Chapter 3

An Outline of Theories of Diffusion in the Lower Layers of the Atmosphere

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LIST OF SYMBOLS

Symbols used frequently in Chaps. 3 and 4 are listed here. (The dimensions mass, length, time, and temperature are abbreviated as *M*, *L*, *T*, and *D*, respectively. Equation or section numbers indicate where the symbol first appears or where additional clarification may be found.)

A	Turbulent or eddy viscosity or "Austausch" coefficient ($ML^{-1}T^{-1}$), Eq. 3.7
A	Refers to average concentration value in discussions of peak-to-average ratio, Eq. 3.137
A	Area within a concentration or exposure isopleth (L^2), Eq. 3.150
C_x, C_y, C_z	Sutton's virtual diffusion coefficients ($L^{n/2}$), Eqs. 3.78 and 3.80
c_p	Specific heat of air at constant pressure ($L^2T^{-2}D^{-1}$), Eq. 3.26
D	Distance along y- or z-axis of center of meandering plume (L), Eq. 3.119
D	Depth of laminar sublayer (L), Eq. 3.11
d	Zero-plane displacement (L), Eq. 3.16
$d_{max.}$	Distance to point of maximum ground concentration from an elevated source (L), Sec. 4-4.1.2
F(n)	Lagrangian eddy-energy-spectrum function (T), Eq. 3.66
f	Coriolis parameter (T^{-1}), Eq. 3.34
g	Gravitational acceleration (LT^{-2}), Eq. 3.18
H	Eddy heat flux [$(ML^2T^{-2})L^{-2}T^{-1}$], Eq. 3.26
h_i	Depth of the mixing layer (L), Eq. 3.133
h	Height of a source above the ground (L), Eq. 3.115
i_u, i_v, i_w	Intensity of turbulence in the x-, y-, and z-directions (dimensionless), Eq. 4.26, Chap. 2, Sec. 2-6.2.2
K	Eddy diffusivity coefficient (L^2T^{-1}), Eq. 3.45
K_H	Eddy heat conductivity coefficient (L^2T^{-1}), Eq. 3.26
K_M	Kinematic eddy-viscosity coefficient (L^2T^{-1}), Sec. 3-1.2.5
k	Von Karman's constant ≈ 0.4 (dimensionless), Eq. 3.12
L	Stability-dependent length introduced by Lettau, Monin, and Obukhov (L), Eq. 3.28
$\mathscr{L}$	Lagrangian integral time scale (T), Eq. 3.70
l	A length scale (L), Table 3.2
n	Frequency (T^{-1}), Eq. 3.66
n	Sutton's parameter associated with stability (dimensionless), Eq. 3.76
P	Refers to peak-concentration value in discussions of peak-to-average concentration, Eq. 3.137
p	Atmospheric pressure ($ML^{-1}T^{-2}$), Eq. 3.17
Q	Source strength; total amount of material released from a point source (M or other units of quantity), Eq. 3.49
Q′	Source strength; time rate of material emission from a continuous point source (MT^{-1}), Eq. 3.91
Q_L	Source strength; total amount of material emitted per unit length from a line source (ML^{-1}), Sec. 3-3.5.8
Q_L'	Source strength; time rate of material emission per unit length from a continuous line source ($ML^{-1}T^{-1}$), Sec. 3-3.5.3
q	Mean value of a conservative air property per unit mass of air, Eq. 3.45

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$R(t)$	Velocity autocorrelation coefficient (dimensionless), Eq. 3.65
Re	The Reynolds number (dimensionless), Eq. 3.5
R_f	Flux form of the Richardson number (dimensionless), Eq. 3.27a
Ri	The Richardson number (dimensionless), Eq. 3.22
T	Temperature (D), Eq. 3.17
t	Time (T); appears in various equations; in special applications T is also used , Eq. 3.108
u,v,w	Components of the wind in the x-, y-, and z-directions, respectively (LT^{-1}), Eq. 3.2
$\vec{V}$	Total wind-motion vector (LT^{-1}), Eq. 3.1
v_d	Deposition velocity (LT^{-1}), Eq. 4.14, Chap. 5, Sec. 5-3.2.1
v_*	Friction velocity (LT^{-1}), Sec. 3-1.2.6
x,y,z	Positions in a Cartesian coordinate system which is usually oriented so that the x-axis is in the direction of the mean horizontal vector wind, the y-axis is crosswind, and the z-axis is vertical (L), Eq. 3.2
Y	A distance between particles (L), Eq. 3.96
z_0	Roughness length (L), Eq. 3.14
β	Lagrangian–Eulerian time-scale ratio (dimensionless), Eq. 3.104
Γ, γ	Dry adiabatic temperature lapse rate and existing temperature lapse, respectively (DL^{-1}), Eq. 3.21
ϵ	Rate of eddy energy transfer (L^2T^{-3}), Sec. 3-2.2.6
ζ	Dimensionless ratio = z/L, Eq. 3.32
Θ	Potential temperature (D), Eq. 4.19
θ	Lateral wind-direction angle or width of sector (expressed as degrees, radians, etc.), Eq. 3.122
μ	Dynamic viscosity coefficient ($ML^{-1}T^{-1}$), Eq. 3.4
ν	Kinematic viscosity (L^2T^{-1}), Eq. 3.5
ρ	Atmospheric density (ML^{-3}), Eq. 3.5
σ_y, σ_z	Standard deviation of the distribution of material in a plume in the y- and z-directions (L), Eq. 3.113
σ_{xI}, σ_{yI}, σ_{zI}	Standard deviation of the distribution of material in a puff in the x-, y-, and z-directions (L), Eq. 3.154
σ_θ	Standard deviation of lateral wind-direction distribution (degrees or radians), Sec. 3-3.4.1
σ_φ	Standard deviation of vertical wind-direction distribution (degrees or radians), Table 4.2
τ	Tangential stress on a unit area of fluid ($ML^{-1}T^{-2}$), Eq. 3.4
τ_0	Tangential stress in the lowest air layers ($ML^{-1}T^{-2}$), Eq. 3.8
φ	Geographical latitude, vertical wind-direction angle or width of sector (degrees or radians), Sec. 3-1.3
$\chi(x,y,z,t)$	Concentration at a point (x,y,z) at time t (ML^{-3}), Eq. 3.89
$\bar{\chi}$	Average concentration (ML^{-3}), Eq. 3.91
χ_{CWI}, $\bar{\chi}_{CWI}$	Crosswind integrated concentration (ML^{-2}), Eq. 3.143
$\bar{\chi}_F$	Fumigation concentration (ML^{-3}), Eq. 3.133
$\bar{\chi}_{max.}$	Maximum concentration on the ground from an elevated source (ML^{-3}), Eq. 3.135
χ_p, $\bar{\chi}_p$	Peak or center-line concentration values (ML^{-3}), Eqs. 3.155 and 3.91
ψ	Exposure (MTL^{-3}); subscripts p and CWI have the same meaning as for concentration (also referred to as the concentration time integral), Eq. 3.156
ω	Angular velocity of earth's rotation [(radians)T^{-1}], Sec. 3-1.3
$^-$ (overbar)	Space, time, or statistical average, Sec. 3-2.3
$\langle\ \rangle$	Running mean average, Eq. 3.75
$'$ (prime)	Superscript referring to deviation from the mean, i.e., $x \equiv \bar{x} + x'$, Eq. 3.1; also used in source-strength notation to indicate a release rate, Eq. 3.91
A, P	Subscripts referring to conditions surrounding a parcel of air and within the parcel, respectively, Eq. 3.18
G	Subscript referring to geostrophic flow
x,y,z	Subscripts referring to coordinate axes

Other subscript notation accompanies the notation found in the preceding portion of this table.

3-1 MEAN FLOW IN THE LOWER LAYERS OF THE ATMOSPHERE

3-1.1 Introduction

The problem to be considered is the description, by means of mathematical–physical models, of the role of the earth's lower atmosphere in redistributing and diluting the radioactive gases and particles that may be introduced into it as a result of various activities of the atomic energy industry. Although most interest is centered on the problem of isolated, more or less continuously emitting sources at or near the ground level, such as fixed nuclear reactors and their associated chemical processing plants, the problem of quasi-instantaneous sources, such as might result, for example, from a nuclear rocket launch-pad accident, will also be considered. The special problems created by the radioactive nature of these various sources are most conveniently dealt with separately. Therefore the results described in this chapter apply equally to nonradioactive air contamination, such as that created by large conventional power plants and many other activities of an industrial society.

The symbols most frequently used in this chapter and in Chap. 4 are listed and defined in the List of Symbols at the beginning of the chapter.

The atmosphere disperses gases and particles rapidly because it is turbulent. Turbulence is the property, easy to recognize but difficult to define, of irregular, chaotic motion possessed by almost all natural fluid flows. In fact, for practical purposes we can best define a turbulent fluid flow as one that has the ability to disperse particles embedded within it quite rapidly, at a rate orders of magnitude greater than can be accounted for by molecular diffusion. Most of the meteorological problems (as well as certain other technical fluid-flow problems, such as heat transfer) of the power, chemical, and atomic energy industries center themselves around the phenomenon turbulent diffusion.

Osborne Reynolds (1895) suggested in 1883 a device by which such a complex phenomenon as a turbulent flow could be reduced to a relatively manageable mathematical form. Reynolds' idea was that the total wind-motion vector, $\vec{V}$, can be thought of as being composed of a constant mean part $\bar{\vec{V}}$ and a fluctuating, or turbulent, part $\vec{V}'$, such that

$$\vec{V} = \bar{\vec{V}} + \vec{V}' \tag{3.1}$$

or, considering the three orthogonal wind components separately,

$$\begin{aligned} u &= \bar{u} + u' && \text{(in the x-direction)} \\ v &= \bar{v} + v' && \text{(in the y-direction)} \\ w &= \bar{w} + w' && \text{(in the z-direction)} \end{aligned} \tag{3.2}$$

Components of the natural wind can be measured by a sensitive anemometer. Figure 3.1

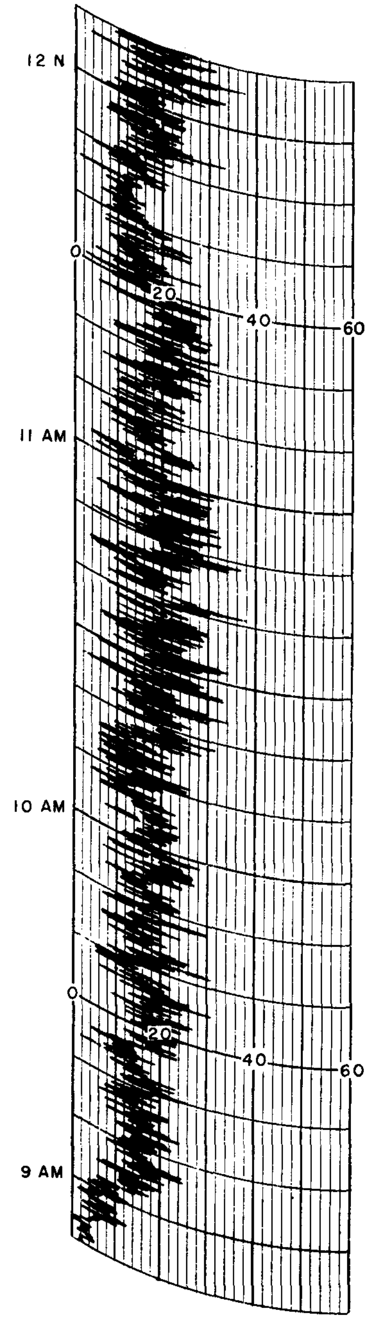


Fig. 3.1 — Portion of anemometer record, u(t), made at an elevation of 35 m in the atmosphere.

is a sample of one such measurement made at 35 m above the ground surface. Notice that the fluctuating, or turbulent, component of the wind is of the same order of magnitude as the fixed, or mean, part; i.e., u' is about as large as $\bar{u}$. This is characteristic of atmospheric turbulence and distinguishes it sharply from wind-tunnel turbulence, where ū is more likely to be 10^2 to 10^3 times u′.

Equation 3.1 or 3.2 tells us that to specify atmospheric turbulence we must first be able to specify the mean state of atmospheric motions. Some consideration of the mean state of the atmosphere is also necessary because the energy supply for atmospheric turbulence lies in the organized large-scale mean atmospheric motions. Moreover the strength of the mean wind is directly related to the capacity of the atmosphere for diluting pollutant materials injected into it. Finally, in the layers of air nearest the earth's surface, extending to elevations of several kilometers, the mean wind pattern itself is determined primarily by turbulence arising from frictional drag at the air–earth interface. Thus it appears that the analysis of atmospheric turbulence is deeply involved with the mean field of motion on four separate counts. A discussion of the mean wind structure in the lower layers of the atmosphere is clearly required as a preliminary to the treatment of the diffusion problem. Although it is convenient to proceed as though this mean wind can actually be defined over some suitable space or time domain, it should be noted that in the atmosphere, in contrast to the wind tunnel, the method of doing this is neither simple nor obvious. Fluctuations of the wind having a wide range of periods can and do occur, but just how to define the average value is not always clear. This problem is discussed at some length in Sec. 3-2.3.

3-1.2 The Mean State of the Wind in the Lowest Layers

3-1.2.1 Viscosity. Assuming for the moment a horizontal, straight, parallel, steady mean wind flow, ū(z), at some level z, fairly near the surface (just how near will be determined subsequently), let us try to determine the mean wind structure. Upon what quantities should it depend? Obviously ū must increase with height, z, for at least some distance above the earth's surface since just at the surface it must equal zero. This means that adjacent horizontal layers of air must be in motion relative to one another, and so certainly ū(z) must be expected to depend also on the viscosity of the atmosphere.

Imagine a small volume of air next to the surface to be symbolized by a deck of smooth new playing cards resting on a table (Fig. 3.2). If the top card is slid parallel to the deck while the cards are held firmly in contact, the bottom card remains fixed, but the remainder of the deck is tilted forward, or sheared, in such a way as to deform the deck into a uniform parallelepiped. The horizontal force on the top card represents the horizontal (in general, tangential) shearing stress on any small air volume. The resistance of the cards to vertical hand pressure symbolizes ordinary (normally directed) air pressure, and the resistance to horizontal slippage of cards symbolizes the viscosity of the air. Just as the bottom card sticks to the table, so the lowest air layer sticks to the surface of the earth.

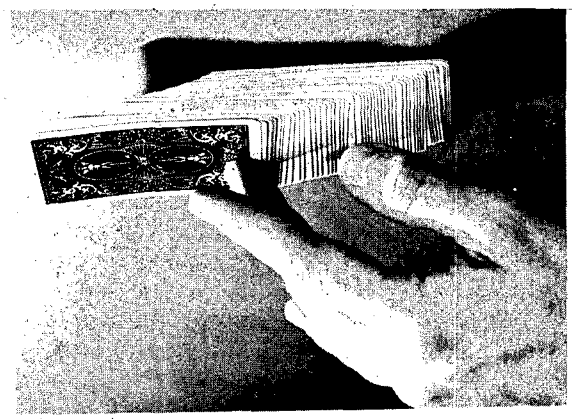


Fig. 3.2—Illustration of shear due to a tangential force.

3-1.2.2 Shearing Stress. A tangential shearing force, or stress, applied for a certain time to the top card produces a certain deformation of the deck, symbolizing a vertical shear of the horizontal wind [dū(z)/dz]; the smaller the viscosity is the greater this effect will be. It is reasonable to suppose that

$$\frac{d\bar{u}}{dz} \propto \frac{\text{shearing stress}}{\text{viscosity}} \qquad (3.3)$$

or, using the symbols ordinarily assigned to these quantities and rearranging terms,

$$\tau = \mu \frac{d\bar{u}}{dz} \qquad (3.4)$$

This is Newton's law for molecular fluid viscosity; τ is the tangential stress on a unit area of the fluid, and μ is called the dynamic viscosity coefficient because it is a measure of the resistance of the fluid to volume distortion resulting from the stress.

3-1.2.3 Mechanical Turbulence. Fluid viscosity, which is described in Sec. 3-1.2.1, depends on molecular structure; it is a bulk property of the fluid, which, however, is determined by internal microscopic fluid characteristics. In fact it is sometimes called internal friction, and its detailed nature can best be elucidated by the arguments of kinetic theory, involving transfer of momentum from layer to layer of the fluid by individual molecules. Molecular viscosity accounts adequately for transfer of fluid properties very near flow boundaries and, in general, in any small volume of fluid. Considering the situation for the flow as a whole, however, we have to deal in the lower atmosphere with a structure much more complex than the simple layered parallel flow we have so far conceived, namely, a turbulent flow. For the time being let us regard this turbulence next to the surface as purely mechanical in origin and as deriving its energy somehow from the mean flow of the air at greater elevations in a way that does not depend on the action of thermal buoyancy forces.

3-1.2.4 The Reynolds Number. What is the nature of low-level turbulent atmospheric motions? In a nonturbulent flow, such as water issuing at low velocity from a tap, paths of adjacent fluid "particles" are essentially parallel, as illustrated in (a) of Fig. 3.3. (By a fluid particle, we have in mind a small volume of the fluid. Such a volume would contain a very large number of molecules, but the molecular nature of the fluid does not concern us here. We regard the fluid as being microscopically continuous, an assumption that permits us to apply the definitions and limiting processes of ordinary differential calculus to the fluid motions.) This nonturbulent flow is called laminar, the connotation being that adjacent layers of fluid remain distinct and identifiable (laminated) and do not intermix. Physically, the laminar stream of water appears smooth and coherent; small irregularities remain small or are rapidly damped. Under these conditions Newton's law for viscosity would be obeyed. If the velocity of the stream is increased slowly, no change may occur at first, but at some point the nature of flow will be observed to change radically and suddenly. The smooth appearance turns to a rough, irregular one, as shown in (b) of Fig. 3.3. It is obvious that adjacent

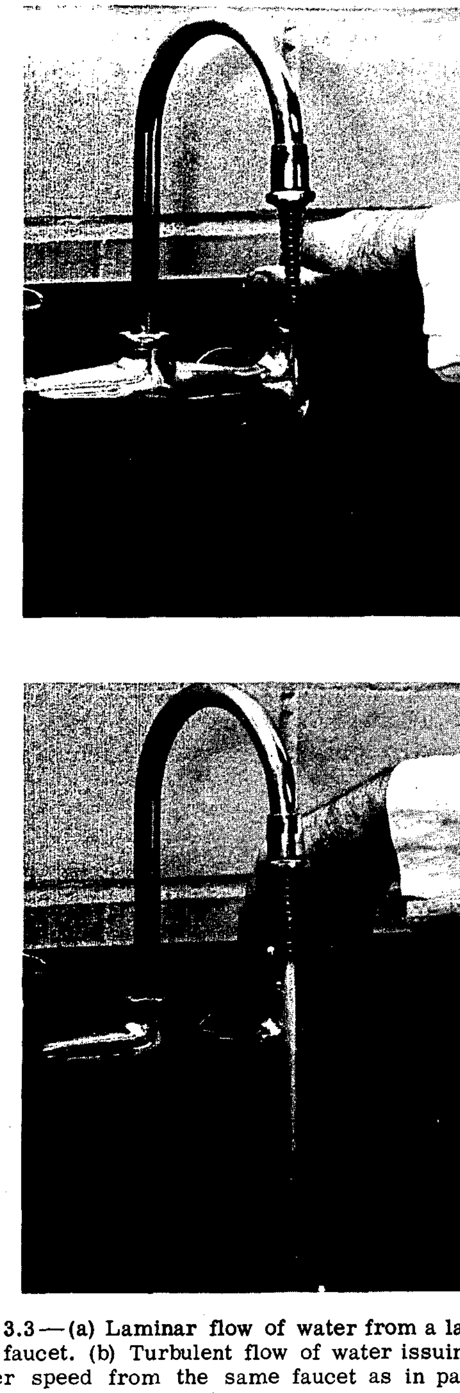


Fig. 3.3—(a) Laminar flow of water from a laboratory faucet. (b) Turbulent flow of water issuing at a higher speed from the same faucet as in part (a). (Courtesy J. E. Westcott)

particle paths no longer are parallel but are intermingled in a highly irregular way. The water churns and splatters. We immediately recognize the new state of the fluid as turbulent and can readily grasp the importance of the turbulent fluid state in problems involving the transfer of such properties as heat and momentum.

It is possible to learn more from this simple experiment. By repeating it with taps having openings of different sizes, we would find that the water flow issuing from a smaller opening stays laminar up to a higher velocity. For example, the very fine stream of water issuing from a laboratory wash bottle is nearly always smooth, i.e., laminar, for considerable distances from its tip [(a) of Fig. 3.4]. On the other hand, the flow from a fire hydrant is invariably turbulent [(b) of Fig. 3.4]. Now consider an experiment otherwise identical but performed with some thick liquid, e.g., heavy oil or molasses, substituted for water as the fluid. We would find that the velocity required to produce turbulent flow is in each case higher than for water.

We have now in a highly qualitative way established the facts about turbulent flows that were found to be significant by Reynolds, who made the first systematic study of the onset of fluid turbulence:* to change a laminar flow into a turbulent flow one must either increase the velocity, increase a characteristic reference length associated with the flow, or decrease the viscosity of the fluid. These factors can be combined into a dimensionless ratio known as the Reynolds number, Re:

$$\begin{aligned} \text{Re} = &\ (\text{a characteristic flow length}) \\ &\times (\text{a characteristic flow velocity}) \\ &\times (\text{dynamic viscosity/density})^{-1} \end{aligned} \tag{3.5}$$

The denominator of this expression is called the kinematic viscosity, ν (square centimeters per second), and is related to the dynamic viscosity by $\nu = \mu/\rho$, where ρ is air density. The kinematic viscosity is a measure of how the intrinsic fluid stickiness, i.e., the dynamic viscosity, affects the overall flow geometry, and consequently it must depend on the inertia

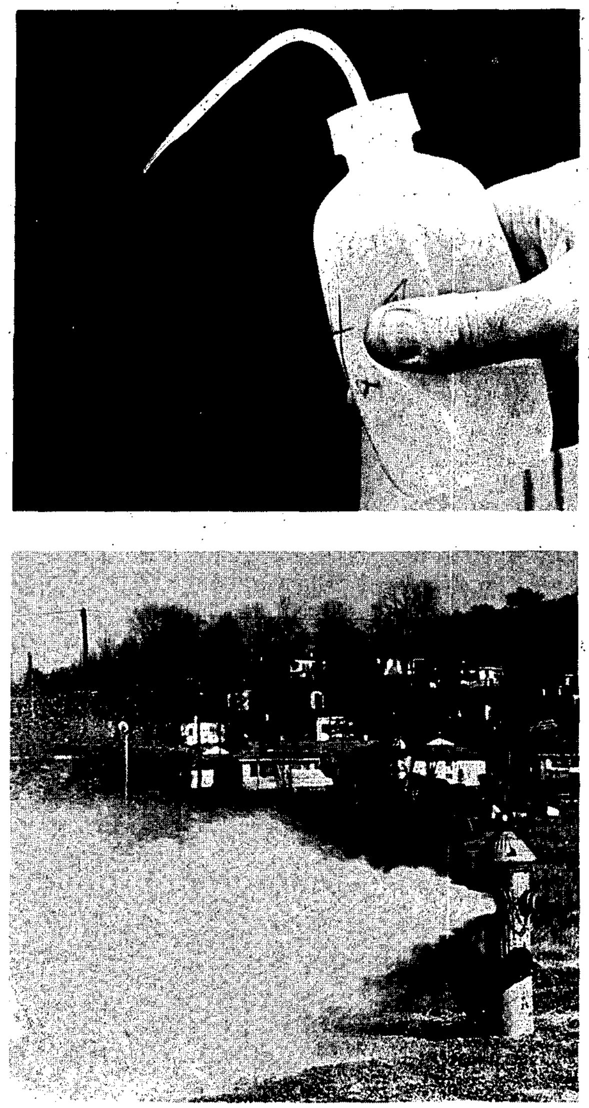


Fig. 3.4—(a) Laminar flow of water from laboratory wash bottle with a small nozzle. (b) Turbulent flow of water from a fire hydrant. (Courtesy J. E. Westcott)

of the fluid and hence on the density. The Reynolds number can also be thought of as the ratio of the inertial to the viscous forces acting on a small volume of fluid,

$$\text{Re} \propto \frac{\text{inertial force}}{\text{viscous force}} \tag{3.6}$$

The inertial force on a unit volume of fluid is equal to the product of density and acceleration; the viscous force equals the viscous stress per unit area.

The way in which a length characteristic of the flow should be specified in the atmosphere

*The word "turbulence," designating a state of fluid flow exceeding a certain critical threshold, was introduced by Lord Kelvin in 1887 according to Rouse and Ince (1957).

is not by any means self-evident. In pipe flows, such as those Reynolds studied, the pipe diameter provides a natural reference length, as did the size of the opening in the example with which the discussion in this section began. In airfoil theory the wing chord is a suitable reference length; in wind tunnels the size of a wire mesh or grid used to induce turbulence likewise defines a length. With any of these definitions, flows characterized by Reynolds numbers above about 10^2 or 10^3 are always turbulent.

What length scale applies in the atmosphere? In the atmosphere there is no particularly obvious macroscopic external length scale associated with the turbulence phenomenon. In fact it seems that fluctuating motions can occur over a very wide size range. For any reference length we might choose as a matter of expediency (height above the ground, for example), we are bound to conclude that the Reynolds number of the atmosphere will be very large because typical velocities are of the order 10^2 cm/sec and ν equals about 0.15 cm²/sec. Consequently we find that atmospheric flows are ordinarily turbulent. The degree of turbulence of the atmosphere can vary over wide limits and depends primarily on the vertical temperature structure, i.e., upon stability. Nevertheless the atmosphere is normally turbulent. In laminar flow the rate of diffusion of molecules or particles is proportional to the coefficient of molecular viscosity, or diffusivity, μ. Because atmospheric flows are turbulent, diffusion in the atmosphere occurs at a rate which is rarely less than several orders of magnitude greater than the molecular rate and which may be many orders of magnitude greater than this value.

3-1.2.5 Eddy Viscosity. By analogy with the Newtonian law of molecular viscosity (Eq. 3.4), Boussinesq (1877) proposed that the effect of turbulent viscosity be taken into account by introducing an augmented viscosity,

$$\tau = (\mu + A)\,\frac{d\bar{u}}{dz} \qquad (3.7)$$

The turbulent, or eddy, viscosity, A, was termed by Schmidt (1925) an "Austausch," or exchange, coefficient. It was, of course, realized much earlier than this that the Newtonian, or molecular, viscosity, μ, was orders of magnitude too small to account for the observed transfer of heat or momentum in fluids. An interesting historical account of the subject by Bateman (1956) mentions the studies by Dalton in 1799 and Count Rumford in 1806. Convection over heated ground was described as early as 1749 by Benjamin Franklin, according to Middleton (1965). The meteorologist Espey in 1840 also used the idea of convective mixing of wind currents to explain the diurnal variation of the wind. In line with the molecular analogy, it is convenient to define a kinematic eddy-viscosity coefficient K_M in terms of A and ρ, $A = \rho K_M$; K_M thus has the same dimensions as ν but is, as we shall see, normally several orders of magnitude larger. The subscript M indicates that this eddy-exchange process involves transfer of momentum.

Notice that a significant new concept, that of an eddy,* has just been introduced. An eddy is thought of as an irregular but somehow identifiable material wind structure, perhaps similar to a "puff of wind" or to a "cat's paw" over open water, having the ability to transfer air properties across the flow in a way that can conveniently be thought of as analogous to transfer by the air molecules on a much smaller scale. At this point we need not try to make this idea very much more precise; indeed to do so will turn out to be impossible in most respects although we shall freely discuss from time to time various properties of eddies. To prevent such looseness of argument from becoming too great a mental or aesthetic obstacle, we need only recollect how much of physical theory can be rationalized by assuming that molecules behave like little hard balls.

Assume then an eddy viscosity K_M that controls, through the properties of turbulent eddies, the mean structure of the wind over the earth's surface. Very near the surface, height above the ground must limit the vertical size of eddies. At greater and greater elevations, eddies that are larger and larger in their

*The concept of an eddy is new, of course, only at this point in the present discussion. The intuitive idea of relating turbulent fluid motion to such an entity seems to be very old. Rouse and Ince (1957) reproduced a sketch drawn by Leonardo da Vinci clearly illustrating eddies in the wake of an obstacle in a water channel (he, however, attributed properties to these eddies that we would not today).

vertical dimension can be present, and so it is reasonable to expect that K_M should depend upon z. Moreover K_M is related to the tangential shearing stress by Eq. 3.7, which may be rewritten

$$\tau_0 = (\rho K_M) \frac{d\bar{u}}{dz} \quad (3.8)$$

where the subscript zero indicates that this is to apply in the lowest air layers and μ is neglected as small.

3-1.2.6 The Logarithmic Wind Profile. In general, the tangential shearing stress, τ_0, will vary with height in the lower layers of the atmosphere. As a simplification we can, however, limit the discussion to a layer of air just next to the surface through which the vertical variation of τ_0 is small enough that τ_0 can be considered constant. Then the vertical structure of the mean wind, $\bar{u}$, for the flow we are considering appears to depend on the following quantities, which can be chosen as fundamental to the flow: the kinematic air viscosity, ν; height above the surface, z; air density, ρ; and frictional stress, τ_0. The quantity K_M can be expressed in terms of these through Eq. 3.8.

In particular, the vertical gradient of the mean velocity in a uniform, straight, parallel flow of air next to the ground, for which the energy of the air turbulence is purely mechanical in origin, involves relations among five dimensional quantities: $d\bar{u}/dz$, ν, z, ρ, and τ_0. If ordinary principles of dimensional analysis are applied to this problem (see Bridgman, 1931), the well-known Π theorem tells us that two independent dimensionless ratios can be formed from these (namely, the number of quantities, 5, minus the number of fundamental dimensions involved, 3) and that a single unknown function of these two ratios may be set equal to zero. Of these remaining quantities, only τ_0 and ρ involve mass, and so they must enter any dimensionless product as the ratio τ_0/ρ, which has the dimensions of velocity squared. Its square root is often termed the friction velocity and is given the special symbol v_*. One dimensionless ratio can thus be written as $(d\bar{u}/dz)(z/v_*)$, and the second, as zv_*/ν. Consequently the dimensional analysis provides the following result:

$$f_1\left(\frac{d\bar{u}}{dz}\frac{z}{v_*}, \frac{zv_*}{\nu}\right) = 0 \quad (3.9)$$

Since our object is to determine $d\bar{u}/dz$, we can solve for the ratio containing it:

$$\frac{d\bar{u}}{dz} = \frac{v_*}{z} f_2\left(\frac{zv_*}{\nu}\right) \quad (3.10)$$

Although simpler than Eq. 3.9, Eq. 3.10 nevertheless contains a function f_2 about which it is not yet possible to speculate on the basis of the assumptions made so far; f_2 may be a simple linear function, or it could equally be highly transcendental. This is a common impasse when dimensional analysis is applied to a complicated problem and can only be resolved by invoking some additional principle or physical understanding of the problem. Let us see what can be accomplished.

Very close to the earth's surface the vertical structure of $\bar{u}$, the vertical velocity profile, must mainly be governed by molecular viscosity because close enough to the surface the turbulence due to eddies must become negligible, there being insufficient height for the eddies to come into play. Let us assume that this situation holds up to some fixed elevation, z = D, and attempt to estimate D. If we substitute D into the second of the dimensionless ratios, which we should recognize as a form of Reynolds number, we see that for a given flow (i.e., a constant value of v_*)

$$Re = Dv_*/\nu = \text{constant} \quad (3.11)$$

Because flows for which $Re \geq 10^2$ are ordinarily turbulent, it follows that an upper limit to D, the depth of the laminar sublayer of the atmosphere, will be of the order of a millimeter since v_* is known from observations to be of the order of 100 cm/sec and ν equals about 10^{-1} cm^2/sec.

This means that blades of grass, grains of dirt, sticks, twigs, people, and so forth, all protrude through the laminar sublayer. In fact, except perhaps for flow over very smooth ice or still water, we may ignore the effect on $d\bar{u}/dz$ of zv_*/ν. Then the equation for the gradient of the wind profile near the earth's surface simplifies to

$$\frac{d\bar{u}(z)}{dz} = \frac{v_*}{kz} \quad (3.12)$$

which will apply to fully turbulent flow over a rough surface, i.e., one whose roughness elements protrude through the laminar sublayer. Such a surface is sometimes called aerodynamically rough; an aerodynamically smooth surface is, in contrast, one whose roughness elements are contained within the laminar sublayer, a case that does not as a rule apply in the atmosphere. The universal proportionality constant, k, is called von Karman's constant and has been found by experimentation to equal 0.4.

Equation 3.12 can be intégrated to obtain the wind profile near the earth's surface in the constant-stress layer:

$$\bar{u}(z) = \frac{v_*}{k} \ln z + \text{constant} \qquad (3.13)$$

The integration constant is usually defined so as to introduce the effect of surface roughness by requiring that $\bar{u} = 0$ when $z = z_0$; z_0 is called the roughness length because it expresses the effect of varying ground surface roughness on the wind profile:

$$\bar{u}(z) = \frac{v_*}{k} (\ln z - \ln z_0) = \frac{v_*}{k} \ln \left(\frac{z}{z_0}\right) \qquad (3.14)$$

Written in this form, the result indicates the role of the integration constant, which is to translate the wind profile without changing its form. This equation is valid only for $z \geq z_0$ since the dimensional argument applies only above the laminar sublayer.

Sometimes z_0 is chosen so that $\bar{u}(z) = 0$ when $z = 0$. If this is done, the wind profile equation takes the form

$$\bar{u}(z) = \frac{v_*}{k} \ln \left(\frac{z + z_0}{z_0}\right) \qquad (3.15)$$

Since, as a practical matter, interest is ordinarily centered on wind at heights where $z \gg z_0$, the two forms are substantially equivalent. It is sometimes necessary to take account of the possibility that the actual zero-plane datum level used in an experiment may differ from the zero-plane implied by Eq. 3.14 either arbitrarily for experimental convenience or because the level down to which the effect of the wind profile extends (e.g., over thick vegetation) does not coincide with the ground surface. This is done by formally introducing a zero-plane displacement, d,

$$\bar{u}(z) = \frac{v_*}{k} \ln \left(\frac{z - d}{z_0}\right) \qquad (3.16)$$

Such precision is not often required in field work. The existence of a logarithmic wind profile next to the earth, such as these equations predict, has been confirmed in numerous experiments for the type of flow that we have specified, i.e., purely mechanical turbulence. Values of z_0 and v_* found from such experiments appear in Table 3.1.

Table 3.1— TYPICAL VALUES OF PARAMETERS GOVERNING THE LOGARITHMIC WIND PROFILE NEAR THE EARTH'S SURFACE*

Type of surface	z_0, cm	v_*, m/sec†
Smooth mud flats; ice	0.001	0.16
Smooth snow	0.005	0.17
Smooth sea	0.02	0.21
Level desert	0.03	0.22
Snow surface; lawn to 1 cm high	0.1	0.27
Lawn, grass to 5 cm	1–2	0.43
Lawn, grass to 60 cm	4–9	0.60
Fully grown root crops	14	1.75

*Based on Sutton (1953), Priestley (1959), and Pasquill (1962).

†For $\bar{u}(2\text{m}) = 5$ m/sec.

3-1.2.7 Effect of Buoyancy. The effect on the wind profile of departures from purely mechanical turbulence must be considered, and we should begin by trying to clarify what is meant by mechanical turbulence. Because of the weight of air, i.e., because of the vertical force exerted on any air volume by gravity, the pressure of the atmosphere decreases with elevation. This vertical pressure variation implies a certain vertical temperature structure governed by the atmosphere's equation of state,

$$p = \rho RT \qquad (3.17)$$

where ρ is the density, R is the gas constant for air, and T is the Kelvin temperature. Specifically, the temperature of a volume of dry air displaced upward by a process that does not add or remove sensible heat will decrease at the linear rate of 1°C per 100 meters, the so-called "dry adiabatic lapse rate."

The mean vertical temperature structure of the lower layers of the atmosphere may under certain circumstances happen to possess a dry adiabatic lapse rate; if so, a small isolated volume of air, an air parcel, that is undergoing adiabatic vertical motion will at all times adjust itself so that it will experience no buoyancy force tending to restore it to its original elevation. It will always possess just the temperature of its environment. Mechanical turbulence in the atmosphere is conceived to have just this essential property that, no matter how irregular the individual eddy motions of which it consists may appear, there are no net buoyancy forces on fluid elements, or eddies, due to departure from an average adiabatic lapse rate.

This restriction to mechanical turbulence has simplified the analysis considerably up to this point, but in the lower atmosphere an adiabatic lapse rate is present only a small fraction of the total time. This seriously restricts the utility of the results obtained thus far. Figure 2.19 of Chap. 2, illustrating the normal clear-day diurnal variation of temperature structure of the lower atmosphere, demonstrates that the adiabatic state can ordinarily be expected only just after dawn and at dusk and will last perhaps for a few moments. The reason is that the flow of heat to and from the underlying surface by radiation, conduction, and convection causes the lapse rate in the lower air layers to vary from day to night over wide limits. During the day vertically displaced volumes of air undergoing adiabatic expansion must be acted upon by positive buoyant forces, and as a result turbulence is enhanced. During the night the converse effect tends ordinarily to suppress turbulence sharply. Of course, when the normal vertical heat flux in the lower layers is restricted markedly, for instance by a thick low cloud layer, the adiabatic state can persist for longer periods of time.

The buoyant force, F, on an air parcel is easily calculated, being equal to the weight of the displaced air volume, W_A, minus the weight of the air parcel, W_P, i.e.,

$$F = W_A - W_P = g\,V(\rho_A - \rho_P) \qquad (3.18)$$

where positive F indicates upward buoyancy, g is the gravitational acceleration, and V is the volume in question. The resulting acceleration, a, of the parcel, i.e., F divided by its mass, is

$$a = g\,\frac{\rho_A - \rho_P}{\rho_P} \qquad (3.19)$$

which, from the equation of state, can be written (bearing in mind that $p_A = p_P$)

$$a = g\,\frac{T_P - T_A}{T_A} \qquad (3.20)$$

where T_A and T_P are the air and parcel temperatures, respectively, in degrees Kelvin. Since the air parcel is conceived of as acquiring buoyancy by changing temperature dry adiabatically in a diabatic (i.e., nonadiabatic) environment, the last equation can also clearly be written as follows:

$$a = \frac{dw}{dt} = g\,\frac{(\gamma - \Gamma)\Delta z}{T_A} \qquad (3.21)$$

where γ = existing (in general, diabatic) lapse rate in the surrounding air
Γ = dry adiabatic lapse rate
Δz = height through which this process operates
w = vertical velocity acquired by the air parcel

The adiabatic lapse rate thus emerges as a natural standard of vertical temperature stratification in the atmosphere. It is of fundamental interest in connection with problems related to the turbulent structure of the lower layers of air because vertical displacements of air parcels, such as occur in turbulent flow, have the following character: (1) Vertical displacements have neutral stability, and displaced air parcels tend neither to fall nor to rise when

$$\gamma = -\frac{dT}{dz} = \Gamma;\ (T_P = T_A)$$

(2) Vertical displacements are unstable and are amplified by buoyancy when

$$\gamma > \Gamma;\ (T_P > T_A)$$

(3) Vertical displacements are strongly damped when

$$\gamma < \Gamma;\ (T_P < T_A)$$

3-1.2.8 The Richardson Number. Previous discussion has indicated that the energy of purely mechanical turbulence is associated with vertical wind shear, $d\bar{u}/dz$, through the agency of an eddy stress, τ_0. In the presence of a diabatic lapse rate, it now appears that turbulent energy is also strongly affected by buoyancy forces. Richardson (1920) suggested that turbulence should occur in the atmosphere when the production of turbulent energy by the wind shear is just large enough to counterbalance its consumption by buoyancy forces. He proposed as a measure, or criterion, of the effect the dimensionless number Ri that has been given his name:

Ri ∝ (rate of consumption of turbulent energy by buoyancy forces) × (rate of production of turbulent energy by wind shear)$^{-1}$ (3.22)

There are several ways to derive the Richardson number. Let us once again view the problem from the dimensional standpoint, asking what are the relevant variables. We have seen that mechanical turbulence is controlled by the vertical shear, or gradient, of the mean horizontal wind, $d\bar{u}/dz$. From Eq. 3.19 we conclude that the effect of consumption of turbulent energy by buoyancy will be governed by gravity, g, air density, ρ, and the vertical density gradient. This follows because Eq. 3.19 can be rewritten in the essentially equivalent form

$$a = \frac{g}{T_A}\frac{d\Theta}{dz} \qquad (3.23)$$

where a is now to be interpreted as the restoring force on a unit mass of air resulting from its unit vertical displacement ($\Delta z = 1$) from an equilibrium position. From these four quantities, involving three fundamental dimensional units, a single dimensionless ratio can be formed, namely,

$$Ri = \frac{g}{T_A}\frac{(d\Theta/dz)}{(d\bar{u}/dz)^2} \qquad (3.24)$$

This can also be written

$$Ri = \frac{g}{T_A}\frac{(\gamma - \Gamma)}{(d\bar{u}/dz)^2} \qquad (3.25)$$

which follows from Eqs. 3.21 and 3.23 and shows the relation of the Richardson number to the departure from an adiabatic lapse rate.

A second form of the Richardson number is often used. In diabatic turbulent shear flow, the significant phenomenon has been shown to be the departure of the temperature of the eddies from that of the surrounding air in which they are conceived as embedded. It follows that, in addition to momentum, the eddies act to transport heat across the flow. An expression for this eddy heat transport, or flux, H, can be written by analogy with Eq. 3.8 for the momentum flux, or stress,

$$H = \rho\, c_p\, K_H\, (\gamma - \Gamma) \qquad (3.26)$$

where c_p is the specific heat capacity of the air at constant pressure and K_H is a coefficient of eddy heat conductivity. If Eqs. 3.8 and 3.26 are substituted into Eq. 3.25 for Ri, we find that

$$Ri = \frac{g\,H}{c_p\,T_A\,\tau_0\,(d\bar{u}/dz)}\,\frac{K_M}{K_H} \qquad (3.27)$$

Thus an alternate definition is the so-called "flux form" of the Richardson number, R_f, where

$$R_f = Ri\,\frac{K_H}{K_M} = \frac{g\,H}{c_p\,T_A\,\tau_0\,(d\bar{u}/dz)} \qquad (3.27a)$$

3-1.2.9 The Diabatic Wind Profile. The Richardson number, Ri, or R_f, has come to be used as a characteristic turbulence parameter rather than as an absolute criterion of turbulence. That is, it is regarded as broadly indicating the nature and to some extent the intensity of the turbulence rather than specifying an exact criterion for turbulence to occur. As such, the Richardson number indicates the quantities upon which the velocity profile will depend in the diabatic case, namely, $\bar{u}$ will involve z, z_0, v_*, and k, as before, and, in addition, the parameters characterizing the diabatic effects, g, ρ, c_p, H, and T_A. A direct dimensional attack on this problem by the method we have been employing will evidently be fruitless because of the large number of dimensionless ratios that can be formed from the quantities involved. An elegant simplification is, however, possible following the suggestion made (independently) by Lettau (1949) and by Monin and Obukhov (1953).

We are considering uniform, straight, parallel turbulent flow near the surface with constant stress and heat flux. We assume that this kind of flow will extend to some elevation above the surface, an elevation that has not yet been directly specified but will be approximated later on. For the time being the depth of the layer can be regarded as that depth through which the assumptions of constant heat and momentum fluxes are applicable. Within this region of applicability, which Lettau calls the surface layer, the turbulence properties, including the wind and temperature profiles, will at any point be under the control of the various physical parameters just enumerated. It is known from the form of the governing equations of motion (Lumley and Panofsky, 1964, for example) that the effect of thermal buoyancy enters this problem through the buoyancy parameter, g/T_A. The conditions of constant momentum flux and heat flux likewise lead (Monin and Obukhov, 1953) to dependence of the flow on the dimensional parameters v_* and $H/c_p\rho$, respectively. From these three parameters, which uniquely characterize the velocity and temperature profiles in the surface layer, Monin and Obukhov formed the unique length, L,

$$L = \left[\frac{v_*^3}{k(g/T_A)}\right]\left(\frac{-H}{c_p\rho}\right)^{-1} \tag{3.28}$$

The quantity L is a constant, characteristic length scale for any particular example of the flow; it is negative for unstable conditions (upward heat flux), positive for stable conditions, and approaches infinity as γ approaches Γ. Of course L, being formed from the same parameters as the Richardson number, is closely related to Ri or R_f,

$$R_f = \frac{K_H}{K_M}\,\mathrm{Ri} = \frac{v_*}{k}\,\frac{1}{L(d\bar{u}/dz)} \tag{3.29}$$

as can easily be verified by substitution. It is more convenient to use L than Ri as a stability parameter characterizing the diabatic velocity profile because Ri must, from Eq. 3.29, vary with height.

Since all quantities having the dimension of length associated with this problem must be proportional to L, the diabatic wind profile is by dimensional analysis found to be

$$\bar{u}(z) = \frac{v_*}{k}\,f_3\left(\frac{z}{L},\frac{z_0}{L}\right) \tag{3.30}$$

Bearing in mind the boundary condition at the ground, $\bar{u} = 0$ when $z = z_0$ provided $z_0 > D$, the depth of the laminar sublayer, we usually express Eq. 3.30 in the equivalent form

$$\bar{u}(z) = \frac{v_*}{k}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right] \tag{3.31}$$

because the role of z_0 as a constant of integration is only to shift the velocity profile without changing its form (Eq. 3.14).

In the past few years, a large amount of research has gone into evaluating the form of the universal function f for various regimes of atmospheric stability. Such a function can be evaluated by means, for example, of a carefully planned program of measurements of $\bar{u}(z)$, or auxiliary physical or mathematical assumptions and principles can be invoked. As experience, in the form of detailed observational and theoretical studies of the vertical transport of heat and momentum in the surface layer, has been accumulated, it has become clear that three physically more-or-less distinct regimes are involved in this problem: forced convection, free convection, and the inversion or stable regime. The forced-convection regime is characterized by the fact that buoyancy does not contribute appreciably to the vertical momentum or heat diffusivities, these being completely dominated by mechanical turbulence and accompanied by a nearly adiabatic lapse rate. In this kind of turbulence, both heat and momentum are transferred by the action of the mechanically driven eddies, and these might be expected to occur at approximately equal rates, i.e., $K_H \approx K_M$.

In free convection, on the other hand, the vertical flux is mostly produced by buoyant motions. Strictly speaking, the term "free convection" should be reserved for the case of no mean wind shear, i.e., the case in which turbulence arises solely from the action of buoyant eddies. In practice, the term is commonly used to describe a turbulence regime that is characterized by the presence of a certain amount of forced, or mechanical, turbulence, i.e., by some shear. As Webb (1962) pointed out, this kind of turbulence should probably be called mixed convection in recognition of its composite nature. The term "free convection" will be retained in this discussion on the grounds that it conforms to current usage, but the point is well taken. The immediate sources of the energy that drive both mechanical and free convection are located at the earth's surface, but, as Scorer (1958, pp. 140-141) points out, the character of the turbulent air motions involved is necessarily quite different. In forced convection the turbulent eddies appear to be most vigorous within the surface layer near the ground. These eddies feed turbulent kinetic energy both upward and downward but always in the direction of smaller fluctuations. The picture is probably similar to that advanced by Townsend (1956, p. 236) in describing the energy flow in a wind-tunnel boundary layer. On the other hand, buoyant elements associated with free convection characteristically grow larger as they ascend from the ground.

The entire subject of the structure of the diabatic surface layer is under study by a number of investigators, and it is possible to give here only a brief sketch of the main results that are available. As an asymptotic approximation to Eq. 3.31, valid under conditions sufficiently near adiabatic, the well-known log plus linear law has been derived by several workers,

$$\bar{u}(z) = \frac{v_*}{k}\left(\ln\frac{z}{z_0} + \alpha\zeta\right) \tag{3.32}$$

where $\zeta = z/L$. This equation implies that $f(\zeta)$ in nearly adiabatic conditions is given by

$$f(\zeta) = \ln \zeta + \alpha\zeta \tag{3.33}$$

Although it is usually associated with the similarity theory, the log plus linear wind profile was also deduced independently in the surface-layer study by Lettau (1949). In fact, a wind profile of this form was suggested by Halstead (1943) on empirical grounds.

Interpolation formulas for velocity profiles that provide a smooth transition between the forced-convection and the free-convection cases have been suggested by Kazansky and Monin (1958), Ellison (1957), Yamamoto (1959), Sellers (1962), and Businger (1959). Panofsky, Blackadar, and McVehil (1960) recently showed that Ellison's diabatic profile agrees well with observations made in unstable air. The results of these studies were summarized in the form of the so-called "KEYPS" function described in detail by Lumley and Panofsky (1964). The case of great stability remains, on the other hand, something of an enigma from the theoretical standpoint. The general conclusion from the above studies is that the log plus linear velocity profile, Eq. 3.32, agrees well with observations in both stable and forced convection conditions for $|\zeta| < 0.1$ if the value $\alpha = 6.0$ is used, but, on the side of considerable stability, this good agreement seems to fail. The suggestion made by Panofsky, Blackadar, and McVehil (1960) is that under very stable conditions the velocity profile no longer will depend simply on distance from the ground as is assumed by the similarity theory. Under very stable conditions there seems to be a decoupling of the direct linkage assumed in the similarity theory between the structure of surface-layer turbulence and the physical presence of the ground with the result that the surface-layer flow properties are primarily determined by the nature of the air flow at still higher elevations in the planetary boundary layer.

3-1.3 Wind Variation in the Planetary Boundary Layer

Restricting consideration to steady, straight, and parallel flow with constant stress and introducing complications serially makes it possible to analyze the average wind structure in the surface layer of the atmosphere in some detail and to isolate and emphasize the crucial phenomenon involved, i.e., that of eddy turbulence. When this has been done, the mean wind has been regarded as a given condition superimposed on the flow, so to speak, from above. A wind-speed profile showing an increase with height above the surface as a result of a net downward transport of momentum by turbulent eddies was found.

On the other hand, it is a matter of common experience (e.g., on airplane flights) that the effect of turbulence decreases with elevation in the lower atmosphere and is usually negligible above several thousand feet. Moreover the eddy stress has been found by analysis of wind-fluctuation observations to decrease with height above the surface layer. Furthermore the mean wind does not increase indefinitely with height.

In the model adopted here, however, both turbulence and the mean wind as a result of the assumption of constant stress must, according to Eqs. 3.8 and 3.12, increase with height. Consequently this model can apply only in the lowest part of this region, and the theoretical picture requires some modification and elaboration.

The equations of motion (i.e., the form of Newton's law, $\Sigma F = ma$) applicable near the earth's surface to a steady horizontal wind flow with parallel isobars, according to texts on dynamic meteorology, are

$$f\bar{v} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial z}\tau_{zx} = 0 \tag{3.34}$$

$$-f\bar{u} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}\tau_{zy} = 0 \tag{3.35}$$

where f is equal to $2\omega \sin\varphi$ and is called the Coriolis parameter (ω is the angular velocity of the earth's rotation and φ is geographical latitude), p is pressure, and the subscript zx on the eddy stress, τ, indicates that it acts to transport x-directed momentum in the vertical or z-direction. Stated in words, the average air motion is governed by the sum of three accelerations: the Coriolis acceleration (or the apparent acceleration due to the earth's rotation), the pressure-gradient acceleration, and the frictional acceleration. The sum of these is zero because the flow is assumed steady (unaccelerated). These equations are supposed to apply to a unit mass of the atmosphere, and so the accelerations are equally likely to be referred to as forces.

By analogy with Eq. 3.8, we might suppose that

$$\tau_{zx} = \rho K \frac{\partial \bar{u}}{\partial z} \tag{3.36}$$

and

$$\tau_{zy} = \rho K \frac{\partial \bar{v}}{\partial z} \tag{3.37}$$

In other words, the eddy viscosity can be generalized by breaking it up into x- and y-components. The K's will, in general, depend on height, z. Then Eqs. 3.34 and 3.35 become

$$f\bar{v} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(K\frac{\partial \bar{u}}{\partial z}\right) = 0 \tag{3.38}$$

$$-f\bar{u} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(K\frac{\partial \bar{v}}{\partial z}\right) = 0 \tag{3.39}$$

Since we expect the effect of turbulent friction to decrease with elevation, the third term in these equations, which represents the accelerations due to eddy turbulence, should become negligible at some height in the atmosphere. If we orient the x-axis in the direction of the wind at this level, $\partial p/\partial x = 0$, the above system simplifies to the following:

$$f\bar{u}_G = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.40}$$

where the subscript G is introduced to designate the level in question. The term $\bar{u}_G$ is called the geostrophic wind, from the Greek words meaning "earth" and "turning," and Eq. 3.40 is known as the geostrophic wind equation.

Assuming, as the simplest useful approximation, that below the geostrophic wind level the effect of eddy viscosity on the mean wind structure can be expressed by letting diffusivity be constant and that the pressure gradient is independent of height, the solution is

$$\bar{u}(z) = \bar{u}_G(1 - e^{-az}\cos az) \tag{3.41}$$

$$\bar{v}(z) = \bar{u}_G e^{-az}\sin az \tag{3.42}$$

where $a = (f/2K_M)^{1/2}$. This can easily be verified by substituting into Eqs. 3.38 and 3.39 and taking into account Eq. 3.40. Figure 3.5 is a plot of this wind distribution, which shows that near the ground friction causes the air to flow across the lines of constant pressure (isobars) in the direction of low pressure. This effect decreases with increasing height and disappears at the geostrophic wind level z_G, which can conveniently be defined as the lowest level at which $\bar{v} = 0$ and therefore the lowest level at which the wind is parallel to $\bar{u}_G$. This must occur when $az = \pi$, from which we can conclude since $f \approx 10^{-4}$ sec^{-1} and K_M is known to be of the order of 10^4 cm^2/sec, that the depth of the layer of frictional influence in the atmosphere is of the order of hundreds of meters. This layer is called the planetary boundary layer. Notice also that the eddy viscosity, K_M, is five

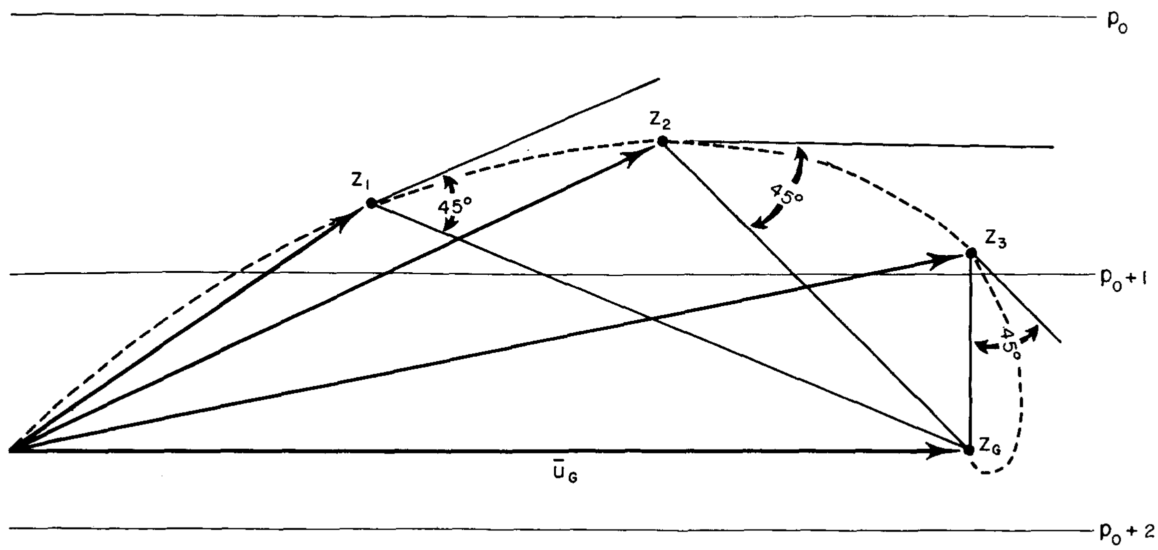


Fig. 3.5—Schematic wind distribution (Ekman's spiral) in the planetary boundary layer, assuming K_M = constant, according to Eqs. 3.41 and 3.42. Wind vectors are plotted from a common origin at increasing heights, z_i, i = 1, 2, etc.

orders of magnitude larger than ν, a fact which justifies neglect of the influence of molecular viscosity.

The frictional acceleration in the equations of motion is known to be of the same order of magnitude as the Coriolis acceleration, about 10^{-1} cgs units; i.e., $|(1/\rho)\ (\partial\tau/\partial z)| \approx 10^{-1}$. Consequently $\partial\tau \approx 10^{-4}\ (\partial z)$, which equals 0.1 dyne/cm^2 in 10 m of height. This will usually amount to about 10% of τ_0 as the following calculation shows. The magnitude of the total eddy stress is given by the sum of Eqs. 3.36 and 3.37 for the components. The value at the surface, τ_0, is, assuming constant K,

$$\tau_0 = \rho\, K_M(\partial\bar{u}/\partial z|_{z=0} + \partial\bar{v}/\partial z|_{z=0}) \tag{3.43}$$

where the wind shears are to be evaluated at the surface. Substituting Eqs. 3.41 and 3.42 into Eq. 3.43, carrying out the differentiations, and letting z = 0, we find that

$$\tau_0 = 2^{1/2} a K_M \rho\, \bar{u}_G = \frac{z_G}{2^{1/2}\pi} f\, \rho\, \bar{u}_G \tag{3.44}$$

Estimating that z_G is of the order of 10^4 cm and $\bar{u}_G \approx 10^3$ cm/sec, we find that τ_0 is of the order of 1 dyne/cm^2.

From this it can be concluded that the depth of the surface layer, i.e., the layer just next to the ground through which the stress τ_0 may be considered to be constant, is of the order of tens of meters, or about 10% of the depth of the planetary boundary layer. Through the surface layer the mean wind direction is approximately constant, and the speed increases with height according to equations derived in the preceding sections. Above the surface layer the mean wind turns to the right (in the northern hemisphere) and attains, provided the overlying flow is geostrophic (i.e., governed by Eq. 3.40), the direction and speed of the geostrophic wind at elevations of the order of hundreds of meters.

The solution to the wind distribution in the planetary boundary layer given by Eqs. 3.41 and 3.42 was first obtained by Ekman in 1902 and is known as Ekman's spiral. It provides a reasonably good qualitative explanation of the wind structure and an order of magnitude estimate of such quantities as z_G and K_M. But we have seen that eddy viscosity, K_M, must vary with height in the planetary boundary layer, increasing just above the ground as larger eddies become effective and then decreasing at greater elevations as the general influence on the airflow of the surface frictional drag decreases. Moreover we can expect, on the basis of analysis of the diabatic surface layer, that the eddy structure in the planetary boundary layer will also be strongly influenced by buoyant heat flux. In addition, for many important practical situations, it cannot be assumed that the governing forces are in balance. The sea breeze and mountain and valley winds are examples of such accelerated flows. All these complicating factors and others are the subject of active research studies, and considerable progress has been made, which, however, would take us too far afield to summarize. As a practical matter, such effects are not normally evaluated quantitatively in connection with estimates of atmospheric diffusion, and so the qualitative discussion given in Chap. 2 provides an adequate guide. Readers interested in further development of this important subject will find useful discussions in the papers by Estoque and Yee (1963), Blackadar, Panofsky, McVehil, and Wollaston (1960), and Lettau (1962).

3-2 DIFFUSION THEORIES

Small particles or droplets released into the atmosphere will separate more or less rapidly from one another under the influence of turbulent eddies, a phenomenon called diffusion. We have already investigated the vertical diffusion of such intrinsic air properties as momentum and heat in Sec. 3-1. Indeed the two phenomena are closely related, differing only in that for particle diffusion the possibility exists of effects arising from the size and inertia of the particles involved. In Sec. 3-2 we are concerned with diffusion of particles from isolated sources in the lower atmosphere.

The problem of turbulent diffusion in the atmosphere has not yet been uniquely formulated in the sense that a single basic physical model capable of explaining all the significant aspects of the problem has not yet been proposed. Instead there are available two alternative approaches, neither of which can be categorically eliminated from consideration since each has areas of utility that do not overlap the other's. The two approaches to diffusion are the gradient transport theory and the statistical theory. Diffusion at a fixed point in the atmosphere, according to the gradient transport theory, is proportional to the local concentration gradient. Consequently it could be said that this theory is Eulerian in nature in that it considers properties of the fluid motion relative to a spatially fixed coordinate system. On the other hand, statistical diffusion theories consider motion following fluid particles and thus can be described as Lagrangian. Diffusion theories may be classified as either continuous-motion or discontinuous-motion theories, depending on whether this particle motion is postulated to occur continuously or as discrete events. There must necessarily be a close connection among all these approaches to the diffusion problem since obviously there is only one atmosphere. We will consider here those aspects of each of these approaches which have found application in the atmosphere.

3-2.1 The Gradient Transport Approach

3-2.1.1 Fickian Diffusion. Adolph Fick, a German physiologist, published a paper in 1855 (*Ann. Physik Chem.*, [2] 94: 59-86) entitled "Über Diffusion." These details are given because, although Fickian diffusion is spoken of quite familiarly by research workers in many disciplines, few appear to know who Fick was. His idea stated in his own words [*Phil. Mag.*, [4] 10: 30-39 (1855)] is: "It is quite natural to suppose that this law for the diffusion of salt in its solvent must be identical with that according to which the diffusion of heat in a conducting body takes place; upon this law Fourier founded his celebrated theory of heat, and it is the same which Ohm applied, with such extraordinary success, to the diffusion of electricity in a conductor." The mathematical statement of this hypothesis, Fick's law, has (in the one-dimensional case) the form of the classical equation of conduction,

$$\frac{d\bar{q}}{dt} = K \frac{\partial^2 \bar{q}}{\partial x^2} \tag{3.45}$$

where K (in the atmosphere) is a constant eddy-diffusivity coefficient and $\bar{q}$ refers to the mean value of some conservative air property per unit mass of air. One of the many interesting applications of this useful equation is to describe the diffusion of thermal neutrons in a nuclear reactor.

The more general case of diffusion in three dimensions in which the diffusion coefficients, which are not necessarily equal, can vary with the three spatial coordinates, i.e.,

$$\frac{d\bar{q}}{dt} = \frac{\partial}{\partial x}\left(K_x \frac{\partial \bar{q}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \bar{q}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \bar{q}}{\partial z}\right) \tag{3.46}$$

was first investigated independently by Richardson (1926) and by Schmidt (1925). The problem of atmospheric diffusion reduced to that of solving Eq. 3.45 or 3.46 under appropriate boundary conditions is often called the K theory. If K_x, K_y, and K_z are constants, the diffusion is called Fickian. The K can be thought of as measuring the flux of a passive scalar quantity $\bar{q}$, such as smoke [flux is defined as $K_x(\partial\bar{q}/\partial x)$ or by a similar expression in y or z]. This quantity, by definition, does not affect the dynamics of the air motions but is merely carried along by them. Consequently, when the turbulence is largely mechanical, $K = K_M$; but, when there is strong thermal convection, $K = K_H$ would be the better approximation. In view of our present limited ability to specify K_H, due largely to the difficulty of determining atmospheric heat flux, this distinction is somewhat academic. In practice K values are usually determined by reference to observed diffusion data.

For a stationary medium Eq. 3.45 in one dimension becomes

$$\frac{\partial\bar{q}}{\partial t} = K\frac{\partial^2\bar{q}}{\partial x^2} \tag{3.47}$$

Boundary conditions specifying a point source are

(1) $\bar{q} \rightarrow 0$ as $t \rightarrow \infty$ $\quad(-\infty < x < +\infty)$

(2) $\bar{q} \rightarrow 0$ as $t \rightarrow 0$

(for all x except x = 0) (3.48)

where $\bar{q} \rightarrow \infty$ such that

$$\int_{-\infty}^{\infty} \bar{q}\, dx = Q \tag{3.49}$$

where Q is the source strength (total release of $\bar{q}$). The solution may be obtained by various mathematical devices, in particular by the method of Fourier series. In fact it is probably fair to say that the existence of a large variety of solutions to Eq. 3.45 with various boundary conditions, from the classical theory of heat conduction, has been one of the greatest incentives to development of the K theory.

The fundamental solution of this problem is known to be a Gaussian function, i.e., it has the form

$$\frac{\bar{q}}{Q} = \frac{1}{at^{1/2}} \exp\left(-\frac{bx^2}{t}\right) \tag{3.50}$$

Notice that the factor x^2 implies a symmetrical cloud and that the factor t^{-1} in the exponent satisfies condition (2) of Eq. 3.48. By partial differentiation of Eq. 3.50, we can easily show that it satisfies Eq. 3.47, and, making use of the continuity condition, Eq. 3.49, we find that $a = (4K\pi)^{1/2}$ and $b = (4K)^{-1}$. Since condition (1) corresponds to an instantaneous point source at $t = 0$, the solution to Eq. 3.45 for an instantaneous point source of $\bar{q}$ with strength Q is

$$\frac{\bar{q}}{Q} = \frac{1}{(4\pi Kt)^{1/2}} \exp\left(-\frac{x^2}{4Kt}\right) \tag{3.51}$$

This solution would apply to an atmosphere in which $\bar{u}$ = constant, $v = w = 0$, and for which the coordinates are thought of as moving with the mean wind, $\bar{u}$.

Equation 3.51 may be extended to three dimensions and generalized to the case (nonisotropic diffusion) where $K_x \neq K_y \neq K_z$. The resulting solutions to Eq. 3.46 are, for $K_x = K_y = K_z = K$ and $x^2 + y^2 + z^2 = r^2$,

$$\frac{\bar{q}(r,t)}{Q} = (4\pi Kt)^{-3/2} \exp\left(-\frac{r^2}{4Kt}\right) \tag{3.52}$$

and, for the nonisotropic case,

$$\frac{\bar{q}(x,y,z,t)}{Q} = (4\pi t)^{-3/2}\,(K_xK_yK_z)^{-1/2} \times \exp\left[-\frac{1}{4t}\left(\frac{x^2}{K_x} + \frac{y^2}{K_y} + \frac{z^2}{K_z}\right)\right] \tag{3.53}$$

These are the fundamental building blocks of Fickian diffusion theory. Integration of one of these instantaneous-point-source solutions with respect to space yields equations for instantaneous volume sources (bomb bursts, for example). Integration of the instantaneous-point-source equation with respect to time gives the continuous-point-source solutions. These may, in turn, be integrated with respect to, say, the y-axis to give the crosswind infinite-line-source equation, or they may be integrated with respect to the horizontal plane, and so on. Probably because of the essentially tractable nature of the mathematics involved, almost every laborer in the vineyard of atmospheric diffusion theory has worked out a solution or two for the Fickian case. As a result, this branch of the subject is now fairly complete.

3-2.1.2 The K Theory. The assumption of constant eddy diffusivity, although it may be of considerable use in the free atmosphere, can hardly apply to the planetary boundary layer, which, as we have seen, is characterized by pronounced shear of the mean wind and large variations in vertical temperature gradients due to heat flux. The K theory of diffusion has addressed itself to these problems.

Equation 3.46 may be simplified by assuming the steady state, i.e., $\partial\bar{q}/\partial t = 0$. If we take an infinite crosswind line source, for which, at ground level,

$$\frac{\partial}{\partial y}\left(K_y \frac{\partial \bar{q}}{\partial y}\right) = 0 \qquad (3.54)$$

(recalling that $\bar{w} = \bar{v} = 0$ if the mean wind blows along the x-axis) and assume, as is reasonable, that $\partial(K_x\partial\bar{q}/\partial x)/\partial x \ll \bar{u}\,\partial\bar{q}/\partial x$, i.e., the x-transport by the mean flow greatly outweighs the eddy flux in that direction, then we can reduce Eq. 3.46 to

$$\bar{u}\frac{\partial \bar{q}}{\partial x} = \frac{\partial}{\partial z}\left(K_z \frac{\partial \bar{q}}{\partial z}\right) \qquad (3.55)$$

This equation, together with the boundary conditions

(1) $\bar{q} \rightarrow 0$ as $z \rightarrow \infty$

(2) $\bar{q} \rightarrow 0$ as $x \rightarrow 0$ for all $z > 0$ but $q \rightarrow \infty$ as $x \rightarrow 0$, $z \rightarrow 0$ such that $\lim\limits_{x\rightarrow 0} \int_0^\infty \overline{uq}\, dz = Q$

(3) $K_z\, \partial\bar{q}/\partial z \rightarrow 0$ as $z \rightarrow 0$ for all $x > 0$ (3.56)

the latter implying zero flux at the ground, has been used as the basis for many investigations.

The effect of shear of the mean wind was taken into account by Roberts, who solved Eqs. 3.55 and 3.56 together with a power-law form of K_z; the solution can be found in Sutton's (1953) book. On the basis of the assumption that the surface layer is about 10 m deep, it has been supposed that power-law solutions to Eq. 3.55 would be strictly valid to a distance of about 100 m from a ground-level source since beyond that the diffusing cloud would be likely to be growing out of the surface layer. It now appears from DeMarrais' (1959) study that as a practical matter such solutions may be valid to considerably greater distances. DeMarrais shows that wind profiles can be fit by power functions to elevations of about 100 m, which implies that K_z can also be represented by a power function to this height.

Extension of the K theory to account for surface-roughness effects was undertaken by Calder (1949), who assumed the power-law wind profile

$$\bar{u} = v_* r' \left(\frac{z}{z_0}\right)^{\alpha'} \qquad (3.57)$$

and chose the constants r′ and α′ so as to give the best fit to the logarithmic wind profile. His solutions are complicated, but over level uniformly rough ground good experimental verification is obtained to distances up to a kilometer from the line source in adiabatic conditions.

Varying atmospheric stability was introduced into the problem by Deacon (1949), who gave a solution for an infinite line source based at the surface, using

$$K(z) = kv_*\, z_0 \left(\frac{z}{z_0}\right)^{\beta}$$

$$\bar{u}(z) = v_* r^* \left(\frac{z}{z_0}\right)^{\alpha^*} \qquad (3.58)$$

and determining β and α* from observed (diabatic) wind profiles. A solution for an infinite elevated crosswind line source has also been given, as has a solution for a finite line source oriented along the mean wind. Finite and infinite plane sources are considered extensively in evaporation theory; a review of much of this material can be found in the monograph by Anderson, Anderson, and Marciano (1950). Lettau (1952) developed a shearing advection correction to the K theory which takes into account the apparent diffusion that results from the presence of shear of the mean wind in the planetary boundary layer. Davies (1954), Gee and Davies (1963), and Saffman (1962, 1963) have also discussed the effect of shear. Some progress has recently been made on solutions to Eq. 3.46 for a continuous point source, both at the ground and aloft, by Rounds (1955) and

Smith (1957). This work has been extended in the papers by Godson (1958) and Davidson and Herbach (1962) to include stable conditions, elevated point sources, and the effect of particle settling.

The K theory has great appeal to research workers in atmospheric turbulent diffusion, judging by the papers just cited as well as many related ones found in the bibliographies that the papers contain. Because the fundamental differential equation involved, Eq. 3.46, can be considerably simplified by eliminating one or more of the space coordinates, K theory is widely applied in studies of evaporation and heat conduction from the earth's surface, which is considered to be an extended, horizontal, plane source. Study of the momentum distribution in the planetary boundary layer has likewise suggested the use of K theories. The abundant literature on this phase of the subject was reviewed by Priestley (1959) [see also Priestley, McCormick, and Pasquill (1958)].

Since, in planetary-boundary-layer heat conduction, the source, or driving term, is a sinusoidal time function, the mathematical complexity of some of these solutions is considerable. Staley (1956) described certain K theories quite accurately as "a mathematical extravaganza." It seems that the attraction exerted by the K theory may stem as much from the opportunity it provides for obtaining mathematically explicit results as from its intrinsic physical correctness. All ramifications of the K theory depend ultimately on the validity of the assumption of simple gradient transport, which is the notion that the flux of a quantity is proportional to the gradient of this quantity. Priestley (1959) points out that there is no precise physical basis for the use of this assumption as the foundation for a description of turbulent diffusion in the atmosphere, and consequently the validity of the K theory "is normally judged from the degree of success achieved in ... predicting particular diffusion phenomena." Calder (1965) studied the applicability of the diffusion equation to the atmospheric case and concluded that the standard K-theory form, Eq. 3.46, cannot be generally valid. Russian workers, e.g., Monin (1959), refer to K theory as a semiempirical theory of diffusion. The basic nature of K theory must be kept in mind as the chain of deductions from the original equation grows longer and more involved.

This being said, it must hastily be added that K theory provides many useful, practical results. For example, an approach to the difficult problem of the deposition of polydisperse aerosols (Davidson and Herbach, 1962) can be made via K theory. Barad (1951) presented a K theory of the complicated problem of diffusion of a bent-over stack plume in very stable atmospheres. There are many other examples. Corrsin has aptly summarized the situation by pointing out that K theory is not useful in principle but only in practice.

3-2.2 Statistical Theories of Turbulent Diffusion

Today the statistical theory of fluid turbulence comprises a large and important body of literature, and its results are applied in many areas from oceanography to cosmology. The study of turbulence by this method actually began, however, with the investigation of turbulent diffusion by Taylor (1921). The statistical approach to the diffusion problem differs considerably from K theory. Instead of studying the material or momentum flux at a fixed space point, one studies the histories of the motion of individual fluid particles and tries to determine from these the statistical properties necessary to represent diffusion.

3-2.2.1 Diffusion by Discontinuous Motion. Many of the essential characteristics of statistical diffusion theory can be introduced by the following classroom experiment in diffusion by discontinuous motion. The instructor takes a number of pennies and distributes them to the class as follows. He tosses one and, according to whether it comes up heads or tails, passes it out to the student on his right or on his left in the middle of the first row. The student, in turn, repeats this, passing the penny over his right or left shoulder, and so on, until finally the penny reaches the back row. The instructor continues tossing more pennies and passing them out. Of course, after a time the students in the back row of the classroom will receive pennies in some more or less regular pattern with most of the pennies going to students near the middle of the row and fewest to those near each end. This experiment, simple and obvious as it is, nevertheless brings out a number of important features of the diffusion problem:

1. The stochastic, or probabilistic, nature of diffusion: this is illustrated by the process used to distribute the pennies.

2. Continuity: the diffusion process must satisfy a continuity condition (i.e., all the pennies should be returned at the end of the experiment).

3. Deposition: occurs if a penny is dropped.

4. Attenuation: at any step, a penny might be removed permanently from the diffusion process (for radioactive particles the analogy is radioactive decay).

5. Effect of sampling: the actual distribution of pennies at the back row is not a perfectly symmetrical distribution. It could be skewed or perhaps bi- or multimodal. Since only a relatively small sample (just a few pennies) was used, the observed distribution will depart from the ideal, symmetrical pattern.

This experiment can be formalized (see Chandrasekhar, 1943). The probability, P, that a penny will move right or left equals $\frac{1}{2}$. After n steps, the penny can be at any of the points $-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n$. The number of possible paths in n steps is 2^n, and $P = 2^{-n}$, i.e., all are equally probable. Let $x = mh$ and $t = nk$; then the probability of a penny's reaching any given point mh, at step nk is $P(mh,nk) = 2^{-n}$ (number of possible paths). The grid spacing, h and k, can be chosen as unity and ignored. Let r equal the number of steps right and l equal the number of steps left in a path. Then $l = r - m$ (number to left = number to right minus total lateral distance), and $l = (n - r)$ (total number of steps minus number to right), i.e., $r - m = n - r$, or $m + n = 2r$, and $r = \frac{1}{2}(m + n)$. The number of paths equals $\binom{n}{r}$, i.e., the number of combinations of r elements or n; so

$$P = \frac{1}{2^n}\binom{n}{r} = \frac{1}{2^n}\frac{n!}{r!(n-r)!} = \frac{1}{2^n}\frac{n!}{\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!} \tag{3.59}$$

which is Bernoulli's distribution. For large values of n, this distribution approaches the normal distribution, normal error curve, or Gaussian distribution:

$$P(m,n) = \left(\frac{2}{\pi n}\right)^{1/2} \exp\left(-\frac{m^2}{2n}\right) \tag{3.60}$$

If Eq. 3.60 is plotted for successive values of n, the familiar bell-shaped curves of the normal error law result. It is of interest that the coin-tossing, or Monte Carlo, method was originally developed by von Neuman and Ulam in connection with complex problems arising in the calculation of diffusion of neutrons through absorbing and shielding media.

The simple discrete-step stochastic diffusion model (sometimes called "the drunkard's walk") implied by the above discussion is far from irrelevant to the atmosphere. Its molecular analog describes Brownian diffusion. On the other hand, actual turbulent atmospheric motions tend to be rather highly self-correlated, in marked contrast with Brownian motion. The approximation that successive diffusion events are uncorrelated is not a good one in the case of atmospheric turbulence except when the time scale of the problem is large compared with the time scale of the diffusion process. The consequences of a direct application of the Brownian-motion analogy to atmospheric diffusion have been investigated by Obukhov (1959), Lin (1960), and Chadam (1962).

The uncorrelated kind of diffusion process described by Eq. 3.60 corresponds closely to Fickian diffusion; consequently it must be governed by a parabolic type of differential equation, such as Eq. 3.47. Physically, parabolic differential equations characterize equalization processes, of which the heat-conduction problem provides the classical example. Solutions of parabolic equations have the character that some effect is felt everywhere except at the initial instant, $t = 0$, as is shown by Eq. 3.51. The implication is that diffusion proceeds in some sense with infinite velocity. Generalizations to more realistic discrete-step diffusion models in which successive events are correlated (drunkard's walk with a memory) have been discussed by Taylor (1921), Goldstein (1951), Davies and Diamond (1954), Davies, Diamond, and Smith (1954), and Monin (1955). These studies indicate that atmospheric diffusion should obey the "telegrapher's equation" rather than a simple parabolic equation of the heat-conduction type. Since, in the diffusion application, the telegrapher's equation is hyperbolic like the wave equation rather than parabolic, it describes diffusion that proceeds at a finite velocity. Thus there will be a definite limit to the distance that fluid particles can disperse in a given amount of time in contrast to the conclusion from Eq. 3.51 that the effect of diffusion is felt everywhere to some extent for all values of $t > 0$. It cannot be denied that finite diffusion is physically more realistic although the practical difference, as shown in Sec. 3-3, is probably not great.

3-2.2.2 Diffusion by Continuous Motion. Taylor (1921) derived a fundamental diffusion theorem that has had very great influence on all subsequent work in this field, both theoretical and practical. Taylor's result applies to diffusion in one space dimension or to the projection onto a single space axis of two- or three-dimensional diffusion in a stationary, homogeneous turbulent flow. A homogeneous turbulent flow is one in which the statistical properties are independent of position. Stationary turbulence is homogeneous in time. Properties of the turbulence, such as the transverse (to the mean flow direction) root-mean-square velocity, $(\overline{v'^2})^{1/2}$, would be expected to be invariant anywhere in such a flow. Turbulence in the upper portion of the planetary boundary layer may approximate the homogeneous type, but surface-layer turbulence is decidedly inhomogeneous. The idea of turbulence homogeneity is a simplification introduced into the theory to permit further progress to be made.

Taylor's calculation involves the motion (continuous) of a fluid particle, which is assumed to be somehow identified or tagged. On the other hand, we might consider a dynamically and chemically inert particle of negligible size and mass which is being transported by the atmosphere. The distance, y, that this particle is carried away from an origin by turbulent wind fluctuations, v', during a time interval, t, is equal to

$$y(t) = \int_0^t v'(t_1)dt_1 \qquad (3.61)$$

(We will not introduce a separate symbolism to distinguish the particle-attached, or Lagrangian, motion from the fixed-point Eulerian motion since this would greatly complicate the notation. The distinction should always be kept clearly in mind, however.) This straightforward process is pictured in Fig. 3.6. By the transformation $t = x/\bar{u}$, we can also visualize the motion as taking place relative to a fixed space axis extending downwind from the origin. A physical example of this phenomenon would be the motion of a smoke particle emitted from a chimney in a steady mean wind.

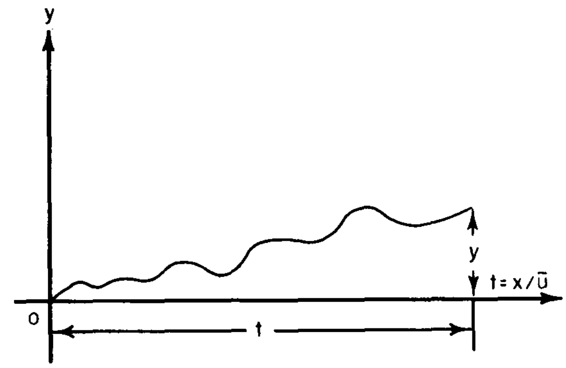


Fig. 3.6 — Path of a tagged particle displaced a distance y in time t by the action of random turbulence.

The simplest meaningful statistical measure of this irregular, random process that we can compute is the mean-square diffusion that would result from a large number of independent repetitions, i.e., the variance, or second moment, of the resulting distribution of particles along the y-axis. By squaring both sides of Eq. 3.61 and taking the average over many repetitions of the experiment (the statistical, or ensemble, average), we are led to Taylor's result:

$$\overline{y^2}(t) = 2\,\overline{v'^2} \int_0^t \int_0^{t_1} R(\xi)\, d\xi\, dt_1 \qquad (3.62)$$

The mathematical steps are given in many references, e.g., Pasquill (1962).

The function $R(\xi)$ is called the one-point Lagrangian velocity correlation coefficient, Lagrangian because it refers to the velocity of a particle rather than the velocity at a fixed space point, and coefficient because it has been normalized, i.e., adjusted, by dividing by $\overline{v'^2}$ so that $R(0) = 1$:

$$R(\xi) = \frac{\overline{v'(t)\, v'(t+\xi)}}{\overline{v'^2}} \qquad (3.63)$$

Batchelor (1949) generalized Eq. 3.62 to three dimensions. In this form the mean-square diffusion becomes a tensor, δ_{ij}^2, with indices ranging from 1 to 3, and $\overline{y^2} \equiv \delta_{22}^2$. Computation of higher order moments, such as $\overline{y^3}$, could be carried out by the same straightforward process although with rapidly increasing complexity.

Since $R(0) = 1$ and, for sufficiently small diffusion time, $R(t) \approx 1$ because R is a correlation coefficient, it follows that when t is small

$$\overline{y^2}(t) \approx \overline{v'^2}\, t^2 \qquad (3.64)$$

When t is large, it may be supposed that the autocorrelation function, R, must approach zero sufficiently rapidly that

$$\overline{v'^2} \lim_{t\to\infty} \int_0^t R(t_1)\, dt_1 = K_1 \qquad (3.65)$$

where K_1 is some constant. The particle must ultimately "forget" its original motion. There are several ways to see why this should be true. Perhaps the most obvious is the one discussed in the following paragraphs.

Consider the Fourier cosine transform F(n) of R:

$$F(n) = 4 \int_0^\infty R(t) \cos(2\pi\, nt) dt \qquad (3.66)$$

where F(n) is called the Lagrangian eddy-energy spectrum. The eddy-energy spectrum expresses the distribution as a function of frequency, n, of turbulent kinetic energy corresponding to the various Fourier components of the (in this case) one-dimensional Lagrangian turbulent velocity field. The simpler term "eddy" will be used from now on to denote a Fourier component of the turbulent velocity field characterized by a certain time or length scale. Although it is always more convenient to speak of an eddy of some size (or of some time scale proportional to n^{-1}), it should not be inferred that such a Fourier component necessarily has a separate, identifiable material existence such as we have previously imagined is possessed by eddies. It must be remembered that two meanings of the term "eddy" exist and that they are often confused in the meteorological literature. The interested reader should consult Corrsin (1959) for a concise discussion of the meaning of the spectrum representation of a turbulent velocity field; a complete discussion of atmospheric energy-spectrum properties is contained in the books by Pasquill (1962) and by Lumley and Panofsky (1964).

At zero frequency ($n = 0$), we can see from Eq. 3.66 that

$$F(0) = 4 \int_0^\infty R(t)\, dt \qquad (3.67)$$

Even without further discussion of the properties of the energy spectrum, it seems clearly to be required that $F(0) \propto K_1 < \infty$. Otherwise the eddy kinetic energy would be in some sense infinite. Consequently we may derive the limit of Eq. 3.62 for large diffusion times:

$$\overline{y^2}(t) \approx 2K_1 t \qquad (3.68)$$

where K_1 is a constant.

The derivative of Eq. 3.68, i.e., $\frac{1}{2}\, d\overline{y^2}/dt$, has the dimensions of a diffusivity. It might be argued therefore that K_1 plays a part similar to that of the K of Fickian theory and that

$$\frac{1}{2}\frac{d\overline{y^2}}{dt} = K_1 = K \qquad (3.69)$$

where K has the original meaning assigned to it, an eddy diffusion coefficient. Comparing Eqs. 3.69 and 3.68, the conditions for the applicability of the K theory in the atmosphere can be appreciated. The quantity $\int_0^\infty R\, dt$ defines a time-scale characteristic of the turbulence called the Lagrangian integral time scale, $\mathscr{T}$:

$$\mathscr{T} = \int_0^\infty R(t)\, dt \qquad (3.70)$$

This argument makes it appear reasonable that Fickian theory, in which K is constant, should apply when the diffusion time, t, is large compared to $\mathscr{T}$.

There appears to be no basic way of evaluating the precise points at which the limits of Taylor's diffusion theorem for small and large times will apply in the atmosphere. If it were possible to measure the Lagrangian autocorrelation function, R, with precision, the applicable diffusion times could be determined, but this is very difficult to do. In fact most of the reliable knowledge of the form of R has been inferred, by applying Eq. 3.62 inversely, from diffusion experiments (Panofsky, 1962; Mickelsen, 1955: and Baldwin and Mickelsen, 1961).

Taylor's theorem can also be written in terms of the eddy-energy spectrum by combining Eq. 3.62 and the inverse transform of Eq. 3.66. It then follows that the mean-square diffusion, $\overline{y^2}$, is

$$\overline{y^2} = \overline{v'^2}\, t^2 \int_0^\infty F(n) \frac{\sin^2(\pi nt)}{(\pi nt)^2}\, dn \qquad (3.71)$$

It can be seen from this not only that $\overline{y^2}(t)$ depends on the entire energy spectrum, F(n), for any value of t but also that the larger t is, the more the diffusion is dominated by the low-frequency contributions to F(n). This follows because the spectrum in Eq. 3.71 is weighted by the function

$$W(n;t) = \left(\frac{\sin \pi nt}{\pi nt}\right)^2 \qquad (3.72)$$

which is largest for small values of nt and rapidly approaches zero for other values. The larger t is, the smaller n must become in order that this weighting factor differ much from zero. In other words, it appears that the large eddies (Fourier components of the motion having low frequency) dominate atmospheric diffusion when this is calculated with reference to a fixed source or axis. In this day and age of high-fidelity sound equipment, there will be general understanding of the statement that the diffusion process acts like a filter for high-frequency spectrum components, having the band-pass characteristic of Eq. 3.72.

3-2.2.3 Method of Moving Averages. Hay and Pasquill (1959) noticed that the integrand of Eq. 3.71 is similar in form to the expression by which a computed turbulence energy spectrum is corrected for the effect of averaging the raw data over a time interval a;

$$F(n)\left(\frac{\sin \pi na}{\pi na}\right)^2 = F_a(n) \qquad (3.73)$$

where $F_a(n)$ is the observed spectrum obtained from a wind-velocity-fluctuation record that has been averaged over the time interval a. The averaging might, for example, reflect the response characteristic of the particular anemometer used or the interval between diffusion measurements.

If the averaging interval is selected to be equal to the time of travel, or diffusion time, t, it follows that

$$\overline{y^2}(t) = \overline{v'^2}\, t^2 \int_0^\infty F_t(n)\, dn \qquad (3.74)$$

By definition $\overline{v'^2} \int_0^\infty F_t(n)\, dn$ is just the total turbulence energy contained in a velocity signal that has been subjected to a moving average over the time t. Thus Eq. 3.74 can be written

$$\overline{y^2}(t) = \langle\overline{v'^2}\rangle_t\, t^2 \qquad (3.75)$$

a form fully equivalent to the autocorrelation and spectrum forms, Eqs. 3.62 and 3.71. The symbol $\langle\,\rangle_t$ indicates that (one component of) the single-point Lagrangian velocity, v′, is to be subjected to a moving average over t prior to computation of the variance.

3-2.2.4 Sutton's Diffusion Model. From the limiting cases for small and large diffusion times of Taylor's theorem, Eqs. 3.64 and 3.68, it appears that the limit for large diffusion time may not be attained very rapidly since there is room in the atmosphere, at least in the horizontal direction, for quite large eddies to come into play. This fact led Sutton to propose his well-known model of averaged plume diffusion. Sutton (1953) reasoned that the Lagrangian single-particle autocorrelation function, $R(\xi)$, must depend only on the intensity of turbulence, $\overline{v'^2}$, on viscosity, ν, and on ξ. Since R(0) = 1 and $R(\infty) = 0$, he proposed on dimensional grounds the following simple interpolation formula for R:

$$R(\xi) = \left(\frac{\nu}{\nu + \overline{v'^2}\xi}\right)^n \qquad (0 < n < 1) \qquad (3.76)$$

If Eq. 3.76 is combined with Eq. 3.62 and if terms of the order of ν are ignored, it develops that

$$\overline{y^2}(t) = \frac{2\nu^n}{(1-n)(2-n)\overline{v'^2}}\,(\overline{v'^2}t)^{2-n} \qquad (3.77)$$

Defining a constant C_y^2, called by Sutton a virtual diffusion coefficient,

$$C_y^2 = \frac{4\nu^n}{(1-n)(2-n)\bar{u}^n}\left(\frac{\overline{v'^2}}{\bar{u}^2}\right)^{1-n} \qquad (3.78)$$

we find that

$$\overline{y^2} = \frac{1}{2}\, C_y^2(\bar{u}t)^{2-n} \qquad (3.79)$$

Sutton further introduced the concept of macroviscosity, $N = v_* z_0$, to replace the molecular viscosity, ν, for flow in the atmosphere in which the effect of molecular viscosity can be ignored.

Sutton originally studied diffusion in the lower few meters, in what we now call the surface layer. Since this region of the planetary boundary layer is characterized by marked vertical shear of the mean wind, the question might be raised whether Sutton's application of Taylor's result, which is based on the assumption of turbulence homogeneity, can conceivably be correct. Certainly the assumption of horizontal turbulence homogeneity at a fixed level is a reasonable one. Therefore it is also reasonable to expect that an expression of the form of Eq. 3.79 might apply in the atmosphere.

Sutton also assumed that similar expressions hold for $\overline{x^2}$ and $\overline{z^2}$. For example,

$$C_z^2 = \frac{4\nu^n}{(1-n)(2-n)\bar{u}^n}\left(\frac{\overline{w'^2}}{\bar{u}^2}\right)^{1-n} \tag{3.80}$$

and

$$\overline{z^2} = \frac{1}{2} C_z^2(\bar{u}t)^{2-n} \tag{3.81}$$

Notice that since $n > 0$ Sutton's expressions for $\overline{y^2}$ and $\overline{z^2}$ grow with time at a rate much more rapid than is true for Fickian diffusion (Eq. 3.68). In view of Eq. 3.71, such behavior could very well be a generally desirable property for an atmospheric diffusion model to have in some suitably restricted range of t, as Batchelor (1949) pointed out.

In order to introduce the effect of stability on the wind profile, it was originally assumed that n could be determined from the following relation:

$$\frac{\bar{u}_1}{\bar{u}_2} = \left(\frac{z_1}{z_2}\right)^{n/(2-n)} \tag{3.82}$$

where the subscripts refer to two different elevations. The justification for identifying n as a stability factor is that this exponent does exhibit a marked variation with stability. On the other hand, no satisfactory direct relation between n as defined by Eq. 3.82 and as defined by Eq. 3.76 is apparent since Eq. 3.76 assumes a homogeneous turbulence field that is true for Eq. 3.82 only if $n = 0$ and Eq. 3.76 involves Lagrangian wind statistics. Equation 3.82 involves the Eulerian wind field. Furthermore, for very large diffusion times, the autocorrelation defined by Eq. 3.76 must be questioned by the same argument that was used in deriving the limit for large time of Taylor's theorem. According to this argument the Lagrangian integral scale of turbulence corresponding to Eq. 3.76 is

$$\mathscr{L} = \int_0^\infty \left(\frac{\nu}{\nu + \overline{v'^2}t_1}\right)^n dt_1 = \infty \tag{3.83}$$

which by Eq. 3.67 implies infinite eddy energy density at zero frequency, i.e., $F(0) = \infty$. This is in conflict with Eq. 3.68 as well as with observed power spectra.

Notwithstanding these purely theoretical difficulties, Sutton's model has been widely proved in practice and sanctioned by usage. It should certainly be regarded as something better in the sense of being more useful, theoretically oriented, or physically motivated than, say, a purely empirical interpolation formula. But it should not be accorded the unequivocal status of a law of nature; it should be used with due regard for its several ad hoc features, and verification over some restricted range of distance and meteorological conditions should not be taken as an open invitation to an uncritical, universal application. Good verifications of diffusion predictions by Sutton's method have been obtained for distances of the order of several kilometers under neutral or unstable conditions.

Attempts have been made to extend the applicability of Sutton's scheme empirically to greater distances by introducing the separate parameters, n_y and n_z, for each direction (Schmidt, 1960; Leonard, 1957; and Barad and Haugen, 1959). Barad and Haugen were able to improve agreement considerably with data on diffusion from a source very near the ground while at the same time emphasizing the basically empirical nature of such extensions to Sutton's formulation.

3-2.2.5 A Similarity Theory of Diffusion in the Surface Layer. The statistical diffusion methods discussed so far depend on stationary, homogeneous turbulence. The planetary boundary layer, particularly the surface layer, however, is characterized by marked inhomogeneity of turbulence in the vertical direction as a result of wind shear and stability. Vertical inhomogeneity of the surface layer is taken into account in the K theories of Calder (1949), Deacon (1949), Frost (1948), Rounds (1955), and Smith (1957) by assuming some variation of

$\bar{u}(z)$ and consequently of K(z), usually a power law. This amounts to recognizing the problem of vertical inhomogeneity without solving it since the coefficients of the assumed power laws, or some related parameters of the problem, are invariably left to be determined from suitable observations. Ellison (1959), prompted by a remark by Batchelor (1959), applied a dimensional method to the determination of the diffusion downwind from a continuous point source in a logarithmic (adiabatic) surface layer. Batchelor (1959a) obtained the same results as Ellison (unpublished note; see also Batchelor, 1964). Subsequently Gifford (1962) attempted to extend the method to the diabatic surface layer, and Cermack (1963), Calder (1963), and Yaglom (1965) presented further results. The remaining paragraphs in this section outline the reasoning involved in these studies, which are important because they treat surface-layer diffusion without postulating a diffusivity.

The Eulerian (spatially fixed) characteristics of surface-layer turbulent flow are, as we have seen in the mean-wind field, particularly simple, being completely characterized by the friction velocity, v_*, and the stability length, L. Since any characteristic surface-layer velocity must therefore be proportional to v_* times a universal function of the dimensionless length, $\zeta = z/L$, Kazansky and Monin (1957) and Monin (1959) reasoned that the maximum vertical velocity of a smoke particle in a diffusing plume emanating from a source at ground level, w_*, must be given by

$$\frac{dz}{dt} = w_* = \lambda' v_* \, \varphi(\zeta) \qquad (3.84)$$

where $\varphi(\zeta)$ is a universal function, λ' is a universal constant, and z refers to the motion of a smoke particle at the upper boundary of the plume. It is reasonable to suppose that the equation for the horizontal velocity of a smoke particle at the upper plume boundary is $dx/dt = \bar{u}$; the shape of the upper boundary of the plume can be described (recalling Eq. 3.31) by

$$\frac{dx}{dz} = \frac{1}{k\lambda'} \frac{[f(\zeta) - f(\zeta_0)]}{\varphi(\zeta)} \qquad (3.85)$$

Monin evaluated the function $\varphi(\zeta)$ from the turbulent-energy-balance equation and found

$$\varphi(\zeta) = \left[1 - \frac{1}{f'(\zeta)}\right]^{1/4} \qquad (3.86)$$

Using Eq. 3.86 and suitable equations for $f(\zeta)$, Monin integrated Eq. 3.85 numerically to obtain the shape of the upper boundary of the plume from an infinite crosswind line source at ground level as a function of stability. A result similar to Monin's was obtained by Kao (1960) with, however, differences in the numerical values involved.

The concentration distribution of a diffusing plume is a statistical function of the Lagrangian (particle-attached) fluid velocities. If Lagrangian statistical properties of the surface-layer flow are assumed to obey the hypothesis of dynamical similarity, as do the above Eulerian properties, then we may proceed as follows.

Let the mean position of a particle be $\bar{x}(t)$, $\bar{y}(t)$, $\bar{z}(t)$. (If we choose the downwind direction to coincide with $\bar{x}$, then $\bar{y} \equiv 0$.) Assume that

$$\frac{d\bar{x}}{dt} = \bar{u}(\bar{z}) \qquad (3.87)$$

i.e., that at any point the horizontal part of the particle's motion equals the average wind speed (see, however, the discussion of this point by Yaglom, 1965). Following a dimensional line of reasoning, we can also conclude that the mean vertical velocity of a particle, $\bar{w}$, is given by

$$\frac{d\bar{z}}{dt} = \bar{w} = b \, v_* \, \varphi_1\left(\frac{\bar{z}}{L}\right) \qquad (3.88)$$

where φ_1 is a universal function that has usually been assumed to coincide with φ of Eq. 3.86 although this cannot be justified a priori and b is a universal constant. As Ellison (1957) pointed out, the role of z_0 in surface-layer turbulence is restricted to that of a horizontal translation of the mean flow, as in Eq. 3.87. Consequently neither $\bar{z}$ nor the concentration depend on z_0.

Now apply dimensional reasoning to determining the probability of a particle's reaching some distance $r = (x^2 + y^2 + z^2)^{1/2}$ from the mean particle position $(\bar{x},0,\bar{z})$. This is the same as inquiring what the concentration distribution, χ, would be following the instantaneous release of Q particles from the coordinate origin, where χ is measured from the mean particle position and averaged over a very large number of repetitions of the experiment and the x-axis is oriented along the mean wind direction. In addition to χ and Q, the relevant variables are the displacements, $x - \bar{x}$, $y - \bar{y} = y$, $z - \bar{z}$; the parameters characterizing the turbulence, v_* and L; and the mean time of particle travel to point $(\bar{x},0,\bar{z})$, t.

For the special case of the axial ground concentration, y = z = 0. From the remaining variables we can form the following:

$$F_3\left(\frac{\chi \bar{z}^3}{Q}, \frac{x-\bar{x}}{\bar{z}}, \frac{\bar{z}}{L}, \frac{v_* t}{L}\right) = 0 \qquad (3.89)$$

Solving for the ratio containing χ, we find

$$\frac{\chi}{Q} = \frac{1}{\bar{z}^3} F_4\left(\frac{x-\bar{x}}{\bar{z}}, \frac{\bar{z}}{L}, \frac{v_* t}{L}\right) \qquad (3.90)$$

To find the continuous-point-source axial concentration, $\bar{\chi}_p$, we would have to integrate Eq. 3.90 with respect to time from 0 to ∞. Since all the remaining dimensionless ratios are functions of time and nothing whatsoever is known about F_4, this becomes a difficult problem. In the adiabatic case, however, L = ∞ and $\varphi = 1$, and the integration can readily be performed:

$$\frac{\bar{\chi}_p}{Q'} = \int_0^\infty \frac{F_4\left(\frac{x-\bar{x}}{\bar{z}},0,0\right) dt}{\bar{z}^3} \qquad (3.91)$$

where Q′ is the continuous-source strength. By changing the integration variable, we find that

$$\frac{\bar{\chi}_p}{Q'} = \int_{-\infty}^{\infty} \frac{F_4\left(\frac{x-\bar{x}}{\bar{z}},0,0\right) d\left(\frac{x-\bar{x}}{\bar{z}}\right)}{bv_*\bar{z}^2\left(\frac{x-\bar{x}}{\bar{z}} + \frac{d\bar{x}}{d\bar{z}}\right)} \qquad (3.92)$$

At a sufficient distance downwind, the diffusing particles will be swept past any point rapidly compared with the time taken to reach that point, and we may assume that $x \approx \bar{x}$. With this simplification it is easily shown that

$$\frac{\bar{\chi}_p}{Q'} b\, v_* \propto \frac{1}{\bar{z}^2[(\bar{x}/\bar{z}) + (1/kb)]} \qquad (3.93)$$

where, from Eqs. 3.87 and 3.88,

$$\bar{x} = \frac{1}{kb}\left[\bar{z}\left(\ln\frac{\bar{z}}{z_0} - 1\right)\right] \qquad (3.94)$$

This is the result found by both Ellison and Batchelor.

By evaluating Eq. 3.93, these authors showed that in the adiabatic surface layer the downwind concentration from a continuous point source varies as x^p, where p varies approximately in the range −1.8 to −1.9. For a continuous, infinite crosswind line source, the downwind concentration was found to vary as x^{-1}. These results are quite interesting. They provide an alternative to Sutton's solution that leads to essentially the same result and, in the adiabatic surface layer, is known to be in excellent agreement with data. It is also interesting to note that the surface-layer diffusion proceeds at a rate quite close to the limiting prediction for homogeneous turbulence, Eq. 3.64, i.e., as x^{-2} (since $x = \bar{u}t$).

In the diabatic case the last two of the dimensionless ratios of Eq. 3.90 cannot be expected to disappear so conveniently. By, in effect, assuming that even in the diabatic case the function F_4 does not depend strongly on these two ratios, Gifford (1962) proposed that

$$\frac{\bar{\chi}_p}{Q'} \propto [\bar{u}(\bar{z})\,\bar{z}^2]^{-1} \qquad (3.95)$$

which is the counterpart to Eq. 3.93 for a nonadiabatic surface layer. For a relation between axial ground concentration and downwind distance, $\bar{x}$, to be obtained, a relation between $\bar{x}$ and $\bar{z}$ must be established. This follows from integration of Eq. 3.85 for $d\bar{x}/d\bar{z}$. Details have been given by Gifford (1962), and the results are in reasonably close agreement with detailed experimental atmospheric-concentration measurements. Wind-tunnel diffusion studies by Cermak (1963) have provided additional verification.

3-2.2.6 Relative Atmospheric Diffusion. Taylor's expression for diffusion measured from a fixed origin or axis, Eq. 3.62, is completely characterized by the statistics of the motion of a single fluid particle. The statistical averaging that Taylor had in mind was independent of any single particular realization of the experiment, that is, of any single set of initial conditions of the turbulent flow. The motions of any two or more fluid particles during a diffusion time t should be completely independent from the point of view of Eq. 3.62. But, if one is interested in the spreading out of an isolated cloud of fluid particles, this requirement cannot hold. Because the particles all start out together, the motions of particles in a cloud, puff, or cluster will at first be strongly correlated. In fact, if it is required that particles start out infinitely close together (an instantaneous-point-source condition), they will (in principle) never separate since at all times they are acted upon by the same fluctuation. Thus Richardson concluded that a spreading dot is unsuitable as a model for cloud diffusion. A second mode of diffusion, based on the rate of spreading of a cluster of fluid particles relative to their mutual center of gravity, must be calculated.

Consider two dispersing fluid particles, the simplest case of the diffusion of a cloud of n particles (Fig. 3.7). Stationary and homogeneous turbulence conditions are again assumed. One can calculate $\overline{Y^2}$, the mean-square value of the spreading, or the relative, diffusion. The procedure is exactly the same as for the calculation of $\overline{y^2}$ (Eq. 3.62), the single-particle dispersion parameter. The distance between the particles, Y, is given by

$$Y = y_1 - y_2 = Y_0 + \int_0^t v_1'(t_1)dt_1 - \int_0^t v_2'(t_1)dt_1 \qquad (3.96)$$

where Y_0 is the initial separation between the particles and the subscripts on y and v′ refer to the particles. The corresponding mean-square relative diffusion is

$$\overline{Y^2} = Y_0^2 + 2\,\overline{v'^2} \int_0^t \int_0^t R(t_2-t_1)\,dt_1\,dt_2 - 2 \int_0^t \int_0^t \overline{v_1'(t_1)\,v_2'(t_2)}\,dt_1\,dt_2 \qquad (3.97)$$

The detailed steps are essentially the same as those for the one-particle case. Generalization to a cloud of particles is given by Batchelor (1952).

Comparison of Eqs. 3.97 and 3.62 shows that the separation between two particles depends on two factors in addition to the single-particle Lagrangian time correlation, R(ξ). These are the initial separation, Y_0, and the relative (two-particle) Lagrangian correlation term, $\overline{v_1'(t_1)\,v_2'(t_2)}$. Notice that if two particles initially

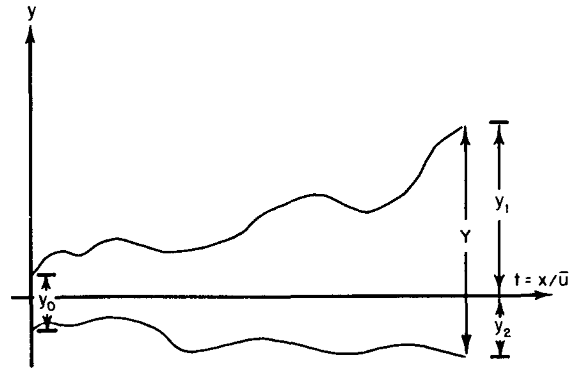


Fig. 3.7—Relative diffusion of two tagged particles (see Eq. 3.96).

occupy the same position in the fluid then $Y_0 = 0$ and $v_1' = v_2'$. As a result $\overline{Y^2} = 0$, and the particles will never disperse relative to one another. By Taylor's diffusion theorem, Eq. 3.62, they will, however, disperse on the average with respect to a fixed axis.

From the foregoing arguments it appears that relative diffusion, that is, the spreading out of a cloud of fluid particles or the spreading of a plume from its center line, is described by the joint Lagrangian statistics of two dispersing particles. Single-particle Lagrangian statistics, on the other hand, describe the average spreading of a plume about a fixed axis. The plume photographs shown in Fig. 3.8 (Culkowski, 1961) may make the distinction between relative and average dispersion clearer. Part a, Fig. 3.8, is an instantaneous (1/50 sec) exposure of a plume. The spreading of this plume relative to its irregular, undulating center line is described by Eq. 3.97. Part b, Fig. 3.8, is a 5-min time exposure of the same plume. The average diffusion about the horizontal plume center line, which is obviously oriented in the direction of the mean wind, is appropriately described by Taylor's diffusion equation (Eq. 3.62).

Qualitatively, relative diffusion should depend on the action of eddies approximately as large as a puff or, as in (a) of Fig. 3.8, as large as the width of the instantaneous plume. We have, on the other hand, noticed (Eq. 3.71) that average plume diffusion rapidly becomes dependent on quite large eddies. This distinction, first made by Richardson, was reemphasized by Yudine (1946), Brier (1950), and particularly by Batchelor in his definitive theoretical treatment (Batchelor, 1949, 1950, 1952). Richardson

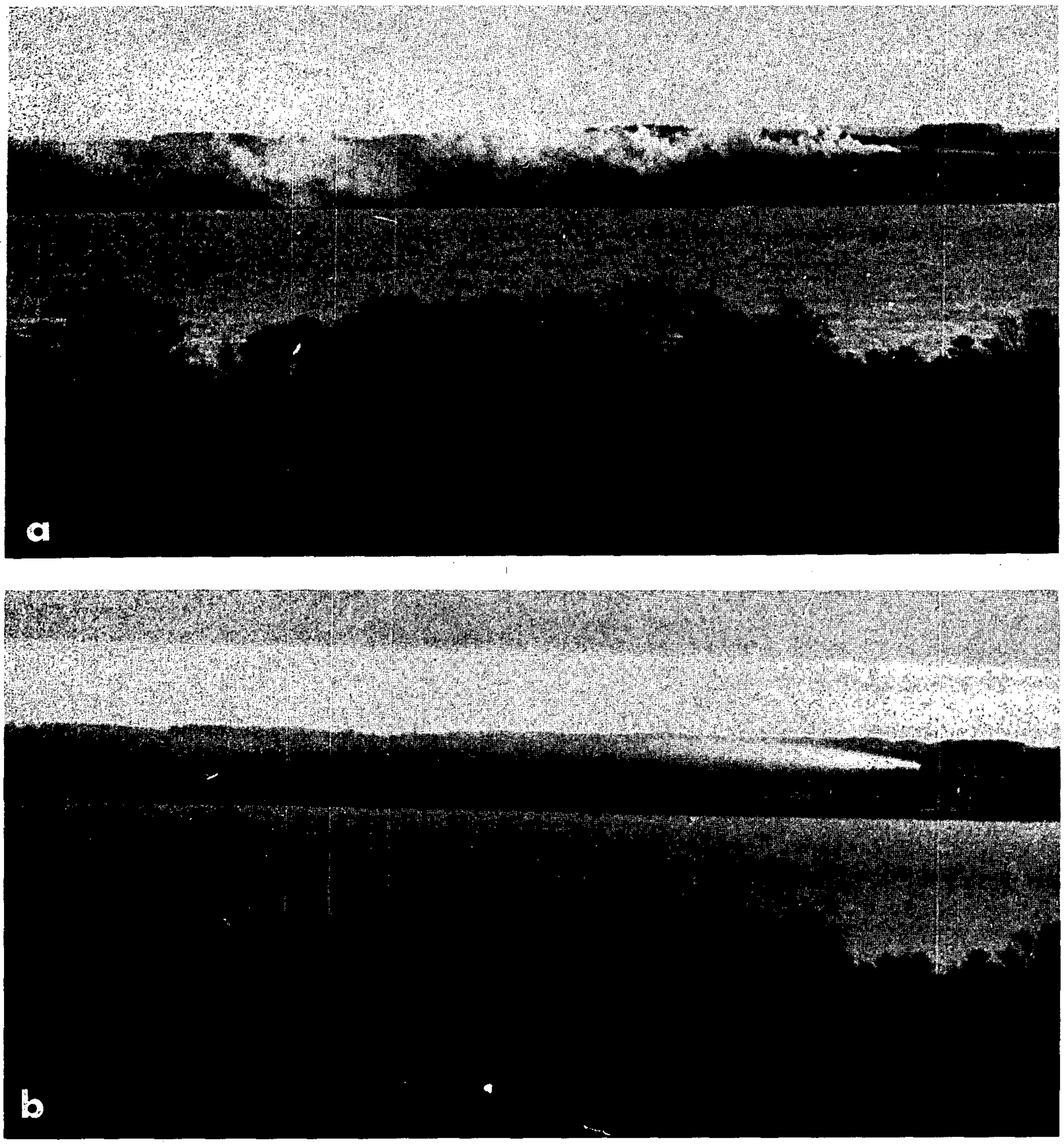


Fig. 3.8—Plume photographs. (a) Instantaneous ($^1/_{50}$-sec) exposure photograph of a plume. (b) Time exposure (5-min) of same plume. (From Culkowski, 1961.)

(1926) observed horizontal eddy diffusivities, K, of particle clusters with widely divergent sizes and arranged his K values according to a length scale, l, corresponding to the size of the clusters involved, as shown in Table 3.2. The K values clearly increase with l, and Richardson proposed the empirical equation $K = 0.2l^{4/3}$ to describe these observed values. Somewhat unexpectedly, the spreading of puffs or clusters appears to depend upon the scale of the diffusion event, i.e., on the separation between representative dispersing particles. A formal explanation for Richardson's discovery was given by Obukhov (1941), who pointed out that the law $K \propto l^{4/3}$ follows by a dimensional argument from the assumption that in the inertial range the structure of eddies governing cloud or cluster spreading is controlled by the rate of eddy-

Table 3.2—VALUES OF HORIZONTAL EDDY DIFFUSIVITIES AT VARIOUS SCALES*

Scale (l), cm	K, cm²/sec	Source of data
5×10^{-2}	1.7×10^{-1}	Molecular diffusion
1.5×10^{3}	3.2×10^{3}	Low-level wind shear
1.4×10^{4}	1.2×10^{5}	Low-level wind shear
5×10^{4}	6×10^{4}	Pilot balloons, 100 to 800 m
2×10^{6}	1×10^{8}	Manned and unmanned balloons
5×10^{6}	5×10^{8}	Volcanic ash
1×10^{8}	1×10^{11}	Cyclonic storms

*From Richardson, 1926.

energy transfer, ϵ (cm²/sec³); K (cm²/sec) must be proportional to $\epsilon^{1/3} l^{4/3}$.

The useful concept of an inertial range of eddies, i.e., a range of eddy sizes in which the properties of turbulence are dominated by the transfer of energy by inertial forces and are independent of viscous dissipation, was introduced by Kolmogorov (1941) and Obukhov (1941). Like so much of significance in atmospheric turbulence and diffusion theory, it has its origin in Richardson's early work. Richardson stated the basic idea in characteristically unorthodox form, the frequently quoted (Shaw, 1942; Sutton, 1949; and Batchelor, 1950, to give but a few examples) quatrain:

Great whirls have little whirls
That feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity.

These lines, a parody of a well-known verse by Swift, represent in all probability the only example of the statement of a fundamental physical principle in doggerel. There is ample room in the atmosphere for whirls (eddies) that are quite large, at least in their horizontal dimensions. Between the largest of these (possibly comparable to the scale of great cyclonic storms) and the smallest (the very small scale of viscous dissipation), there is a wide range of eddy sizes available for the eddy-energy cascade process described so neatly by Richardson's rhyme. In a sufficiently restricted portion of this size range, the turbulence properties must be independent of both the manner of energy supply to the large-scale eddies and the manner of eddy-energy dissipation at very small scales by viscosity. Consequently the turbulence properties in this range must be determined only by the rate of eddy-energy transfer, ϵ (cm²/sec³). This range of small-scale high-frequency eddies, lying in size just above the dissipative range of eddy sizes, is called the inertial range. The available evidence on the limits of the inertial range of eddy sizes in the atmosphere has been summarized by MacCready (1962). From his work we can conclude that eddies ranging in size from several times height above the surface down to well below the resolving power of ordinary wind-measuring equipment should be inertial in character.

Batchelor argued that in the inertial range the rate of relative diffusion, $d\overline{Y^2}/dt$, can depend only on the initial separation, Y_0, on the diffusion time, t, and on the rate of eddy-energy transfer per unit mass, ϵ. Furthermore, for t greater than some value t*, the diffusion rate will be independent of the initial separation. Provided this also occurs within the inertial range, purely dimensional considerations result in the following predictions concerning the rate of relative diffusion:

$$\frac{d\overline{Y^2}}{dt} \propto (\epsilon Y_0)^{2/3} t \qquad (t < t^*) \qquad (3.98)$$

$$\frac{d\overline{Y^2}}{dt} \propto \epsilon\ t^2 \qquad (t > t^*) \qquad (3.99)$$

$$t^* \approx Y_0^{2/3} \epsilon^{-1/3} \qquad (3.100)$$

Integrating Eqs. 3.98 and 3.99, we get

$$\overline{Y^2}(t) - \overline{Y^2}(0) \propto t^2 \qquad (t < t^*) \qquad (3.101)$$

$$\overline{Y^2}(t) \propto t^3 \qquad (t > t^*) \qquad (3.102)$$

Not much credence was attached at first to these predictions of relative diffusion since the extent of the inertial range of eddy sizes was thought to be quite small. Until very recently researchers have attempted to explain the diffusion of puffs near the surface or the instantaneous spreading of a plume by applying Eq. 3.62. Note that Eq. 3.64 should never predict diffusion any faster than t^2, in marked contrast to the prediction of Eq. 3.102. The reanalysis of data on puff spreading and concentration from several experiments by Gifford (1957) has confirmed the validity of the relative-diffusion predictions in the atmosphere.

By studying the relative accelerations rather than the velocities of particle pairs, Lin (1960, 1960a) derived a relative diffusion law of the same form as Eq. 3.102 without, however, making explicit use of the inertial-range concept. Lin found that

$$\overline{Y^2} = \frac{2}{3} Dt^3 \qquad (3.103)$$

where D is a quantity having the dimensions of energy dissipation (ϵ), i.e., cm^2/sec^3. The proof is quite similar to that of Taylor's diffusion theorem. Since Eq. 3.103 is not restricted to the inertial range, it is presumably valid over a greater spatial domain and may contain the explanation for Richardson's empirical diffusion law, which can be derived from it. Furthermore D is a Lagrangian parameter, i.e., it arises in a Lagrangian description of diffusion; whereas ϵ is Eulerian in nature. Consequently Lin's result appears to be a conceptual improvement. Smith and Hay (1961) also studied relative diffusion by assuming a certain form of the relative-velocity correlation and assuming that the material distribution in a cluster is Gaussian. They derived a particularly simple relative-diffusion formula that has been useful in several field studies (see Chap. 4).

It is interesting that the concept of relative diffusion has been employed by oceanographers to explain the spreading of dye patches on the sea surface (see the excellent summary of this area of research by Okubo, 1962). Relative diffusion has also been invoked in discussing the spreading of sodium vapor trails in the lower ionosphere (e.g., Coté, 1963, and Zimmerman and Champion, 1963).

3-2.3 The Problems of Averaging

Two distinct kinds of averages have so far been used in discussing diffusion: the time average of the instantaneous turbulent velocity field, on which Reynolds' average-wind definition was based, and the statistical, or ensemble, average, which was introduced in connection with Taylor's diffusion theory. In addition, a distinction must be made between two distinct systems of reference before averaging can be performed. These are the spatially fixed, or Eulerian, system and the particle-attached, or Lagrangian, system.

Eulerian coordinates can be fixed with respect to a certain location, e.g., an anemometer. Consequently the Eulerian fixed-point system is the natural system for experimentalists to use. On the other hand, the Eulerian reference frame can be thought of as attached to and moving along with the mean wind. In this system the mean wind components vanish, and only the turbulent components remain. Of these, fluctuations in either space or time or a combination of both can be discussed. Most of the statistical theory of turbulence is developed in the Eulerian space or space–time system.*

3-2.3.1 Taylor's Hypothesis. Because the mean wind speed in wind-tunnel flows is very large compared with the root mean square of the turbulent fluctuations, Taylor (1938) proposed transforming from the experimentally convenient Eulerian fixed-point coordinates to the Eulerian space scheme by introducing the transformation $x = \bar{u}t$, where x is the distance covered in t seconds. This transformation is valid in wind-tunnel work, where $\bar{u} \gg (\overline{v'^2})^{1/2}$. In

*A suitable designation of these various Eulerian frames of reference has caused meteorologists some difficulty. For example, it has been suggested that what we here call the "Eulerian time" system, in accordance with the usage of fluid-turbulence theoreticians, should be called a "pseudo-Lagrangian" system (Pasquill, 1963) or a "pseudo-Eulerian" system (Frenkiel, 1948). Adding to the confusion, what is here termed the "Eulerian fixed-point" system has commonly been designated the "Eulerian time" system in meteorological literature; Pasquill (1963), on the other hand, proposes calling it the "quasi-Eulerian" system.

Without pretending that it solves all possible problems of turbulence nomenclature, I urge that meteorologists use the system suggested in the text above for the following reasons:

1. The terms "Eulerian" and "Lagrangian" should refer only to the basis of the coordinate system. If this is particle attached, the term "Lagrangian" is appropriate; the term "Eulerian" is properly applied to all other cases without the need for qualifying prefixes of questionable relevance.

2. The term "fixed point" unambiguously characterizes the commonest type of Eulerian reference or measurement system, that in which the measuring probe is located at a fixed point in space.

3. The term "Eulerian time" should mean the same thing to specialists in both atmospheric and wind-tunnel turbulence; moreover, it should be intelligible as a special case of the term "Eulerian space–time," which is generally understood to apply to a reference system that is at rest with respect to the mean flow.

the atmosphere, where $\bar{u} \approx (\overline{v'^2})^{1/2}$ and the mean wind may vary, the applicability of this transformation is not so obvious although it has been widely employed. It is probably valid, at least for turbulence fluctuations of comparatively high frequency.

3-2.3.2 Eulerian-Lagrangian Averages. Before wind-fluctuation statistics can be applied to the diffusion problem, a method must be devised to convert these statistics, measured at a point by an anemometer, bivane, or other device, into the corresponding Lagrangian values that apply to the motion of a fluid particle. This problem, which in principle is a purely mathematical one, is notoriously difficult. Hay and Pasquill (1959) suggested as a working approximation that the Langrangian time, ξ, is approximately linearly related to the time, t, of the Eulerian fixed-point reference system, i.e.,

$$\xi = \beta t \qquad (3.104)$$

where β is a dimensionless Lagrangian–Eulerian time-scale ratio. This proposal is closely related to the result of an earlier study by Gifford (1955), who showed that

$$\frac{n_E}{n_L} = \frac{\xi}{t} = \frac{1.12\,\bar{u}}{(\overline{v'^2})^{1/2}} + 1 = \beta \qquad (3.105)$$

where n_E and n_L are Eulerian fixed-point and Lagrangian frequencies, respectively, referring to the corresponding energy spectra. It appears that by an order-of-magnitude approximation of the turbulence intensity, $(\overline{v'^2})^{1/2}/\bar{u}$, we can expect $2 \le \beta \le 12$. In fact Hay and Pasquill (1959), on the basis of a series of eight short-range low-level diffusion observations, computed values of β ranging from 1.1 to 8.5.

Values of β can be computed directly from diffusion observations by introducing the scale transformation Eq. 3.104 into Eq. 3.71,

$$\overline{y^2}(t) = \overline{v'^2}\, t^2 \int_0^\infty F_E(n) \left[\frac{\sin(\pi n t/\beta)}{\pi n t/\beta}\right]^2 dn \qquad (3.106)$$

where $F_E(n)$ is the Eulerian energy spectrum corresponding to velocities v′(t) measured at a fixed point. The integrand in Eq. 3.106 is evidently equivalent to the spectrum of a fixed-point Eulerian velocity record that has been averaged over a time interval t/β. If the total Lagrangian turbulence energy, $\overline{v'^2}$, were equal to the corresponding value for the fixed-point Eulerian velocities, Eq. 3.106 would be equivalent to

$$\overline{y^2}(t) = \langle \overline{v'^2}(t) \rangle_{t/\beta}\, t^2 \qquad (3.107)$$

by the same argument that led to Eq. 3.75. For incompressible, stationary, and homogeneous turbulence conditions, the equality was proved by Lumley (1957). Observations of diffusion can be compared to $\overline{y^2}(t)$ as calculated from the Eulerian fixed-point wind-fluctuation moving-average variances of Eq. 3.107, $\langle \overline{v'^2} \rangle_{t/\beta}$, for various values of t/β. Since the diffusion time, t, is known from the distance involved in the experiments ($x = \bar{u}t$), this procedure determines β.

In addition to data on diffusion over a length scale of several hundreds of meters, Hay and Pasquill (1959) also examined diffusion data on a scale of a thousand miles (Durst, Crossley, and Davis, 1957) and on the very small scale of wind-tunnel diffusion (Mickelsen, 1955). For both these sets of data, the computed β values lie in the range 1 to 10, and so it appears that β "is evidently at least of the same order for an enormous range in the scale of turbulence" (Hay and Pasquill, 1959). Thus the practical utility of this simple Lagrangian–Eulerian transformation seems on the whole to be quite well established. Further comparisons of atmospheric diffusion and wind-fluctuation data indicate that a lower limit, e.g., $\beta = 1$, should be employed in unstable conditions and a β value approaching an upper limit of 10, in stable conditions. Haugen (1960) has reported a tendency for computed β values to increase with distance from the source in stable conditions. Wippermann, Gburcik, and Klug (1962) found β values less than unity for diffusion on a hemispheric scale. Fortunately, as Pasquill (1962) pointed out, moderate departures from the average value of $\beta = 4$ have little practical effect on diffusion estimates.

3-2.3.3 Finite Sample–Infinite Sample Averages. It would seem, ideally, that samples obtained over very long times or throughout very large volumes might be substituted for the ensemble averages demanded by statistical turbulence theory provided the turbulence fluid flow possesses stationary, homogeneous statistical properties. Practically, however, such ideally long samples are not ordinarily obtained in the atmosphere, either because of the difficulty of making and analyzing extensive observations or because some change in the external flow situation violates the stationarity condition. Frenkiel (1952a) discussed the latter problem and provided an interesting example of the marked changes that can actually occur. The lower layers of the atmosphere rarely maintain a state of turbulence that approximates a statistically stationary condition for more than, perhaps, a few hours. Even if gross changes in the large-scale wind field (passage of a front, onset of a sea breeze, etc.) do not occur, the duration of a quasi-stationary condition is limited to a few hours by the marked diurnal variation of low-level turbulence. For this reason it is essential to be able to form some idea about the effect of finite sampling periods on turbulence statistics.

The effect of finite sampling on turbulence statistics has been studied by Ogura (1957), Kahn (1957), Pasquill (1962), and Smith (1962) in terms of correlations or spectra. The principal result of these studies, given by both Ogura and Pasquill, is the following expression for the ensemble average diffusion over a diffusion time t computed with respect to a sampling period T:

$$\overline{y_T^2}(t) = \overline{v'^2}\, t^2 \int_0^\infty F(n) \left\{ \left[1 - \frac{\sin^2 \pi nT}{(\pi\, nT)^2} \right] \times \frac{\sin^2 \pi nt}{(\pi nt)^2} \right\} dn \qquad (3.108)$$

As T becomes large, this equation reduces to Eq. 3.71. Pasquill has pointed out that the term multiplying F(n), amounts to a filtering of the spectrum, which effectively suppresses the contributions to diffusion from spectral frequencies much higher than 1/t and lower than 1/T (see Pasquill, 1962, Sec. 1-4 for a complete discussion). The interpretation of finite diffusion in terms of spectrum filtering (Eq. 3.108) suggested to Jones and Pasquill (1959) the so-called "sigma meter," a very useful and practical device for estimating diffusion from wind-fluctuation records (Chap. 6, Sec. 6-4.2.2).

Some sort of assumption about the functional form of the turbulence statistics, i.e., autocorrelation, spectrum, or running-mean variance, must be made before more specific theoretical results can be obtained. (The reader is referred to Ogura's study for an idea of what can be accomplished along this line.) In such practical applications as the analysis of diffusion experiments, the opportunity to perform an averaging that corresponds even remotely to the ensemble average usually does not exist. Nevertheless the above discussion gives at least a qualitative idea of the effect on the mean-square diffusion, $\overline{y^2}$, of the relation between diffusion time and time of sampling so far as departures from ideal diffusion statistics are concerned.

The experimentalist, of course, wants to know how to interpret this discussion in practical terms. There seems to be no better guide than ordinary statistical sampling practice combined with common sense. For example, suppose we have a record of the transverse wind-velocity fluctuation, v′, over a period of time equal to T from which we wish to estimate the Eulerian fixed-point autocorrelation, $R_E(t)$. We would probably restrict t to values of about $\frac{1}{20}$ or $\frac{1}{10}$ of T at most in order to have reasonably well-behaved statistics; even so our confidence in the computations of R_E for t in the neighborhood of T/10 would be very low indeed. Naturally we would also require that no gross changes in the character of the turbulence had occurred during this period T that would violate the stationary condition. This means that the record should not reflect the passage of frontal systems, changes from land to sea breeze or valley to mountain wind regimes, changes from mechanical to convective turbulence or from stable to unstable conditions, or any other marked disturbance of the external forces driving the turbulence.

Similarly, we might wish to interpret measurements of the path of a single floating balloon in terms of Lagrangian turbulence statistics, as has recently been proposed by Angell (1963) and by Pack (1962). For this purpose we would probably employ segments no longer than $\frac{1}{20}$ to $\frac{1}{10}$ of the total length of an observed balloon trajectory so that the estimated autocorrelation, R(ξ), or diffusion, $\overline{y^2}$, for example, would have reasonable statistical stability. We would likewise make certain that none of the occurrences that affect the external forces driving the turbulence had taken place during the run. Furthermore, to combine various segments of a single trajectory statistically, we would have to ascertain whether or not the segments were located over similar underlying terrain. All these factors impose very serious and very real limitations on our ability to perform reproducible turbulence or diffusion experiments in the lower atmosphere, particularly at large scales, and it is essential to keep them in mind when planning, making, or interpreting such experiments.

3-3 ATMOSPHERIC DIFFUSION MODELS AND APPLICATIONS

3-3.1 The Gaussian Plume Diffusion Model

The object of all the preceding discussion has been to arrive at useful mathematical formulas describing atmospheric diffusion. The main theories of atmospheric diffusion have now been mentioned, and we have seen that the well-known normal, or Gaussian, distribution function provides a fundamental solution to the Fickian diffusion equation. The Gaussian distribution has been assumed as a continuous-source diffusion model by Sutton (1932), Frenkiel (1953), and many other workers. Combination of the Gaussian assumption with one of the following expressions for the mean-square particle diffusion,

$$\overline{y^2} = 2Kt \qquad (3.109)$$

$$\overline{y^2} = \overline{(v')^2}\, t^2 \qquad (3.110)$$

$$\overline{y^2} = \frac{1}{2} C_y^2 (\bar{u}\, t)^{2-n} \qquad (3.111)$$

(and similar expressions for $\overline{x^2}$ and $\overline{z^2}$) forms the basis for most of the practical plume-diffusion formulas that are found in the literature on applications.

Strictly speaking, the Gaussian diffusion model applies only in the limit of large diffusion time and for homogeneous, stationary conditions, for which, we have observed, the diffusion problem may be stated in the form of the simple Fickian differential equation. Batchelor (1949) conjectured, however, that the Gaussian function may provide a general description of average plume diffusion because of the essentially random nature of this phenomenon by analogy with the central limit theorem of statistics. Lin and Reid (1963) pointed out that for very small diffusion times the distribution of particles should take the same form as the wind-fluctuation distribution since the particle trajectories coincide with the instantaneous wind; in the atmosphere this approximates a Gaussian distribution fairly closely. Moreover recent experimental diffusion studies by Hay and Pasquill (1957), Cramer, Record, and Vaughan (1958), and Barad and Haugen (1959) indicate that the Gaussian plume formula should have a wide area of practical applicability in the atmosphere.

The usual way of deriving average-plume-diffusion formulas starts with the assumption of an instantaneous point source of material diffusing in three dimensions. The source strength is Q in grams or curies; the concentration is $\chi = \chi(x,y,z,t)$; x, y, and z are the usual coordinate axes, the point (0,0,0) being a fixed origin; and t is the time of travel of the cloud. If σ_y^2 is the variance of the distribution and if it is assumed that $x = \bar{u}t$, which makes the σ's functions of x, then the Gaussian formula for an instantaneous point source of material is

$$\chi(x,y,z,t) = Q(2\pi\sigma_y^2)^{-3/2} \exp(-r^2/2\sigma_y^2) \qquad (3.112)$$

where $r^2 = [(x-\bar{u}t)^2 + y^2 + z^2]$, and it is assumed, for the moment, that $\sigma_y = \sigma_x = \sigma_z$, i.e., that the diffusion is isotropic. If this may not be assumed, as is clearly the case under stable meteorological conditions or in the presence of boundary effects, it is usually assumed that the diffusion takes place independently in the three coordinate directions. Then

$$\chi(x,y,z) = \frac{Q(2\pi)^{-3/2}}{(\sigma_x\sigma_y\sigma_z)} \exp\left\{-\left[\frac{(x-\bar{u}t)^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right]\right\} \qquad (3.113)$$

Equations 3.112 and 3.113 have been written in terms of the standard deviation symbols σ_x, σ_y, and σ_z to stress the following point. According to its derivation Taylor's diffusion function, $\overline{y^2}$, specifically applies to the one-dimensional problem. When it is used to describe the average diffusion of a real three-dimensional cloud, it correctly describes the diffusion of the marginal projection on the y-axis of this cloud. Likewise, $\overline{x^2}$ and $\overline{z^2}$ must be regarded as applying to marginal distributions on their respective axes. Consequently Eqs. 3.112 and 3.113 contain the implicit assumption that the distribution of the diffusing cloud is, in the terminology of mathematical statistics, jointly as well as separately normal. This is equivalent to the assumption that cross-product terms such as $\overline{yz}$ do not contribute to diffusion. As indicated in Sec. 3-2, Batchelor extended Taylor's theory formally to provide a general theoretical expression for the diffusion tensor, including such terms as $\overline{yz}$; but such terms will naturally depend on Lagrangian correlations more complicated than R(ξ). If we assume joint normality, we may write the diffusion equations with $\sigma_y^2 = \overline{y^2}$, and so on.

A further restriction to the applicability of Eqs. 3.112 and 3.113 in connection with Eqs. 3.109, 3.110, and 3.111 follows from the discussion of the phenomenon of relative diffusion in Sec. 3-2. In principle these formulas may not be conceived as describing the spreading of a single puff of material or of an ensemble of puffs relative to their centers of mass. Application of these equations to such puff, or cluster, spreading is valid only when the average diffusion is calculated over an ensemble of puff experiments relative to a fixed axis. By Eq. 3.64 the maximum rate of average diffusion from a fixed axis is proportional to t^2, but by Eq. 3.102 the average diffusion about the center of mass of a puff can be as great as t^3 when it occurs in the inertial range. Consequently, in principle, quantitative errors can result if the two phenomena are confounded. In fact data summarized in Fig. 4.38 show that puffs do have a somewhat greater growth rate, particularly at shorter distances (i.e., at smaller times), than do plumes (Fig. 4.21).

The method of obtaining a continuous-point-source diffusion formula from Eq. 3.112 or 3.113 proceeds according to the principle of superposition. The plume is regarded as resulting from the addition of an infinite number of overlapping averaged puffs, carried along the x-axis by the mean wind, $\bar{u}$, as in (a) of Fig. 3.9. Each puff is in reality composed of the average over an ensemble of puffs which have diffused for a time t and consequently have reached the position (x,0,0). Mathematically this corresponds to integration of Eq. 3.113 with respect to t from 0 to ∞. This integration is not convenient because the values of σ, in general, depend on t and hence on x because $x = \bar{u}t$. As a practical matter, diffusion along the x-axis is always neglected by comparison with the gross transport along the x-axis by the mean wind, producing what Frenkiel (1953) has termed the spreading-disk diffusion model for a continuous point source, (b) of Fig. 3.9. With this simplifi-

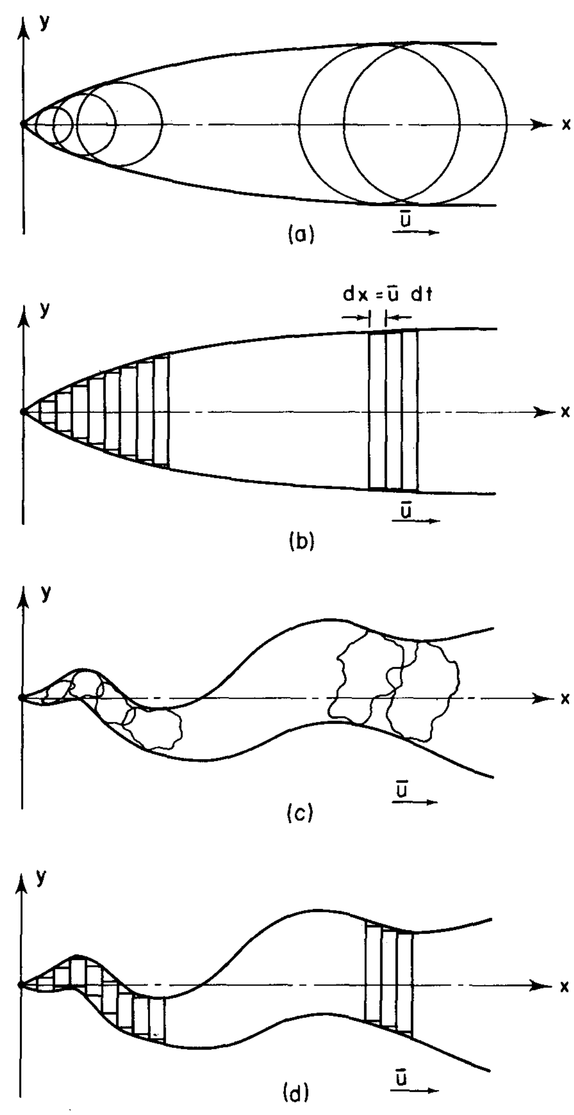


Fig. 3.9—(a) Schematic formation of plume from superposition of individual averaged elements. (b) Schematic spreading-disk plume model obtained by neglecting x-diffusion. (c) Appearance of naturally occurring plumes, with "real" puff elements indicated. (d) Fluctuating plume model.

cation integration of the equation can readily be carried out:

$$\frac{\bar{\chi}(x,y,z)}{Q'} = (2\pi\sigma_y\sigma_z\bar{u})^{-1} \times \exp\left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right)\right] \quad (3.114)$$

where $\sigma_y \equiv (\overline{y^2})^{1/2}$ and $\sigma_z = (\overline{z^2})^{1/2}$. The continuous-source strength, Q', is in grams or curies per second, and the quantities σ_y and σ_z can now be regarded as functions of x.

Since most isolated continuous sources are located at or near the earth's surface, it is necessary to account for the presence of this physical barrier to the flux. This has usually been done by the technique, borrowed from heat-conduction theory, of assuming an image source located symmetrically, with respect to the ground plane, to the actual source. The result is

$$\frac{\bar{\chi}}{Q'} = (2\pi\sigma_y\sigma_z\bar{u})^{-1} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) \times \left\{\exp\left[-\frac{(z-h)^2}{2\sigma_z^2}\right] + \exp\left[-\frac{(z+h)^2}{2\sigma_z^2}\right]\right\} \quad (3.115)$$

where h is the elevation of the source above the ground plane. If the receptor is located at the ground level ($z = 0$), then

$$\frac{\bar{\chi}}{Q'} = \frac{1}{\pi\sigma_y\sigma_z\bar{u}} \exp\left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{h^2}{2\sigma_z^2}\right)\right] \quad (3.116)$$

which is the form of the Gaussian plume model most usually used.

By combining Eqs. 3.109 through 3.111 with Eq. 3.114, we can obtain the continuous-point-source diffusion formulas of Roberts, Frenkiel, and Sutton. The same generalized Gaussian plume equation, moreover, serves as a useful interpolation formula for the interpretation of field diffusion trials, as in Cramer's studies (1957, 1959). Cramer combined Eq. 3.116 with the assumed power laws

$$\sigma_y \propto x^p \quad (3.117)$$

and

$$\sigma_z \propto x^q \quad (3.118)$$

and obtained best fits to the Prairie Grass and Round Hill diffusion data (Chap. 4, Sec. 4-4.2.1).

Barad and Fuquay (1962) compared several detailed plume-concentration measurements made under very stable atmospheric conditions with the bivariate normal distribution function. In their study the usual implicit assumption, discussed previously, that the distribution of diffusion in the y- and z-directions is jointly as well as separately normal, was not made. Their results indicate that the plume distribution was not jointly normal in this extreme case but that the vertical and horizontal distributions are separately normal.

3-3.2 A Fluctuating Plume Model

The spreading-disk plume model of Eq. 3.116 describes diffusion averaged over some period of time. Practical experience indicates that this period of time is at least several minutes (see Fig. 3.8). The appearance of real plumes is quite different from that predicted by this assumed model, especially during unstable conditions when the entire plume, at any instant, meanders or fluctuates about some mean position, as in (c) of Fig. 3.9. Accordingly, a fluctuating plume model has been proposed (Gifford, 1958, 1959a) which differs from Eq. 3.114 in that the centers of the disk elements are conceived of as distributed at random around their mean position, (d) of Fig. 3.9. The basic Gaussian equation for the instantaneous concentrations is

$$\frac{\chi}{Q'} = (2\pi\overline{Y^2}\bar{u})^{-1} \times \exp\left[-\frac{(y-D_y)^2 + (z-D_z)^2}{2\overline{Y^2}}\right] \quad (3.119)$$

In this equation D_y and D_z are distances to the center of the instantaneous plume from the axis; D_y and D_z are assumed also to possess Gaussian distributions with variances $\overline{D^2}$. As defined in Sec. 3-2, $\overline{Y^2}$ is a relative diffusion parameter.

The mean value of χ/Q', assuming $\overline{D_y^2} = \overline{D_z^2} = \overline{D^2}$, is found to be

$$M\left(\frac{\chi}{Q'}\right) = [2\pi\bar{u}(\overline{Y^2} + \overline{D^2})]^{-1} \times \exp\left[-\frac{r^2}{2(\overline{Y^2} + \overline{D^2})}\right] \quad (3.120)$$

where $r = (y^2 + z^2)^{1/2}$. A similar result was obtained by Hilst (1957). Extension to the case where $\overline{D_y^2} \neq \overline{D_z^2}$ is straightforward. The mean value of χ/Q' has exactly the same form as Eq. 3.114, but the diffusion has been separated into a mean and a fluctuating part. In addition to the mean values, more complex statistics, in particular the variance and the distribution of χ/Q', can be calculated (Gifford, 1959a). Further results based on this model have been presented in papers by Moore (1963) and Scriven (1965).

3-3.3 Remark on Non-Gaussian Diffusion Models

The virtues of the Gaussian distribution function are considerable, and the temptation to employ it exclusively is correspondingly great. Statistically it is completely determined by its second moment, i.e., by σ^2. It has many highly useful purely mathematical properties; for instance, it possesses a self-reciprocal Fourier transform. Moreover, as noted earlier, it agrees reasonably well with much, although not all, of presently available atmospheric diffusion data. Non-Gaussian diffusion distributions arise from the various K theories and also from the statistical diffusion theories of Goldstein, Monin, and Davies mentioned briefly in Sec. 3-2. It is natural to ask whether these may not be better than the Gaussian model discussed at some length in the previous section. For example, we might ask whether non-Gaussian diffusion models agree better with diffusion observations. Elliott (1960) compared the Prairie Grass data with Calder's non-Gaussian K theory diffusion model and with Sutton's Gaussian model. His conclusion is that, although Sutton's model gives a slightly better fit to the Prairie Grass data, the differences are quite small from any practical point of view. Pasquill (1962) also showed that the resulting plume center-line concentration formulas of Calder's model and Monin's (1959) limited-diffusion velocity model, differ but little.

On the other hand, the basic theoretical point emphasized by Monin (1959) and others, namely, that the speed of a real diffusion event like the spreading out of a smoke plume must necessarily be less than some finite value, such as the speed of sound, is certainly correct in principle. It should be clearly understood, however, that anomalies in diffusion data, such as those arising from the presence of marked wind shear (Barad and Fuquay, 1962) or the irregular departures from smooth concentration contours noted by Elliott (1959), are not to be explained as an effect of finite speed of diffusion. Because Gaussian plume models have proved to be, by and large, reasonably successful in explaining observed concentration patterns, it seems reasonable to continue to employ them in practice.

3-3.4 Estimation of Diffusion Coefficients

For practical use to be made of diffusion formulas numerical values for the diffusion coefficients σ_y and σ_z must be determined. Various theoretical expressions were derived for this purpose, particularly Eqs. 3.64, 3.68, 3.75, 3.79, and 3.81. Equation 3.64, corresponding to the limiting case of Taylor's formula for small diffusion times, has been used by Frenkiel (1952, 1952a, and 1953). It undoubtedly gives reliable predictions for diffusion times up to at least a few minutes. At the opposite limit Eq. 3.68, corresponding to the case of large diffusion times, has been used to solve the problem of diffusion on scales ranging from continental to global (Machta, 1958). Values of K appropriate to various scales were given in Table 3.2. Equations 3.79 and 3.81, Sutton's model, have frequently been applied in reactor-hazard analyses and air-pollution studies, and there has been considerable experience with Sutton's diffusion coefficients. The theoretical limitations of this model have been discussed in Sec. 3-2, and examples of observed parameter values are noted in Chap. 4.

Equation 3.75, the moving average variance method, seems a promising development in that (1) it specifies diffusion coefficients by a detailed analysis of atmospheric turbulence measurements, (2) it does not involve adjustable empirical constants, (3) it is comparatively free from debatable physical assumptions, and (4) it is not, in principle, limited to a particular range of diffusion times. It is, however, limited as are all applications of Taylor's theorem, Eq. 3.62, to stationary, homogeneous turbulence conditions. The condition of homogeneity in particular limits its effectiveness in estimating vertical diffusion from sources near the ground. It also appears practically desirable to be able to estimate diffusion coefficients from meteorological data more universally available than detailed wind-fluctuation measurements or even just these 10% values. The 10% value is only an estimate. It may or may not apply to smoke plumes in general; the point has never been studied exhaustively. Moreover, since many of the clouds and plumes that interest us are in-

Table 3.3—RELATION OF TURBULENCE TYPES TO WEATHER CONDITIONS

A— Extremely unstable conditions
B— Moderately unstable conditions
C— Slightly unstable conditions
D— Neutral conditions*
E— Slightly stable conditions
F— Moderately stable conditions

Surface wind speed, m/sec	Daytime insolation			Nighttime conditions	
	Strong	Moderate	Slight	Thin overcast or $\geq 4/8$ cloudiness†	$\leq 3/8$ cloudiness
<2	A	A–B	B		
2	A–B	B	C	E	F
4	B	B–C	C	D	E
6	C	C–D	D	D	D
>6	C	D	D	D	D

*Applicable to heavy overcast, day or night.
†The degree of cloudiness is defined as that fraction of the sky above the local apparent horizon which is covered by clouds.

to make diffusion estimates based on only a general knowledge of a location. The various series of field-diffusion experiments described in Chap. 4 provide considerable guidance for such estimates.

3-3.4.1 Pasquill's Diffusion Curves. On the basis of available data, including the Prarie Grass experiments, and guided by theoretical expectations, Pasquill suggested in an unpublished note in 1958 a practical scheme for the estimation of diffusion which is particularly suitable for practical applications. The substance of this note is contained in the papers by Meade (1959, 1960) and by Pasquill (1961, 1962). The general idea can, as well, be expressed in terms of σ_y and σ_z; moreover it can be related to results derived earlier in this chapter.

The visible edge of a diffusing cloud has often been assumed to coincide roughly with the lateral point at which the concentration falls to 10% of its axial value and could, in any event, be defined as this point, as was done by Pasquill (1961) and Holland (1953), for example. For smoke screens the visible smoke-plume edge is approximated by this figure (Gifford, 1959). Pasquill and Meade define a smoke-plume elevation, H, and an angular spread, θ, which are visible, there appears to be no special virtue to this definition. If, instead, we define plume concentration distributions in terms of their standard deviations, we find, in Pasquill's notation, that,

$$H = 2.14\, \sigma_z \qquad (3.121)$$

and, for fairly small values of θ,

$$\theta = \frac{4.28\, \sigma_y}{x} \qquad (3.122)$$

The numerical coefficient 2.14 is just the 10% ordinate of the normal error curve.

Figures 3.10 and 3.11 exhibit families of curves of σ_y and σ_z for various stability categories, based on the values of H and θ given originally by Pasquill. The manner of relating these curves to prevailing conditions of average wind speed and to the estimated radiation balance is set out in Table 3.3, which was also presented in the papers by Pasquill and Meade. An evaluation of Eq. 3.116 for various values of stack height, employing the σ_y and σ_z values of Figs. 3.10 and 3.11, has been carried out by Hilsmeier and Gifford (1962) (these results are reproduced in Sec. A.3 of the Appendix). The studies by Beattie (1961), Couchman (1961),

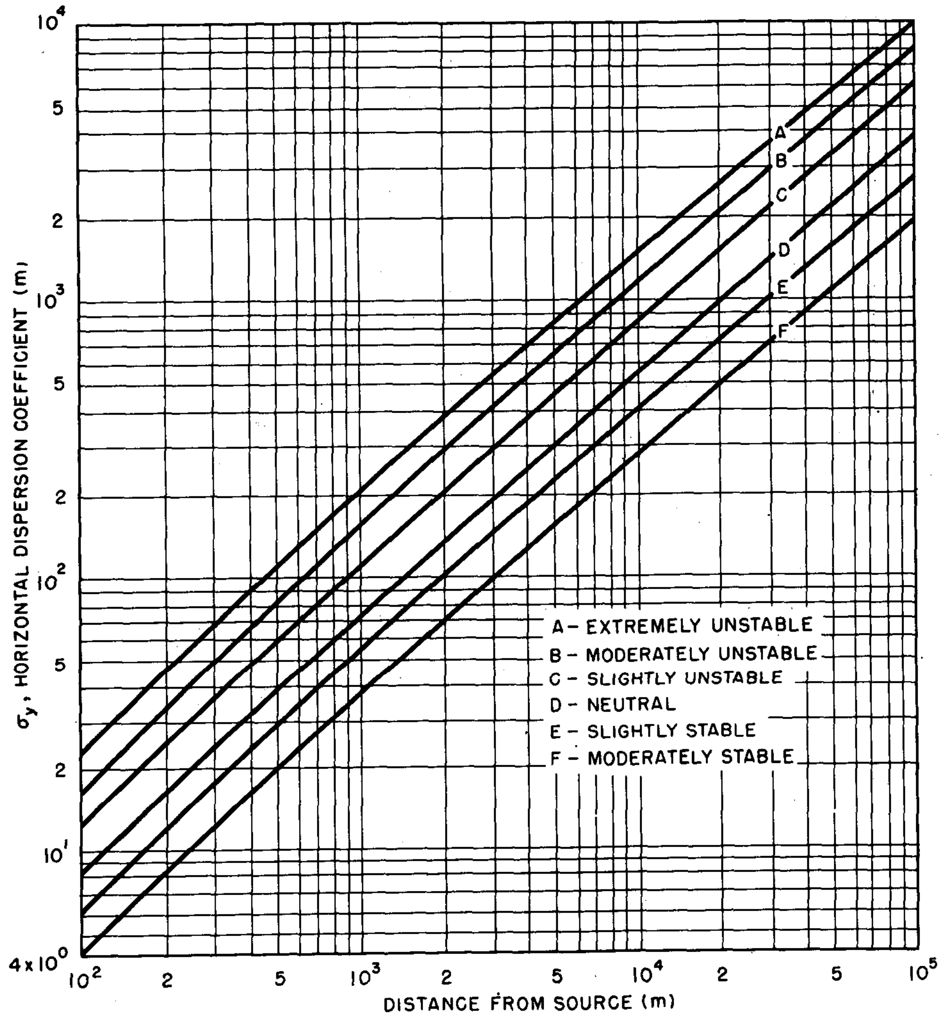


Fig. 3.10—Lateral diffusion, σ_y, vs. downwind distance from source for Pasquill's turbulence types.

and Bryant (1964) have also employed these values of σ in plume diffusion analyses.

Sections 4-4.3 and 4-4.4 of Chap. 4 indicate that Pasquill's curves fit the experimental data collected since the Prairie Grass experiments quite well. Furthermore the experimental data discussed in these sections demonstrate that the standard deviation of the horizontal wind direction, σ_θ, for a short averaging time and for the sampling times used in these experiments (10 min to 60 min) can be related empirically to the measured values of plume width or to normalized average concentration or exposure from continuous sources. On the basis of these data, Pasquill's stability categories can be relabeled approximately in terms of measured values of σ_θ as follows:

Pasquill stability categories	σ_θ
A, extremely unstable	25.0°
B, moderately unstable	20.0°
C, slightly unstable	15.0°
D, neutral	10.0°
E, slightly stable	5.0°
F, moderately stable	2.5°

Pasquill's method of estimating diffusion is well suited to field use because a simple recording wind vane and anemometer erected at a proposed site can, when used with the wind-direction range theory (Chap. 2, Sec. 2-6.2.3),

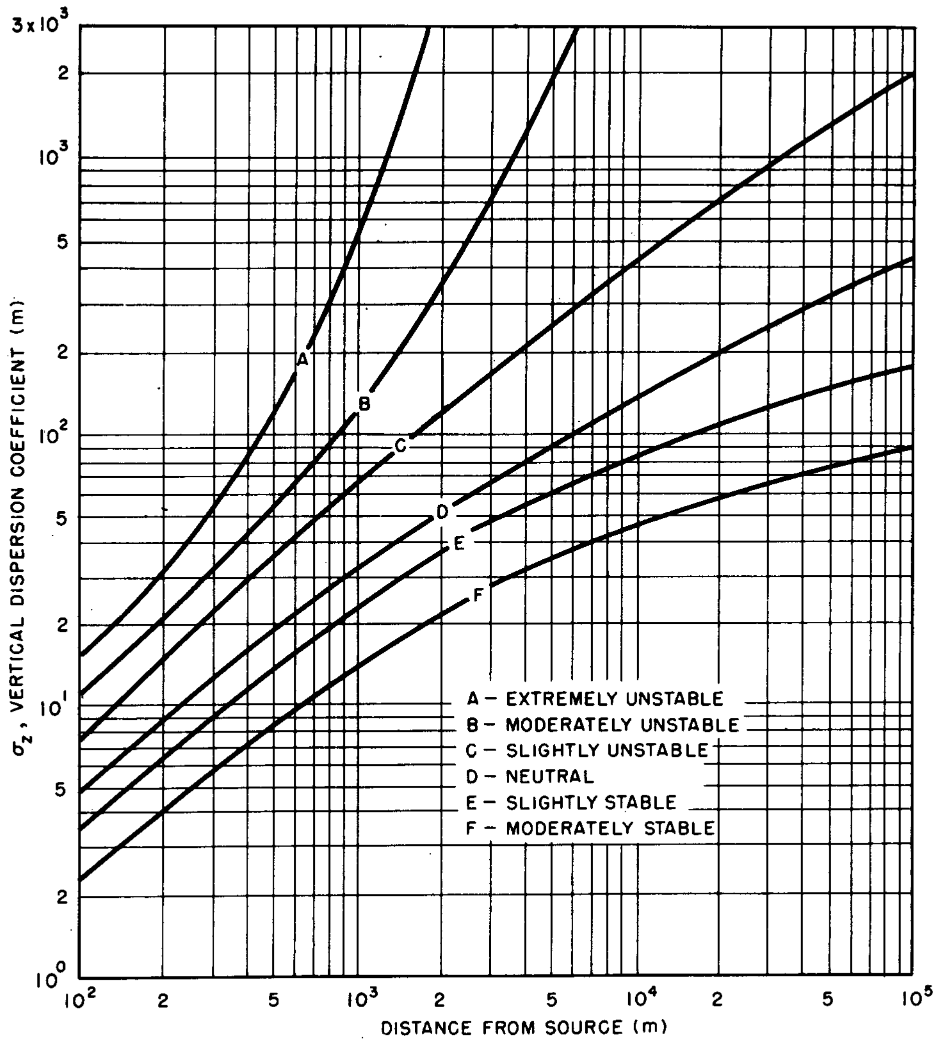


Fig. 3.11—Vertical diffusion, σ_z, vs. downwind distance from source for Pasquill's turbulence types.

furnish climatologically useful estimates of σ_θ rapidly. Only simple manual data processing is necessary. The wind-measuring system will, moreover, furnish data for other climatological wind statistics for the site, such as wind roses, and will also serve as the necessary wind-velocity monitoring equipment for permanent installation when the reactor or other plant is in operation.

3-3.4.2 Quantitative Use of Smoke Observations to Determine Diffusion Coefficients. Visual and photographic observations of smoke plumes and puffs have always appealed to workers in atmospheric diffusion as a useful research tool. Characteristically Richardson (1920) worked with time-exposure photographs of smoke puffs very early in the history of diffusion study. The use of smoke as a diffusion index continues to be widespread to this day. Quantitative interpretations of smoke observations (Sutton, 1932, Holland, 1953, Kellogg, 1956, Frenkiel and Katz, 1956, Gifford, 1957, 1959, Saissac, 1958, Inoue, 1960, and Högström, 1964) have usually exploited Roberts' (1923) opacity theory in which the visible edge of the smoke plume or puff is supposed to represent a constant threshold density of smoke particles along the line of sight.

The total density of smoke particles is obtained, according to the opacity idea, by integration of the concentration-distribution equation along a line of sight. For the generalized Gaussian plume distribution, Eq. 3.116, assuming that the plume is being viewed from a fairly great vertical distance, this procedure would give

$$\int_0^\infty \bar{\chi}\, dz = \frac{Q' \exp(-y^2/2\sigma_y^2)}{2\pi\sigma_y\sigma_z\bar{u}} \times \int_0^\infty \exp\left(\frac{-z^2}{2\sigma_z^2}\right) dz \quad (3.123)$$

If there is a fixed threshold value of the integrated concentration, χ_e, corresponding to the visible plume edge and located a distance $y_e(x)$ from the plume axis, then it can be shown, using the condition for a maximum value, that

$$(\sigma_y^2)_m = y_m^2 \quad (3.124)$$

where y_m is the maximum value of $y_e(x)$. Then

$$(\sigma_y^2)_m = y_e^2 \left\{ \ln \left[\frac{e y_m^2}{\sigma_y^2(x/\bar{u})} \right] \right\}^{-1} \quad (3.125)$$

where e is the base of natural logarithms. Figure 3.12 illustrates the meaning of the various lengths used.

An equation equivalent to Eq. 3.125 for the case of plume observations made at a great horizontal distance, following a line of sight integration in the y-direction, is

$$(\sigma_z^2)_m = z_e^2 \left\{ \ln \left[\frac{e z_m^2}{\sigma_z^2(x/\bar{u})} \right] \right\}^{-1} \quad (3.126)$$

Corresponding equations for smoke puffs based on Eq. 3.125 have also been given (Gifford,

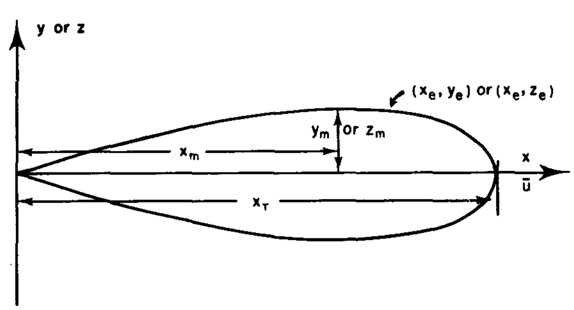


Fig. 3.12—Meaning of various quantities used in smoke-plume analysis.

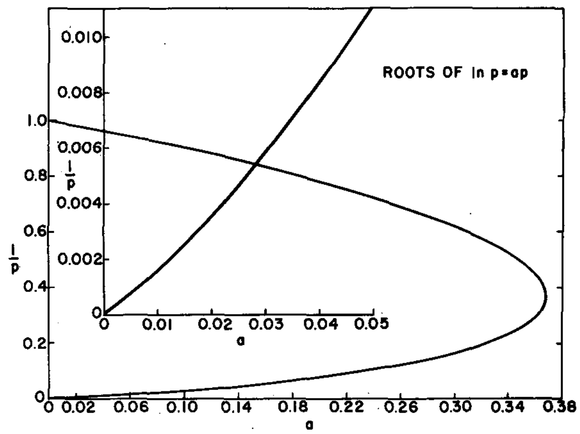


Fig. 3.13—Plot of the equation ln p = ap.

1958). All these transcendental equations can be solved for $\sigma_y(x/\bar{u})$ or $\sigma_z(x/\bar{u})$ if there are visual or photographic observations of smoke plumes from which y_e and z_e, the plume half-width and half-height, can be determined. The computations are facilitated by Fig. 3.13, which is a graphical solution of the equation ln p = ap. No assumption about the analytical form of σ_y or σ_z is necessary.

A simpler procedure when, as in the smoke studies by Sutton (1932), Kellogg (1956), Frenkiel and Katz (1956), Holland (1953), Moses and Clark (1956), and others, one is willing to choose in advance a specific form for the diffusion function, was suggested by Gifford (1959). It is to combine, for example, Eq. 3.109, 3.110, or 3.111 directly with Eq. 3.124. Systematic exploitation of this idea leads to a number of particularly simple pairs of formulas for diffusion coefficients:

$$K_y = \bar{u}\left(\frac{e}{2}\right)\left(\frac{y_m}{x_T}\right) y_m$$

$$K_z = \bar{u}\left(\frac{e}{2}\right)\left(\frac{z_m}{x_T}\right) z_m \quad (3.127)$$

$$\overline{v'^2} = \bar{u}^2 e \left(\frac{y_m}{x_T}\right)^2$$

$$\overline{w'^2} = \bar{u}^2 e \left(\frac{z_m}{x_T}\right)^2 \quad (3.128)$$

$$C_y^2 = 2x_T^n e \left(\frac{y_m}{x_T}\right)^2$$

$$C_z^2 = 2x_T^n e \left(\frac{z_m}{x_T}\right)^2 \quad (3.129)$$

$$K_y = \left(\bar{u}\,\frac{y_m}{2}\right)\left(\frac{y_m}{x_m}\right)$$

$$K_z = \left(\bar{u}\,\frac{z_m}{2}\right)\left(\frac{z_m}{x_m}\right) \qquad (3.130)$$

$$\overline{v'^2} = \bar{u}^2\left(\frac{y_m}{x_m}\right)^2$$

$$\overline{w'^2} = \bar{u}^2\left(\frac{z_m}{x_m}\right)^2 \qquad (3.131)$$

$$C_y^2 = 2x_m^n\left(\frac{y_m}{x_m}\right)^2$$

$$C_z^2 = 2x_m^n\left(\frac{z_m}{x_m}\right)^2 \qquad (3.132)$$

In these equations x_T is the total plume length and x_m is the distance downwind from the source at which the maximum plume width or height, y_m or z_m, occurs. The utility of these formulas lies in the fact that many of the significant plume dimensions that need to be determined from visual observations or photographs appear as ratios and so do not need to be measured absolutely but only relatively; and certain of the remaining distances, e.g., x_T and x_m in Eqs. 3.129 and 3.132, appear as *n*th powers (roughly as fourth roots) and consequently need only to be approximated. A procedure somewhat similar to this but involving the K theory was used by Richardson and Proctor (1925), and further interesting results were derived by Inoue (1960, 1961), using an equivalent method in connection with his similarity theory of diffusion.

Examples of diffusion-coefficient determinations at nuclear-reactor sites by the ratio method, i.e., by one of Eqs. 3.127 through 3.132, have been given by Bowne (1961), Culkowski (1961), Gifford, Culkowski, and Hilsmeier (1963), and Hewson, Gill, and Walke (1963). When this method is applied to plume photographs, some form of time averaging of the smoke-plume observations is desirable. Time-exposure photographs of plumes through neutral density filters were described by Culkowski (1961), who has experimentally determined the necessary film reciprocity factors for quite long exposure times. Bowne (1961), Shorr (1952), Saissac (1958), Richardson (1920), and Inoue (1960) have also reported long-time-exposure smoke-plume photographs. It appears, on the basis of Culkowski's example, that a close approximation to the effect of time averaging can be achieved by estimating the smoothed envelope of the instantaneous photograph and basing the plume measurements on this envelope. In view of its real simplicity and economy and the measure of agreement with direct diffusion measurements reported in various of the above references, the ratio method seems a promising way to obtain plume diffusion coefficients.

3-3.5 Equations for Calculating Concentration and Exposure

The equations presented in the first edition of *Meteorology and Atomic Energy* for dealing with various practical diffusion problems that arise in reactor-hazard analysis and in other air-pollution problems were based on the widely used diffusion model formulated by Sutton. A list of these equations appears in the Appendix Sec. A.4. Many of these equations were first presented by Holland (1953). There is a need for a corresponding list of diffusion equations based on the simple Gaussian formula, Eq. 3.116. In this section a number of such equations will be considered. Most can be converted to the equivalent Sutton form by means of Eqs. 3.79 and 3.81.

3-3.5.1 Characteristic Continuous-source Plume Equations. Equations for the five characteristic continuous-source plume types described in Chap. 2, namely, fanning, fumigation and trapping, looping, coning, and lofting, can be developed as follows.

3-3.5.1.1 Fanning. Fanning is characterized by very slow vertical diffusion during stable conditions. Concentrations can be estimated from Eq. 3.116 with σ values corresponding to stable conditions, for which the horizontal diffusion, σ_y, considerably exceeds the vertical diffusion, σ_z. Figure 3.14 illustrates (a) fanning that commenced a very short distance from the source and (b) fanning that did not begin for some considerable distance from the source.

3-3.5.1.2 Fumigation and Trapping. Hewson and Gill (1944) introduced the term "fumigation" to describe the rapid mixing downward to the ground of material that has accumulated aloft during a period of atmospheric stability,

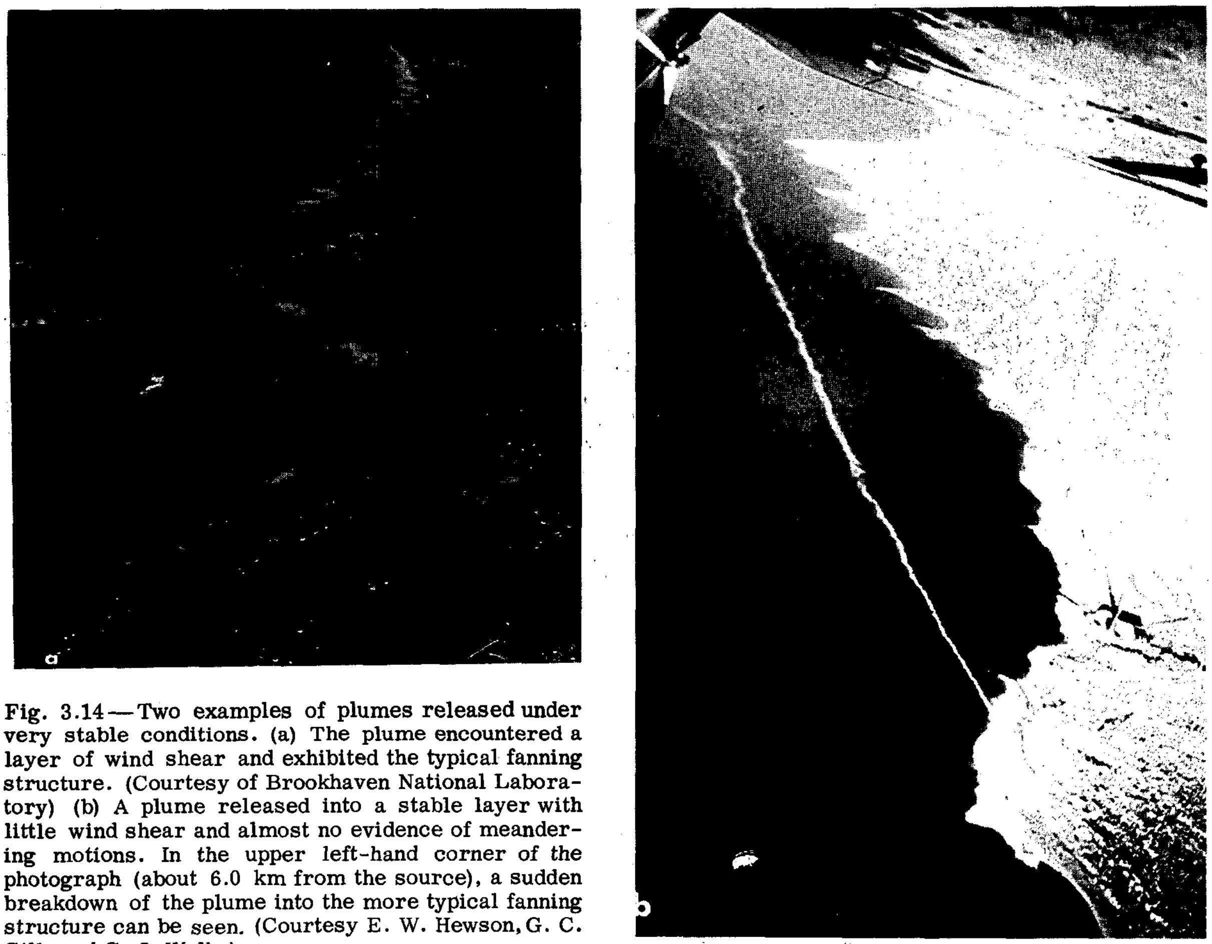


Fig. 3.14—Two examples of plumes released under very stable conditions. (a) The plume encountered a layer of wind shear and exhibited the typical fanning structure. (Courtesy of Brookhaven National Laboratory) (b) A plume released into a stable layer with little wind shear and almost no evidence of meandering motions. In the upper left-hand corner of the photograph (about 6.0 km from the source), a sudden breakdown of the plume into the more typical fanning structure can be seen. (Courtesy E. W. Hewson, G. C. Gill, and G. J. Walke)

an occurrence that is common after dawn when the nocturnal temperature inversion is rapidly dissipated by warming due to solar heating of the ground. Concentrations due to the fumigation effect can be estimated by integrating Eq. 3.115 with respect to z from 0 to ∞ and then considering the material in the cloud to be distributed uniformly through a layer of height h_i. The equation for the fumigation concentration, $\overline{\chi}_F$, is accordingly

$$\overline{\chi}_F = \frac{Q'}{(2\pi)^{1/2}\bar{u}h_i\sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) \qquad (3.133)$$

This equation can also be used to describe the trapping condition during which the effluent diffuses rapidly below the base of an elevated inversion but is prevented by the stable layer from diffusing to greater heights. For example, h_i could be taken as the height of the base of a persistent inversion aloft, such as the West Coast (California) subsidence inversion. Or h_i might be the height of the top of the planetary boundary layer or of the base of some other distinct inversion layer, such as a frontal inversion. Scorer (1959) says that a stable layer approximately equal to the elevation of ridge tops often marks the upper boundary of smoke diffusion in valleys; h_i could be identified with this level.

In the fumigating plume shown in Fig. 3.15, fumigation occurred within the stable air of a lake breeze, a situation analagous to that which might occur during a sea breeze. Plume trapping from an open burn and from a stack is shown in Fig. 3.16. Another instance of trapping, in this case of the combined detritus from natural and man-made processes operating over a large area, is shown in Fig. 3.17.

By considering both the ground and the inversion base to be reflecting barriers, Hewson,

Fig. 3.15 — An illustration of a fumigating plume near the shore of Lake Michigan. The plume was embedded in a very stable air flow originating over the lake during late afternoon on a summer day. As the cool stable air moved inland, it was heated from below, and a fumigation pattern was created. (Courtesy E. W. Hewson, G. C. Gill, and G. J. Walke)

Gill, and Bierly (1959) derived the following formula for trapping:

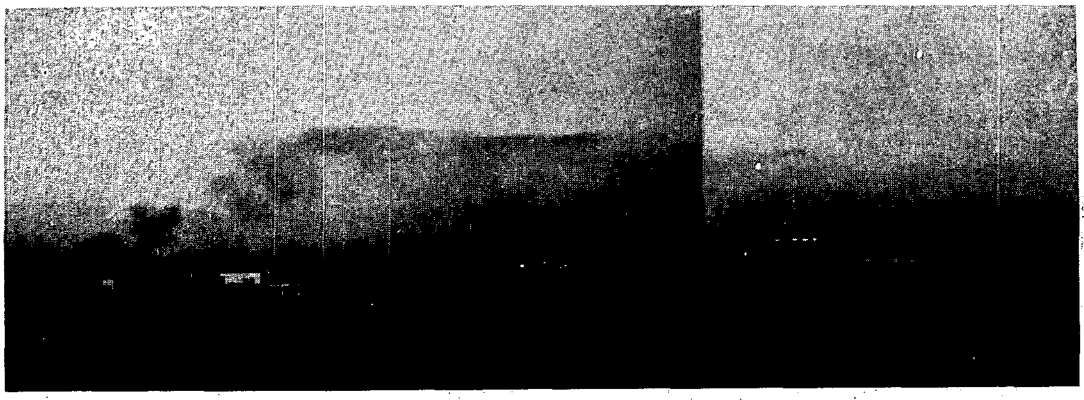
$$\frac{\overline{\chi}}{Q'} = \frac{1}{\pi\, \sigma_y \sigma_z \bar{u}} \left\{ \exp\left(-\frac{h^2}{2\sigma_z^2}\right) + \exp\left[-\frac{(2h_i - h)^2}{2\sigma_z^2}\right] + \exp\left[-\frac{(-2h_i - h)^2}{2\sigma_z^2}\right] \right\} \qquad (3.134)$$

where h_i is the height of the inversion base and the result is expressed in terms of σ_y and σ_z instead of the corresponding Sutton formula as given by these authors. This equation is a special case of a more general formula developed earlier by Hewson, which contains an infinite series of exponential terms corresponding to the plume reflections.

A similar result was developed by Albracht (Lindackers, Bresser, and Albracht, 1965). Gifford (1961), following proposals by Meade (1959) and Pasquill (1961), suggested that, in the event vertical diffusion is restricted by a strong inversion lid at some height h_i, diffusion could be computed directly from Eq. 3.116 by assuming that the value of σ_z involved is constant at distances beyond the point where $\sigma_z = h_i/2.15 \approx h_i/2$. This suggested treatment of trapping, offered purely on the basis of its simplicity, agrees very closely with Eq. 3.133, differing only by a small constant factor, at all downwind distances greater than a few stack heights. There are, as Lindackers, Bresser, and Albracht (1965) have shown by carrying out the calculations, differences between the results of trapping calculations based on such simple assumptions as these and the results based on the assumption of multiple reflections of the plume. Without experimental evidence it is not possible to make a choice between these alternatives now.

Two problems may be encountered in the application of Eq. 3.133 or Eq. 3.116 to trapping or fumigation calculations. First, there is no direct indication of the minimum distance from the stack beyond which these equations may be applied. Judging by qualitative discussion in the literature (for example, Bierly and Hewson, 1962, and Pooler, 1965), there seems to be some uncertainty about this point although there is no obvious reason why the distance should exceed a few stack heights. The second problem is encountered in the attempt to specify a value for σ_y. During the fumigation process the plume is mixed through the increasingly unstable layer below the inversion. Therefore the effective value of σ_y to be used in estimating fumigation concentrations should probably be somewhat greater than the inversion value to account for this augmented mixing.

3-3.5.1.3 Looping. Looping is the most spectacular of plume conditions in appearance.



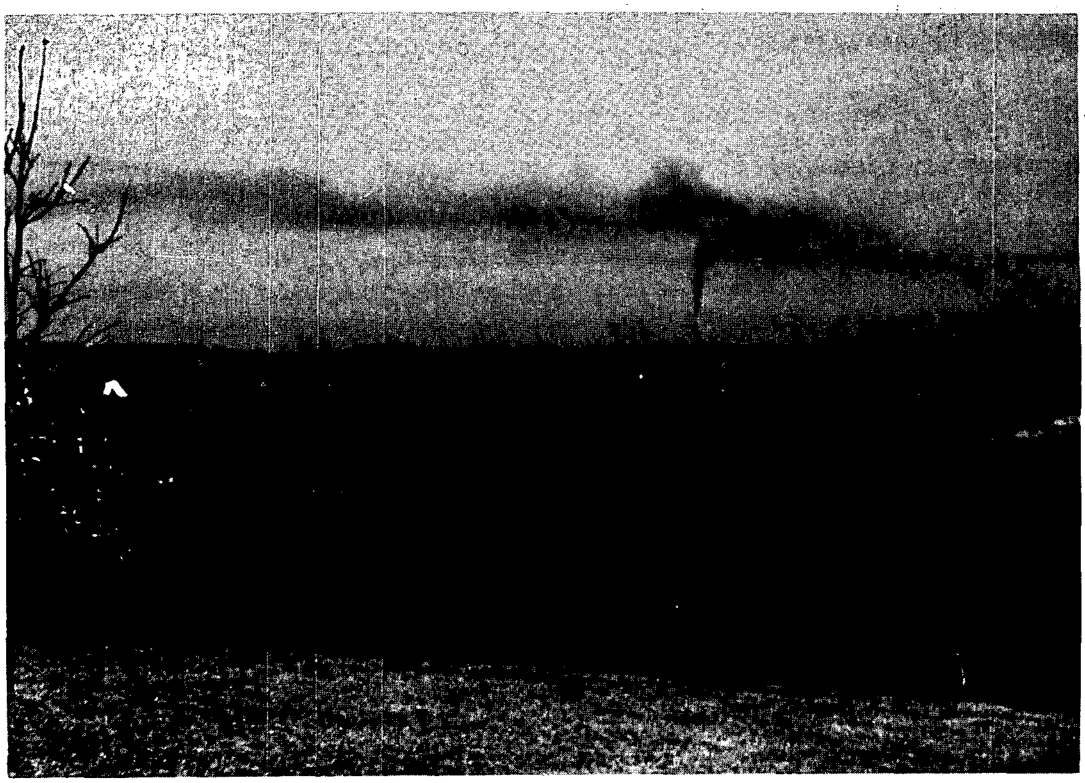


Fig. 3.16—Two illustrations of the trapping of smoke from an isolated source within the planetary boundary layer. (Courtesy D. H. Slade and W. M. Culkowski)

Large loops of the plume are carried down to the ground and cause momentary bursts of high concentration, only to be replaced by effluent-free air as corresponding loops go aloft. The average concentration during plume looping probably corresponds to a Pasquill type A condition and can thus be estimated from Eq. 3.116. Figure 3.18 illustrates looping. Holland (1953) suggested that the maximum ground concentration during looping, $\bar{\chi}_{max.}$, could be estimated from the usual plume equation expressed for the axial concentration, y = 0, due to a ground-level source, h = 0, where, however, the downwind distance from the source is redefined as $x' = (x^2 + h^2)^{1/2}$, x being the actual distance and h, the source height. Expressed in terms of Eq. 3.116, Holland's equation for the maximum concentration during looping is

$$\bar{\chi}_{max.} = \frac{Q'}{\pi\, \sigma_y(x')\, \sigma_z(x')\, \bar{u}} \qquad (3.135)$$

Fig. 3.17—Trapping of smoke and haze, which has originated over a broad area, beneath the West Coast (California) subsidence inversion. (Courtesy W. M. Culkowski)

Because of the great variability of instantaneous concentration during looping, it is helpful to be able to estimate the peak to average concentration ratio. This can be done by means of the fluctuating plume model described in Sec. 3-3.2. In the fluctuating plume model, the mean-square diffusion, $\overline{y^2}$, is separated into two portions: a part due to the instantaneous spreading out of the plume, $\overline{Y^2}(t)$, and a part attributable to the meandering or looping, $\overline{D^2}(t)$, i.e.,

$$\overline{y^2}(t) = \overline{Y^2}(t) + \overline{D^2}(t) \tag{3.136}$$

It can be shown (Pasquill, 1962, and Gifford, 1960) that for large travel times $\overline{D^2}$ approaches some constant value but, according to Batchelor (1952), $\overline{Y^2}$ will increase indefinitely with time. This follows from Eq. 3.97. From the ratio of Eq. 3.119 to Eq. 3.120, it is possible to compute the peak to average concentration ratio. The peak concentration occurs when $y = D_y$ and $z = D_z$, i.e., when the receptor is at the center line of the instantaneous plume. The result is

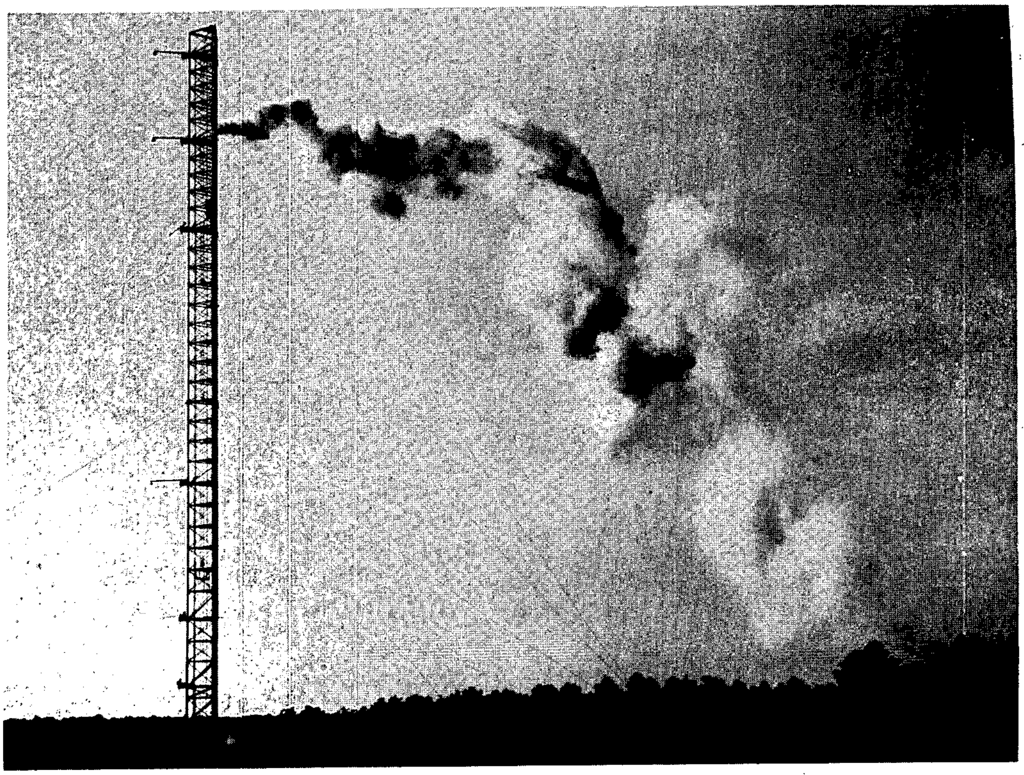
$$\frac{\text{Peak}}{\text{Average}} = \frac{P}{A} = \frac{(\overline{Y^2} + \overline{D^2})}{\overline{Y^2}} \exp\left[\frac{y^2}{2(\overline{Y^2} + \overline{D^2})} + \frac{z^2}{2(\overline{Y^2} + \overline{D^2})}\right] \tag{3.137}$$

If $y = z = 0$, i.e., on the mean plume axis (or, equivalently, at the ground at a moderate distance downwind from the source),

$$\frac{P}{A} = \frac{\overline{Y^2} + \overline{D^2}}{\overline{Y^2}} = 1 + \frac{\overline{D^2}}{\overline{Y^2}} > 1 \tag{3.138}$$

Since $\overline{D^2} \rightarrow$ constant, $P/A \rightarrow 1$ for large travel times.

The effect of stack height on P/A depends on the term $\exp [z^2/2(\overline{Y^2} + \overline{D^2})]$, which involves the total vertical plume diffusion $(\overline{Y^2} + \overline{D^2}) = \overline{z^2}$. This exponential can be estimated, for example, from Eq. 3.81 or by reference to observations. For stack heights of interest, reasonable values of vertical diffusion indicate that P/A values at the ground fairly near the stack base may be one or two orders of magnitude greater



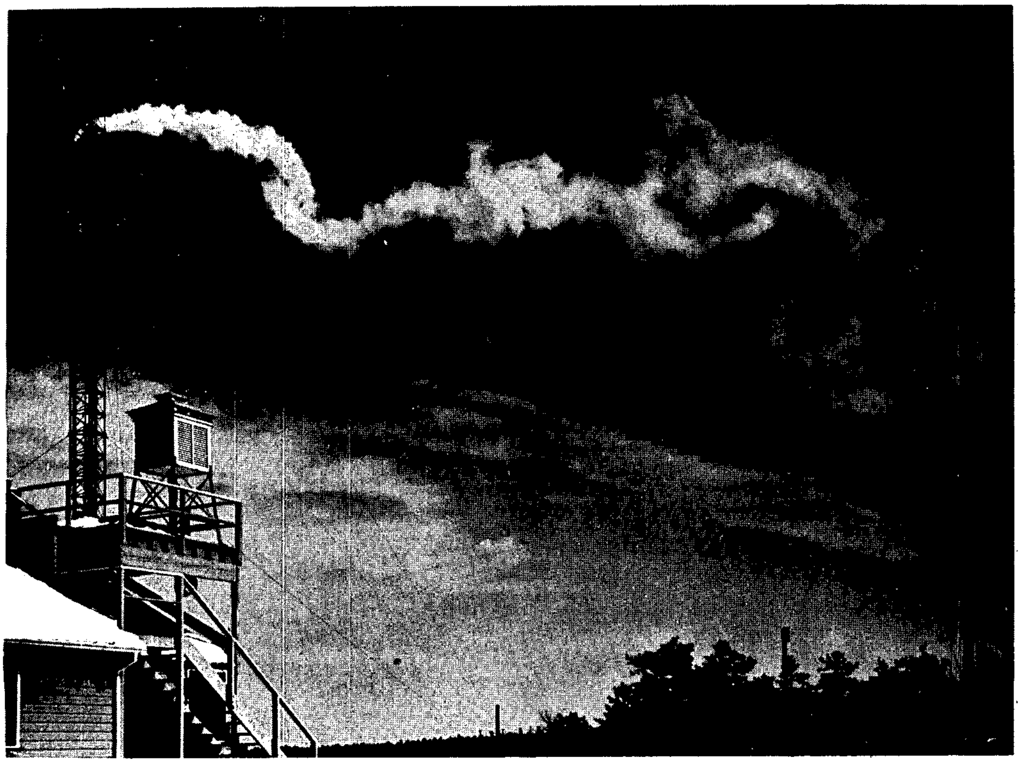


Fig. 3.18—Two illustrations of looping plumes observed at the Brookhaven National Laboratory.

than those on the mean plume axis. Various observations of P/A as a function of distance from the source, the relative level of source and receptor, and the times over which the peak and average concentrations were obtained are discussed in Chap. 4.

In principle the theoretical results of Sec. 3-3.2 on fluctuating plumes and the above paragraphs apply equally to other diffusion conditions and not just to looping. The looping condition, however, makes visually evident the separation between plume spreading and meander.

3-3.5.1.4 Coning. Coning is the straightforward, relatively uncomplicated case of diffusion in a neutral or slightly stable atmosphere and is handled by means of Eq. 3.116, evaluated for the Pasquill type C or D conditions. Figure 3.19 shows an instantaneous photograph and a time exposure of a coning plume.

3-3.5.1.5 Lofting. Since a ground-based inversion prevents material from reaching the surface, lofting is of practical importance largely as the possible precursor of a fumigation. A reasonable scheme for estimating concentrations in the lofting plume might simply be to treat the inversion base as the level z = 0 and to apply Eq. 3.116 (with h = 0 to obtain concentrations along the plume center line) although there are no concentration observations confirming this suggestion.

3-3.5.2 Volume-source Formulas. Because of the possible emission of airborne radioactive material through leaks in a reactor-containment structure, Eq. 3.116 should be modified for the effect of a volume source. In a reactor-hazard analysis, the source generally consists of some fraction of the fission products contained in the reactor core, and the source material is assumed to be distributed uniformly throughout the volume of the building enclosing the reactor. For many power reactors the enclosure is a large pressure-tight dome designed to have, at most, some specified leakage rate under the postulated accident conditions. The source strength, Q′, is defined, but the location of the leak and the effect of the building on the source geometry must be determined.

Reasoning that a reactor building must have a turbulent wake in its lee, Fuquay (1960) suggested treating the building effect as an initial dilution factor, D_B,

$$D_B = cA\bar{u} \tag{3.139}$$

where A is the cross-sectional area of the building normal to the wind. In other words, any material escaping from the containment building is assumed to be dispersed rapidly into a volume equal to c times the building cross-sectional area times the wind speed. The

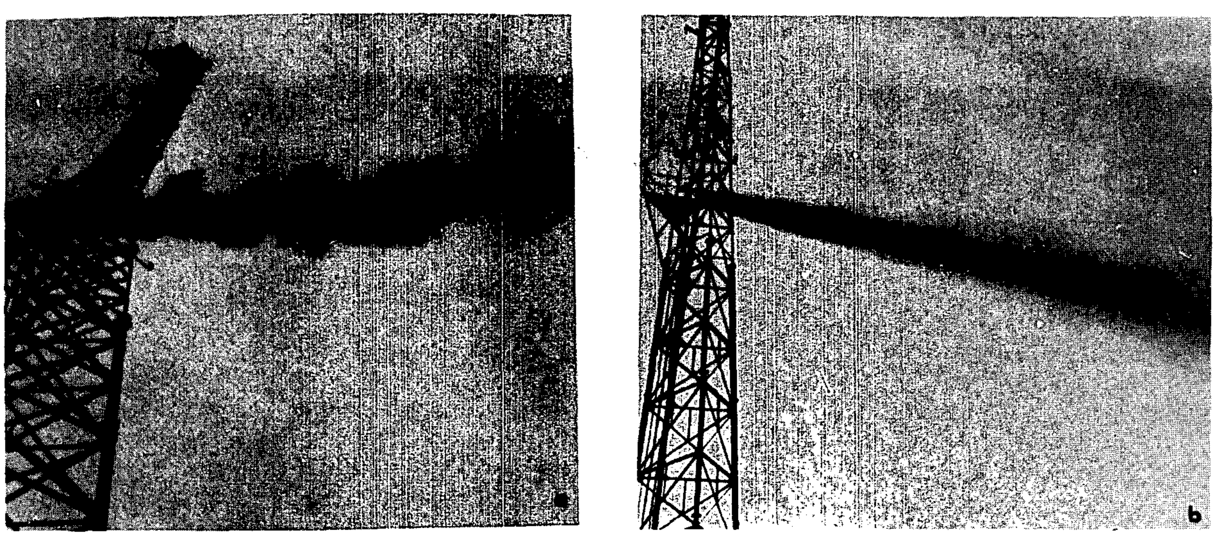


Fig. 3.19 — (a) Coning plume using an exposure of 1/25 sec at the meteorological tower of the Big Rock Point reactor site near Charlevoix, Mich. (b) The same coning plume photographed with a time exposure of 5 min. (Courtesy W. M. Culkowski)

factor c represents an estimation of the relation of the cross-sectional area of the building to the size of observed pressure wakes, and its exact numerical value will have to be determined by suitable experiments. Gifford (1960) suggested that, as a reasonable estimate, $\frac{1}{2} \le c \le 2$. The reason for choosing these particular bounds, which were actually no more than a guess, was to provide, in the absence of suitable experimental data, usable numbers for concentration estimations. According to Barry (1964), who made an interesting and useful summary of the results of a number of recent experiments, studies with wind-tunnel models have suggested values of c near the lower of these limits, namely, c = 0.50 to 0.67. Of course, it is not impossible that larger values of c may be found if suitable full-scale atmospheric experiments are performed, particularly in unstable light-wind conditions. A comprehensive summary of relevant wind-tunnel measurements of building dilution effects is given in Chap. 5. A few atmospheric experiments have been reported by Islitzer (1965) and J. E. Martin (1965). A photograph from Martin's paper, Fig. 3.20, illustrates the building effect on the plume.

The building dilution factor, D_B, is combined with the atmospheric dilution factor, $D_A = Q'/\bar{\chi}$, in a way similar to Fuquay's (1958) handling of stack dilution,

$$D_{total} = D_B + D_A \qquad (3.140)$$

Combining Eqs. 3.116, 3.139, and 3.140, one can reasonably assume that, as suggested by Davidson (1965),

$$\frac{\bar{\chi}}{Q'} = (\pi \Sigma_y \Sigma_z \bar{u})^{-1} \exp\left[-\left(\frac{y^2}{2\Sigma_y^2} + \frac{h^2}{2\Sigma_z^2}\right)\right] \qquad (3.141)$$

where Σ_y and Σ_z are total diffusion factors given by

$$\Sigma_y = (\sigma_y^2 + cA/\pi)^{1/2}$$

$$\Sigma_z = (\sigma_z^2 + cA/\pi)^{1/2} \qquad (3.142)$$

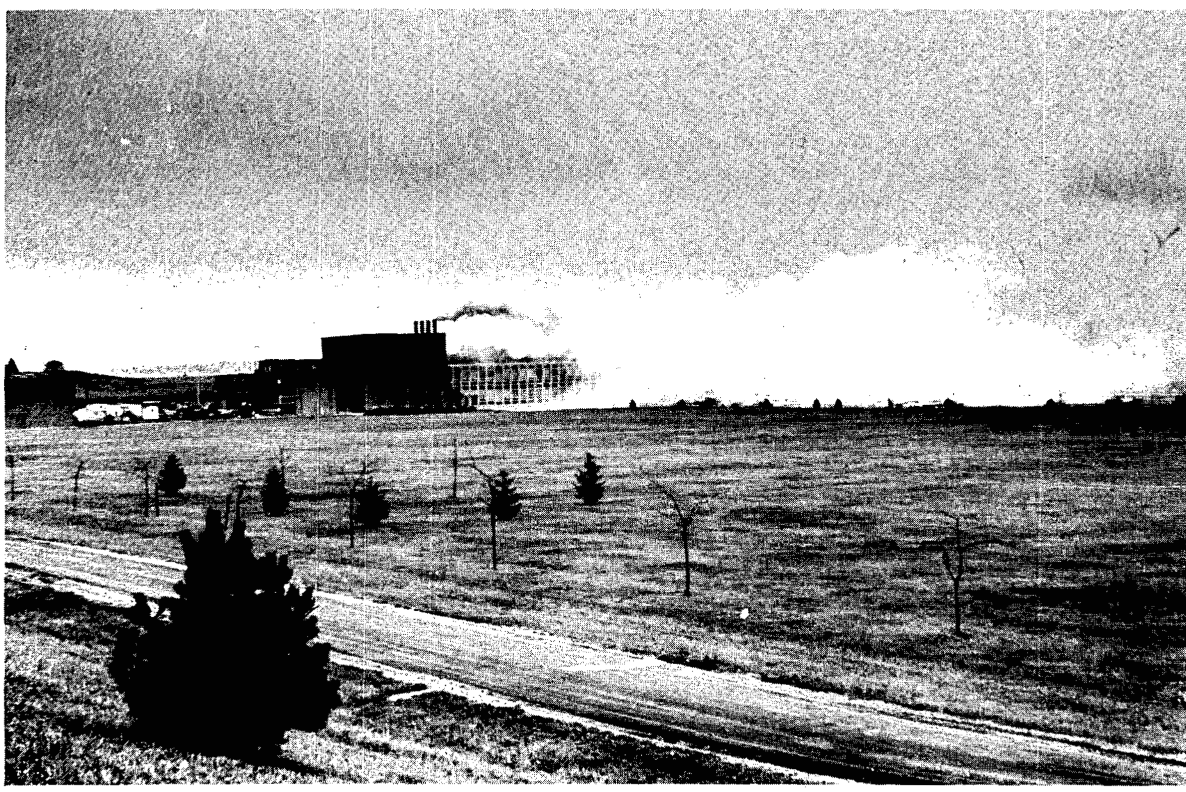


Fig. 3.20—A photograph of a smoke plume released from the top of a building during neutral conditions. (Courtesy J. E. Martin, 1965).

Equation 3.141 resembles the volume-source treatment proposed by Holland (1953) in that the volume effect is taken into account by adding a correction to the diffusion term of a Gaussian distribution. Holland achieved much the same result by defining a "virtual point source" that would produce a Gaussian plume, or puff, having a "width" equal to that of the actual volume source at the initial point.

Various other initial volume-source distribution functions were compared by Gifford (1955a) with the Gaussian initial distribution. The conclusions from this study are that the Gaussian initial volume leads to much simpler diffusion expressions and that it is conservative; that is, it leads to downwind concentrations slightly greater than for other volume-source distributions that were considered.

3-3.5.3 Crosswind Integrated Concentration. The crosswind integrated concentration, $\overline{\chi}_{CWI}$, from a continuous source is obtained by integrating Eq. 3.116 with respect to y from $-\infty$ to ∞:

$$\overline{\chi}_{CWI} = \frac{2^{1/2} Q'}{(\pi)^{1/2}\sigma_z \overline{u}} \exp\left(-\frac{h^2}{2\sigma_z^2}\right) \qquad (3.143)$$

This equation is particularly useful as an interpolation formula in connection with field-diffusion trials because it contains only one diffusion parameter, σ_z. The same equation describes the concentration due to a continuous infinite crosswind line source (source strength, Q_L'), which might be realized in practice by, for example, a heavily traveled highway. Some discussion of the problem of a line source of finite length has been presented by Elliott and Barad (1964) among others, and the problem of a crosswind line source oriented at some angle to the mean wind direction was discussed by Barad, Haugen, and Fuquay (1960). Equation 3.143 has also been used as an approximation for an area source (Turner, 1964).

3-3.5.4 Long-period Average Concentration. Over a period of time, the direction of the mean wind shifts. The wind rose, which gives the joint wind-speed and direction-frequency distribution, is therefore a useful indicator of the characteristic features of the climate of a particular place. To obtain an estimate of the average concentration over a period that is very long compared with that over which the mean wind is computed, multiply the integrated concentration formula, Eq. 3.143, by the frequency with which the wind flows toward a given sector and divide by the width of that sector at the distance of interest:

$$\overline{\chi}_{\text{long-term av.}} = \left(\frac{2}{\pi}\right)^{1/2} \frac{0.01\, f\, Q'}{\sigma_z \overline{u}(2\pi x/n)} \times \exp\left(-\frac{h^2}{2\sigma_z^2}\right) \qquad (3.144)$$

where the frequency, f, is expressed in per-cent, $2\pi x/n$ is the sector width, and Q', σ_z, and $\overline{u}$ are averages over the long time period. An expression equivalent to this forms the basis for the calculations by Meade and Pasquill (1958) of annual SO_2 concentrations in the vicinity of the Staythorpe Power Station (using the corresponding Sutton formula) and is similar to one proposed by Culkowski (1960) (see also Lowry, 1951)

3-3.5.5 Maximum Concentration and Its Distance from the Continuous Elevated Source. Because σ_y and σ_z are not necessarily the same functions of x, in general, it is not possible to obtain simple explicit formulas for the maximum ground concentration and its distance from the source. However, in the special case $\sigma_y = \sigma_z$, i.e., for neutral or slightly unstable conditions, these maximum values can be specified. Differentiating Eq. 3.116 with respect to x and setting the result equal to zero in the usual way gives

$$\overline{\chi}_{\text{max.}} = \frac{2Q'}{\pi h^2 e \overline{u}} \qquad (3.145)$$

when $h^2 = 2\sigma_z^2$. In the slightly more general case characterized by $\sigma_y = a\sigma_z$, $d\sigma_y/dx = a\,d\sigma_z/dx$, i.e., where the vertical and horizontal cloud growths are simply proportional, which again occurs when $h^2 = 2\sigma_z^2$, the result is

$$\overline{\chi}_{\text{max.}} = \frac{2Q'}{\pi h^2 e \overline{u}} \frac{\sigma_z}{\sigma_y} \qquad (3.146)$$

Because the maximum concentration occurs when $h = 2^{1/2}\sigma_z$, this formula may also be written in the following form

$$\overline{\chi}_{\text{max.}} = \frac{2^{1/2} Q'}{h e \overline{u} (\sigma_y)_{\text{max.}}} \qquad (3.147)$$

where the notation indicates that the value of σ_y to be used is the one applying at the maximum concentration distance.

Alternatively, Fig. A.4 of the Appendix, which presents evaluations of Eq. 3.116 with the σ_y and σ_z values of Figs. 3.10 and 3.11, can be used.

3-3.5.6 Concentration Isopleths: Plume Width and Height Formulas. Practical computations with diffusion formulas often require the construction of concentration isopleths, for example, in connection with calculation of total population dosage (Gomberg, 1958). For this purpose it is convenient to know the distance z_p or y_p where the concentration has dropped to p% of its value on the plume axis, For the generalized Gaussian plume model, the following formulas are an obvious application of Eq. 3.116:

$$y_p = \left(2\sigma_y^2 \ln \frac{100}{p}\right)^{1/2} \qquad (3.148)$$

and

$$z_p = \left(2\sigma_z^2 \ln \frac{100}{p}\right)^{1/2} \qquad (3.149)$$

From these, with p = 10%, it can be seen that H, as defined by Eq. 3.121, is equal to $2z_p$ and that a similar relation exists between y_p and θ.

On the basis of plots of concentration isopleths, Hilsmeier and Gifford (1962) have computed areas enclosed by various concentration values, $\bar{\chi}$, for sources located at the surface. They followed the generalized Gaussian formula, Eq. 3.116, and used the diffusion-parameter values of Figs. 3.10 and 3.11. The results are shown in Fig. 4.32 of Chap. 4 together with observed isopleth areas from the Prairie Grass and Green Glow programs, as analyzed by Elliott (1959) and by Elliott and Nickola (1961).

An extensive numerical computation of areas within concentration isopleths, based on Sutton's diffusion model, Eq. 3.116 taken in combination with Eqs. 3.79 and 3.81, was undertaken by Rosinski (1958). Rosinski's computation allowed for the effect of varying deposition rates. A similar computation using Sutton's model was performed by Velez (1961), who allowed for the effects of varying source heights and radioactive decay of mixed fission products. Nishiwaka (1959) likewise employed Sutton's model to estimate concentration-isopleth areas and, in addition, provided several useful approximations to the isopleth area based on the areas of equivalent ellipses. His formulas, which give the area A within a concentration isopleth, $\bar{\chi}$ = constant, for a surface-level source, are:

$$A_1 \approx \frac{2\pi}{(2-n)}\, C_y \exp\left[-\frac{2-(n/2)}{2-n}\right] (\gamma)^{2-(n/2)} \qquad (3.150)$$

$$A_2 \approx \frac{\pi}{2}\, C_y\, e^{-1/2}\ (\gamma)^{2-(n/2)} \qquad (3.151)$$

$$A_3 \approx \frac{\pi}{4}\, 2^{n/2} C_y (\ln 2)^{1/2}\ (\gamma)^{2-(n/2)} \qquad (3.152)$$

where

$$\gamma \equiv \left(\frac{2Q'}{\bar{\chi}\bar{u}\pi C_y C_z}\right)^{n-2} \qquad (3.153)$$

and the other parameters have their usual meanings. The error of these useful approximations is ≤3% for A_2 and A_3 and ≤6% for A_1, as compared with areas calculated directly from Sutton's formula.

3-3.5.7 Multiple and Area Sources. Many of the large nuclear installations already face the problem of emissions from several isolated sources. If there are only a few sources, it is a simple matter to compute their concentrations individually and sum these to obtain their joint effect. The arithmetic can in some cases be simplified by taking advantage of a circumstance that seems first to have been pointed out by Bosanquet and Pearson (1936). Because of symmetry with respect to the x-axis, Gaussian diffusion models possess the property that, if the source and receptor locations are interchanged, the numerical value of the concentration is not affected. This means that the concentration at a point downwind from a number of isolated sources can be computed by imagining all the sources to be combined and located at the receptor point and summing the resulting (computed) concentrations at the actual source points after reversing the mean wind direction, $\bar{u}$. Culkowski (1960) has shown that this scheme can also be applied to annual average concentrations from multiple sources. A plastic overlay, or template, of concentration isopleths expedites the calculation.

On the other hand, sources may be so numerous that they can be considered most effectively as an area source, and the point-source plume formula may be integrated over this area. This procedure was followed by Lucas (1958). If there are a very large number of individual sources, it may be desirable, as Turner (1964) has done, to combine them into a smaller number of virtual area sources and then to sum the concentrations that result from these.

3-3.5.8 Instantaneous-source Diffusion Equations. In addition to procedures for average plume diffusion, a procedure for calculating the diffusion from sudden, explosive, or very short term releases of material to the atmosphere is often required. Although the so-called "hot-cloud" accident, an instantaneous release of all the nuclear and chemical energy of a reactor to the atmosphere, is no longer considered credible because of reactor-containment features, other possibilities for generating sources of this kind exist. Some examples are the short-term controlled release of fission products from a contained accident, explosive accidents occurring during nuclear-fuel reprocessing, accidental criticalities, launching-pad accidents involving nuclear (or chemical) rockets, and nonnuclear explosions of all kinds.

Equation 3.113 for the instantaneous puff concentration, χ, can be written

$$\chi(x,y,z) = Q\,2^{-1/2}\pi^{-3/2}(\sigma_{xI}\sigma_{yI}\sigma_{zI})^{-1} \times \exp\left\{-\left[\frac{(x-\bar{u}t)^2}{2\sigma_{xI}^2} + \frac{y^2}{2\sigma_{yI}^2} + \frac{h^2}{2\sigma_{zI}^2}\right]\right\} \quad (3.154)$$

Here Eq. 3.113 has been multiplied by 2 to account for the assumed ground reflection so that it will be consistent with the plume equation, Eq. 3.116. For reasons discussed in Sec. 3-2.2.6, it is to be expected that, in general, the puff standard deviations σ_{yI} and σ_{zI} will differ from the corresponding plume σ_y and σ_z. Appropriate values based on recent experimental data are presented in Sec. 4-10.3.

Since the processes of puff creation are frequently associated with some degree of violent expansion (typically an explosion or short and rapid burning), it will usually be necessary to consider the diffusion of a puff that has some finite initial volume. This can be done by combining an initial volume dilution with the atmospheric value. In this case the equation for the concentration at the puff center, where $x = \bar{u}t$, $y = z = 0$, and V is the initial volume, is

$$\chi_p(x,y,z) = 2^{-1/2}\pi^{-3/2}Q\,(\sigma_{xI}\sigma_{yI}\sigma_{zI} + V)^{-1} \quad (3.155)$$

The consequence to a receptor subjected to the passage of an airborne cloud of radioactive or other contaminants is frequently expressed in terms of the integrated concentration, sometimes called the exposure, ψ. The exposure is the integral of the concentration over a specified time interval,

$$\psi_s = \int_{T_0}^{T_0+T} \chi_s\, d\tau \quad (3.156)$$

where T refers to the time of exposure and the subscript s is introduced as a reminder that sources of different types may be involved. The average concentration over the interval T is found by dividing ψ_s by T. This concept would be too obvious to belabor were it not for the following interesting fact. Consider a puff, i.e., an instantaneous point source, as specified by Eq. 3.154. The total exposure that would be experienced by a receptor at a point (x,y,0) when the puff passes by is given by

$$\psi = \int_{x/\bar{u}}^{\infty} \chi(x-\bar{u}t,y,0)\, dt \quad (3.157)$$

If the usual assumption is made that the puff passes rapidly overhead so that σ_{yI} and σ_{zI} will be effectively constant during the time of puff passage (compare with Eqs. 3.113 and 3.114), it follows that

$$\psi = Q(\pi\sigma_{yI}\sigma_{zI}\bar{u})^{-1} \exp\left[-\left(\frac{y^2}{2\sigma_{yI}^2} + \frac{h^2}{2\sigma_{zI}^2}\right)\right] \quad (3.158)$$

Thus the equation for the exposure is seen to have the same mathematical form as the equation for the continuous-plume concentration, and methods of calculation that provide the latter also can be used to compute the former.

The crosswind integrated concentration from an instantaneous point source may be obtained by integrating Eq. 3.142 with respect to y from $-\infty$ to ∞. The resulting equation is

$$\chi_{CWI} = \frac{Q}{\pi\sigma_{xI}\sigma_{zI}} \exp\left\{-\left[\frac{(x-\bar{u}t)^2}{2\sigma_{xI}^2} + \frac{h^2}{2\sigma_{zI}^2}\right]\right\} \quad (3.159)$$

The equation for the exposure from a crosswind integrated instantaneous point source follows from integration of Eq. 3.158:

$$\psi_{CWI} = \left(\frac{2}{\pi}\right)^{1/2} \frac{Q}{\sigma_{zI}\bar{u}} \exp\left(-\frac{h^2}{2\sigma_{zI}^2}\right) \quad (3.160)$$

A diffusion equation of possible interest is that for an instantaneous infinite crosswind line source. Although an actual source of this type can be conceived only by a considerable exercise of the imagination, this mode of release is approximated by the effluent from a rapidly traveling rocket or an airplane or by the exhaust from an automobile traveling along a highway. The equations for concentration and exposure from the instantaneous infinite crosswind line source are identical with Eqs. 3.159 and 3.160, respectively, with the exception that the source strength, Q (amount), must be replaced by the appropriate line-source value, Q_L (amount per unit length).

3-3.5.9 Nonideal Characteristics of Atmospheric Diffusion. In all the previous discussion, it has been assumed that particles or gases diffusing in the atmosphere behave as if they were identical with ideal fluid particles or points. The diffusing material is, in effect, identical in its physical properties with the assumed fluid continuum, possessing neither extension, inertial, nor buoyant properties of its own. For practical purposes gases and submicron particles can be assumed to behave in this way and can therefore be expected to obey laws of diffusion calculated on such a basis.

On the other hand, many diffusing particles of interest lie in a size range that does not encourage the ideal fluid-point assumption. Moreover there are certain removal processes, e.g., deposition, washout, and radioactive decay, that can significantly affect diffusion, and it is desirable to give these processes some consideration. In most cases it will not be possible to go much farther than a qualitative description of the significant physical processes, which are often characterized by considerable complexity and subtlety and for which in many cases an extended theoretical treatment is not yet available.

A heavy diffusing particle (i.e., one that does not follow the ideal fluid-point assumption) falls under the action of gravity. In the absence of turbulent mixing, the particle reaches a terminal velocity given quite accurately by Stokes' law. In the presence of turbulence, however, this orderly settling process is markedly changed. This conclusion follows from the readily observable fact that particles of various kinds are present in the atmosphere in equilibrium amounts, having diameters such that they would rapidly settle out if Stokes' law applied.

The problem of the diffusion of heavy particles in a turbulent fluid turns out to be very difficult in theory. Physically, the reason is that the path of such a particle is not a function of any particular set of boundary or initial conditions. Rather, the problem has to be formulated in a way that recognizes that the path of the particle at any instant depends continuously on its trajectory during its prior travel through the turbulent medium. The integro-differential equations resulting from this formulation are not easily simplified. In addition to gravitational settling, the effects of particle inertia, the inertia of the displaced air, and the possible boundary-layer effects on the particles come into play. It is not surprising that there are few reliable, practically useful results on the turbulent diffusion of heavy particles. The reader interested in fundamental aspects should be aware of Tchen's (1947) formulation and Lumley's (1957) discussion. Applied studies have been presented by F. B. Smith (1959), Yudine (1959), and Liu (1956).