here is the nucleon, $1/2^+$. Through this J value, we pass a trajectory of odd signature, positive intrinsic parity. (This can be based on the model of a nucleon as a composite of itself and a π meson.) This trajectory, with a slope of unity (1 BeV^{-2}), would give a $5/2^+$, T = 1/2 resonance about 1.9 BeV. There is such a πN scattering resonance, at 900 MeV pion lab energy, which may lie on this trajectory. In fact, there is some evidence for a third recurrence, $9/2^+$ at 2645 MeV.

Two more pairs of πN resonances are candidates for common trajectories; [$3/2^+$(1236), $7/2^+$(1924)] (T = 3/2) with an odd-signature, positive-intrinsic-parity trajectory, and [$3/2^-$(1518), $7/2^-$(2190)] (T = 1/2) with an odd-signature, negative-intrinsic-parity trajectory. All three of these trajectories are compatible with an average slope of unity. There are no other candidates for recurrences at present. (See Fig. 1.)

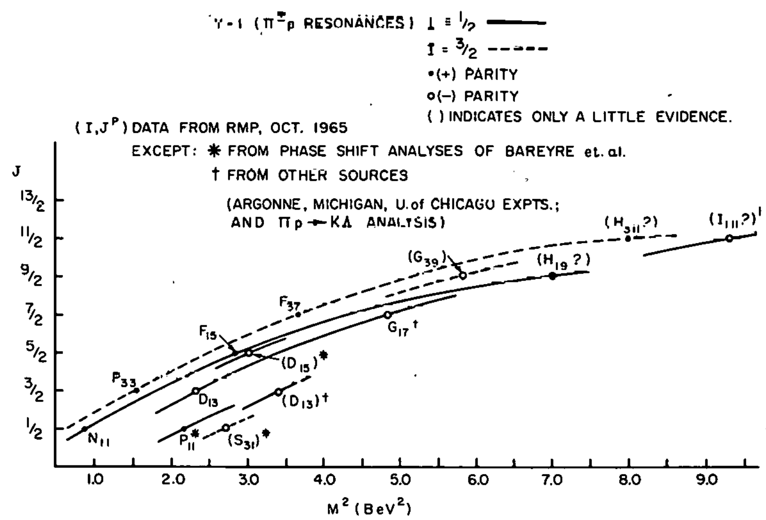


Fig. 1

Nonstrange Baryon Resonances and Possible Trajectories

These trajectories have additional physical implications for backward πN scattering, as we shall see in Part Two.

Next we consider the Y = 0 states: Λ, Σ, Y_0^*s, and Y_1^*s. There are two candidates for recurrences: $Y_0^*(1815)$, if $5/2^+$, can belong to the Λ_0 trajectory; and $Y_1^*(2065)$, if $7/2^+$, can belong to the $Y_1^*(1385)[3/2^+]$ trajectory.

Finally, turning to the Y = −1 states (Ξ and Ξ^*), we find one possibility: if $\Xi^*(1933)$ is T = 1/2 and $5/2^+$, it can be a recurrence of Ξ.

VIII. Concluding Remarks, Part One

No rigorous check on the Regge-pole ideas can be obtained by looking for recurrences, as long as no detailed theoretical models are employed to calculate trajectories. The recurrence idea must be regarded only as a rough guide to the possibilities of higher resonant states on the basis of empirical knowledge of low-lying states, and a conceptual framework for systematizing our knowledge of existing states.

We will show in Part Two, however, that in a sense, there is some possibility of checking the Regge-pole ideas if we confine our attention to the poles in the upper ℓ plane near zero total energy (S = 0). At the same time, if valid, there are predictive powers inherent in the Regge-pole concepts. Many of these predictions stem from the factorization property of residues.

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PART TWO

APPLICATIONS OF REGGE POLES IN THE ANALYSIS OF HIGH-ENERGY REACTIONS

I. Relativistic Description of Scattering and Reaction Processes and Crossing Relations

A. Kinematics

Consider the elastic scattering of two spinless particles, called here π and N, with masses μ and M. Let (k_1, P_1) be the four-vector momenta of the incoming (μ, M) particles, respectively, and (k_2, P_2) their outgoing momenta. We define two relativistic invariants,

$$s \equiv (k_1 + P_1)^2$$

and

$$t \equiv (P_1 - P_2)^2 = (k_1 - k_2)^2 .$$

We assume $P_1^2 = P_2^2 = M^2$, $k_1^2 = k_2^2 = \mu^2$, and $P_1 + k_1 = P_2 + k_2$; units are chosen so that $c = 1$. It can be shown (cf. Chapter 5 of Omnès and Froissart[1]) that all components of the momenta k_1, P_1, k_2, and P_2 are determined in any given reference frame if μ, M, s, and t are specified; i.e., the scattering event is uniquely specified by s and t in an invariant way.

In the center of mass (c.m.) frame, where $\vec{P}_1$ lies along the Z axis, we evaluate s and t in terms of energy and scattering angle as follows:

$$s = (k_{10} + P_{10})^2 = (E_\pi + E_n)^2 = W^2$$

where W is the total c.m. energy;

$$t = (k_{10} - k_{20})^2 + 2\vec{k}_1 \cdot \vec{k}_2 - 2\vec{k}_1^2 - 2\vec{k}_2^2$$

or

$$t = -2k^2(1 - \cos\theta),$$

where θ is the c.m. scattering angle, and k is the c.m. spatial momentum of the nucleon, related to W by

$$W = E_\pi + E_n = (k^2 + \mu^2)^{1/2} + (k^2 + M^2)^{1/2} .$$

For the equal-mass case $(\mu = M)$, relevant to pp scattering, for example, we obtain the simpler relation

$$s = 4(M^2 + k^2) .$$

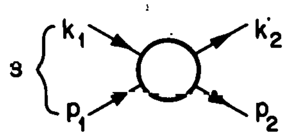
Note that for fixed s, the physical scattering region $-1 \leq \cos\theta \leq +1$ corresponds to the t interval

$$-4k^2 < t < 0.$$

Further discussions of kinematics are contained in Ref. 2.

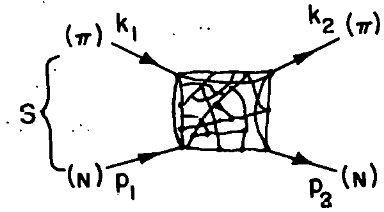
B. Crossing, and Definition of Invariant Amplitudes

We can draw a diagram for the scattering process as follows:

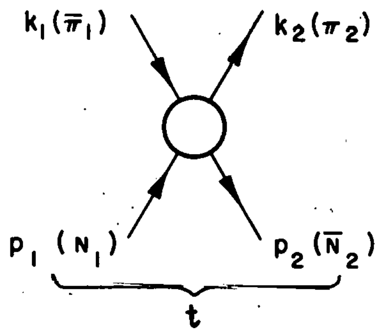


Here arrows indicate the movement of charge or baryon number and indicate the sense in which the spatial momenta $\vec{k}_i$, $\vec{P}_i$ are defined.

If we consider any field-theoretic perturbation-theory diagram contributing to the scattering, as indicated schematically here (the braces indicate the incoming pair of particles),

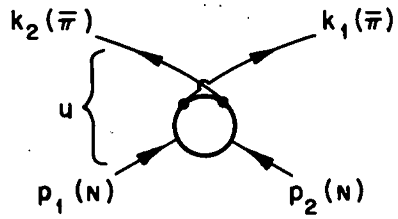


it is apparent that the same diagram also occurs in two other reactions, whose diagrams may be obtained from this one by interchanging certain external lines, or by considering the diagram from a different direction. Let us call the μ-M elastic scattering reaction I. Then it is related to the process $N\bar{N} \rightarrow \pi\pi$, which we call reaction II, as follows:



The incoming particle No. 2 now is an antiparticle $\bar{N}$ with momentum $(-\vec{P}_2)$ but positive energy $[+(\vec{P}_2^2 + M^2)^{1/2}]$. In the Feynman point of view, we have changed the direction of time for this particle. (Alternatively, we see that the arrows do not change their orientation with respect to the internal diagrams; hence, they indicate opposite sign for baryon number flow and spatial momentum.) The outgoing particle No. 1 also must have reversed quantum numbers and momentum $-\vec{k}_1$.

We obtain a third reaction from I by considering k_2 to be an incoming line and k_1 an outgoing line. Then both the mesons' quantum numbers and momenta are reversed, leaving the nucleon states as before. We may call this crossed reaction III. We denote the associated c.m. energy variable as u, defined by $u = (P_1 - k_2)^2$. The diagram may be drawn as follows:



If $\mu = M$ (e.g., as in pp scattering), $u = -2k\,(1 + \cos\theta)$ in terms of channel I quantities.

If reaction I was π^+p elastic scattering, then reaction III will be π^-p scattering, while reaction II will be $p\bar{p} \to \pi^+\pi^-$, with $(\text{c.m. energy})^2 = t$.

Suppose we describe the scattering process I with a function $A^{(I)}(s,t)$ proportional to the scattering amplitude when s and t are in the physical regions for reaction I. If we consider A as given by the sum of all Feynman diagrams, we can show that a corresponding reaction amplitude $A^{(II)}(t,s)$ for reaction II must be the same function of s and t since both amplitude are obtained by summing the same set of diagrams. The relation of these invariant amplitudes A to the nonrelativistically defined scattering amplitude $f_k(\theta)$ may be deduced by examining the terms of the covariant perturbation expansion, and requiring simple relations under interchange of s and t. The proper choice is (for spinless particles)

$$A^{(I)}(s,t) = s^{1/2}f_k(\theta), \qquad [(k^2 + \mu^2)^{1/2} + (k^2 + M^2)^{1/2}]^2 = s.$$

Then

$$\left(\frac{d\sigma}{d\Omega}\right)^{(I)} = s^{-1}\left|A^{(I)}\right|^2.$$

in the physical region for reaction I, while in the physical region for reaction II,

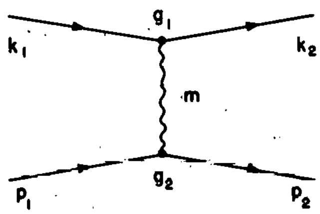
$$\left(\frac{d\sigma}{d\Omega}\right)^{(II)} = t^{-1}\left|A^{(II)}\right|^2.$$

The crossing relation connecting $A^{(I)}$ and $A^{(II)}$ then is

$$A^{(I)}(s,t) = A^{(II)}(t,s), \tag{37}$$

where the convention is followed of writing first the variable (t or s) that represents the square of the c.m. energy for the physical reaction process identified by the superscript on A.

A concrete example of crossing relations is a one (scalar)-particle exchange diagram



which contributes

$$[A^{(I)}(s,t)]_{OME} = \frac{g_1 g_2}{M^2 - t}.$$

For reaction II we obtain the same, i.e.,

$$\left(\frac{d\sigma}{d\Omega}\right)^{(II)}_{OME} = \frac{1}{t}\left|\frac{g_1 g_2}{M^2 - t}\right|^2,$$

where $t = -2k^2(1 - \cos\theta)$ for reaction I, but for reaction II, t is the square of the c.m. energy and is greater than $4M^2$.

In general, crossing is severely complicated by the presence of spin; less severe are treatments of isospin and other nonspatial quantum numbers.

Observe that the crossing relations between reactions I and II never connect physical regions of reaction I directly to physical regions of reaction II, since these do not overlap. They are summarized by the following table:

Reaction I	Reaction II
$-4k_s^2 \le t \le 0$	$t > 4M^2$
$s > (M + \mu)^2$	$(s < 0)$

(The precise limits on s in reaction II are not given since they are somewhat complicated.)

Thus, to predict one reaction in terms of another, we need to introduce theoretical knowledge concerning A to allow extrapolation. The most naive methods usually fail, as we now illustrate (to motivate the subsequent introduction of Sommerfeld-Watson transform).

C. Example

Suppose we analyze reaction I in partial waves and write

$$A^{(I)}(s,t) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, A_\ell^{(I)}(s)\, P_\ell(\cos\theta), \qquad (38)$$

$$\text{where} \begin{cases} \cos\theta = 1 + t/2k_s^2; \\ k_s^2 = \dfrac{[s - (M+\mu)^2][s - (M-\mu)^2]}{4s}. \end{cases}$$

(A similar decomposition can be written for $A^{(II)}$ terms of channel II variables.) In terms of phase shifts,

$$A_\ell^{(I)}(s) = \rho^{-1}(s)\, e^{i\delta_\ell(s)} \sin\delta_\ell(s),$$

where

$$\rho(s) = 2k_s/s^{1/2}.$$

This series represents the physical scattering amplitude well for strong interaction problems (short-range potentials), and terms are experimentally undetectable for, say, $\ell > 10\ k/\mu_\pi$, where μ_π is the pion mass. As a function of (complex) $Z_s = \cos\theta$, this series converges in an ellipse with focii ±1 and a major axis determined by the range of the potential effective in channel I scattering; for a practical expectation of range $(2\mu_\pi)^{-1}$, the ellipse reaches to about $|Z_s| \sim 1 + 2\mu_\pi^2/k^2$. We can fit some rational function in s to each $A_\ell(s)$ and satisfactorily interpolate to $s < 0$. However, if we attempt to use the series for large $|Z_s|$, the series diverges. This is the case for high c.m. energy in reaction II; e.g., for fixed channel II reaction angle, $|Z_s|$ grows linearly with t, the square of the channel II reaction energy.

If only channel I s-wave scattering were important, this might not be serious; but these are not the interesting cases, and in general we find untenable results.

The interesting cases, in general, are those in which a resonance or bound state appears in the crossed channel. In such a case, we might expect only one partial-wave amplitude in that channel to dominate the reactions, if we use the series representation (38).

For example, consider the $f^0(1250)$ $\pi\pi$ resonance, presumably 2^+. This will appear, in any channel that has the same quantum numbers, as a Breit-Wigner form for the D-wave ($\ell = 2$), partial-wave amplitude involved. Thus, in $N\bar{N} \to \pi\pi$ we expect a contribution (ignoring nucleon spin), by analogy with (38),

$$[A^{(II)}(t,s)]_{f^0} = \sum_{\ell} (2\ell + 1)[A_{\ell}^{(II)}(t)]_{f^0} P_{\ell}(\cos\theta_t) = 5 \cdot \frac{k_t^4 \Gamma_{N\bar{N}}}{M_{f^0}^2 - t - i\Gamma_{tot}} \cdot P_2(\cos\theta_t),$$

where a k_t^{2J} factor has been used in the resonance formula to insure correct threshold behavior of A^{II} at the threshold $t \to 4M^2$, Γ_{tot} is the full width of f^0, $\Gamma_{N\bar{N}}$ is a coupling factor to $N\bar{N}$, and θ_t is the c.m. reaction angle involved in channel II. If we consider now the physical region of reaction I, $t < 0$, and we can ignore the imaginary part of A relative to the real part; thus, we have

$$[A^{(I)}(s,t)]_{f^0} = [A^{(II)}(t,s)]_{f^0} \cong 5\Gamma_{N\bar{N}} \cdot \frac{P_2(\cos\theta_t)}{M_{f^0}^2 - t} \cdot k_t^4.$$

Now for high-energy scattering in channel I, $s \to \infty$, we find

$$P_J(\cos\theta_t) \to (\cos\theta_t)^J \to \left(\frac{s}{2k_t^2}\right),$$

so we can write (with $J = 2$)

$$[A^{(I)}(s,t)]_{f^0} \xrightarrow[(s\to\infty)]{} \frac{\gamma(s/s_0)^2}{M_{f^0}^2 - t},$$

where γ and s_0 are some real constants.

This would predict that at high energies, in πp elastic scattering,

a. $|A^{(I)}| \sim o\ (s/s_0)^2$, hence $\frac{d\sigma}{dt} \sim o\ (s/s_0)$ for fixed $-t$.

b. Behavior in t like $\frac{d\sigma}{dt} \sim \left(\frac{1}{M_{f^0}^2 - t}\right)^2$.

c. $A^{(I)}$ becomes real.

All three of these are definitely in contradiction with experimental high-energy πp scattering; we find instead

A. $d\sigma/dt \sim$ constant for fixed (small) $-t$.
B. Exponential forward peak with width of order $4\mu_\pi^2$.
C. $A^{(I)}$ is mostly imaginary.

This extrapolation, however, has used only one term of the series such as (38), and as we have seen that the series diverges in the channel I physical region, our results do not contradict the physical hypothesis that the existence of f^0 in channel II leads to interesting consequences for channel I. We have to use for $A^{(II)}$ a different representation, which converges properly. Such a representation can be obtained by converting the series (38) into a contour integral in the complex ℓ plane, the Regge representation.

II. Regge Representation for Invariant Amplitudes

If there exists any function $A_\lambda(s)$ analytic in the complex λ plane in a neighborhood of the positive real axis and sufficiently well behaved as $|\lambda| \to \infty$, we can define the integral

$$I(s,z) = \frac{1}{2\pi i} \oint_C \frac{A_\lambda(s)(2\lambda + 1)\ P_\lambda(-z)}{\sin \pi\lambda}\ d\lambda, \tag{39}$$

where the contour C encloses the positive real axis in the λ plane but no other poles of the integrand.

The legendre function $P_\lambda(-z)$ for complex λ can be represented as a hypergeometric function,

$$P_\lambda(-z) = F\left(\lambda + 1,\ -\lambda;\ 1;\ \frac{1 + z}{2}\right). \tag{40}$$

[For $|x| < 1$ the hypergeometric function can be represented as a power series

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{\Gamma(c)\ \Gamma(a+n)\ \Gamma(b+n)}{\Gamma(a)\ \Gamma(b)\ \Gamma(c+n)\ \Gamma(1+n)}\ x^n,$$

and for $|x| > 1$ there exist integral representations enabling its computation for general x.]

The integrand of (39) has poles at $\lambda = 0, 1, 2, \ldots$, and we can evaluate $I(s,z)$ by evaluating the residues at these poles. The result is

$$I(s,z) = \sum_{\ell=0}^{\infty} (2\ell + 1)\ A_\ell(s)\ P_\ell(z),$$

where the relation $P_\ell(z) = (-1)^\ell\ P_\ell(-z)$ has been employed.

Now if $A_\ell(s)$ for $\ell = 0, 1, 2, \ldots$, are the physically invariant partial-wave amplitudes for scattering, if z is identified as the cosine of the scattering angle, z_s, we can identify I with the invariant amplitude,

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell + 1)\ A_\ell(s)\ P_\ell(z_s). \tag{41}$$

Thus if there exists a function $A_\lambda(s)$, analytic in the λ plane in a neighborhood of the positive real λ axis, which coincides with the physical scattering amplitude for λ = zero or positive integer, and dropping off fast enough as $|\lambda| \to \infty$ so that the integrals under consideration converge, we can write the following integral representation for $A(s,t)$:

$$A(s,t) = \frac{1}{2\pi i} \oint_C \frac{(2\lambda + 1)\, P_\lambda(-z)\, d\lambda}{\sin \pi\lambda} A_\lambda(s). \tag{42}$$

The passage from (41) to (42) is called the Sommerfeld-Watson (SW) transform. Now consider the singularities of $A_\lambda(s)$ in the complex λ plane, to the right of the line $\mathrm{Re}\,\lambda = -1/2$. If there are only poles, we can distort the contour C to run along a vertical line, say $\mathrm{Re}\,\lambda = -1/2$, and pick up the residues of the poles α_n in the right-hand λ plane over which the contour had to be distorted. The result is

$$A(s,t) = \frac{1}{2\pi i} \int_{-i\infty-1/2}^{+i\infty-1/2} d\lambda\, \frac{(2\lambda + 1)\, P_\lambda(-z)}{\sin \pi\lambda} A_\lambda(s)$$

$$+ \sum_{(\alpha_n)} \frac{(2\alpha_n + 1)\, P_{\alpha_n}(-z)}{\sin \pi\alpha_n} \beta_n. \tag{43}$$

The α_n (pole locations) and β_n (pole residues) depend on s, since the singularities in λ of $A_\lambda(s)$ will (in general) depend on s. We call this representation (43) the <u>Regge representation</u>.

In the crossed channel, $|z|$ will be large, but this does not destroy the convergence of (43), and we can employ (43), as a tool to exploit crossing relations in general.

Now the (nonrelativistic) partial-wave amplidudes $A_\ell = \sqrt{s}\, f_\ell$ for complex ℓ obtained from Part One of these lectures, as interpolators between bound states and resonances, satisfy the conditions necessary for $A_\lambda(s)$. This is clear, except for the convergence properties as $|\lambda| \to \infty$ in the right-hand λ plane. These convergence problems (for superpositions of Yukawa potentials) are treated with the determinantal method in Ref. 5 of Part One.

Thus the terms in the sum in (43) are exactly the Regge poles (at $\lambda = \alpha_n$) which occur in the right-hand half of the λ plane, whose locations are determined by the condition

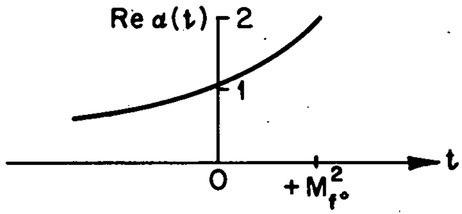
$$D_{\alpha_n(s)}(S) = 0. \tag{44}$$

As s is varied continuously from energies above threshold (where resonances are found) to negative values (which lie in the physical region for scattering in the crossed channel), the poles follow their trajectory functions $\alpha(s)$ and exhibit residues $\beta(s)$.

The integral in (43) is called the <u>background integral</u>. We will now exhibit the asymptotic high-energy limit for the crossed-channel reaction.

As $|z| \to \infty$, the $P_\alpha(z)$ have asymptotic behavior $(-z)^\alpha$, whose magnitude is determined by $(-z)^{Re\alpha}$.

Thus the background integral behaves asymptotically like $(-z)^{-1/2}$, while the Regge poles (α_n) behave like $(-z)^{Re\alpha_n(s)}$. If there are some α_n for which $Re\alpha_n > -1/2$ for physical s values in crossed-channel scattering, then these poles will asymptotically dominate the crossed-channel amplitude as $t \to \infty$. As a concrete illustration, consider the physical example examined in the preceding section. Here we interchange the roles of s and t channels, considering Regge poles in channel II. There will be a trajectory $\alpha_0(t)$ passing near the f^0 resonance region, i.e., $Re\alpha_0(M_{f^0}^2) = 2$, in channel II, where t is the $(\text{energy})^2$. Suppose this trajectory, for $t < M_{f^0}^2$, behaves as follows:



i.e., $\alpha(0) \sim 1.0$, $\alpha(t) \approx \alpha(0) + t\alpha'(0) + \ldots$ for small (-t).

Then in channel I, for $t < 0$ and $s \to \infty$, we obtain, keeping only this pole in (43),

$$A^{(I)}(s,t) = A^{(II)}(t,s) \cong \frac{[2\alpha_0(t) + 1]\, P_{\alpha_0(t)}(-z_t)\, \beta_0(t)}{\sin[\pi\alpha_0(t)]}. \quad (45)$$

Assuming for the moment that

$$\beta_0(t)/\sin[\pi\alpha_0(t)]$$

is not singular near $t = 0$, we obtain as $s \to \infty$ for small (-t) (comparing the derivation in the previous section,

$$A^{(I)}(s,t) \to \frac{[2\alpha_0(t) + 1]\, s_0^{\alpha_0(t)}}{\sin[\pi\alpha_0(t)]} \cdot \frac{\beta_0(t)}{(2k_t^2)^{\alpha_0(t)}} \cdot \left(\frac{s}{s_0}\right)^{\alpha_0(t)} \quad (46)$$

$$\equiv F(t) \cdot (s/s_0)^{\alpha_0(t)},$$

where s_0 is any scale parameter (note that F depends on the choice of s_0).

Now we find the following behavior of $A^{(I)}$:

(a) $(d\sigma/dt) \sim \frac{1}{k_s^2 \cdot s} \cdot |A^{(I)}|^2 \sim$ constant for fixed t, $s \to \infty$,

and if F(t) is slowly varying for small -t, then for fixed s

(b) $(d\sigma/dt) \sim e^{tR^2}$, where $R^2 = 2\alpha'(0) \log (s/s_0)$;

so that under such circumstances R^2 grows logarithmically with s.

These are, in fact, the features experimentally found for high-energy πp scattering, except that R^2 apparently grows at a negligible rate.

However, we have yet to establish a phase for the poles, and we must avoid having a singularity at $t = 0$ due to $\sin [\pi\alpha(t)]$ vanishing. These points depend on the introduction of a signature factor for the trajectory.

III. Signature and Phase of Pole Terms

It is clear that symmetry considerations appear in this problem, since the $\pi\pi$ state (due to Bose statistics) have only even ℓ bound states and resonances when their isospin is even. This shows we cannot have a physical pole at $\ell = 1$ with $t = 0$ contributing to any scattering amplitude involving $\pi\pi$.

To explicitly take symmetry into account, and include the possibility of exchange potentials in the determination of pole parameters α and β, we define even and odd scattering amplitudes, in the nonrelativistic formalism as follows:

$$f^+(s,z) \equiv \frac{f(s,z) + f(s,-z)}{2} = \sum_{\substack{\ell \\ \text{(even)}}} (2\ell + 1) f_\ell(z) \left[\frac{P_\ell(z) + P_\ell(-z)}{2}\right];$$

and

$$f^-(s,z) \equiv \frac{f(s,z) - f(s,-z)}{2} = \sum_{\substack{\ell \\ \text{(odd)}}} (2\ell + 1) f_\ell(z) \left[\frac{P_\ell(z) - P_\ell(-z)}{2}\right].$$

Then $f = f^+ + f^-$; even signature poles will yield bound states and/or resonances in f^+, while odd signature poles contribute to f^-.

If exchange potentials are present, the even ℓ and odd ℓ scattering amplitudes will be obtained by solving two different Schrödinger equations, with potentials

$$V^+(r) = [V(r) + V^{exch}(r)]/2$$

and

$$V^-(r) = [V(r) - V^{exch}(r)]/2,$$

respectively.

Then we have two (off-shell) Born approximations $V_\ell^+(k,k')$ and $V_\ell^-(k,k')$ each analytic in complex ℓ.

These will yield two distinct scattering amplitudes, each an analytic function of ℓ, i.e., $f_\ell^+(s)$ and $f_\ell^-(s)$, such that we can write

$$f^+(s,z) = \sum_\ell (2\ell + 1)\ f_\ell^+(s) \left[\frac{P_\ell(z) + P_\ell(-z)}{2}\right] \tag{47}$$

and

$$f^-(s,z) = \sum_\ell (2\ell + 1)\ f_\ell^-(s) \left[\frac{P_\ell(z) - P_\ell(-z)}{2}\right],$$

where the sums are allowed to run over all (even as well as odd) ℓ values.

Now we can apply SW transforms separately to $\sqrt{s}\ f^+ = A^+$ and $\sqrt{s}\ f^- = A^-$, which yield the pair of Regge representations,

$$A^+(s,t) = \frac{1}{2\pi i} \int_{-i\infty-1/2}^{+i\infty-1/2} d\lambda\ \frac{2\lambda + 1}{\sin(\pi\lambda)} \left[\frac{P_\lambda(z_s) + P_\lambda(-z_s)}{2}\right] A_\lambda^+(s)$$

$$+ \frac{1}{2} \sum_{\substack{\text{even} \\ \text{signature poles}}} (2\alpha_n + 1)\ \beta_n \left[\frac{P_{\alpha_n}(z) + P_{\alpha_n}(-z)}{\sin(\pi\alpha_n)}\right]$$

and

$$A^-(s,t) = \frac{1}{2\pi i} \int_{-i\infty-1/2}^{+i\infty-1/2} d\lambda\ \frac{2\lambda + 1}{\sin(\pi\lambda)} \left[\frac{P_\lambda(z_s) - P_\lambda(-z_s)}{2}\right] A_\lambda^-(s)$$

$$+ \frac{1}{2} \sum_{\substack{\text{odd} \\ \text{signature poles}}} (2\alpha_n + 1)\ \beta_n \left[\frac{P_{\alpha_n}(z_s) - P_{\alpha_n}(-z_s)}{\sin(\pi\alpha_n)}\right], \tag{48}$$

with $A = A^+ + A^-$ being the invariant scattering amplitude. We will utilize these by considering Regge poles in channel II, where t is the square of the mass of the resonance or bound state; thus, instead of z_s we use z_t, and the arguments of α's and β's will be t.

Now for $|z| \to \infty$ (high energy in the crossed channel), we find that asymptotically A becomes the sum over all even and odd signature poles. Odd signature poles contribute terms of the form

$$(2\alpha + 1)\ \frac{P_\alpha(z) - P_\alpha(-z)}{2 \sin[\pi\alpha(t)]}\ \beta(t) = (2\alpha + 1)\ \frac{\beta(t)\ P_\alpha(z)}{\sin[\pi\alpha(t)]} \left[\frac{1 - e^{-i\pi\alpha(t)}}{2}\right] \tag{49}$$

[since $P_\alpha(-z) = e^{-i\pi\alpha} P_\alpha(z)$] while even signature poles yield

$$(2\alpha + 1)\ \frac{P_\alpha(z) + P_\alpha(-z)}{2 \sin[\pi\alpha(t)]}\ \beta(t) = (2\alpha + 1)\ \frac{\beta(t)\ P_\alpha(z)}{\sin[\pi\alpha(t)]} \left[\frac{1 + e^{-i\pi\alpha(t)}}{2}\right]. \tag{50}$$

Note the odd signature terms now are not singular (i.e., not poles at all) when $\alpha = 0, \pm 2, \pm 4, \ldots,$ since the signature factor $(1 - e^{-i\pi\alpha})$ cancels the zeros of the denominator. Similarly, the even signature terms are not singular when $\alpha = \pm 1, \pm 3, \pm 5, \ldots$.

Assuming α and β are real for $t < 0$, we see the phase of each pole term is the phase of the signature factor $(1 \pm e^{i\pi\alpha})$.

We can exhibit this phase explicitly as follows:

Odd signature:

$$\frac{1 - e^{-i\pi\alpha}}{2 \sin(\pi\alpha)} = \frac{e^{-i\pi\alpha/2}}{2} \cdot \frac{e^{i\pi\alpha/2} - e^{-i\pi\alpha/2}}{2 \sin\frac{\pi\alpha}{2} \cos\frac{\pi\alpha}{2}} = ie^{-i\pi\alpha/2}\left(\frac{1}{2 \cos\frac{\pi\alpha}{2}}\right)$$

Even signature:

$$\frac{1 + e^{-i\pi\alpha}}{2 \sin(\pi\alpha)} = \frac{e^{-i\pi\alpha/2}}{2} \cdot \frac{e^{i\pi\alpha/2} + e^{-i\pi\alpha/2}}{2 \sin\frac{\pi\alpha}{2} \cos\frac{\pi\alpha}{2}} = e^{-i\pi\alpha/2}\left(\frac{1}{2 \sin\frac{\pi\alpha}{2}}\right). \qquad (51)$$

In the nonrelativistic formalism for calculating the poles, we can first show that $\alpha(t)$ is real for $t < 0$. This can be seen from the fact that K_ℓ is real for $k^2 < 0$ and real ℓ, but complex for complex ℓ [see Eq. (9)]. The eigenvalues of K_ℓ then (since K is not hermitian) are real only for real ℓ, and hence zeros of $\det(I - K_\ell)$ exist only for real ℓ, when $k^2 < 0$. [This can also be seen from the series representation (19).]

Similarly, we can see that β_n is real for $k^2 < 0$ since, when K_ℓ is real, $N \sim \frac{-\delta}{\delta k} \det(I - K)$ is also real [see (15) and discussion following]; the residue β_n is proportional to $\left[N_\lambda \Big/ \left(\frac{\partial D}{\partial\lambda}\right)\right]_{\lambda=\alpha_n}$ and both numerator and denominator are real for $k^2 < 0$.

Thus the phase of each pole contribution, as seen in high-energy, crossed-channel reactions, is given by $e^{-i\pi\alpha(t)/2}$, or i times this. This relates the phase to the asymptotic high-energy behavior, independent of details.

As an important example, consider the f^0 problem as before. Now $J^P = 2^+$ implies even signature for this trajectory; so this pole will contribute [instead of (45)] a term such as (50), as follows:

$$[A^{(I)}(s,t)]_{f^0} = [A^{(II)}(t,s)]_{f^0} = [2\alpha_0(t) + 1]\, \beta_0(t)\, P_{\alpha_0(t)}(z_t)\, \frac{1 + e^{-i\pi\alpha_0(t)}}{2 \sin[\pi\alpha_0(t)]}. \qquad (52)$$

For high energies in channel I, $z_t \to s/2k_t^2 \to \infty$, and we obtain the asymptotic form,

$$[A^{(I)}(s,t)]_{f^0} \rightarrow \frac{2\alpha_0(t) + 1}{2 \sin\left[\frac{\pi\alpha_0(t)}{2}\right]} \frac{\beta_0(t)\ s_0^{\alpha_0(t)}}{(2k_t^2)^{\alpha_0(t)}} \left(\frac{s}{s_0}\right)^{\alpha_0(t)} e^{-i\pi\alpha_0(t)/2}, \qquad (53)$$

where s_0 is any suitable scale factor. Now if (for some s_0) the expression in brackets is a slowly varying function of t near t = 0, we obtain the same results as discussed after (46) above, but in addition we have determined the phase at t = 0; it is purely imaginary. Thus, to (a) and (b) following (46) above, we add: (c) $A^{(I)}(s,0)$ is asymptotically purely imaginary. (To fit the asymptotic variation of R^2 with energy, which is very slow in the πp scattering reaction, it is necessary to have $s_0 \lesssim 0.30$ BeV^2.)

This pole, then, gives a satisfactory qualitative description for the asymptotically high-energy πp scattering, since (a), (b), and (c) now agree with properties evident from present experimental data above 6 BeV and for $-t < 0.30$ (BeV)^2; the real part of A in the forward direction is small compared with the imaginary part.

IV. Discussion of Poles in πp Scattering and Charge Exchange

The choice $\alpha(0) = 1$ for the f^0 trajectory essentially determines the s-independence of values of $d\sigma/dt$ near the forward direction in πp scattering. This number is essentially determined, then, from the high-energy data. Such a pole, whose trajectory passes through $\alpha = 1$ at $t = 0$, has even signature, isospin zero, and G parity positive (which we have introduced as associated with f^0), is called a "Pomeranchon." Its existence guarantees equality and constancy of particle and antiparticle cross sections in the asymptotic limit. Such equality and constancy were first strongly suggested by Pomeranchuk on the basis of forward-scattering dispersion relations combined with intuitive ideas about diffraction scattering (inelastic processes) at high energies. Note that total cross sections (σ_T) are related to forward elastic-scattering amplitudes by the optical theorem

$$\sigma_T = \frac{4\pi}{k^2} \text{ Im } f(s,0)$$

and hence linearly to A(s,0). Thus the dominant poles determine the energy dependence of σ_T. The qualitative features of P (the Pomeranchon pole) are similar to scattering from an absorbing disc, except that the radius varies with s.

Of course, other poles nearby in the λ plane may compete with the Pomeranchon (P) at nonasymptotic energies. If the energy dependence of the πp elastic scattering and total cross-section data (above, say, 4 BeV) is analyzed on the assumption that only one other pole contributes, a second pole with $\alpha(0) \cong 0.50$ is found. Since π^+p and π^-p scattering are very similar, the dominant contribution to this "correction" pole must have T = 0 in the channel II reaction (otherwise there would be a change of sign in dominant part of π^-p correction compared to π^+p correction). Since the t-channel reaction involves two π's, this pole must have even signature; so it must have the same quantum numbers as P. It has been called therefore P'.

A possible physical resonance lying on the P' trajectory has been found at 1.67 BeV and has been called $f^{0'}$.

There is, in addition, a small but significant difference between $\sigma(\pi^+p)$ and $\sigma(\pi^-p)$ in the energy region 4-20 BeV, which seems to have about the same energy dependence as the P' contribution. A pole that can account for this must have T = 1 in channel II. The only known isovector $\pi\pi$ resonance is the ρ, which has $J^P = 1^-$ at 760 MeV. The ρ trajectory then must have $\alpha_\rho(0) \approx 0.50$; this is confirmed by charge-exchange data. The charge-exchange reaction $\pi^-p \to \pi^0 n$ can be represented as the difference between T = 3/2 and T = 1/2 elastic πp scattering amplitudes. The P and P', being isoscalars in channel II, do not contribute to this difference, and the ρ is the only pole known that contributes. This is the most clear-cut test known of Regge pole applicability, and analysis of the data seems to bear out the pole conjecture very well (see Ref. 3).

A good discussion of the energy dependence of total cross sections may be found in Udgaonkar's article;[4] we will not pursue this further in these lectures. The phase of forward scattering is discussed in Ref. 5.

All the discussion presented so far has been oriented toward the influence of mesonic states in the t channel (e.g., f^0,ρ) on high-energy πN scattering, particularly near the forward direction where $|-t|$ is small. However, channel III is yet to be considered. The Regge poles associated with that channel are πN resonances and bound states (we consider the nucleon to be a bound state of π and N with binding energy equal to the pion mass).

There are crossing relations connecting the channel III invariant amplitudes with channel I; referring to the discussion in Section I-B of the three channels connected by the same four-leg Green's function, we define a channel III invariant amplitude $A^{(III)}(u,t)$ such that (ignoring spin for the moment)

$$\frac{d\sigma^{III}}{d\Omega} = u^{-1} \cdot |A^{(III)}(u,t)|^2.$$

The crossing relation then reads

$$A^{(I)}(s,t,u) = A^{(III)}(u,t,s),$$

where only two of the three variables are independent; we have written all three explicitly to obtain a symmetrical notation.

Channel III represents π^+p elastic scattering, if channel I is π^-p scattering, with (c.m. energy)^2 = u. The dominant Regge trajectories in the π^+p channel are associated with the T = 3/2 resonances. If the trajectory picture in Fig. 1 is essentially correct with respect to this point, there should be only one important trajectory near u = 0, namely the one that passes through the $P_{3/2\ 3/2}(1238)$ resonance. Let us denote this one by α_*. Then we expect that when (and if) $\alpha_* > -1/2$, in the physical region for channel I scattering, we get a contribution to the asymptotic behavior for channel I of the form

$$\frac{d\sigma^I}{d\Omega} \sim |S^{\alpha_*}|^2.$$

Now (from Fig. 1) α_* is presumably above zero when u is close to zero. In terms of t,

$$u = 2M^2 + 2\mu^2 - s - t.$$

Thus, for u to be small and s large, we need large (-t), i.e., large angles in the center of mass. If $s \gg 2(M^2 + \mu^2)$, we get

$$u \approx -2k^2(1 + \cos\theta)$$

(as in the equal mass case), so we need $\cos\theta \approx -1$ to obtain the Regge asymptotic behavior from α_*. (Note that for large -t, the channel II poles presumably retreat into the lower half of the ℓ plane and do not contribute to asymptotic behavior.)

If a relativistic treatment of nucleon spin is considered, it is found that α_* (or any fermion Regge pole trajectory) and β_*, the associated residue, should be considered as analytic functions of $W_u = \sqrt{u}$ (instead of t, for example, when the t channel contains boson Regge poles) and that for $u < 0$, we obtain complex conjugate pairs of Regge poles: $\alpha_*(W)$, $\alpha_*^{(2)}(W) = [\alpha_*(W)]^*$.

As a result, the phase relations are not as simple for fermion poles as for boson poles; in addition to signature factors of the form

$$\frac{1}{2}\left[1 \pm e^{-i\pi(\alpha-1/2)}\right],$$

we have contributions to the phases from the complex nature of the trajectories and residues. It is still true, however, that a constraint exists connecting the phase and energy dependence of each pole.

A complete theoretical discussion has been given by Singh[6] and the most important points are discussed by Kinoshita.[7] A phenomenological analysis by Chew and J. D. Stack[8] has shown that the energy dependence and backward peak width observed in π^-p scattering are consistent with bootstrap calculations of N* parameters. As Chew and Stack point out, however, more accurate data over a wide range of energies near $\cos\theta = -1$ are necessary, as in the π^-p charge-exchange reaction (near $\cos\theta = +1$), to provide a crucial test of the dominance of single α_* pole.

V. Variation of Residues and Diffraction Peak Widths

If we represent pole terms as in (53) by (for even signature)

$$[A(s,t)]_{pole} \cong F_{s_0}(t)\left(\frac{s}{s_0}\right)^{\alpha(t)} e^{-i\pi\alpha(t)/2}, \qquad (54)$$

(or with an additional factor of i for odd signature) where F depends on the choice of scale parameter s_0, it is apparent that we must determine under what circumstance F could be a slowly varying function of t. This circumstance depends on the behavior of the residue $\beta(t)$ (we assume $\alpha(t)$ can be reasonably approximated by a straight line for small -t).

The only point in t we can investigate simply (in a nonrelativistic formalism at least) is a threshold for the channel II reaction, where the channel II c.m. momentum k_t vanishes. At such points we know the threshold behavior of the residues is

$$\beta_n(t) \sim (k_t R)^{2\alpha_n} (k_t \to 0)$$

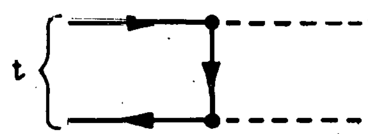
(see Part One, Section IV), where R is the effective range of the potential that acts in channel II scattering. Thus, if we define a reduced residue

$$b_n(t) \equiv \beta_n(t)/(k_t R)^{2\alpha_n(t)}$$

we know that $b_n(t)$ is slowly varying near a threshold in t. Applying this to (53), we write F from (54) as

$$F_{s_0}(t) = \frac{2\alpha(t) + 1}{2 \sin(\pi\alpha/2)} \cdot b(t) \cdot \left(\frac{R^2 s_0}{2}\right)^{\alpha(t)} . \qquad (55)$$

If s_0 is chosen to equal $(R^2/2)^{-1}$, we see that all terms in $F_{s_0}(t)$ are slowly varying near a threshold in t. This is as much as can be done in a model-independent sense; we must then assume that the extrapolation from t = (threshold in channel II) down to t = 0 is a smooth one, and that $F_{s_0}(t)$ is also slowly varying near t = 0. This clearly depends on the details of the channel II potential, for example, and may not be true in all cases. It appears to be an adequate assumption for P, but not for ρ. We estimate the range R by examining the largest-range effective potential contributed by simple one-particle exchange diagrams. For example, in $N\bar{N} \to \pi\pi$, we find one-nucleon exchange



contributes the longest range $R^2 \sim 2\mu M$; this figure is consistent with the s_0 values needed to match the slow logarithmic dependence of the πp diffraction width.

This analysis of residues was first applied by Desai (Ref. 8 of Part I) to πp and pp scattering.

The range R, and hence the appropriate value of s_0, will be different for different reactions. Thus the diffraction peak width, and its rate of shrinkage with s, will differ. Estimates of R_{pp} indicate that pp scattering should exhibit much stronger shrinkage than πp, which agrees qualitatively with experiment.

If one does not rely on F being essentially constant, it will be necessary to parametrize the residues in some way such as an exponential (or sum of exponentials) in t. Then one may choose s_0 arbitrarily, e.g., $1(\text{BeV})^2$. This was done by Phillips and Rarita in their detailed fit to high-energy meson-nucleon scattering,[9] which involves many free parameters.

For other reactions, e.g., Kp and pp scattering, poles other than P, P', and ρ are possible since the $\pi\pi$ selection rule forbidding G = -1 trajectories is absent. Trajectories associated with ω, and an isovector G = -1 (R) pole, discussed in the next section, are required in phenomenological elastic-scattering analyses (see Ref. 9).

VI. Exchange and Inelastic Reactions

The phenomenological fits to total and differential elastic cross sections fail in the case of $\bar{p}p$ scattering in the energy range presently available. For other scattering processes they are reasonably successful, especially for small momentum transfer (see, for example, Binford and Desai[10]). All such phenomenological fits require many parameters. The relevance of simple poles to high-energy processes is much more striking in exchange reactions or inelastic (two-body) reactions, where only one (or perhaps two) poles are allowed in certain favorable cases by strong-interaction selection rules. We have already remarked on π^-p charge exchange. Another case that is very restrictive is the "isospin-exchange" reaction $\pi^-p \to \eta n$. The only poles allowed in channel II, which here is $p\bar{n} \to \pi^+\eta$, have isospin 1, G = -1, and even signature with positive intrinsic parity; associated physical resonances could have JP = 2^+ then (but not 1^-). The leading such trajectory has been called R, and the A_2 may be a physical resonance lying on this trajectory.

The data for this reaction between 4 and 18 BeV/c are clearly consistent with such a single pole; the situation has been analyzed thoroughly by Phillips and Rarita.[11] An analogous case is $\pi^-p \to X^0$ n, but this reaction seems to be rare and in any case has not been studied with care above 6 BeV/c.

A third case in which only one known pole can contribute is $\pi^+p \to \pi^0 N^{*++}_{3/2}$, which allows only ρ in the t channel. Unfortunately the experimental situation is not so clear-cut for this reaction, but data are consistent at present with the hypothesis of a simple pole with trajectory such that $\alpha(0) \cong 0.5$, which is true for ρ as seen in the charge-exchange reaction.

The reactions $\pi^+p \to \eta N^{*++}_{3/2}$, $\pi^+p \to X^0 N^{*++}_{3/2}$ involve only the R trajectory, but no significant data are as yet available on these reactions.

This exhausts the cases in which only one pole contributes. Several reactions involve only two known poles; the simplest examples are

$K^-p \to \overline{K^0}n$	ρ,R	KN, $\overline{K}N$ charge exchange
$K^+n \to K^0p$	ρ,R	

These have been discussed in Ref. 9.

$K^-p \to \pi^0\Lambda$	K*,Q	$\pi^-p \to K^0\Sigma^0$	K*,Q	PS-N hypercharge exchange
$\pi^-p \to K^0\Lambda$	K*,Q	$\pi^+p \to K^+\Sigma^+$	K*,Q	
$K^-p \to \pi^0\Sigma^0$	K*,Q	$K^-p \to \pi^-\Sigma^+$	K*,Q	

These are discussed in Ref. 12.

Here the Q is a pole whose resonances would have quantum numbers $(I,J^P) = (1/2,2^+)$; the K* pole has a resonance with $(I,J^P) = (1/2,1^-)$ at 880 MeV.

The recently observed K*(1410), if $J^P = 2^+$, provides a trajectory suitable for Q.

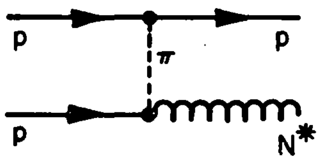
Trajectories with $I = 3/2$ would also contribute to such reactions; but there is no known meson or meson resonance with this isospin quantum number, so unless they would definitely be called for by data (for example in $K^-p \rightarrow \pi^+\Sigma^-$ at high energies), these possibilities are ignored at present.

Many other reactions, producing baryon resonances and/or meson resonances in the final state, involve only two poles. But the number of parameters required is large when the number of spin states involved is large, and fits to data are difficult to carry out in an unambiguous way. This is partially compensated for in some cases where the decay-density matrices can be determined experimentally since these provide a great deal of information concerning the reaction.

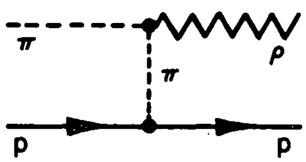
The factorization property of residues (Part One, Section VI) is important in connecting one reaction with another, especially when spin is taken into account.

VII. Peripheral Inelastic Reaction Model as Special Case: Comparison in General

There have been moderately successful explanations of some high-energy reactions, especially baryon resonance production $pp \rightarrow pN^*$, from a field-theoretic point of view using the idea that one-meson exchange diagrams dominate such reactions. In the above case, one-pion exchange contributes:



Another case which has been at least qualitatively successful is $\pi p \rightarrow \rho p$, also involving pion exchange:



(These models work very well if absorptive corrections are applied. We will discuss such modifications of pole terms in Section IX of these lectures.)

Other processes such as $\pi N \to \pi N^*_{3/2}$, which in such a picture might proceed by vector meson exchange (e.g., like ρ), have been successful to a more limited extent; isospin ratios and N* decay-density matrix elements are correct, but energy dependence and angular distributions are not very good. An extensive survey and bibliography are presented in Ref. 13.

We now indicate how "elementary" (i.e., field theoretic with Feynman diagram interpretation) particles can be represented as a limiting case of Regge poles. As a consequence, we see a connection between the Regge-pole approach for inelastic reactions and the peripheral models. Underlying our discussion will be the idea that every particle is composite, in the sense that it can be obtained by solving some relativistic bound-state problem.

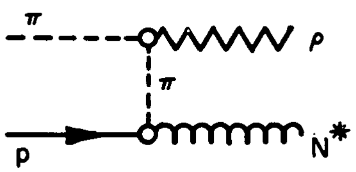
Consider for example a Regge-pole model (ignoring spin) for $\pi^- p \to \rho^0 n$. In the t channel, we have $\bar{p}p \to \pi\rho$; the relevant poles must have G = -1, isospin nonzero; there is no signature restriction. No known resonances or particles have isospin >1, so we look for I = 1, G = -1 particles. The only well-established state is the π meson; if A_1 is a true resonance, it would also be a candidate. (See Ref. 8 of Part One for classification of meson trajectories.) Assuming only π, we see the physical region for channel I (s channel) $\pi^- p \to \rho^0 n$ near the forward direction involves t values within one or two μ_π^2 of the point where $\alpha_\pi = 0$, i.e., the physical pion pole. Since this is a small interval, compared to the characteristic dimension 1 $(\text{BeV})^2$ we have seen in trajectory slopes, we can approximate the (pion trajectory) pole term

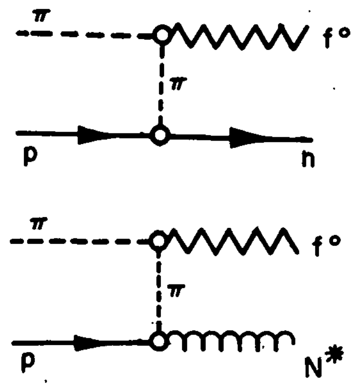
$$\frac{(2\alpha + 1)\ \beta(t)}{\sin \pi\alpha(t)}\ P_\alpha(-z) \left(\frac{1 + e^{-i\pi\alpha(t)}}{2}\right) \tag{56}$$

by its behavior near t = 0, $\alpha = 0$,

$$\frac{\beta(0)}{\sin\,[\pi(t-\mu_\pi^2)\ \alpha']} \cong \frac{\beta(0)/\pi\alpha'}{t - \mu_\pi^2}, \tag{57}$$

which is just the form of the elementary-particle result (ignoring spin); the dominator is just the propagator of the (virtual) pion evaluated on the mass shell for the physical reaction in question, where $[\beta(0)/\pi\alpha']$ takes on the significance of a product of pion-nucleon and pion-rho-pion coupling constants, gf. We observe finally that the residue factorization property guarantees that $\beta(0)$ can be factored into such a product. In comparing (for example) similar reactions (again ignoring spin) such as





assuming all these are dominated by the same pole, putting $\beta_{\rho p} = \beta(0)$ for the reaction $\pi p \to \rho p$ and similar notation for the other reactions, we find the relations

$$\frac{\beta_{\rho p}}{\beta_{\rho N*}} = \frac{\beta_{f^0 p}}{\beta_{f^0 N*}}, \tag{58}$$

which are the same as if we considered elementary pion exchange with coupling constants at the vertices.

Note that such a correspondence in detail (as a function of s and t) depends essentially on the small mass of the pion, which allows an extrapolation to take place over only a small interval in t from the crossed-channel physical region to the pion pole. We expect that more massive resonances (e.g., ρ, ω) differ in greater detail when considered as elementary (i.e., fixed spin), compared to treatment as a Regge pole.

The most striking overall difference between the Regge pole-formulation and the covariant perturbation theory (with fixed-spin particles) appears in the energy dependence of the inelastic (or exchange) reactions. The former predicts asymptotically

$$\frac{d\sigma}{d\Omega} \sim s^{2\alpha-1} \quad \text{or} \quad \frac{d\sigma}{dt} \sim s^{2(\alpha-1)},$$

where α is always less than 1, while the latter predicts

$$\frac{d\sigma}{dt} \sim s^{2(J-1)},$$

where J is the spin of the physical exchanged particle or resonance.

For the light, spin-0 mesons this does not make much difference, but for the 1^- and 2^+ resonances it is a big change. Note that the phase of the inelastic processes is also quite different from the (real) prediction of fixed-spin perturbation theory; however, the phases are not easy to measure.

The relative importance of various poles, if more than one contribute, depends on the energy (s) in general. Although the lightest mass state (e.g., π) may dominate at low energies because the factor $\mu_\pi^2 - t$ is small, at high energies this consideration gives way to the factor s^α, which is greater for higher trajectories (e.g., ρ).

Remark. SU_3(and other higher symmetries) predictions may be developed in a way analogous to the relations between coupling constants in perturbation theory, through the use of appropriate Clebsch-Gordan coefficients. This is discussed in Refs. 9, 11, 12, and 14. The latter reference, in particular, using only residues evaluated at $t = 0$ (by analyzing only total cross sections) avoids some of the ambiguities associated with symmetry breaking. These arise, for example, because the functions $\beta(t)$ will not be the same functions of t, even within a multiplet of SU_3 due to the mass differences.

VIII. Spin, Polarization, and Decay-density Matrices

The introduction of spin for the incoming and outgoing particles complicates the practical application of Regge poles enormously. One of the worst difficulties appears in the crossing relations. The scattering amplitudes for given spin (or alternatively helicity states) in channel I are obtained as linear combinations of spin or helicity amplitudes in channel II, with coefficients that depend on s and t. Thus one requires first of all (in principle) a determination of this crossing matrix, which is an involved problem in general. If this matrix is known (as it is in the simplest cases such as πp and pp scattering), the problem remains of determining the many parameters required to specify the various residues involved. The situation in both channel I and channel II may be described with a multichannel formalism (such as that of Section VI, Part One) in which different spin or helicity states are represented by different channels. (Only some of these will be actually coupled, because angular-momentum, time-reversal, and parity selection rules will forbid some transitions.) The residues for each spin state will (in general) have different behavior as functions of t.

Some (mostly formal) results concerning crossing relations for helicity amplitudes have been proved by Trueman and Wick[15] and employed by Gottfried and Jackson[16] in their discussion of spin in exchange models (including Regge poles). They show, for example, that for certain cases (e.g., $Kp \to K^*p$), in the spin-parity analysis of exchanged mesons (using evidence from the decay-density matrix of final-state resonances), one may generalize immediately from fixed-spin (e.g., elementary K*) exchange to Regge poles. In general, however, the situation as regards spin is much more complicated when Regge trajectories are used than in the elementary-exchange peripheral model.

The essential formalism for Regge poles in πN scattering and charge exchange may be found in Ref. 17, a pioneering paper on the subject of Regge poles and their phenomenological application. (This formalism, as far as kinematic factors are concerned, may be used for KN reactions also, if we include poles of both signatures for each isospin state instead of only one in πN reactions.) A summary is given in Ref. 9. Discussion of pp scattering (and pn charge exchange) is given in Ref. 18. This formalism for $\pi p \to \pi N^*$ and $KP \to KN^*$ has been worked out by Hara.[19]

Following the approach of Refs. 15 and 16, we decompose a general reaction amplitude in channel II (for the reaction $a + \bar{c} \to \bar{b} + d$) in terms of covariant helicity amplitudes

$$\langle\lambda_3\lambda_4|A^{II}(t,s)|\lambda_1\lambda_2\rangle,$$

such that the differential cross section for the reaction $a + \bar{c} \to \bar{b} + d$ with incoming $(a,\bar{c})$ helicities (λ_1,λ_2) and outgoing $(\bar{b},d)$ helicities (λ_3,λ_4) is given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{II}_{\lambda_1\lambda_2;\lambda_3\lambda_4} = \frac{1}{t}\left|\langle\lambda_3\lambda_4|A^{II}(t,s)|\lambda_1\lambda_2\rangle\right|^2. \qquad (59)$$

Similarly, in channel I, the reaction $a + b \to c + d$ is described by channel I helicity amplitudes

$$\langle\nu_3\nu_4|A^{I}(s,t)|\nu_1\nu_2\rangle,$$

such that the differential cross section for the channel I reaction with incoming (a,b) helicities (ν_1,ν_2) and outgoing (c,d) helicities (ν_3,ν_4) is given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{I}_{\nu_1\nu_2;\nu_3\nu_4} = \frac{1}{s}\left|\langle\nu_3\nu_4|A^{I}(s,t)|\nu_1\nu_2\rangle\right|^2. \qquad (60)$$

Then, according to Ref. 15, there exists an orthogonal crossing matrix X (whose elements are functions of s and t and are real in the physical regions), such that

$$\langle\nu_3\nu_4|A^{I}(s,t)|\nu_1\nu_2\rangle = \sum_{\mu_1\mu_2} X^{\mu_1\mu_2}_{\nu_2\nu_4}\langle\nu_3\mu_2|A^{II}(t,s)|\nu_1\mu_1\rangle. \qquad (61)$$

The X matrix simplifies to some extent in the $s \to \infty$ asymptotic unit, or if equal mass particles are involved. Further, as $t \to 0$, X is nonsingular, so we can use a limiting form near the forward (channel I) direction if desired.

In the πN case, we have only two independent helicity amplitudes; in channel I, where the helicities refer to initial and final nucleons, we have

$$G_+(s,t) = \langle +1/2|A^{I}|+1/2\rangle = \langle -1/2|A^{I}|-1/2\rangle$$

and

$$G_-(s,t) = \langle -1/2|A^{I}|+1/2\rangle = \langle +1/2|A^{I}|-1/2\rangle, \qquad (62)$$

where the latter equalities follow from time-reversal symmetry and parity conservation.

In channel II, $N\bar{N} \to \pi\pi$, where helicities refer to incoming nucleon and antinucleon, we define

$$F_+(t,s) = \left\langle |A^{II}| +1/2+1/2 \right\rangle = \left\langle |A^{II}| -1/2-1/2 \right\rangle,$$

and

$$F_-(t,s) = \left\langle |A^{II}| +1/2-1/2 \right\rangle = \left\langle |A^{II}| -1/2+1/2 \right\rangle, \qquad (63)$$

Then X can be represented as a 2 x 2 matrix, which is derived in Ref. 15.

The Regge-pole decomposition $F_\pm$, which is appropriate for P,P',ρ in the $N\bar{N} \to \pi\pi$ channel, may be obtained from a Sommerfeld-Watson (SW) transform applied to partial-wave representations of the helicity amplitudes ($z_t = \cos\theta_t$):

$$F_+ = \sum_J (2J+1)\; f_+^J(t)\; d^J_{1/2\;1/2}(z_t),$$

and

$$F_- = \sum_J (2J+1)\; f_-^J(t)\; d^J_{1/2\,-1/2}(z_t). \qquad (64)$$

The d functions here are as defined by Jacob and Wick.[20] Each of these sums must be separated into even and odd J values to obtain analytic continuations in J for the partial-wave amplitudes. Then we continue the d functions analytically in J by using their definitions in terms of hypergeometric functions. After performing the SW transform, we obtain

$$F_+^e = \sum_{\substack{\text{even-}\\ \text{signature}\\ \text{poles}}} \beta^e_{+n}(t)\, \frac{(2\alpha_n^e(t)+1)\; d_{1/2\;1/2}^{\alpha_n^e(t)}(z_t)}{\sin \pi\alpha_n^e(t)} \left(\frac{1+e^{-i\pi\alpha_n^0(t)}}{2}\right) + \begin{pmatrix}\text{even}\\ \text{background}\\ \text{integral}\end{pmatrix},$$

and

$$F_+^0 = \sum_{\substack{\text{odd-}\\ \text{signature}\\ \text{poles}}} \beta^0_{+n}(t)\, \frac{(2\alpha_n^0(t)+1)\; d_{1/2\;1/2}^{\alpha_n^0(t)}(z_t)}{\sin \pi\alpha_n^e(t)} \left(\frac{1-e^{-i\pi\alpha_n^e(t)}}{2}\right) + \begin{pmatrix}\text{odd}\\ \text{background}\\ \text{integral}\end{pmatrix}, \qquad (65)$$

and similarly for F_-^e and F_-^0, with

$$d_{-1/2\;1/2}^{\alpha_n^{e,0}(t)}(z_t);$$

and finally,

$$F_+ = F_+^e + F_+^0, \quad F_- = F_-^e + F_-^0.$$

Note that the same trajectories will in general appear in F_+^e and F_-^e, and in F_+^0 and F_-^0, but with different residues in (+) and (-) amplitudes.

Now when crossing is employed, and $z_t \to \infty$ as $s \to \infty$, we utilize the asymptotic behaviors

$$d^{\alpha_n}_{1/2\ 1/2}(z_t) \to (z_t)^{\alpha_n}$$

and

$$d^{\alpha_n}_{-1/2\ 1/2}(z_t) \to \alpha_n (z_t)^{\alpha_n - 1} \tag{66}$$

and obtain results in the asymptotic region of the form

$$G_\pm(s,t) \cong \sum_n (2\alpha_n + 1)\left[\frac{1 + \varepsilon_n e^{-i\pi\alpha_n(t)}}{2 \sin \pi\alpha_n(t)}\right]\left(X_{\pm,+}\beta_{+n} z_t^{\alpha_n} + \alpha_n X_{\pm,-}\beta_{-n} z_t^{\alpha_n - 1}\right), \tag{67}$$

where $\varepsilon_n = \pm 1$ is the signature factor of the n^{th} pole, and X_{ij} are the crossing-matrix elements. (The $X_{\pm,-}$ have one higher power of s than the $X_{\pm,+}$, so the contributions from β_+ and β_- are comparable in general.)

Defining reduced residues

$$b_n^\pm(t) = \frac{\beta_{\pm n}(t)\ s_0^{\alpha_n(t)}}{(2k_t^2)^{\alpha_n(t)}},$$

we obtain finally

$$G_\pm(s,t) \cong \sum_n \left\{(2\alpha_n(t) + 1)\left(\frac{1 + \varepsilon_n e^{-i\pi\alpha_n(t)}}{2 \sin \pi\alpha_n(t)}\right)\left(\frac{s}{s_0}\right)^{\alpha_n(t)}\right\}\gamma_{\pm n}(t), \tag{68}$$

where the $\gamma_{\pm n}$ become independent of $\underline{s}$ (as indicated) in the high-energy limit:

$$\gamma_{+n}(t) = X_{++} b_n^+(t) + \alpha_n(t)\left(\frac{X_{+-}}{s}\right) b_n^-(t)$$

and

$$\gamma_{-n}(t) = X_{-+} b_n^+(t) + \alpha_n(t)\left(\frac{X_{--}}{s}\right) b_n^-(t). \tag{69}$$

Now the differential cross section in channel I is given asymptotically by

$$\frac{d\sigma}{d\Omega} = \frac{1}{s}\left\{|G_+|^2 + |G_-|^2\right\} \sim s^{2\alpha-1},$$

where α is the trajectory of the dominant pole.

The polarization of the final nucleon, if the target is unpolarized, is given by

$$P(\theta) \quad \frac{d\sigma}{d\Omega} = 2\,\mathrm{Im}\{G_+^* G_-\}/s.$$

The polarization thus vanishes if the phases of G+ and G− are equal, as they are when a single pole dominates. With two poles contributing, say α_1 and α_2, we get a polarization that asymptotically has energy dependence $s^{-(\alpha_1-\alpha_2)}$ (assuming $\alpha_1 > \alpha_2$).

These facts remain true in general for other reactions involving spin, e.g., pp scattering. Note that although final nucleon polarization (assuming an unpolarized target) vanishes when only a single pole contributes, this is not true for other spin-correlation parameters, which may not be asymptotically zero.

For the general reaction with spin, (64) can be generalized to[20]

$$\left\langle\lambda_3\lambda_4|A^{II}|\lambda_1\lambda_2\right\rangle = \sum_J f^J_{\lambda_1\lambda_2;\lambda_3\lambda_4}(t)\cdot(2J+1)\cdot d^J_{\lambda_3-\lambda_1,\lambda_4-\lambda_2}(z_t), \tag{70}$$

and, separating into even and odd J, SW transforms can be applied to yield the following generalizations of (65):

$$\left\langle\lambda_3\lambda_4|A^{II}|\lambda_1\lambda_2\right\rangle = \sum_{\text{poles}} \beta^{(n)}_{\lambda_1\lambda_2;\lambda_3\lambda_4}(t)\left\{\frac{(2\alpha_n(t)+1)\, d^{\alpha_n(t)}_{\lambda_3-\lambda_1,\lambda_4-\lambda_2}(z_t)}{2\sin\pi\alpha_n(t)}\left[1+\varepsilon_n e^{-i\pi\alpha_n(t)}\right]\right\} + (\text{background integrals}). \tag{71}$$

The asymptotic forms of the d functions are powers of z_t analogous to (66), and one obtains eventually for high-energy channel I reactions, the general form,

$$\left\langle\nu_3\nu_4|A^I(s,t)|\nu_1\nu_2\right\rangle \cong \sum_{\text{poles}} \frac{[2\alpha_n(t)+1][1+\varepsilon_n e^{-i\pi\alpha_n(t)}]}{2\sin\pi\alpha_n(t)}\cdot\left(\frac{s}{s_0}\right)^{\alpha_n(t)}\cdot\left\{\sum_{\lambda_2\lambda_4} X^{\lambda_4\lambda_2}_{\nu_4\nu_2}\, b^{(n)}_{\nu_1\lambda_2\nu_3\lambda_4}(t)\right\}. \tag{72}$$

More details have been given by Fiset[21] who points out that for evaluating spin-summed cross sections, one does not need to know X since its orthogonality allows its elimination.

Gottfried and Jackson[16] show how the density matrix for final-state resonance decays may be obtained from such helicity matrix elements.

The formula (72) shows that the energy dependence and phase of pole contributions are relatively easily predicted even when many spin states occur in the reaction. The reduced residues $b_{\lambda_1\lambda_2\lambda_3\lambda_4}(t)$, however (at least in a purely phenomenological approach), are in general independent functions in each reaction, although a factorization property holds for these helicity residues (as a special case of multichannel reactions). For an application, see Barmawi.[22,23]

When a reaction is fit by poles it is possible in some cases to check the reasonableness of the residues within the framework of bound-state models, as developed by bootstrap practicioners. An example of a nontrivial case is presented in Ref. 24.

IX. Regge Poles in the Optical Model Potential and Absorptive Corrections

A. Motivation

If we regard the Regge-pole formalism as a whole as a generalization of field-theoretic, i.e., covariant, perturbation-theory, peripheral (one-meson exchange) models, then analogies with more complete field-theoretic (or even potential-theory) models may enable further developments toward more exact agreement with experiment. That such developments are indeed necessary is indicated by the necessity for applying absorptive corrections to processes involving pion exchange.

We have seen that the π pole should, for small momentum transfer, behave like the elementary π exchange pole as calculated from covariant perturbation theory (in lowest order). However, quantitative agreement with experiments such as $\pi p \to \rho p$, including overall normalization and ρ decay-density matrix elements, require strong modification of the elementary formula as provided by the absorptive-correction method. References 25-27 discuss this method and its quantitative results. Generalizing from this, it seems plausible that (at least) normalizations and spin-state properties (e.g., polarization and decay-density matrices) will not be correctly described by a Regge-pole model (at least when simplest hypotheses are employed for the residue behaviors), but some modifications (e.g., "absorptive corrections") are necessary. For inelastic channels, which individually constitute a small part of the total cross sections at high energies, this can be expressed by saying that competition from many other channels should be explicitly taken into account in the formalism. This is provided for if there is some means of insuring unitarity of the S matrix at high energies, where much inelasticity is present.

Although it is possible to develop the following (optical-model) approach from a covariant, field-theoretic viewpoint, the physical ideas are more simply understood with a high-energy (but apparently nonrelativistic), potential-theory formalism, using analogies with nuclear-physics scattering theory in the optical-model approximation.

B. Optical-model "Potential" Definition

Consider a scattering process in which many inelastic channels are open. For the moment, we consider only two-body channels. We can formally describe the system with a multichannel Schrödinger equation as in Part One, Section VI, for the radial-wave functions of the system

$$\left\{\frac{d^2}{dr^2} + k_i^2 - \frac{\ell(\ell + 1)}{r}\right\} U_{\ell i}(k_i,r) = \sum_j V_{ij}(r)\, U_{\ell j}(k_j,r). \qquad (73)$$

(We may assume the V_{ij} are independent of energy k, although this is not necessary.) In Part One, Section VI, we indicate how one could develop a matrix integral equation [cf. Eq. (30)] whose Fredholm factors N and D yielded the multichannel bound states as well as multichannel scattering amplitudes. Our motivation was primarily to obtain low-energy information, i.e., concerning bound states and scattering resonances.

Now, however, we are interested in the high-momentum scattering (large k_i) problem associated with (73), and we will sketch a different formulation for such a purpose, motivated primarily by the optical-model formulation of scattering by a complex nucleus. A nucleus has excited states, and any of these may be excited by a projectile passing through the nucleus in a scattering process; thus the probability for inelastic processes (i.e., excitation) is large, and in fact the excitation properties may dominate the calculation of scattering.

In this section, we will show that by formally eliminating explicit reference to channels $j \neq i$, a complex potential operator $V^{(i)}$ may be constructed, such that the elastic scattering in channel i is given (exactly) by solving a one-channel radial Schrödinger equation with the complex, nonlocal potential $V^{(i)}_{k,\ell}(r,r')$ as follows ($k = k_i$):

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell + 1)}{r^2}\right] U_{\ell i}(k,r) = \int_0^\infty dr'\, V^{(i)}_{k\ell}(r,r')\, U_{\ell i}(k,r'). \qquad (74)$$

[In Section C below we will show that for large momentum (k), we can simplify further to obtain a local potential, and then further to obtain an explicit solution.]

This V(r,r') will be denoted by the (exact) optical potential; it is sometimes called a pseudopotential.

To show this, we proceed as follows: Define the (diagonal matrix) operators (kinetic-energy plus centrifugal-barrier terms) by

$$T_{\ell ij} = \left[\frac{d^2}{dr^2} + k_i^2 - \frac{\ell(\ell + 1)}{r^2}\right] \delta_{ij}.$$

Then (73) may be written

$$T_\ell \vec{U} = V_\ell \vec{U},$$

where

$$\vec{U} = \begin{pmatrix} U_{\ell 1}(k_1,r) \\ U_{\ell 2}(k_2,r) \\ \cdot \\ \cdot \\ \cdot \\ U_{\ell N}(k_N,r) \end{pmatrix}.$$

Suppose we can find a diagonal-matrix Green's function operator $G_{0\ell}$ [analogous to the single-channel $G_\ell(k;\underset{\sim}{r},r')$ of Part One, Section II] which is a right inverse for T, i.e., $T\, G_0\, \vec{U} = \vec{U}$, and such that $(G_0\, \vec{U})$ satisfies outgoing-wave boundary conditions in all channels; and suppose we can find a $\vec{U}_0(r)$ such that

(1) $T\, \vec{U}_0 = 0$,

and

(2) $\vec{U}_0(r)$ satisfies boundary conditions describing only a plane wave in channel I:

$$\vec{U}_0 = \begin{pmatrix} U_1^0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \qquad U_1^0 \propto j_\ell(k,r).$$

Then if $\vec{U}$ is the solution of (73), with plane wave in channel I plus outgoing waves in all channels, $\vec{U}$ satisfies

$$\vec{U} = \vec{U}_0 + G_{0\ell}\, V_\ell\, \vec{U}, \tag{75}$$

which can be readily verified by substituting (75) into (73), and by utilizing the boundary conditions stated above.

Now separate the first component (U_1) of (75) from the other components $(U_m, m > 1)$, and the result is

$$U_1 = U_1^0 + (G_{0\ell}\, V_\ell)_{11}\, U_1 + \sum_{m>1} (G_{0\ell}\, V_\ell)_{1m}\, U_m, \tag{76}$$

and

$$U_m = \sum_{n>1} (G_{0\ell}\ V_\ell)_{mn}\ U_n + (G_{0\ell}\ V_\ell)_{m1}\ U_1. \tag{77}$$

Let H_ℓ be the (N - 1) x (N - 1) matrix of operators

$$(H_\ell)_{mn} = (G_{0\ell}\ V_\ell)_{mn}, \quad m \neq 1, \quad n \neq 1,$$

appearing in (77). Then (77) can be written

$$(I - H_\ell)\ \vec{U}' = \vec{Y}, \tag{78}$$

where $\vec{U}'$ signifies the (N - 1) components

$$\begin{pmatrix} U_2 \\ U_3 \\ \cdot \\ \cdot \\ U_N \end{pmatrix}$$

(i.e., $\vec{U}$ with the first component deleted), and $\vec{Y}$ signifies the (N - 1) components

$$Y_m = (G_{0\ell}\ V_\ell)_{m1}\ U_1, \quad m \neq 1.$$

From the form (78), assuming we can find an inverse operator $(I - H_\ell)^{-1}$, then (77) becomes

$$\vec{U}' = (I - H_\ell)^{-1}\ \vec{Y}, \tag{79}$$

or, in terms of components, for m > 1 we have

$$U_m = \sum_{n>1} (I - H_\ell)^{-1}_{mn}\ (G_{0\ell}\ V_\ell)_{n1}\ U_1. \tag{80}$$

Substituting (80) into (76), we obtain an equation which involves only the channel 1 wave function in an explicit way,

$$U_1 = U_1^0 + (G_0\ V)_{11}\ U_1 + \sum_{\substack{m\neq 1 \\ n\neq 1}} (G_0\ V)_{1m}\ (I - H_\ell)^{-1}_{mn}\ (G_0\ V)_{n1}\ U_1, \tag{81}$$

which can be written

$$U_1 = U_1^0 + K_1 U_1, \tag{82}$$

where the operator K_1 is defined by

$$K_1 = (G_{0\ell}\, V_\ell)_{11} + \sum_{\substack{m\neq 1\\ n\neq 1}} (G_{0\ell}\, V_\ell)_{1m}\, (I - H_\ell)^{-1}_{mn}\, (G_{0\ell}\, V_\ell)_{n1}. \tag{83}$$

Finally, to return to the form of a single-channel Schrödinger equation (74), we multiply by T_{11} and employ $T_{11}\, U_1^0 = 0$; then

$$T_{11}\, U_1 = (T_{11}\, K_1)\, U_1 \equiv V^{(1)}\, U_1, \tag{84}$$

which is just in the form (74), with

$$V_\ell^{(1)} = T_{\ell 11}(G_{0\ell}\, V_\ell)_{11} + T_{\ell 11} \sum_{\substack{m\neq 1\\ n\neq 1}} (G_{0\ell}\, V_\ell)_{1m}\, (I - H_\ell)^{-1}_{mn}\, (G_{0\ell}\, V_\ell)_{n1}. \tag{85}$$

Separating out the terms involving only channel 1, we can write (85) more explicitly as

$$V_\ell^{(1)} = (V_\ell^{(r)})_{11}\, \delta(r - r') + \sum_{\substack{m\neq 1\\ n\neq 1}} (V_\ell)_{1m}\, (I - H_\ell)^{-1}_{mn}\, (G_{0\ell}\, V_\ell)_{n1}, \tag{86}$$

where we have used the facts that T_ℓ and $G_{0\ell}$ are diagonal, and $T_\ell G_{0\ell} = I\delta(r - r')$.

Remarks

1. The restriction to two-body channels was only for convenience of notation; the formal results are true for multiparticle channels as well.

2. $(I - H_\ell)$, and hence $(I - H_\ell)^{-1}$, depend on the channel momenta $k_n (n \neq 1)$, and are complex above the threshold for channels $m \neq 1$, i.e., inelastic thresholds. Thus $V_\ell^{(1)}$ will be complex above the inelastic threshold for scattering in channel 1, and even below such physical thresholds, $V_\ell^{(1)}$ will be energy-dependent, unless only V_{11} is nonzero, (if V_{11}^ℓ itself is energy-independent).

3. A relativistic field-theoretic construction of the optical potential for high-energy physics has been outlined by Blokhintsev et al.[28]

C. High-momentum Scattering with Optical Potential; the Eikonal Approximation

A physical interpretation can be attached to the nonlocal, energy-dependent, potential operator $V_{k\ell}^{(1)}(r,r')$ as follows: Suppose the potentials V_{ij} are very weak, so we can use a first-order approximation to (74), where the unperturbed $U_1(r)$ represents a plane wave; i.e., we use the first iteration of the integral equation (81) or (82).

Then $[V(r,r')\ U_1^0(r')\ dr']$ is the differential source strength [for the wave equation (74)] at r, due to the incident wave function U_1^0 at r', in an interval dr'.

The nonlocality of V(r,r') comes from the physical fact that channel 1 particles can go into channel m ($m \neq 1$) at point r', and reappear in channel 1 by making the transition ($m \to 1$) at point r.

This interpretation allows us to develop approximations good at high energies. In particular, it is plausible that for large incident momentum, i.e., high velocities, the optical potential becomes essentially local, i.e.,

$$V_{k\ell}^{(1)}(r,r') \to \bar{V}_{k\ell}^{(1)}(r)\ \delta(r - r'),$$

where

$$\bar{V}_{k\ell}^{(1)}(r) = \int_0^\infty dr'\ V_{k\ell}^{(1)}(r,r')\ j_\ell(kr')$$

(aside from a normalization factor). In the classical limit, this is intuitively evident, especially if the other channels have larger masses, since in that case particles in channel $n \neq 1$ would travel slower than the particles in channel 1 (by conservation of energy), and so transitions to and from other channels must occur at neighboring points (r,r').

A basically semiclassical approximation for the solution of (74) is convenient for high-momentum scattering and small angles if one assumes V is approximately local in r. This approximation, the Eikonal, has been extensively discussed by Glauber[29] using scattering equations in three dimensions. We can obtain the same results simply by finding an approximation satisfying the following requirements:

a. High partial waves dominate the elastic-scattering cross section.

b. The approximation should yield the complex, energy-dependent Born approximation for the scattering amplitude in channel I in the limit $|V| \to 0$.

c. The channel 1 phase shifts (complex above inelastic threshold) are a linear function of the potential $V^{(1)}$, in the high-momentum unit.

The first two requirements are generally to be expected when the momentum k_1 becomes large compared to the range of the potential, and when only small angle scattering dominates the elastic cross section. This last requirement is the more restrictive one for high-energy physics.

The significance of postulate c becomes apparent when we consider the simple optical model of a partially absorbing sphere. Here the "phase shifts" will be purely imaginary, and the attenuation factor (inverse mean free path) in each partial wave should be proportional to the optical density of the target, which is characterized by a purely imaginary potential V if it is purely absorbing. (Postulate c actually represents the dynamical assumptions involved in the approximation, although in a nontransparent fashion.)

If postulate a is true, we can replace the partial-wave sum

$$f(\theta) = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos \theta) \left[\frac{e^{2i\delta_{\ell}(k)} - 1}{2ik}\right] \tag{87}$$

by an integral over continuous ℓ, and use a large $-\ell$ approximation for the P_{ℓ}'s,

$$f(\theta) = \int_0^{\infty} d\ell (2\ell + 1) J_0[(2\ell + 1) \sin (\theta/2)] \left[\frac{e^{i\chi(\ell,k)} - 1}{2ik}\right],$$

where we have written a continuous function χ instead of the discrete $2\delta_{\ell}$.

If we change variables to $b = (\ell + 1/2)/k$, known as the impact-parameter variable, this becomes

$$f(\theta) = ik \int_0^{\infty} b\, db\, J_0[2k\, b \sin (\theta/2)][1 - e^{i\chi(b,k)}]. \tag{88}$$

Note that

$$2k \sin (\theta/2) = \sqrt{-t},$$

so alternatively we can write

$$f(s,t) = ik \int_0^{\infty} b\, db\, J_0(b\sqrt{-t})[1 - e^{i\chi(b,k)}]. \tag{89}$$

Now if χ is to yield the Born approximation for small $|V|$, and χ (proportional to phase shift) is linear in V, then as $|V| \to 0$, we find $\chi \to 0$, and we must have, to first order in V,

$$f \to f^{Born} = ik \int_0^{\infty} b\, db\, J_0(b\sqrt{-t})[-i\chi(b,k)]. \tag{90}$$

Using the inverse Fourier-Bessel transform, we obtain

$$\chi(b,k) = \frac{1}{k} \int_0^{\infty} x\, dx\, J_0(xb)\, f^{Born}(s,-x^2), \tag{91}$$

where

$$x \equiv \sqrt{-t}.$$

Finally, since χ is to be linear in V, and f^{Born} is linear in V, the relation (91) must be true for all strengths of V and not just $|V| \to 0$.

Thus (91) and (89) define our high-momentum, small-angle approximation. The function X is known as the Eikonal function.

The Eikonal approximation may also be obtained as a linearized form, suggested by the high-energy limit, of the WKB approximation. However, the Eikonal method is actually much better than the WKB method for small-angle scattering; for example, the WKB approximation demands that all phase shifts be large in magnitude, whereas the Eikonal approximation (correctly) obtains the Born approximation for small phase shifts.

In practice, the potentials for the coupled-channel problem (73) are not known, and one tries to estimate the optical potential (in local approximation) directly, either by analyzing the data in a purely phenomenological approach, or by adopting some simplified theoretical model. One example of the latter is found in nuclear physics, where high-energy elastic scattering from a large nucleus is well described by a "grey sphere" model,

$$\begin{aligned}\bar{V}^{(1)}(r) &= iV_0 \qquad r \le R, \\ &= 0 \qquad r > R,\end{aligned}$$

where V_0 is real and positive. Another example is described in the next section.

Remarks

1. The Eikonal method in relativistic field theory is not on firm ground at present, since one does not have an explicit equation of motion analogous to the Schrödinger equation, and yet to investigate the nature of the approximation, one must go beyond the lowest orders of perturbation theory. The best information on its significance available at present may be found in the thesis of Torgerson.[30]

2. The Regge (complex angular momentum in t channel) representation for the amplitudes $A^I(s,t) = \sqrt{s}\, f(s,t)$ obtained from the Eikonal formula may be used, in which case we discover that $A_\lambda^{II}(t)$ has branch points (not only poles) in the complex λ plane whose locations depend on t. At $t = 0$ (with the Pomeranchon in the Eikonal function) they move up to $\lambda = 1$, and appear of equal asymptotic importance as the Pomeranchon pole. For $-t > 0$, they are higher than the Pomeranchon and dominate over the pole asymptotically. Such branch points, consequences of s channel unitarity, were first located by Amati et al.[31] and further discussed by Mandelstam.[32,33] However, the cancellations pointed out by Mandelstam[32] may not occur in the Eikonal formalism; this point is discussed in Ref. 34.

D. Regge Poles and the High-energy, Optical-model Born Approximation

To determine X, it is necessary to know the functional form of the Born approximation for scattering, f^{Born}.

If we have a model that contains a parameter, say Γ, which multiplies the strength of the potential, then f^{Born} will be proportional to Γ.

The amplitude f, determined by the Eikonal expression, then will approach zero as $\Gamma \to 0$. Conversely, if we have a model such that f is proportional to a parameter Γ as $\Gamma \to 0$, we can interpret the limiting function (f/Γ) as the Born approximation f^{Born}, except for normalization.

Now the phenomenological Regge-pole model, described previously in these lectures as a model for the high-energy behavior of scattering amplitudes f, can be formally provided with such a strength parameter simply by multiplying the crossing matrices by Γ; the model as previously described then corresponds to $\Gamma = 1$. But the discussion above then implies that the Regge poles in the crossed channels are to be interpreted in the sense of a Born approximation for the s channel reaction and supply us not with f itself but with f^{Born}, hence the Eikonal function χ.

In the limit of small-momentum transfer and high energy, the Born approximation sometimes gives a satisfactory qualitative description of the scattering amplitude, but in general the use of Regge poles in χ leads to quantitatively different results than using Regge poles in f, particularly in reactions such as charge exchange, which can be considered as small differences of elastic-scattering amplitudes. The difference between f and f^{Born} in such a case can be expressed as follows: If f_{CE} is the difference between πN elastic scattering in isospin states $T = 3/2$ and $T = 1/2$, we have Eikonals $\chi_{3/2}$ and $\chi_{1/2}$, obtained from the Regge poles contributing to these states; ignoring spin, we obtain

$$f_{CE} = ik \int_0^\infty b\, db\, J_0(b\sqrt{-t})[e^{i\chi_{1/2}} - e^{i\chi_{3/2}}]. \tag{92}$$

Now if the elastic scattering is dominated by $T = 0$ poles in the <u>t</u> channel (e.g., P and P'), such that

$$|\chi_{3/2} - \chi_{1/2}| \ll 1, \tag{93}$$

we can write

$$\chi_{3/2} = \bar{\chi} + \frac{\delta\chi}{2}, \quad \chi_{1/2} = \bar{\chi} - \frac{\delta\chi}{2},$$

where

$$|\delta\chi| \ll 1,$$

and $\bar{\chi}$ is the average elastic-scattering Eikonal function; thus

$$f_{EL} \cong ik \int_0^\infty b\, db\, J_0(b\sqrt{-t})(1 - e^{i\bar{\chi}}). \tag{94}$$

Expanding the exponentials in (92) and keeping only first powers of $\delta\chi$, we obtain

$$f_{CE} \cong ik \int_0^\infty b\, db\, J_0(b\sqrt{-t})\, e^{i\bar{\chi}}(-i\delta\chi). \tag{95}$$

Now $(k\delta\chi)$ is just the Fourier-Bessel transform [cf. (91)] of the charge-exchange Born approximation, i.e., the dominant Regge pole in the charge-exchange amplitude. If we determine $\bar{X}$ from elastic-scattering experimental data through (94), then (95) represents exactly the absorptive-correction prescription (omitting spin) of Refs. 25-27 applied to πN charge exchange with a Regge-pole form for f_{CE}^{Born}.

A simplified numerical example of the magnitude of this correction is given in Ref. 34, as well as some additional discussion concerning the foundations of the method in relativistic formalism. Applications to reactions with spin are naturally complicated, and follow the methods described in Refs. 25-27 with Born terms as discussed in Section VIII of Part Two of these lectures, provided the elastic scattering can be well described by a spin-independent scattering amplitude.

E. Spin Flip Amplitudes in the Eikonal Approach

If the above condition is not met, one must use an extension of the Eikonal formalism including spin. The nonrelativistic description for spin 1/2-spin 0 cases is given in Ref. 29, but a relativistic treatment has not been given in the literature. For completeness, and for illustrative purposes, we now discuss the relativistic πN scattering problem (including spin). Our starting point will be the postulates a, b, and c above; development of the Eikonal formalism for helicity nonflip (G_+) and helicity flip (G_-) amplitudes then will follow analogously to the helicity-less problem.

In πN scattering, scattering eigenstates (and hence phase shifts) for definite J and parity (±) may be constructed; the corresponding partial-wave scattering amplitudes $f_{J\pm}$ are linear combinations of the partial-wave helicity amplitudes $g_\pm^J$ occurring in the expansions[20]

$$G_+(s,t) = \sum_J \; J + \frac{1}{2} \; d^J_{1/2\;1/2}(z) \; g^J_+(s),$$

and

$$G_-(s,t) = \sum_J \; J + \frac{1}{2} \; d^J_{1/2\,-1/2}(z) \; g^J_-(s). \tag{96}$$

Explicity, the J^P eigenamplitude expansion for spin 1/2-spin 0 scattering can be written[35]

$$f_1(s,t) = \sum_{\ell=0}^{\infty} f_{\ell+} \, P'_{\ell+1}(z) - \sum_{\ell=2}^{\infty} f_{\ell-} \, P'_{\ell-1}(z)$$

and

$$f_2(s,t) = \sum_{\ell=1}^{\infty} (f_{\ell-} - f_{\ell+}) \, P'_\ell(z), \tag{97}$$

where $z = \cos\theta$; the normalization is fixed by

$$\frac{d\sigma}{d\Omega} = |f_1 + f_2 \cos\theta|^2 + |f_2|^2 \sin^2\theta. \tag{98}$$

The $f_{\ell\pm}$, expressed in terms of phase shifts, are

$$f_{\ell\pm} = \frac{e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm}}{k} = \frac{e^{2i\delta_{\ell\pm}} - 1}{2ik}. \tag{99}$$

Here $\ell = J - 1/2$ for $f_{\ell-}$ and $\ell = J + 1/2$ for $f_{\ell+}$; (ℓ is the orbital angular momentum and is a good quantum number when spin is present in this case, only because there is an unique relation between J, parity, and ℓ for spin 1/2-spin 0 scattering).

Comparing (97) and the normalization appropriate to covariant helicity amplitudes,

$$\frac{d\sigma}{d\Omega} = s^{-1}\{|G_+|^2 + |G_-|^2\}, \tag{100}$$

we find the following relations between $G_\pm$ and f_1 and f_2:

$$G_+ = (f_1 + f_2) \cos (\theta/2)$$

$$G_- = (f_1 - f_2) \sin (\theta/2). \tag{101}$$

Then, if we examine the expressions of the $d^J_{\mu\nu}$ functions in terms of P'_ℓ and sin (θ/2) or cos (θ/2) (see Ref. 20 of appendix), we obtain, by comparing (96) and (97),

$$g^J_+ = \sqrt{s}[f_{\ell+} + f_{(\ell+1)-}] = [i\rho(s)]^{-1} \cdot [e^{2i\delta_{\ell+}} + e^{2i\delta_{(\ell+1)-}} - 2]$$

and

$$g^J_- = \sqrt{s}[f_{\ell+} - f_{(\ell+1)-}] = [i\rho(s)]^{-1} \cdot [e^{2i\delta_{\ell+}} - e^{2i\delta_{(\ell+1)-}}], \tag{102}$$

where $\rho(s) = 2k/\sqrt{s}$, and ℓ is defined to be $J - 1/2$ here. The expansions (96) are now (by postulate a) replaced with the integral over continuous $b = (j + 1/2)/k$ as before, using a small angle approximation for the d function (see appendix of Ref. 27):

$$G_+(s,t) = k^2 \cos (\theta/2) \int_0^\infty b\, db\, J_0(b\sqrt{-t})\, g_+(s,b^2)$$

and

$$G_-(s,t) = k^2 \int_0^\infty b\, db\, J_1(b\sqrt{-t})\, g_-(s,b^2), \tag{103}$$

where, putting $\chi_\pm(s,b^2)$ in place of $2\delta_{\ell\pm}(s)$, we obtain

$$g_+(s,b^2) = (i\rho)^{-1}[e^{i\chi_+(s,b^2)} + e^{i\chi_-(s,b^2)} - 2]$$

and

$$g_-(s,b^2) = (i\rho)^- [e^{i\chi_+(s,b^2)} - e^{i\chi_-(s,b^2)}]. \quad (104)$$

(Analyticity properties in t of $G_\pm$ require that the $g_\pm$ be functions of b^2.)

Now we determine the χ's by examining the Born approximations for G_+ and G_-. Before doing this, we find it convenient to define "nonflip" and "flip" Eikonal functions

$$\chi_0 = \frac{1}{2}(\chi_+ + \chi_-), \quad \chi_f = \frac{1}{2}(\chi_+ - \chi_-);$$

so

$$\chi_+ = \chi_0 + \chi_f, \quad \chi_- = \chi_0 - \chi_f.$$

Then we can write, by rearranging the exponentials in (104),

$$G_+(s,t) = \frac{ik^2}{\rho} \cos(\theta/2) \int_0^\infty b\, db\, J_0(b\sqrt{-t})[1 - e^{i\chi_0} \cos \chi_f]$$

and

$$G_-(s,t) = \frac{k^2}{\rho} \int_0^\infty b\, db\, J_1(b\sqrt{-t})[e^{i\chi_0} \sin \chi_f]. \quad (105)$$

Now expanding to first order in χ_0 and χ_f, we require that the Born approximations be obtained as follows:

$$G_+^B = \frac{k^2}{\rho} \cos(\theta/2) \int_0^\infty b\, db\, J_0(b\sqrt{-t})\, \chi_0(s,b^2),$$

and

$$G_-^B = \frac{k^2}{\rho} \int_0^\infty b\, db\, J_1(b\sqrt{-t})\, \chi_f(s,b^2). \quad (106)$$

Inverting these Fourier-Bessel transforms, we obtain by using (101),

$$\chi_0(s,b^2) = (\rho/k^2) \int_0^\infty x\, dx\, J_0(xb) [f_1^{Born}(s, -x^2) + f_2^{Born}(s, -x^2)]$$

and

$$\chi_f(s,b^2) = (\rho/k^2) \int_0^\infty x\, dx\, J_1(xb) \sin(\theta/2) [f_1^B(s, -x^2) - f_2^B(s, -x^2)], \quad (107)$$

where $x = \sqrt{-t}$; since $x = 2k \sin(\theta/2)$, we can also express χ_f as

$$\chi_f(s,b^2) = (\rho/2k^3) \int_0^\infty x^2 dx\, J_1(xb)[f_1^B(s, -x^2) - f_2^B(s, -x^2)]. \qquad (108)$$

[Remark: Although the cos $(\theta/2)$ factor in (106) is quantitatively irrelevant, since the Eikonal method is good only for small angles, it is essential in the inversion formula leading to (107); otherwise there would be (formally at least) an integration over a region where cos $(\theta/2)$ is imaginary, involving a factor of $[\cos(\theta/2)]^{-1}$.]

Observe that even though the spin-flip amplitude f_2 may be zero in Born approximation, we obtain a nonzero f_2 when the relativistic Eikonal method is used. This is related to the spin-orbit term which appears in the Dirac equation for (certain classes of) central potentials.

If the Born approximations for helicity-flip (G_-^B) amplitudes are small (but not G_+^B), we can expand G_- to first order in χ_f and obtain

$$G_-(s,t) \cong \left(\frac{k^2}{\rho}\right)\int_0^\infty b\, db\, J_1(b\sqrt{-t})\, e^{i\chi_0(s,b^2)}\, \chi_f(s,b^2), \qquad (109)$$

while

$$G_+(s,t) \cong \left(\frac{k^2}{\rho}\right)\cos(\theta/2) \int_0^\infty b\, db\, J_0(b\sqrt{-t})[1 - e^{i\chi_0(s,b^2)}]. \qquad (110)$$

Formula (109) can be interpreted as the absorptive correction formula for the helicity-flip amplitude Born term (χ_f being a linear transform of G_-^B), if we deduce χ_0 from experiment assuming $|G_-|^2 << |G_+|^2$. The purpose of using (109) would be to estimate polarization (necessarily small for most angles in such a case), given a model for G_-^B (e.g., Regge poles).

F. Polarization in πN Scattering

It is possible to obtain large polarizations in πp scattering for some scattering angles even if $|\chi_f| << |\chi_0|$, $|\chi_f| << 1$. This is well known in nuclear-physics applications of the optical model (see, for example, Ref. 36), but is not yet widely appreciated in high-energy physics. We now indicate how this can occur.

Suppose $|\mathrm{Re}\, G_+| << |\mathrm{Im}\, G_+|$ for most angles but that Im G_+ has a simple zero at $z = z_1$, and that Re G_+ and $|G_-|$ are small and slowly varying near $z = z_1$. Then putting $\mathrm{Im}\, G_+(z) \cong \gamma(z - z_1)$, $\alpha = \mathrm{Re}\, G_+(z_1)$, $\beta_1 = \mathrm{Re}\, G_-(z_1)$, and $\beta_2 = \mathrm{Im}\, G_-(z_1)$, we have, for z near z_1,

$$s\,\frac{d\sigma}{d\Omega} = |G_+|^2 + |G_-|^2 \cong \gamma^2(z - z_1)^2 + \delta^2$$

where $\delta^2 = \alpha^2 + \beta_1^2 + \beta_2^2$. If (as assumed) $\gamma^2 >> \delta^2$, the differential cross section will have a sharp minimum at $z = z_1$. Now the polarization near $z = z_1$ has the form

$$P(\theta) = \frac{2\,\mathrm{Im}(G_+^* G_-)}{|G_+|^2 + |G_-|^2} \cong 2\frac{\beta_1\gamma(z - z_1) + \alpha\beta_2}{\gamma^2(z - z_1)^2 + \delta^2}.$$

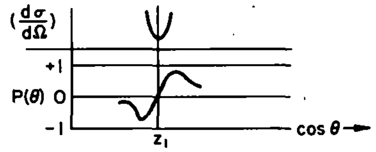
Therefore, if $|\alpha\beta_2| << |\gamma\beta_1|$ (true if α, β_1, and β_2 are all comparable in magnitude), we have

$$P(\theta) \cong 2\beta_1 \frac{\gamma(z - z_1)}{\gamma^2(z - z_1)^2 + \delta^2}.$$

The maxima (and minima) of this occur at $(z - z_1)^2 = \delta^2/\gamma^2$, and at these points:

$$|P|_{MAX} = \beta_1/\delta = \beta_1/(\beta_1^2 + \beta_2^2 + \alpha^2)^{1/2}.$$

Between these maximal points, P has a zero at $z = z_1$. Note that $|P|_{MAX}$ can be essentially unity if $\beta_2^2 + \alpha^2 << \beta_1^2$. In such a case, $P(\theta)$ has the behavior sketched below with $d\sigma/d\Omega$ for comparison:



Such local minima in $d\sigma/d\Omega$, with associated "dispersion curves" for $P(\theta)$, have been observed in π^-p scattering near 2 BeV/c. (There is no direct evidence that $|\chi_f| << 1$ or $|\chi_0|$, however, in this case, and it is not clear whether the Eikonal approximations would be good at this energy. Furthermore, in this problem the energy dependence of the secondary maximum amplitude is too rapid to be associated with leading poles in χ.)

If the Regge-pole Eikonal approach is adopted, one finds then, at sufficiently small angles (where the Born approximation is expected to give qualitatively correct results), a polarization and differential cross section roughly as predicted by Phillips and Rarita[9] whereas at larger angles one can easily obtain large polarizations for some angles if Im G_+ has a zero while $|\text{Re } G_+|$ and $|G_-|$ are small compared to $|\text{Im } G_+|$, for most angles, as indicated by the discussion above.

X. Concluding Discussion

The Regge-pole concept in high-energy physics may be considered a great improvement for high energies on the older ideas of single-particle exchange models, but the underlying physical assumptions (as presented in these lectures) are similar. Regge poles thus provide a semiphenomenological connection between resonances or particles with t channel quantum numbers, and high-energy scattering or exchange processes in the s channel.

The present data on high-energy $\pi^\pm p$ and $K^\pm p$ scattering and charge exchange, as well as less accurate data on other meson-baryon final states, is consistent with a few Regge poles in each case (most of this is exhibited by Ref. 9) provided only small $-t$ values are considered. That this restriction should be present is plausible if we interpret these poles as Born approximations for the amplitude, in an optical-model viewpoint.

The situation in pp scattering and pn charge exchange requires six poles (P, P', ω, ρ, R, and π); the first three were included in order to fit pp elastic scattering only,[10] but ρ and R are necessary to fit the energy dependence of $\sigma_{np} - \sigma_{pp}$;[37] finally, π is necessary to explain the sharp peak for $0 < -t < \mu_\pi^2$ observed in pn charge exchange.[38] However, all these poles are expected to contribute, on the basis of the mesonic-state mass spectrum (cf. Ref. 10 of Part One), so they are not introduced in an ad hoc way.

With $p\bar{p}$ scattering, at momenta below 10 BeV/c at least, the Regge poles apparently do not dominate the scattering amplitude. This is indicated by the energy dependence of the width (in -t) of the $p\bar{p}$ diffraction-scattering peak, which expands with increasing energy, contradicting the simple expectation of logarithmic shrinkage if only one pole is used. In any case, the pp and $p\bar{p}$ behaviors are quite different. The introduction of the optical-model concepts, however, allows these to be reconciled; the $p\bar{p}$ system has a strong absorptive potential due to annihilation channels, and this extra contribution masks out the pp Regge poles at low to moderate energies. This may be thought of as an "absorptive correction" to the elastic-scattering Regge poles, which presumably will become less important at higher energies than presently accessible, thus asymptotically insuring similar behavior for pp and $p\bar{p}$. A similar effect (nonasymptotic absorptive contribution to X) may be responsible for the secondary maximum seen in πp and K^-p scattering between 1.5 and 3 BeV/c, since it disappears rapidly relative to the forward peak with increasing energy.

Although the small-angle phenomenological approach, as outlined in these lectures, is not able to provide a complete dynamical scheme for strong interactions, it does provide a nontrivial model framework wherein constraints between many reactions are present. An example of such constraint is the existence of a single-trajectory function α(t) for each pole, independent of the reaction in which the pole participates, and essentially determining the asymptotic energy dependence and phase (for each t) of any one-pole contribution. The fits already have produced nontrivial predictions for polarization at small -t values, which seem to agree in order of magnitude with experimental results in πN small-angle scattering at momenta as low as 2.5 BeV/c.

Further results on the energy dependence of reactions involving only one or two poles (cf. Section VI) will provide stringent tests of the consistency of the pole analysis, either of the amplitude itself or of the optical potential.

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APPENDIX A

Legendre Functions, Hypergeometric Functions, and the Gamma Function

For convenient reference, we summarize the representations of $P_\lambda(z)$ and Q_λ as hypergeometric functions and their singularities in λ. These representations are contained in Whittaker and Watson.*

$$P_\lambda(z) = F\left(\lambda + 1, -\lambda; 1; \frac{1-z}{2}\right);$$

$$Q_\lambda(z) = \frac{1}{(2z)^{\lambda+1}} \frac{\sqrt{\pi}\,\Gamma(1+\lambda)}{\Gamma(3/2+\lambda)} F\left(\frac{\lambda+1}{2}, \frac{\lambda}{2}+1; \lambda + 3/2; z^{-2}\right).$$

For $|x| < 1$, the hypergeometric function F can be represented by the convergent power series

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(1+n)} x^n.$$

For other values of x, the following (Barnes') integral representation may be used:

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(c+s)} (-x)^s\, ds,$$

where the path of integration is deformed (if necessary) to avoid poles of the integrand depending on a, b, and c.

The Γ function has the integral representation

$$\Gamma(z) = \int_0^\infty e^{-t}\, dt\, t^{z-1},$$

and, alternatively, an infinite product representation (Weierstrass),

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where γ is Euler's constant.

The last representation shows that $\Gamma(z)$ has simple poles when z is at a negative integer, and these are the only singularities for finite $|z|$.

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*E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, England (1963), 4th ed., reprinted, Chapters 12, 14, and 15.

This allows us to conclude, from the above representations of P_λ and Q_λ, that

(A1) $P_\lambda(z)$ is analytic in the entire finite λ plane for any z, and

(A2) $Q_\lambda(z)$ has only poles in the finite complex λ plane, and these occur at $\lambda = -1, -2, -3, \ldots$.

APPENDIX B

Regge Poles for Arbitrarily Weak Potentials

When the strength of the potential is infinitesimal, the FODA (Section III of Part One) becomes exact. Then (for a simple Yukawa potential) the Regge poles are the poles of

$$[f_\lambda(k^2)]_{FODA} = \frac{(g^2/2k^2)\; Q_\lambda(1 + \mu^2/2k^2)}{1 - \frac{g^2}{2}\int_0^\infty \frac{dq}{q^2(q^2 - k^2 - i\varepsilon)}\, Q_\lambda(1 + \mu^2/2q^2)}$$

Now if $g^2 \to 0$, or $|k| \to \infty$, we have just a Q_λ function (the numerator), which has poles at $\lambda = -1, -2, -3, \ldots$; these are the (fixed) Regge poles of the Born approximation. For $g^2 \neq 0$, the poles of Q_λ in the denominator now cancel the poles of the numerator at the negative integers; but the denominator now will vanish for some λ near every such negative integer, since the Q_λ function gets arbitarily large for λ sufficiently close to such a pole. Thus, if g^2 is varied smoothly away from zero, the poles in λ will move smoothly away from the negative integers, but remain close to them for small g^2 (or large $|k|$). In fact, it can be shown (cf. Ref. 5 of Part One) that the trajectories (pole positions in the complex λ plane) $\alpha_n(k)$ can be expanded to first order in g^2 as follows:

$$\alpha_n(k) = -n + ig^2/k + \ldots .$$

Similar results hold for a superposition of Yukawa potentials, where g^2 is then a weighted average strength.