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HIGH-ENERGY BEHAVIOR OF GAUGE THEORY TREE GRAPHS*

by

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ABSTRACT

The unitarity bounded high energy behaviour of Born graphs for production of two massive vector mesons is shown to result from the decomposition of the amplitude into a locally gauge invariant part and a suppressed remainder. In addition, asymptotic helicity conservation is found to emerge.

One of the interesting aspects of recent unified weak-electromagnetic gauge theories is the resultant "good" high energy behaviour of Born graphs for processes like $\nu_e \bar{\nu}_e \to W^+ W^-$. In the present note we point out that this effect has its origin in the gauge invariant couplings of the particles involved, and furthermore leads to an asymptotic helicity conservation for these processes.

Gell-Mann, Goldberger, Kroll, and Low[1] used the electron-exchange Born diagram for $\nu_e + \bar{\nu}_e \to W^+$(longitudinal)$+W^-$(longitudinal) as a test case for illustrating the "bad" high energy behaviour of the usual weak interaction theory. They pointed out that the amplitude f for this process, normalized according to $\frac{d\sigma}{d\Omega} = |f|^2$, behaves like the CM energy of one particle, E at large E. However to avoid an obvious contradiction with unitarity, f should decrease with E at least as fast as $\frac{1}{E}$. While it might be argued that the Born graph does not represent the whole theory it would nevertheless be a desirable feature if the perturbation approach to weak interactions would be as suitable as the analogous approach to electromagnetic interactions (which doesn't suffer from the above difficulty). In the SU(2) x U(1) gauge theory, Weinberg[2] noted that this problem is solved because there is another diagram corresponding to $\nu_e \bar{\nu}_e \to$virtual$\to Z \to W^+ W^-$, where Z is a neutral intermediate boson having a Yang-Mills coupling to $W^+ W^-$. This diagram's contribution just cancels the leading E dependence from the original diagram and results in $f \sim \frac{1}{E}$.

To start things off it is instructive to give the results of an extension of Weinberg's calculation on $\nu_e \bar{\nu}_e \to W^+ W^-$ to the cases where the W^+ and W^- may have any combination of longitudinal or transverse polarizations. In the list below the final states are ordered according to the magnitude of the total helicity, $|h_{tot}| = |h(W^+)+h(W^-)|$ and f is written as f_1 (due to electron exchange) + f_2(due to Z diagram). G is the Fermi constant and θ is the scattering angle.[3]

final states	f_1	f_2
$\|h_{tot}\| = 0$, long.-long.	$\frac{-G}{2\sqrt{2}\pi} E\sin\theta + 0(\frac{1}{E})$,	$\frac{+G}{2\sqrt{2}\pi} E\sin\theta + 0(\frac{1}{E})$
$\|h_{tot}\| = 0$, trans.-trans.[4]	$\frac{\mp G m_W^2}{4\sqrt{2}\pi} \frac{\sin\theta}{E}(\frac{1\pm\cos\theta}{1-\cos\theta}) + 0(\frac{1}{E^3})$,	0
$\|h_{tot}\| = 1$, long.-trans.[4]	$\frac{G m_W}{4\pi}(\cos\theta\pm 1) + 0(\frac{1}{E^2})$	$\frac{-G m_W}{4\pi}(\cos\theta\pm 1) + 0(\frac{1}{E^2})$
$\|h_{tot}\| = 2$, trans.-trans.	$\frac{-G m_W^2}{4\sqrt{2}\pi} \frac{\sin\theta}{E} + 0(\frac{1}{E^3})$,	$\frac{G m_W^2}{4\sqrt{2}\pi} \frac{\sin\theta}{E} + 0(\frac{1}{E^3})$

We see from the above list that in the old weak interaction theory (just f_1) those amplitudes involving longitudinal final particles would violate unitarity at large E. (Essentially this is because the longitudinal polarization vector contains a factor E while the transverse polarization vectors do not.) However, the f_2 terms are seen to bring these amplitudes back in line. Furthermore, there is another cancellation which makes the $|h_{tot}|=2$ amplitudes fall away faster with E. All these results can be summarized in the following formula for the leading energy dependence of f

$$f \sim E^{-1-|h_{tot}|} \tag{1}$$

According to Eq. (1) the amplitudes with $h_{tot} = 0$ final states will dominate at large energy and will behave as $\frac{1}{E}$. Since the initial $\nu_e \bar{\nu}_e$ state has $h_{tot} = 0$ we see that these diagrams are giving <u>asymptotic helicity conservation</u>. This is an amusing feature and may indicate that gauge type theories provide a mechanism for explaining the phenomenological helicity conservation observed in some hadronic interactions.[5]

In order to get some insight into what is going on and to see the role of Yang-Mills couplings in producing the good high energy behaviour for this process we shall now consider the kinematically simpler analogous reaction

$$\pi^+(p_1) + \pi^-(p_2) \to \rho^+(k_1,\varepsilon_1) + \rho^-(k_2,\varepsilon_2)$$

with Yang Mills couplings but with massive ρ-mesons. The three Born diagrams are shown in Fig. 1 and the relevant part of the interaction Lagrangian is

$$\mathcal{L}_{int} = \frac{ig}{2}\,\mathrm{Tr}(\rho_\mu \phi \overleftrightarrow{\partial}_\mu \phi) + \frac{ig}{2}\,\mathrm{Tr}[(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)\rho_\mu \rho_\nu]$$

$$+ \frac{g^2}{4}\,\mathrm{Tr}(\phi\rho_\mu\phi\rho_\mu - \phi\phi\rho_\mu\rho_\mu) + \ldots \tag{2}$$

where g is a coupling constant and the 2 x 2 matrices ϕ and ρ_μ are defined in terms of the Pauli matrices $\underset{\sim}{\tau}$ by $\phi = \frac{1}{\sqrt{2}}(\underset{\sim}{\phi}\cdot\underset{\sim}{\tau})$, $\rho_\mu = \frac{1}{\sqrt{2}}(\underset{\sim}{\rho}_\mu\cdot\underset{\sim}{\tau})$. It is then straightforward to calculate the amplitude:

$$f = \frac{1}{16\pi E}\sqrt{\frac{|\underset{\sim}{k}_1|}{|\underset{\sim}{p}_1|}}\;\bar{\varepsilon}_{1\alpha} M_{\alpha\beta}\,\bar{\varepsilon}_{2\beta} \tag{3a}$$

$$M_{\alpha\beta} = \frac{ig^2}{2}\Bigg\{\frac{1}{p\cdot k_1 - k_1\cdot k_2}\,[p_\alpha p_\beta - k_{1\beta}k_{2\alpha} + p_\alpha k_{1\beta} - k_{2\alpha}p_\beta]$$

$$+ \frac{2}{2k_1\cdot k_2 - m_\rho^2}\,[-p\cdot k_1\,\delta_{\alpha\beta} + p_\alpha k_{1\beta} - k_{2\alpha}p_\beta] - \delta_{\alpha\beta}\Bigg\} \tag{3b}$$

where $p = p_2 - p_1$ and $\bar{\varepsilon}_{1\alpha} = (-1)^{\delta_{\alpha 4}}\varepsilon^*_{1\alpha}$ etc. To evaluate f for various helicities in the final state it is sufficient[3] to make the following explicit choices of momenta:

$$p_1 = (0,0,|\underset{\sim}{p}_1|,iE) \quad , \quad k_1 = (0,|\underset{\sim}{k}_1|\sin\theta,|\underset{\sim}{k}_1|\cos\theta, iE)$$
$$p_2 = -p_1^{\ *} \quad , \quad k_2 = -k_1^{\ *} \tag{4a}$$

and of polarization vectors:

$$\bar{\varepsilon}_1^{\,L} = (0,\frac{E}{m_\rho}\sin\theta,\frac{E}{m_\rho}\cos\theta,\frac{i|\underset{\sim}{k}_1|}{m_\rho}) \quad , \quad \bar{\varepsilon}_1^{\,+} = \frac{1}{\sqrt{2}}(-i,-\cos\theta,\sin\theta,0)$$
$$\bar{\varepsilon}_1^{\,-} = (\bar{\varepsilon}_1^{\,+})^* \quad , \quad \bar{\varepsilon}_2^{\,L} = -(\bar{\varepsilon}_1^{\,L})^* \quad , \quad \bar{\varepsilon}_2^{\,+} = i\bar{\varepsilon}_1^{\,-} \quad , \quad \bar{\varepsilon}_2^{\,-} = -i\bar{\varepsilon}_1^{\,+} \tag{4b}$$

In (4b) the superscripts +, L, and - stand for helicity =1,0, and -1 objects respectively. When Eqs. (4) are substituted into Eqs. (3) we find that miraculous cancellations occur again to give exactly the same energy dependences as a function of total final state helicity as for the $\nu\bar{\nu}\to W^+W^-$ case (Eq. (1)). Thus since the initial state has zero total helicity asymptotic helicity conservation again emerges, in addition to the good high energy behaviour. Note that in the $\pi^+\pi^-\to\rho^+\rho^-$ case the cancellations are taking place among the contributions from three rather than two diagrams. It is thus clear that the explanation for the good high energy behaviour is not most readily to be found in the topological structure of the diagrams involved.

The clue to understanding the matter on more general grounds comes from decomposing $M_{\alpha\beta}$ of Eq. (3b) in the following way:

$$M_{\alpha\beta} = M^{\circ}_{\alpha\beta} + M'_{\alpha\beta} \tag{5a}$$

$$M^{o}_{\alpha\beta} = \frac{ig^2}{2k_1\cdot k_2}\left\{-(x+1)(k_1\cdot k_2\delta_{\alpha\beta}-k_{2\alpha}k_{1\beta}) + \frac{1}{x-1}(p_\alpha - xk_{2\alpha})(p_\beta + xk_{1\beta})\right\} \quad (5b)$$

$$M'_{\alpha\beta} = \frac{ig^2}{2}\frac{m_\rho^2}{k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 - m_\rho^2}(-\delta_{\alpha\beta}p\cdot k_1 + p_\alpha k_{1\beta} - k_{2\alpha}p_\beta) \quad (5c)$$

where

$$x = p\cdot k_1/k_1\cdot k_2 = -p\cdot k_2/k_1\cdot k_2.$$

$M'_{\alpha\beta}$ has the property that it goes to zero as $m_\rho^2 \to 0$. Thus in a completely local gauge invariant theory ($m_\rho=0$) we would only have the $M^{o}_{\alpha\beta}$ term. This term is manifestly gauge invariant, satisfying:

$$k_{1\alpha}M^{o}_{\alpha\beta} = M^{o}_{\alpha\beta}\, k_{2\beta} = 0 \quad (6)$$

First consider the leading energy dependences for the contributions to f from $M'_{\alpha\beta}$. Because the factor $m_\rho^2/k_1\cdot k_2 \sim -m_\rho^2/2E^2$ all these terms are suppressed at large E by $1/E^2$ from what they would otherwise be; explicitly:

final states		non-gauge invariant part of f
$\lvert h_{tot}\rvert = 0$	long.-long.	$0(1/E)$
$\lvert h_{tot}\rvert = 0$	trans.-trans.	$0(1/E^3)$
$\lvert h_{tot}\rvert = 1$	long.-trans.	$0(1/E^2)$
$\lvert h_{tot}\rvert = 2$	trans.-trans.	$0(1/E^3)$

All these contributions have good high energy behaviour! The above pattern of suppression follows generally when it is realized that an overall factor like m_ρ^2/E^2 must be present for the non-gauge invariant part of the amplitude. This is because m_ρ^2 only appears explicitly in a propagator denominator[(6)] which we may write as

$$\frac{1}{s-m_\rho^2} = \frac{1}{s} + \frac{m_\rho^2}{s}\,\frac{1}{s-m_\rho^2}$$

thereby separating its contribution to $M^{o}_{\alpha\beta}$ and to $M'_{\alpha\beta}$.

Next consider the gauge invariant object $M^{o}_{\alpha\beta}$. By covariance, the most general solution of (6) may be written as:

$$M^{o}_{\alpha\beta} = A(k_1 \cdot k_2 \delta_{\alpha\beta} - k_{2\alpha} k_{1\beta}) + B(p_\alpha - x k_{2\alpha})(p_\beta + x k_{1\beta}) \qquad (7)$$

when A and B are functions of the invariants $k_1 \cdot k_2$ and x. In the present case $A = \frac{-ig^2}{2} \frac{x+1}{k_1 \cdot k_2} = (1-x^2)B$ and both A and B fall off as $1/E^2$ at large energies. This fall-off is to be expected for dimensional reasons since A and B should not explicitly contain any particle masses if they come from Born terms of a (zero-mass) Yang-Mills theory. For the sake of generality we may now pretend that we don't know A and B and prepare using (4) and (7), the following list for the leading gauge invariant contribution to the various amplitudes:

final states	gauge invariant part of f
$\lvert h_{tot}\rvert = 0$, long.-long.	$\frac{m_\rho^2}{16\pi E}(A - B\cos^2\theta) = 0(\frac{1}{E^3})$
$\lvert h_{tot}\rvert = 0$, trans.-trans.	$\frac{-iB\sin^2\theta E}{8\pi} = 0(\frac{1}{E})$
$\lvert h_{tot}\rvert = 1$, long.-trans.	$\frac{-\sqrt{2}Bm_\rho \sin\theta\cos\theta}{16\pi} = 0(\frac{1}{E^2})$
$\lvert h_{tot}\rvert = 2$, trans.-trans.	$\frac{i}{16\pi E}[2E^2(A - B\sin^2\theta) + (2Bm_\pi^2\sin^2\theta - Am_\rho^2)] = 0(\frac{1}{E})$

We see that all the gauge invariant contributions to f also fall off fast enough to satisfy unitarity. In this case those amplitudes involving longitudinal particles in the final states are suppressed. This is heuristically reasonable since $M^{o}_{\alpha\beta}$ is essentially the amplitude for production of zero-mass vector particles which of course can only be transverse. Thus, both the gauge invariant and non-gauge invariant parts of the Born amplitude for $\pi^+\pi^- \to \rho^+\rho^-$ have been seen, in a fairly general way, to possess good high energy behavior. The crucial point was the separation (Eg. (5a)) of the amplitude into a locally gauge invariant part and a suppressed remainder.

How can we understand the asymptotic helicity conservation in general terms? From the last two lists we see that the $h_{tot}=0$ final state amplitudes will fall off as 1/E while the $|h_{tot}|=1$ amplitudes will go as $1/E^2$. This fits in nicely with Eq. (1). However the $|h_{tot}|=2$ amplitudes will fall off as 1/E, which would violate Eq. (1) and asymptotic helicity conservation, unless we impose the condition

$$A \sim B\sin^2\theta \text{ (CM system)} \tag{8}$$

Eq. (8) is of course satisfied for $M^o_{\alpha\beta}$ represented by Eq. (5b), since $(1-x^2)\to\sin^2\theta$ at large E. Eq. (8) represents a detailed property of the Yang-Mills coupling scheme.

Incidentally, the general Eq. (7) also gives the Born amplitude for the locally U(1) gauge invariant process $\pi^+\pi^-\to 2\gamma$, when we make the identification $A = -2ie^2/k_1\cdot k_2 = (1-x^2)B$. In this case the final photons can only be transverse. Since Eq. (8) is satisfied there will be asymptotic helicity conservation, the $|h_{tot}|=2$ final state amplitudes being suppressed by $1/E^2$ compared to the $h_{tot}=0$ ones.

Finally, it might be interesting to investigate in detail under what conditions asymptotic helicity conservation and good high energy behaviour also emerge for other processes involving gauge bosons and to attempt to apply these ideas to strong interactions. We would like to thank A. P. Balachandran and H. Rupertsberger for helpful discussons.

FOOTNOTES AND REFERENCES

1. Gell-Mann, Goldberger, Kroll, and Low, Phys. Rev. 179, 1518(1969).

2. S. Weinberg, Phys. Rev. Letters 27, 1688(1971). We have also checked the analogous cancellation for an SU(3)xU(1) gauge theory: J. Schechter and Y. Ueda, Phys. Rev. (to be published).

3. For simplicity we are suppressing the overall phase factor which contains the azimuthal (ϕ) dependence of the amplitude by setting $\phi = \frac{\pi}{2}$.

4. The ± sign corresponds to different transverse polarization assignments within the given case. Note that the E dependence is not affected by this.

5. For example,

Aachen et.al., collaboration, Phys. Rev. 175, 1669(1968).

J. Ballam et.al., Phys. Rev. Letters 24, 960(1970).

6. The term $k_\mu k_\nu/m_\rho^{\ 2}$ in the propagator numerator is easily seen to make no contribution to the Born diagram.

FIGURE CAPTION

Fig. 1: Born diagrams for $\pi^+\pi^- \to \rho^+\rho^-$ in a Yang-Mills Theory.

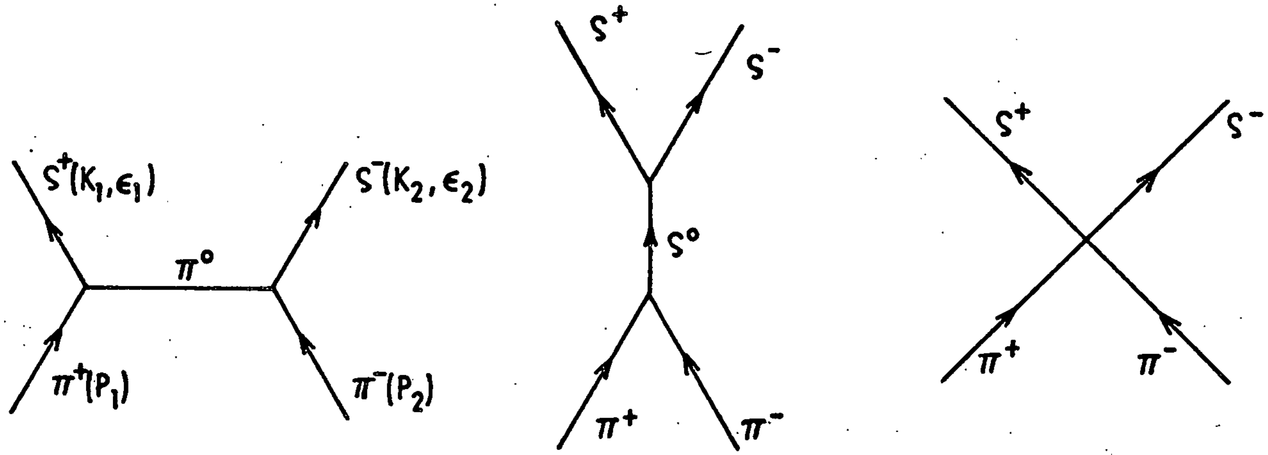

Fig. 1