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MAGNETIC INTERACTIONS ON THE BASIS OF A MODIFIED SHELL MODEL OF COMPLEX NUCLEI

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Magnetic Interactions on the Basis of a Modified Shell Model of Complex Nuclei

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The independent particle shell model of M. G. Mayer and Axel, Jensen and Suess does not give quantitative agreement with the measured static magnetic moments of most odd-even complex nuclei. Furthermore, the model gives qualitative disagreement with certain measured magnetic dipole (M1) isomeric transitions. Recent attempts to obtain better agreement with this magnetic interaction data by the use of exchange moments, i.e. meson effects, has proved only moderately successful. Therefore, an attempt has been made to modify the usual shell model by considering more complicated wave functions.

Specifically, the ground state wave function of odd-even nuclei is assumed to be an admixture of states fairly closely related to the shell model state. These states share the total angular momentum of the nucleus among three particles, in a prescribed manner, as compared to the shell model state in which one particle has the total angular momentum of the nucleus. Two of the particles sharing the angular momentum are in equivalent states and are coupled to an angular momentum of two, and are in turn coupled to a

third particle so as to give the correct angular momentum and parity of the nuclear wave function. In the Mayer-Jensen shell model the two particles in equivalent states are assumed to have zero angular momentum. This particular coupling scheme is suggested by the observed angular momentum and parity of most of the first excited states of even-even nuclei.

The possible admixed states are assumed to be determined by the last filled shells for protons and neutrons in the odd-even nucleus, and these states are assumed to occur with equal probability in the nuclear wave function. However, the contribution of states in which the two particles in equivalent states are of the even type in the odd-even nucleus is limited by a restriction on the coupling to the third odd particle. This limitation on the sharing of the total angular momentum with the even particles is suggested by the experimentally observed near equality of the magnetic moment deviations of odd proton and odd neutron nuclei with the same number of odd particles (the number of even particles is generally different for the nuclei). The modified shell model represents a synthesis of the usual independent particle shell model and the statistical model of Margenau and Wigner.

Magnetic moments are calculated for the simple three particle wave functions for all possible couplings. These values are then used to obtain the magnetic moments of all measured odd-even nuclei. In the calculation all interference effects are neglected. The magnetic moments obtained in this manner represent a considerable improvement in fitting the data as compared to the magnetic moments calculated on the basis of the Mayer-Jensen shell model.

Several magnetic dipole (M1) isomeric transitions have been observed which should be forbidden according to the classification of the nuclear states obtained from the Mayer-Jensen shell model, provided the ordinary magnetic moment operator is assumed to be responsible for the transition (the transition matrix element is zero because of a selection rule on the orbital angular momentum). The transitions are no longer forbidden if the nuclear wave functions are described by the modified shell model. The transition matrix elements are calculated for the five observed transitions and they are found to be of the same order of magnitude as the experimentally observed transition matrix elements, with very good agreement in three transitions.

The improved agreement with magnetic interaction data obtained by use of the modified shell model, indicates that the Mayer-Jensen shell model gives an inadequate description of the nuclear ground state wave functions. The modified shell model may represent a considerable improvement in describing the nuclear wave functions, but it must be emphasized that both interference effects and exchange effects have been neglected in the analysis of the data.

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I. INTRODUCTION

One of the most accurately measured properties of a nucleus is the magnetic moment. Because of this and the relatively simple nature of the magnetic moment operator, it is especially interesting to try to interpret the observed values theoretically. The aim of such a theoretical study would be to learn more about the nature of the wave functions of complex nuclei and from a knowledge of these wave functions to increase our understanding of the nuclear force problem.

Unfortunately the basic simplicity of the theory is marred by various complications, for example, relativistic effects and exchange effects. These effects modify the form of the magnetic moment operator in an ambiguous manner. Thus the magnetic moment data cannot be expected to give unambiguous information concerning the nuclear wave functions. However, certain features of the odd-even (odd Z even N, or even Z odd N) nuclear magnetic moments appear to depend primarily on the nuclear wave functions rather than any modification of the magnetic moment operator. These features of the magnetic moments and their relation to the nuclear wave functions will constitute the main topic of this thesis.

In order to make a theoretical calculation of the magnetic moments of complex nuclei it is necessary to use some specific model of the nucleus. The starting point of this study will be the independent particle shell model as suggested by Goeppert-Mayer[1] and by Haxel, Jensen, and Suess[2]. This model has had great success in accounting for the "magic" numbers 2, 8, 20, 50, 82, and 126 as well as successfully assigning the correct spin to the ground state of odd-even complex nuclei. An essential feature of the model is a specific assignment of the parity of the ground state as well as the spin.

However, the model does more than just assign a given spin to the ground state. The model specifically assigns this spin to only one nucleon of the odd numbered nucleons in the odd-even nucleus. All other nucleons are assumed to couple to zero angular momentum. In its simplest form the model implies that any two identical particles in the same state of angular momentum j and orbital angular momentum l couple to a total angular momentum $j' = 0$.

A calculation of the magnetic moment is very simple for the type of coupling just described. The magnetic moment of the nucleus in this case is just the magnetic moment of a single particle with total angular momentum and orbital angular momentum equal to that of the complex nucleus. For a given orbital angular momentum L, the total angular momentum J can assume the values $J = L \pm \frac{1}{2}$. If a plot is made of

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[1] M. Goeppert-Mayer, Phys. Rev. 78, 16 (1950).

[2] Haxel, Jensen, and Suess, Z. f. Phys. 128, 295 (1950).

the calculated magnetic moment of a proton or of a neutron for various values of J, two lines are obtained for each type of particle, one line for $J = L + \frac{1}{2}$ and the other line for $J = L - \frac{1}{2}$. These lines are usually referred to as the Schmidt lines. These plots, shown in Fig. 1 and Fig. 2, are only meaningful for half integer values of J.

If the shell model were strictly valid, the magnetic moments of all odd-even nuclei would fall on the appropriate Schmidt lines. A plot of the data, Fig. 1 and Fig. 2, shows that such cases are certainly rare, although most of the points lie decidedly closer to one or the other of the Schmidt lines. It is especially interesting to note that with the exception of five nuclei, H^3, He^3, C^{13}, N^{15}, and Os^{189}, all the points fall between the two Schmidt lines, and that three of the exceptions, C^{13}, N^{15}, and Os^{189}, have $J = L - \frac{1}{2} = \frac{1}{2}$ and have only minute deviations from the appropriate Schmidt line. The deviation of the two quite pronounced exceptions, H^3 and He^3, outside of their appropriate Schmidt line is believed to be due primarily to exchange moments[3].

At first sight the magnetic moment data appears to confirm the validity of the independent particle shell model. However, an entirely different model of the nucleus, the Margenau-Wigner[4] model, in which the angular momentum is

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[3]F. Villars, Phys. Rev. 72, 257 (1947); Helv. Phys. Acta 20, 476 (1947); R. G. Sachs, Phys. Rev. 72, 312 (1947); R. Avery and R. G. Sachs, Phys. Rev. 74, 1321 (1948).

[4]Margenau and Wigner, Phys. Rev. 58, 103 (1940).

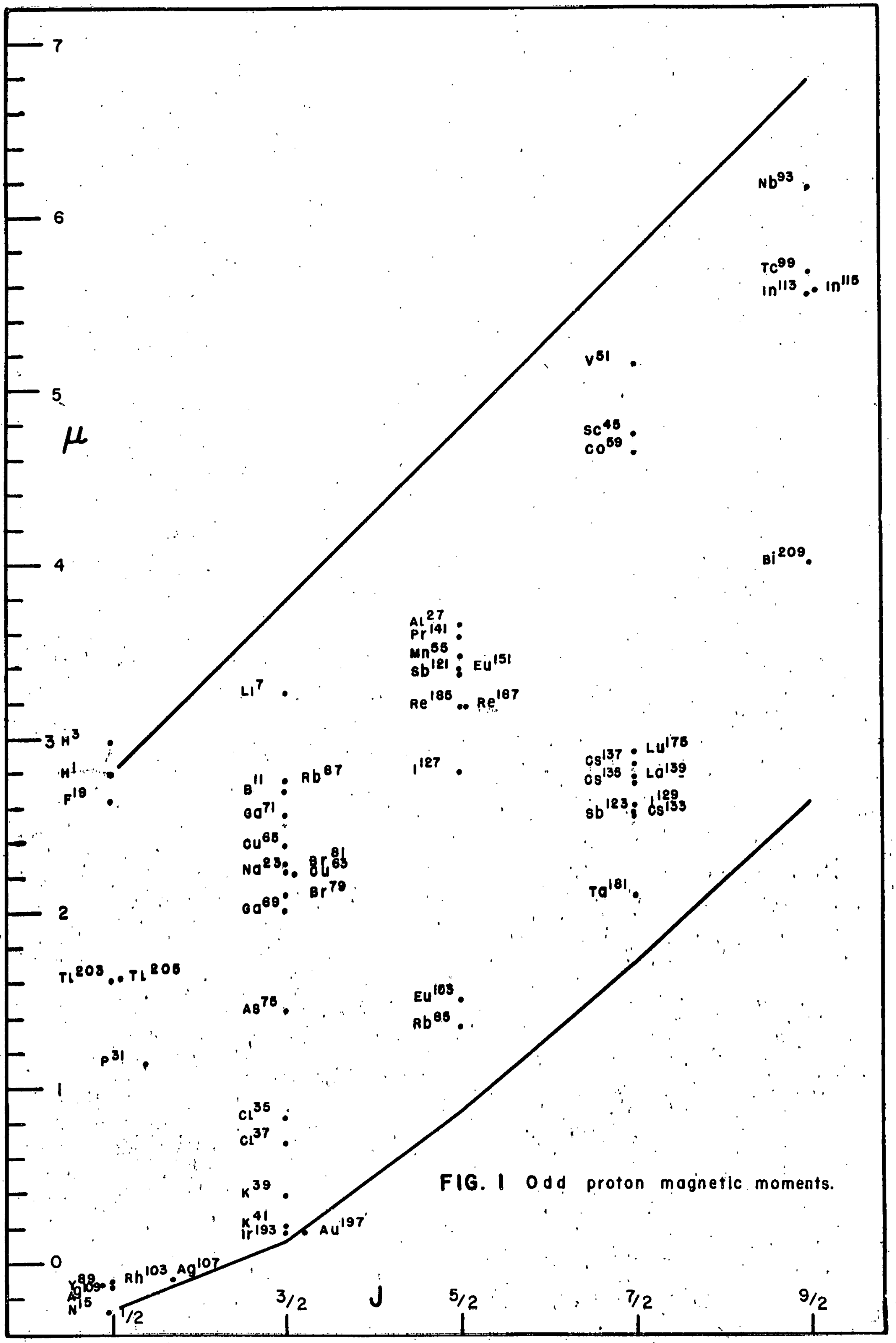

FIG. 1 Odd proton magnetic moments.

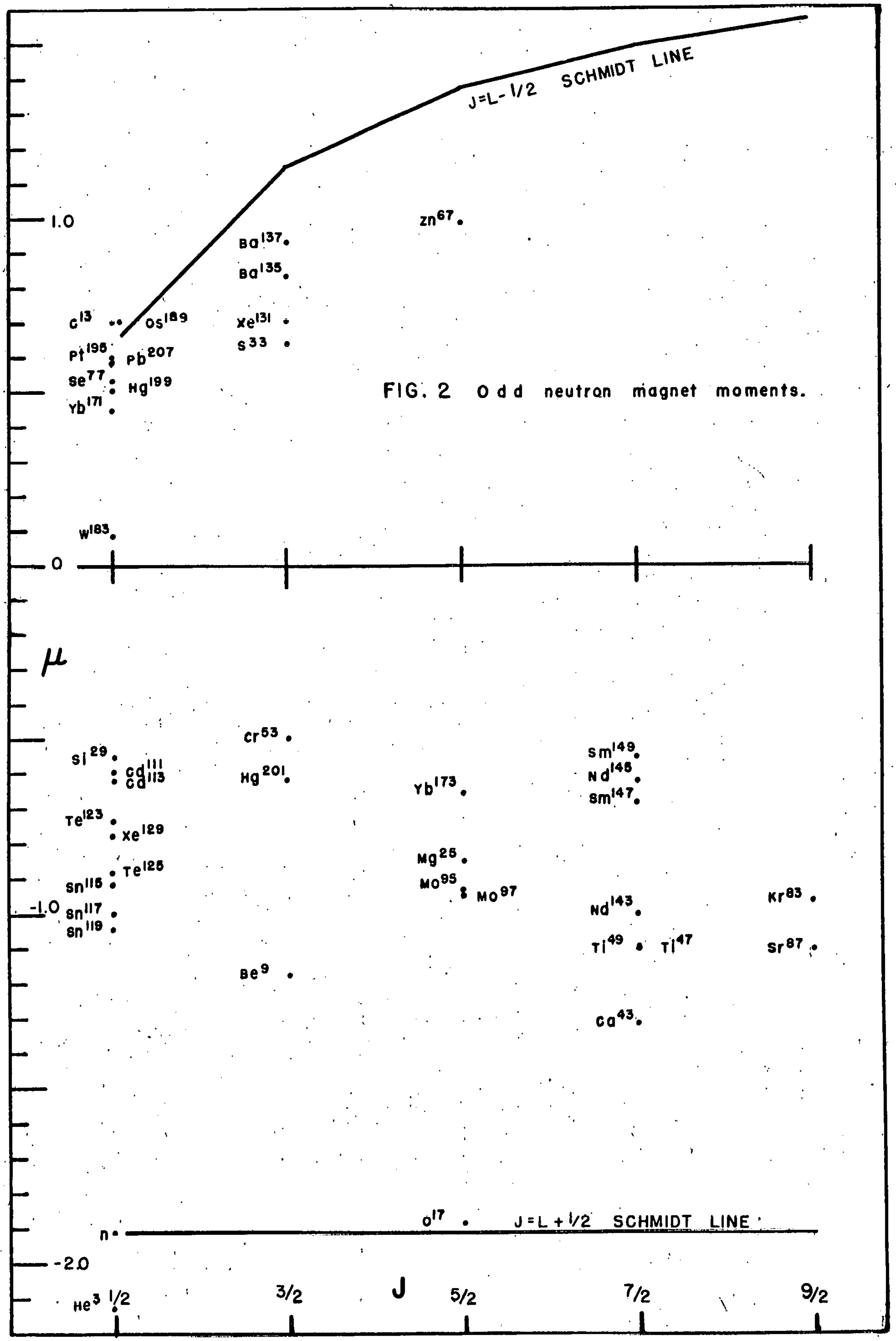

FIG. 2 Odd neutron magnet moments.

shared among all the particles of the nucleus, gives almost as good a description of the magnetic moment data. Thus relatively small deviations of the magnetic moments from the Schmidt lines may indicate wave functions which are radically different from the shell model wave functions.

On the other hand the wave functions may be adequately described by the shell model, and as indicated before, the deviation would then be due to modified magnetic moment operators caused by exchange effects, i.e. the effects of meson currents in the nucleus. This has been the point of view adopted in several investigations[5]. M. Ross considered the question of magnetic moment deviations in great detail with special emphasis on those exchange moments which are capable of explaining the H^3 - He^3 magnetic moment anomalies. The theoretical magnetic moment deviations calculated on the basis of these exchange moment operators were found to be small compared to the experimentally observed deviations, and the calculated deviations were all found to have the incorrect sign (deviations outside the Schmidt lines). Ross finally concluded that, although magnetic moment deviations due to exchange effects may exist in complex nuclei, these deviations are masked by much more important deviations due to distortions of the nuclear wave function away from the pure shell model wave function.

[5] H. Miyazawa, Prog. Theoret. Phys. 6, 263 (1951); A. De Shalit, Helv. Phys. Acta 24, 296 (1951); F. Bloch, Phys. Rev. 83, 839 (1951); A. Russek and L. Spruch, Phys. Rev. 87, 1111 (1952); Marc Ross, Phys. Rev. 88, 939 (1952).

It may be argued that the exchange magnetic moment operators in complex nuclei are quite different from those needed to explain the H^3 - He^3 magnetic moment anomalies. However, if the operators are assumed to be two body operators (corresponding to the assumption of two body forces among nucleons), then it has been demonstrated by Ross that the exchange magnetic moment operators must be of the same form as those considered previously to explain the H^3 - He^3 magnetic moment deviations. Thus simple two body exchange magnetic moment operators seem incapable of explaining the data.

It is possible that the exchange magnetic moment operators are many body operators rather than simple two body operators. However, many body operators would be expected to produce magnetic moment deviations which vary only slightly from nucleus to nucleus, especially in heavier nuclei. While the observed deviations of the heavier nuclei seem to indicate some regularity, the fluctuations of the magnetic moment deviations from nucleus to nucleus, especially in those odd-even nuclei with 41 to 65 odd nucleons, seem much too great to be consistent with this sort of explanation of the magnetic moment deviations.

These qualitative arguments against many body operators are reinforced by consideration of certain isomeric magnetic dipole transitions[6,7]. These particular transitions are forbidden on the basis of shell model wave functions and the

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[6] R. G. Sachs and M. Ross, Phys. Rev. 84, 379 (1951); Marc Ross, Phys. Rev. 88, 935 (1951).

[7] R. L. Graham and R. E. Bell, Can. J. of Phys. 31, 377 (1953).

ordinary magnetic moment operator. A typical isomeric transition of this type is found in Te^{133} (or Te^{135}) where the excited state (according to the shell model) is believed to be a d 3/2 state and the ground state is (also according to the shell model) an s 1/2 state. This transition has angular momentum changes of $\Delta L = -2$ and $\Delta J = -1$, where ΔL indicates the change of orbital angular momentum and ΔJ indicates the change of total angular momentum. The ordinary magnetic dipole operator only connects states in which $\Delta L = 0$.

The exchange magnetic dipole operators do not have this limitation on ΔL and the occurrence of the "forbidden" (subject to the purity of the two states) transition can be explained by the presence of these operators in the total magnetic moment operator. Ross used the same operators as in his magnetic moment deviation calculations and he found that the measured lifetimes of the forbidden transitions are consistent with the exchange moment operators needed to account for the H^3 - He^3 magnetic moment anomalies.

These results, at first, seem to confirm the existence of exchange moments in complex nuclei. However, "forbidden" transitions of the above type can also occur if the wave functions of the two levels are not pure shell model functions but are a suitable admixture of states. Such a possibility is especially interesting since the nuclei concerned in these forbidden transitions have considerable magnetic moment deviations. Both the magnetic moment deviations and the forbidden transitions may be explainable in terms of a suitable admixture of states rather than any exchange magnetic moment operator. This is the point of view taken in this thesis so forbidden transitions will be studied in conjunction with the magnetic moment deviations.

The results of the exchange magnetic moment analysis of the forbidden magnetic moment transitions are still of considerable interest. Although it is not possible to establish the absolute existence of the exchange magnetic moment operators on the basis of the forbidden transitions, interaction effects (including many body operators) capable of explaining the magnetic moment deviations do appear to be excluded since these would be expected to produce shorter lifetimes than measured. Thus the forbidden transitions may be used to set an upper bound on the contribution of exchange effects[8] in heavier nuclei[9].

The question of the nature of the nuclear wave functions is seen to be one of the primary importance in dealing with

[8] This statement is only partially true since only two of four possible exchange interactions considered by Ross give any significant contribution to the transition probabilities. Thus an upper bound is not especially meaningful for the other two exchange interactions, which could increase significantly in heavy nuclei without any contradiction to the above analysis.

[9] The previous discussion indicates the rather tenuous nature of the evidence for the existence of exchange effects in complex nuclei, although there is every reason to believe that these exchange effects do exist on the basis of both the H^3 - He^3 magnetic moment anomalies and meson theory. Any further test for the existence of exchange effects in complex nuclei would be very valuable. Such a test could possibly be offered by an appropriate analysis of angular correlation data.

magnetic dipole moment data. This thesis will disregard all exchange effects and concentrate on the study of static magnetic moment deviations in relation to the nuclear wave functions. The independent particle shell model of Jensen and Mayer is taken as a starting point. However, as previously mentioned, a small deviation of the magnetic moment from the Schmidt limits may indicate a nuclear wave function which differs considerably from that prescribed by the shell model.

In order to treat the problem of magnetic moment deviations systematically, it is necessary to consider a systematic modification of the nuclear wave functions. Because of the nature of the ordinary magnetic moment operator, it is only necessary to consider the angular part of the wave functions in all that follows.

The arrangement of levels in the shell model in terms of a single particle wave function can be considered valid as far as the symmetry properties of the nucleus are concerned. Any modification of the shell model wave function must be consistent with the symmetry properties of that function. Only wave functions with the same total angular momentum and parity can be admixed to form the nuclear state. However, it is entirely possible that the amplitude of the shell model state is quite small in the admixture of states which presumably constitute the nuclear wave function.

A simple modification of the Mayer-Jensen shell model state is obtained by distributing the angular momentum among three particles instead of concentrating it on one. It is generally assumed in the shell model that two identical particles with the same j and l couple to give angular momentum $j' = 0$. This coupling rule seems to be verified by the fact that all even-even nuclei appear to have total angular momentum $J = 0$. While the shell model prescription for coupling is very attractive in its simplicity, there is no reason to believe that the addition of another particle to an even-even nucleus will not rearrange the coupling in a more complicated fashion.

Any recoupling would require the rearrangement of at least two particles in the even-even core of the odd-even nucleus. Thus a simple modification of the shell model scheme would be to uncouple two identical particles which had been coupled to give $j' = 0$, and recouple them to some angular momentum $j' \neq 0$ without changing the j and l values of the nucleons. These two particles would then have to be coupled to the single particle of the shell model, i.e. the angular momentum j' would have to be coupled to the angular momentum J of the single particle so as to give the same angular momentum J for the complex nucleus. This modification of the shell model coupling scheme would of necessity have the same parity as the original state.

In a j-j representation the Pauli principle allows two identical particles in the same j and l states to couple into only even total angular momentum, i.e. $j' = 0, 2, 4$, etc. A reasonable modification of the shell model is to consider states where two identical particles are recoupled to give an angular momentum j' = 2. This choice seems to be justified on the basis of the experimental fact that the first excited states of even-even nuclei nearly always have total angular momentum J = 2 and positive parity[10].

The modification of the shell model state which is used in this thesis is more general than that just described, although the basic features are the same. As before, the angular momentum of the nucleus is shared among three particles, of which two are identical particles coupled to j' = 2 (both particles having the same j and l). However, the third particle is no longer restricted to the total angular momentum J and orbital angular momentum L of the shell model state. The third particle, which now has total angular momentum j" and orbital angular momentum l", is coupled with j' = 2 to give a total angular momentum J (the same J as for the shell model state) for the three particle system. Wave functions of this type will be used to calculate magnetic moments and these calculated moments will be compared with the observed deviations from the Schmidt lines of the magnetic moments of odd-even nuclei[11].

The actual wave function of the nucleus will be assumed

[10]Gertrude Scharff-Goldhaber, Phys. Rev. 90, 587 (1953).

[11]Magnetic moment deviations have been studied with similar but more restricted modifications of the shell model. A. DeShalit, Phys. Rev. 90, 83 (1953); Minoru Umezawa, Prog. Theoret. Phys. 8, 509 (1952).

to be a linear combination of the usual Mayer-Jensen shell model state and states of the type just described in which the angular momentum is shared among three particles. Extensive sharing of the angular momentum among the nucleons of the nucleus (Margenau-Wigner model) would be expected on the basis of the usually assumed short range, two body nuclear forces. However, the shell model, with its success in explaining much of the data concerning complex nuclei, is at the other extreme. The model in this thesis therefore represents a synthesis of the Margenau-Wigner model and the Mayer-Jensen shell model of the nucleus. The extent of the sharing of the angular momentum among different nucleons will depend on the assumed admixture of states and on the allowed values of j, ℓ and j'', ℓ'' used in constructing modified shell model states of three particles.

The two identical particles in equivalent states may be either odd (of the odd number type of particles in the odd-even nucleus) particles or even (of the even number type of particles in the odd-even nucleus) particles. Since these two particles are assumed to be coupled to $j' = 2$, then j'' could in principle assume any one of the possible values $(J + 2,\ J + 1,\ \text{-----}\ |J - 2|)$ allowed by the vector rule with the appropriate value of ℓ'' being determined by considerations of parity.

However, the Mayer-Jensen shell model is still assumed to determine a system of levels which will limit the allowable values of j, ℓ and j'', ℓ''. Specifically, each shell in the shell model is composed of a number of states of roughly the same energy separated from other such groupings of states by a noticeable energy difference. In any given nucleus the allowable values of j, ł and j", ł" are those which occur within the unclosed shells of the nucleus, i.e. sharing of the angular momentum is assumed to only occur in the last shell of odd particles and even particles.

When the two identical particles coupled to j' = 2 are even particles, the value for j" is arbitrarily restricted to the one value j" = J, the angular momentum of the nucleus. The restriction is assumed to hold for the sake of simplicity and on the basis of the following evidence which seems to indicate that the amount of angular momentum shared with the even particles is smaller than that shared among the odd particles[12].

Schawlow and Townes[13] have observed definite correlations (near equality in magnitude) of the deviations from the Schmidt lines of the magnetic moments of pairs of odd-even nuclei, with the same total angular momentum J, where one nucleus has n odd protons and the other has the same number n of odd neutrons. If the angular momentum J is shared only among the n odd particles and if the wave function representing the n odd protons is the same as that representing the

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[12] In the final model of the nucleus considered in this thesis, this particular restriction on the possible values of j" automatically decreases any sharing of the angular momentum with the even particles.

[13] A. L. Schawlow and C. H. Townes, Phys. Rev. 82, 268 (1951).

n odd neutrons (mirror property), then the calculated magnetic moment deviations (appropriately defined) are equal for the two nuclei regardless of the admixture of states contributing to the odd particle wave function. The experimentally observed correlations can therefore be understood qualitively by assuming that the even particles do not share the angular momentum and that odd proton states are the same as odd neutron states.

The explanation just given for the experimentally observed correlations is undoubtedly an oversimplification, especially since none of the experimental deviations for the nuclear pairs are exactly equal. It is also hard to understand how there can be any significant sharing of the angular momentum among the odd particles without some sharing among the even particles. However, the observed correlations seems to suggest that there must be a significant mirroring of the odd particle states regardless of the specific role played by the even particles.

The question of the role played by the even particles in sharing the angular momentum is complicated and will be discussed in greater detail in the main body of the thesis. However, the great difference in the number of even particles in the two nuclei of the pair (the odd proton nucleus would generally have a much larger atomic number A than the corresponding odd neutron nucleus) would seem to indicate that any mirroring of states (odd particle states included) would be improbable if the even particles had a great share of the total angular momentum. If the even particles do not play a too disturbing role, then the mirror property of the odd particle states is fairly reasonable on the basis of the assumed charge independence of nuclear forces.

In Section II of this thesis, the magnetic moments are calculated for three particle wave functions, of the type previously described, for all possible values of j, ł and j", ł". The main qualitative feature of the results is the fact that the great majority of the calculated moments fall between the two Schmidt limits. It then follows that many suitable admixtures of states can be formed to fit the experimental magnetic moments, the great majority of which also fall between the Schmidt limits.

A specific model of complex nuclei is proposed in Section III in which all possible three particle states (and the shell model state), as already prescribed, are assumed to have equal probability in the nuclear wave function. This represents a rather thorough sharing of the angular momentum among the particles of the unclosed shell of the odd particles and a less complete sharing of the angular momentum with the particles in the even unclosed shell. The magnetic moments calculated from this model, using appropriate assumptions for simplification, represent a dramatic improvement in fitting the data as compared to the usual Mayer-Jensen shell model.

Finally, in Section IV, the "statistical shell model" used in the previous section to calculate the static magnetic moments, is used to calculate the transition matrix elements of the "forbidden" magnetic dipole transitions. The transition matrix elements are also found to be in generally good agreement with experimental results.

The statistical shell model represents a rather extensive modification of the Mayer-Jensen shell model. The improved agreement with the experimental data of static magnetic moments and forbidden magnetic dipole transitions therefore indicates that the Mayer-Jensen shell model may give a poor description of the ground state wave functions of complex nuclei. The statistical shell model is closely related to the Mayer-Jensen shell model by the way the three particle states are chosen for admixture and by the choice of spin and parity which is that prescribed by the Mayer-Jensen shell model, but the statistical shell model seems to be more compatible with the assumed two body, short range nature of the nuclear forces.

II. MAGNETIC MOMENTS OF ODD-EVEN NUCLEI.

A. Schmidt Values of the Magnetic Moment.

The ordinary magnetic moment operator is given by

$$\vec{\mu} = \sum_{P} (\vec{j}_P + [\mu_P - \tfrac{1}{2}]\vec{\sigma}_P) + \mu_N \sum_{N} \vec{\sigma}_N \tag{II-1}$$

where μ_p (μ_p = 2.791 n.m.) and μ_n (μ_n = - 1.913 n.m.) are respectively the magnetic moments of a proton and a neutron, $\vec{j}_p$ is the single proton total angular momentum operator, and $\vec{\sigma}_p$ and $\vec{\sigma}_n$ are respectively the usual Pauli spin operators for a proton and a neutron. The magnetic moment of a nuclear system is defined as the expectation value of the z-component, μ^z, of the magnetic moment operator:

$$\mu = \langle \psi_J^J, \mu^z \psi_J^J \rangle \tag{II-2}$$

where ψ_J^M is the wave function of the nuclear system (only the angular part of the wave function is considered in all ordinary magnetic moment calculations) having a total angular momentum J and a z component of angular momentum M.

In the Mayer-Jensen independent particle shell model, a single odd particle assumes the total angular momentum J. The magnetic moments calculated on the basis of this model are the Schmidt values of the magnetic moment

a) $\mu_s^P = \langle \phi^J_{J=L\pm\frac{1}{2}} | j^z + (\mu_P - \frac{1}{2})\sigma^z | \phi^J_{J=L\pm\frac{1}{2}} \rangle$

(II-3)

b) $\mu_s^N = \mu_N \langle \phi^J_{J=L\pm\frac{1}{2}} | \sigma^z | \phi^J_{J=L\pm\frac{1}{2}} \rangle$

where $\phi^M_{J=L\pm\frac{1}{2}}$ is the single particle wave function having a total angular momentum $J = L \pm \frac{1}{2}$, a z component of the angular momentum M, and an orbital angular momentum L. These functions are

a) $\phi^M_{J=L+\frac{1}{2}} = \sqrt{\frac{L+M+\frac{1}{2}}{2L+1}}\, Y_L^{M-\frac{1}{2}} \chi^+ + \sqrt{\frac{L-M+\frac{1}{2}}{2L+1}}\, Y_L^{M+\frac{1}{2}} \chi^-$

(II-4)

b) $\phi^M_{J=L-\frac{1}{2}} = -\sqrt{\frac{L-M+\frac{1}{2}}{2L+1}}\, Y_L^{M-\frac{1}{2}} \chi^+ + \sqrt{\frac{L+M+\frac{1}{2}}{2L+1}}\, Y_L^{M+\frac{1}{2}} \chi^-$

where Y_L^M are the usual surface spherical harmonics[14] and $\chi^\pm$ are the usual Pauli spin functions. The single particle functions can now be used to evaluate the Schmidt magnetic moments as defined by II-3. The essential expectation

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[14] Condon and Shortley, The Theory of Atomic Spectra, Chapt. III, Cambridge University Press (1951).

value for the calculation is

$$\left\langle \phi^{M}_{J=L\pm\frac{1}{2}} , \sigma^{z} \phi^{M}_{J=L\pm\frac{1}{2}} \right\rangle = \frac{\mathcal{E}(J)\,2M}{2L+1} \qquad \text{(II-5)}$$

where $\mathcal{E}(J)$ is defined as + (plus) when $J = L + \frac{1}{2}$ and - (minus) when $J = L - \frac{1}{2}$. The Schmidt values for the magnetic moment are then

$$\begin{aligned}
&\text{a)} \quad \mu_s^{p} = J + (\mu_p - 1/2) && \text{for } J = L + 1/2 \\
&\qquad \mu_s^{p} = J - (\mu_p - 1/2)\frac{J}{J+1} && \text{for } J = L - 1/2 \\
&\text{b)} \quad \mu_s^{n} = \mu_n && \text{for } J = L + 1/2 \\
&\qquad \mu_s^{n} = -\mu_n \frac{J}{J+1} && \text{for } J = L - 1/2
\end{aligned} \qquad \text{(II-6)}$$

These values of the magnetic moments yield the Schmidt lines plotted in Fig. 1 and Fig. 2.

B. <u>Magnetic Moments with Modified Shell Model Wave Functions</u>.

The wave function of the nucleus is assumed to be an admixture of the usual shell model state with a number of states in which the angular momentum J is shared among three particles, i.e.

$$\psi_J^J = a_s \phi^J_{J=L\pm\frac{1}{2}} + \sum_{j,j''} a_{jj''} \psi_J^J(2jj'')$$

$$a_s^2 + \sum_{j,j''} a_{jj''}^2 = 1 \qquad \text{(II-7)}$$

where a_s is the probability amplitude of the shell model (single particle) state and $a_{jj''}$ is the probability amplitude of the three particle state $\psi_J^J(2jj'')$.

The types of three particle states considered are, by assumption, quite limited. Two of the particles are assumed to be identical (either two odd particles or two even particles of the odd-even nucleus). Furthermore, the two particles are each assumed to have the same total angular momentum j and orbital angular momentum ℓ, and are assumed to be coupled to $j' = 2$ (compared to $j' = 0$ as is assumed in the shell model). These two particles are then coupled with the third particle (of necessity an odd particle), which has a total angular momentum j'' and orbital angular momentum ℓ'', so as to give a final angular momentum J for the system of three particles. The vector addition rule of angular momentum allows J to have the values $j'' + 2$, $j'' + 1$, --- $|j'' - 2|$ for every value of j''. The value of ℓ'' is determined for a given j'' by parity considerations.

In an independent particle model, a system of n particles is represented by a product of n single particle wave functions, or by a linear combination of such product functions.

The parity of a product function is defined as $(-1)^{\sum_i \ell_i}$, where ℓ_i is the orbital angular momentum of the i^{th} product of the product function. When the state of a system is represented by a linear combination of product functions, invariance conditions (invariance under coordinate inversions) require that every product function in the linear combination have the same parity.

The parity of a single particle shell model state is given by $(-1)^L$ where L is the orbital angular momentum of the shell model state. The parity of two particles in the same j and ℓ state (regardless of how they are coupled together) is $(-1)^{2\ell} = +1$. Thus for the states $\Psi_J^{\,J}(2jj'')$ described above, the parity is $(-1)^{2\ell + \ell''} = (-1)^{\ell''}$. Since the parity must be the same for all states of a linear combination of states, the orbital angular momentum ℓ'', which is uniquely determined for a given J, L and j'', can be seen to be restricted to one of the values $\ell'' = L \; ; L \pm 2$.

The three particle wave functions can be represented as

$$\Psi_J^{\,J}(2jj'') = \sum_{m,m''} \langle j'',2,m'',m \mid j'',2,J,J \rangle \, J_2^{\,m} \, \phi_{j''}^{\,m''} \qquad \text{(II-8)}$$

$$= \sum_{m} \langle j'',2,J-m,m \mid j'',2,J,J \rangle \, J_2^{\,m} \, \phi_{j''}^{\,J-m}$$

The coefficients $\langle j'', 2, m'', m \mid j'', 2, J, J \rangle$ are the transformation amplitudes for vector addition (Clebsch-Gordon coefficients) of angular momentum[14]. J_2^m is the angular momentum function with total angular momentum $j' = 2$

and z component of angular momentum m formed by coupling two particles having the same j and l.

The correct wave function of the nucleus must be properly antisymmetrized to take into account the Pauli exclusion principle. In isotopic spin notation the wave function must be antisymmetric with respect to the interchange of any two nucleons. In more usual notation, where neutrons and protons are treated as different particles, it is only necessary to antisymmetrize with respect to an interchange among protons and among neutrons separately. Since the magnetic moment operator is a sum of single particle operators, the expectation value of the magnetic moment operator reduces to the corresponding sum of single particle expectation values[15]. Thus the antisymmetrization of the wave function does not need to be considered when calculating the magnetic moment expectation value. However, the Pauli principle must still be considered when determining possible values for j, l, j'', l'' and J, L. In certain cases the values of j'' are more restricted than previously indicated. It is especially important to exercise care when all three particles have the same angular momentum and parity, i.e. $j = j''$ and $l = l''$. The Pauli principle forbids the formation of angular momentum $J = \frac{1}{2}$ for three identical particles in identical states of j and l[16].

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[15] See reference in Footnote 14, p. 171.

[16] See reference in Footnote 14, p. 263.

The magnetic moments can now be calculated using the wave functions just described. First, all three particles are assumed to be of the same kind (odd particles). Also, only expectation values involving the modified wave functions are considered, even though cross terms (interference terms) are in general possible for certain admixtures of modified shell model states. The magnetic moments become, using II-1, II-2, II-8, and II-5[17]

a) $$\mu^P(2jj'';J) = \left\langle \Psi_J^J(2jj''), \sum_P \left(j_P^z + (\mu_P - \tfrac{1}{2})\sigma_P^z\right) \Psi_J^J(2jj'') \right\rangle$$

$$= J + (\mu_P - \tfrac{1}{2})\left[\frac{\varepsilon(j'')\,2J}{2\ell''+1} + 2\left(\frac{\varepsilon(j)}{2\ell+1} - \frac{\varepsilon(j'')}{2\ell''+1}\right)\left(\sum_m m\left|\langle j'',2,J-m,m \mid j'',2,J,J\rangle\right|^2\right)\right]$$

(II-9)

b) $$\mu^N(2jj'';J) = \left\langle \Psi_J^J(2jj''), \sum_N \mu_N \sigma_N^z\, \Psi_J^J(2jj'') \right\rangle$$

$$= \mu_N\left[\frac{\varepsilon(j'')\,2J}{2\ell''+1} + 2\left(\frac{\varepsilon(j)}{2\ell+1} - \frac{\varepsilon(j'')}{2\ell''+1}\right)\left(\sum_m m\left|\langle j'',2,J-m,m \mid j'',2,J,J\rangle\right|^2\right)\right]$$

---

[17] The magnetic moment calculation is quite simple for the state $J_2^m(2j)$ since the two particles are in equivalent states (states with the same j and l). The Clebsch-Gordon coefficients for the state $J_2^m(2j)$ do not have to be known explicitly (most of the coefficients of interest have not been tabulated). Thus

$$J_2^m = \sum_{m''} \langle j,j,J,m \mid j,j,m'',m-m''\rangle\, \phi_j^{m''}(1)\, \phi_j^{m-m''}(2)$$

$$\sum_{m''} \left|\langle j,j,J,m \mid j,j,m'',m-m''\rangle\right|^2 = 1$$

$$\therefore \langle J_2^m \mid \sigma_1^z + \sigma_2^z \mid J_2^m\rangle = \sum_{m''} \varepsilon(j)\, \frac{2(m''+m-m'')}{2\ell+1} \times$$

$$\times \left|\langle j,j,J,m \mid j,j,m'',m-m''\rangle\right|^2 = \varepsilon(j)\,\frac{2m}{2\ell+1}$$

$\Psi_J^J(2jj'')$ represents a function of three protons in II-9a and a function of three neutrons in II-9b. The sum $\sum_m m \left|\langle j'', 2, J - m, m \mid j'', 2, J, J\rangle\right|^2$ depends on the values of j'' and J:

$$\sum_m m \left|\langle j'', 2, J - m, m \mid j'', 2, J, J\rangle\right|^2 =$$

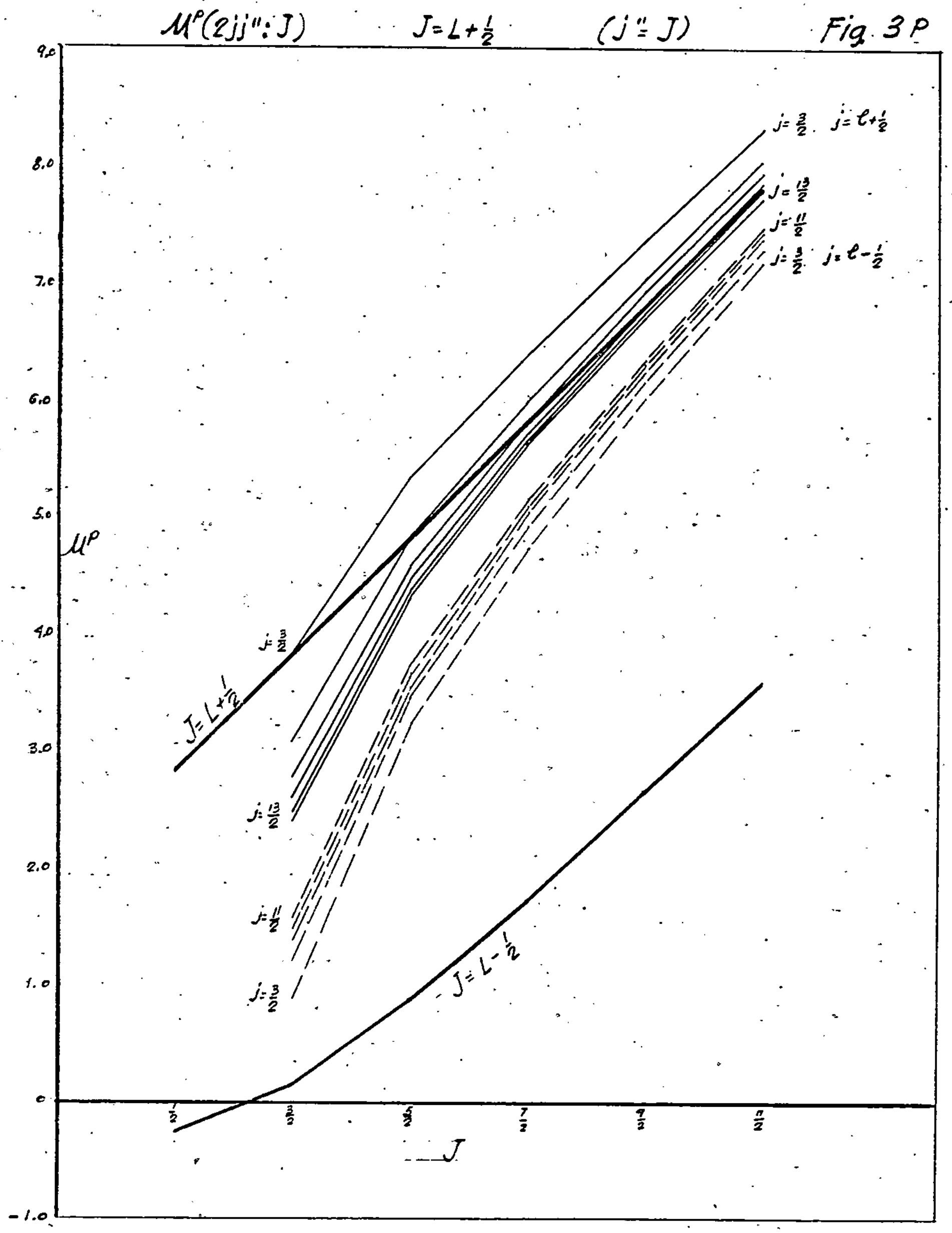
$$
\begin{aligned}
&= -\frac{2J}{J+1} && \text{for } j'' = J + 2 \\
&= \frac{2 - J}{J+1} && \text{for } j'' = J + 1 \qquad\qquad \text{(II-10)} \\
&= \frac{3}{J+1} && \text{for } j'' = J \\
&= \frac{J \quad 3}{J+1} && \text{for } j'' = J - 1 \\
&= 2 && \text{for } j'' = J - 2
\end{aligned}
$$

A large number of possible magnetic moments are represented by the relations II-9a and II-9b. Different values are obtained for $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ for the same choice of j, j'', and J. The smallest allowed value for j is $j = 3/2$ since two $j = \frac{1}{2}$ particles cannot be combined to give an angular momentum $j' = 2$ as required in the modified shell model wave function.

For a given choice of j and J there are in general five possible values for j'', all giving different values for the magnetic moment. In specifying J, it is also necessary to specify whether $J = L + \frac{1}{2}$ or $J = L - \frac{1}{2}$ in order to be able to specify l'' and whether $j'' = l'' + \frac{1}{2}$ or $j'' = l'' - \frac{1}{2}$. Thus for example, if $j'' = J + 1$ and $J = L + \frac{1}{2}$, then in order to have the same parity for the three particle state as for the shell model state $j'' = J + 1 = L + 3/2 = (L + 2) - 1/2 = l'' - 1/2$. Then the required value for l'' is $l'' = L + 2$ and $j'' = l'' - 1/2$. The possibility $j'' = l'' + 1/2$ is excluded since this would require that $l'' = L + 1$ which is a state having a different parity from that of the shell model state. A similar analysis must be undertaken for all values of j, j'' and J, when II-9a and II-9b are used to calculate magnetic moments.

When the angular momentum J of the shell model state has small values ($J = 1/2$ and $J = 3/2$), the possible values of j'' are more limited than previously indicated. When $J = 1/2$, the possibilities $j'' = J - 1$ and $j'' = J - 2$ are obviously excluded since a negative value for j'' is meaningless. The value $j'' = J = 1/2$ is also excluded since $j'' = 1/2$ cannot be combined with $J = 2$ to give the required angular momentum $J = 1/2$. Thus for $J = 1/2$ only two values of j'' ($j'' = J + 1$ and $j'' = J + 2$) are possible. When $J = 3/2$, then only $j'' = J - 2$ is excluded. When $J \geq 5/2$, all values of j'' allowed by the vector addition rule are possible ($j'' = J + 2$, $j'' = J + 1$, ---- $j'' = |J - 2|$). All the excluded possibilities just discussed are implicitely contained in the vector addition rule ($J = j'' + 2$, $J = j'' + 1$, ----- ------ $J = |j'' - 2|$).

The magnetic moments $\mu^p(2jj'':J)$ and $\mu^n(2jj'':J)$ given by II-9a and II-9b are plotted as a function of J in Figures 3p to 12p and Figures 3n to 12n respectively. A separate plot is made for the possibility $J = L + 1/2$ or $J = L - 1/2$ for each possible j''. Since j'' in general may assume five values this results in ten plots for $\mu^p(2jj'':J)$ and ten


$\mu^P(\ell jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J)$
Fig. 3 P
μ^P
$J=L+\frac{1}{2}$
$J=L-\frac{1}{2}$
$j=\frac{3}{2}$ $j=\ell+\frac{1}{2}$
$j=\frac{13}{2}$
$j=\frac{11}{2}$
$j=\frac{3}{2}$ $j=\ell-\frac{1}{2}$
J

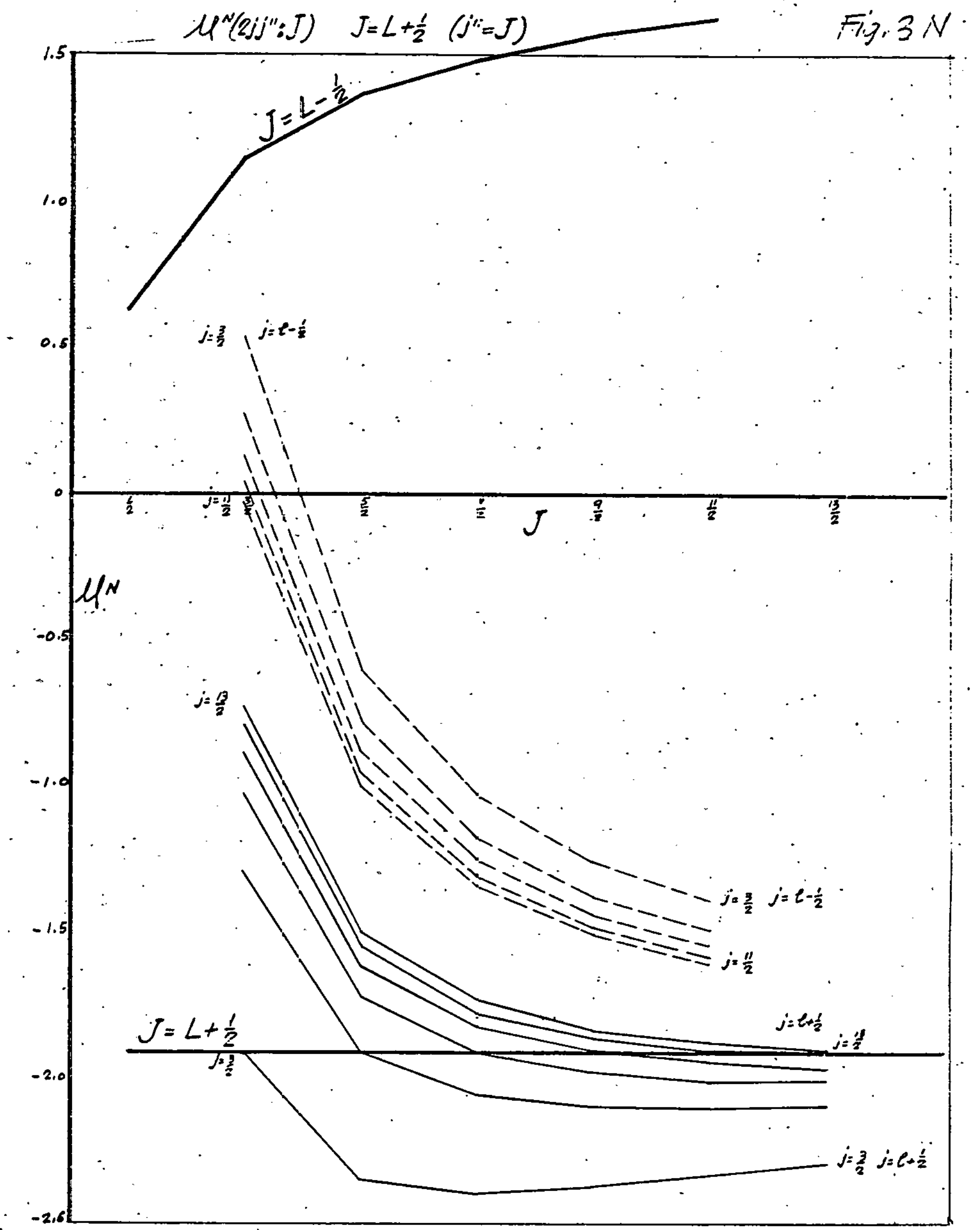

$\mu^N(\ell jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J)$
Fig. 3 N
μ^N
$J=L-\frac{1}{2}$
$J=L+\frac{1}{2}$
$j=\frac{3}{2}$ $j=\ell-\frac{1}{2}$
$j=\frac{13}{2}$
$j=\frac{11}{2}$
$j=\ell+\frac{1}{2}$
$j=\frac{3}{2}$ $j=\ell+\frac{1}{2}$
J

Fig. 4 P

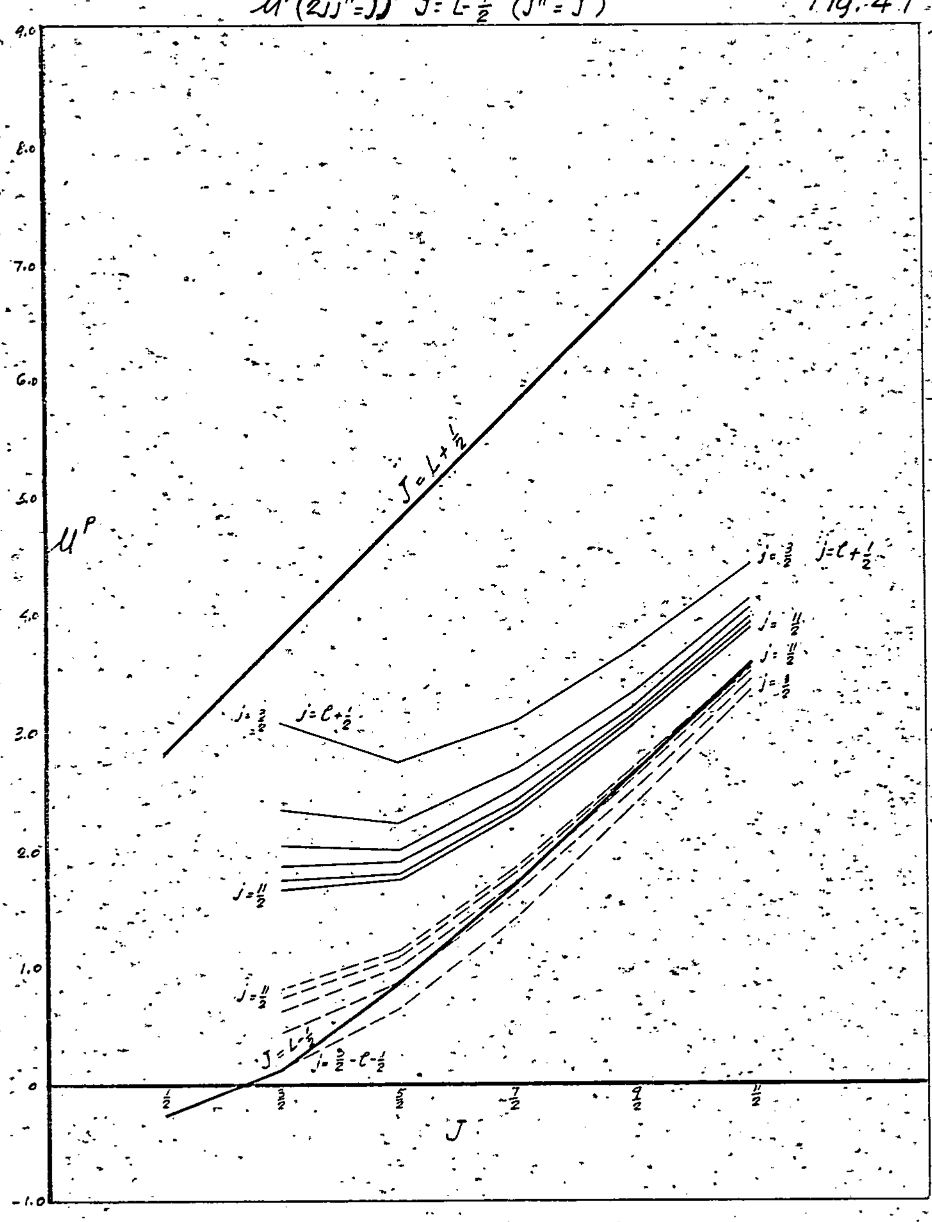
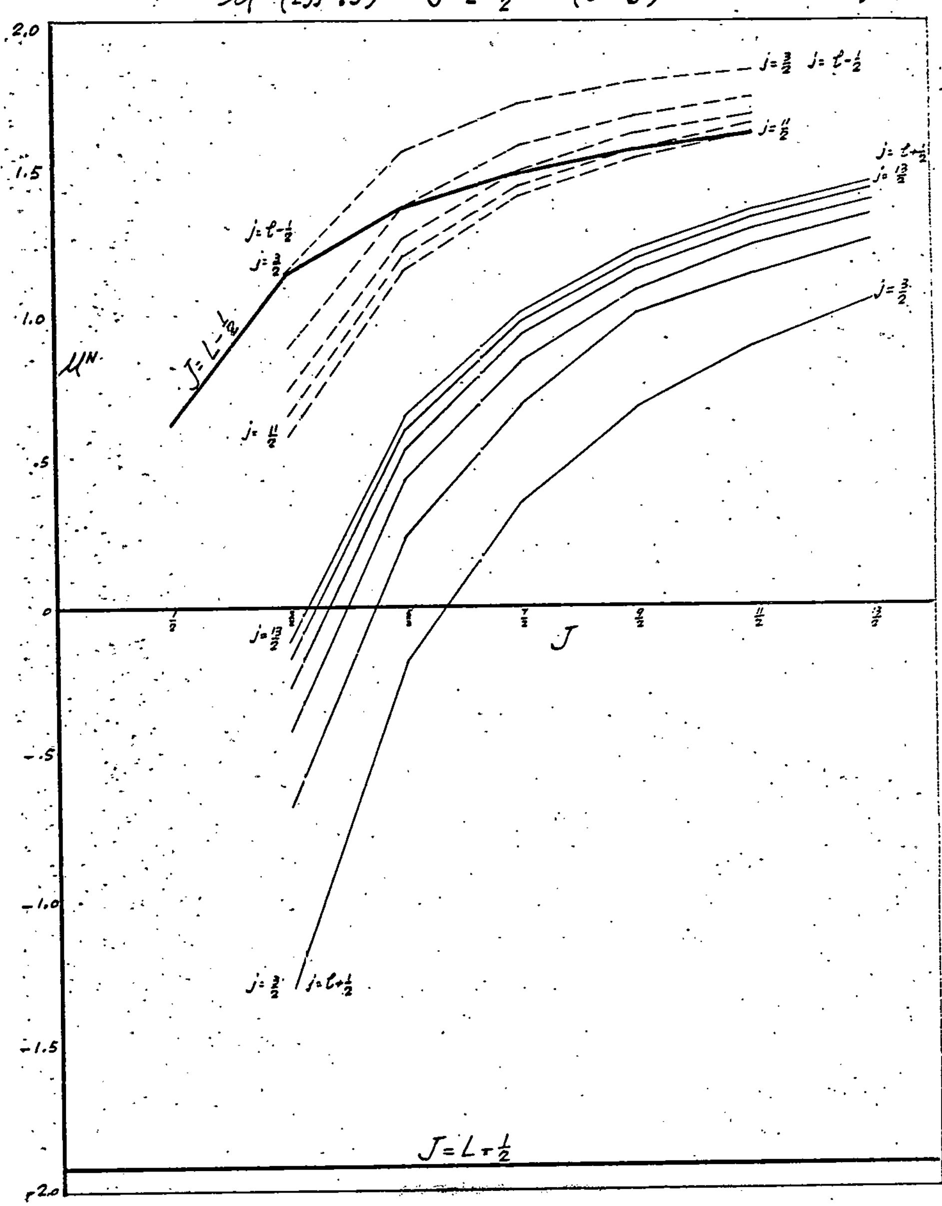
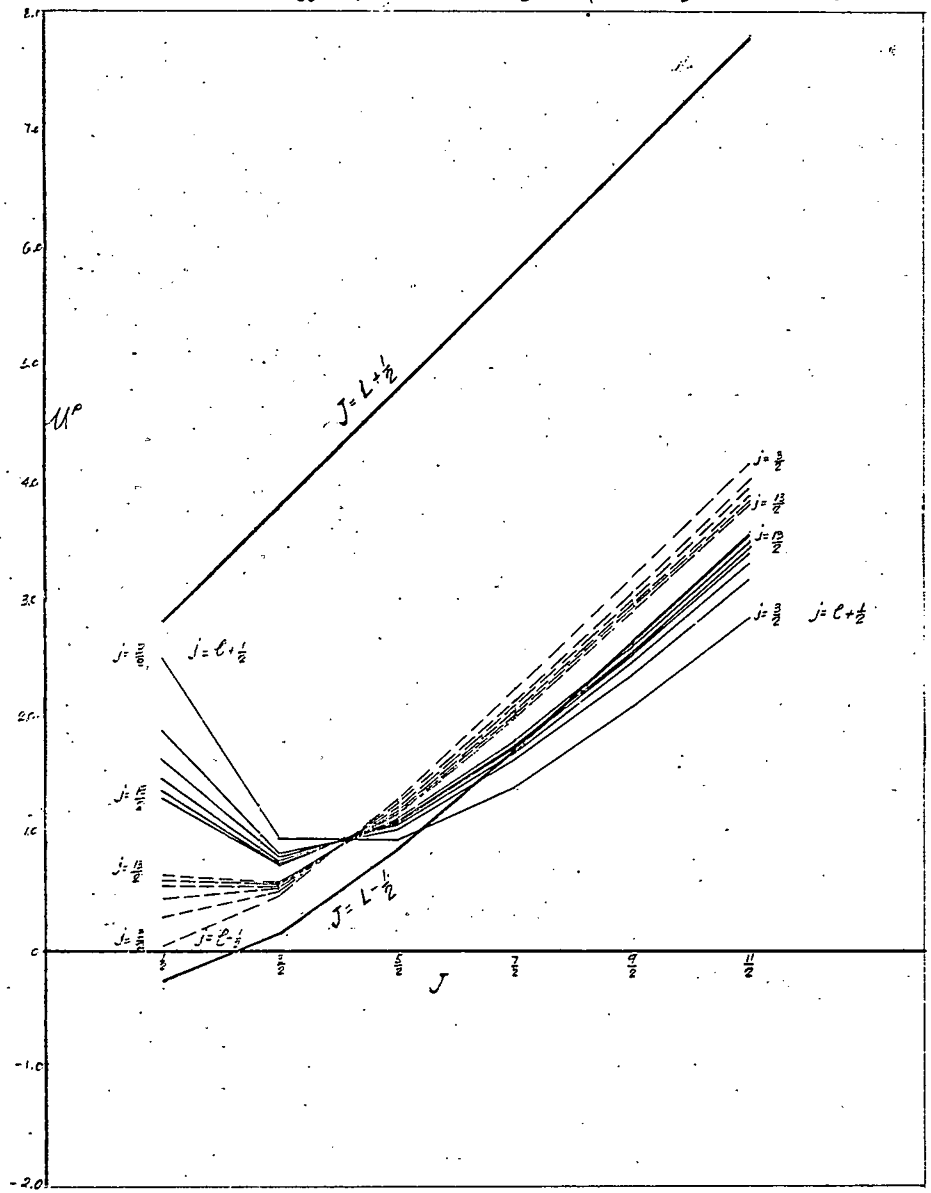


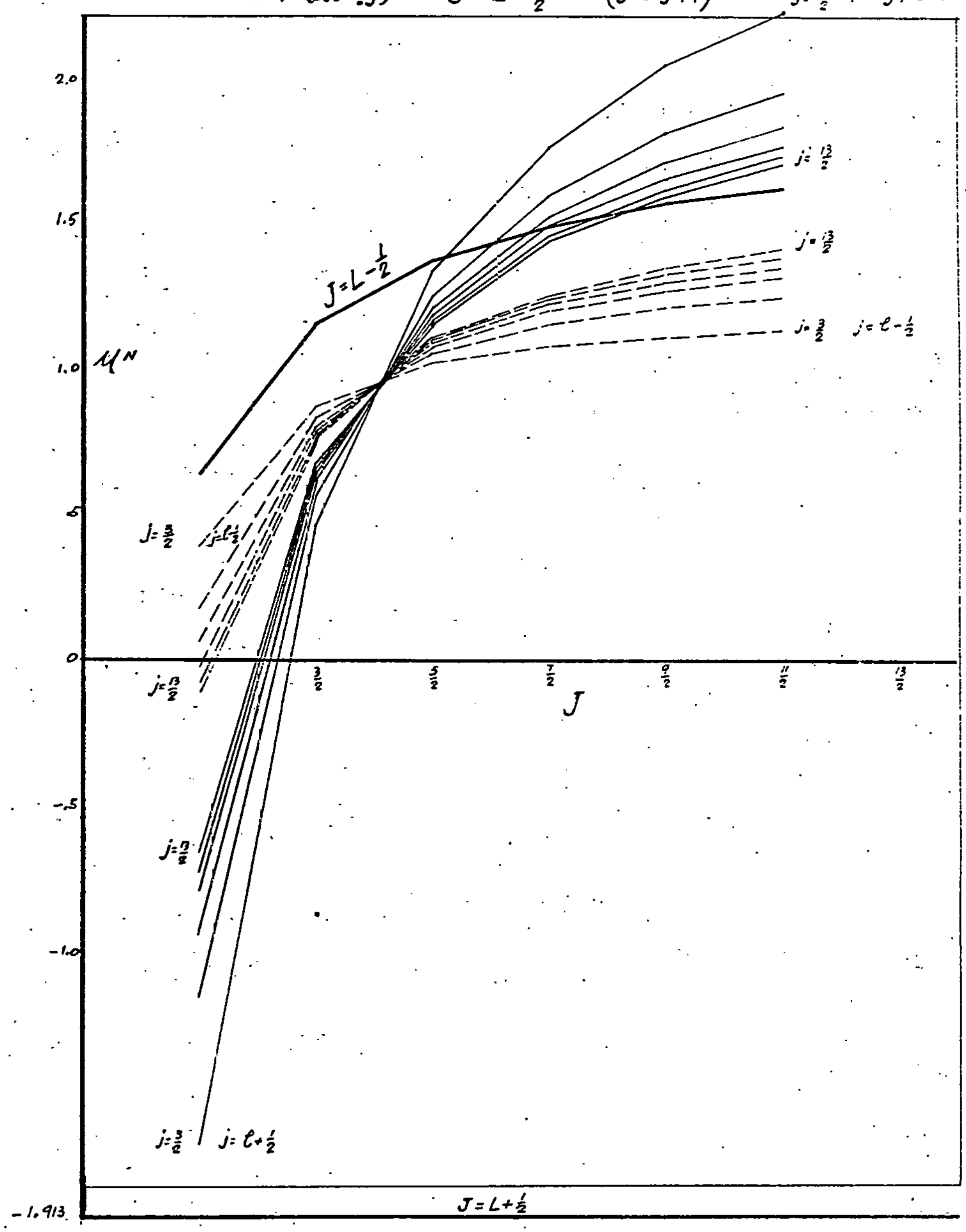
Fig. 4 N



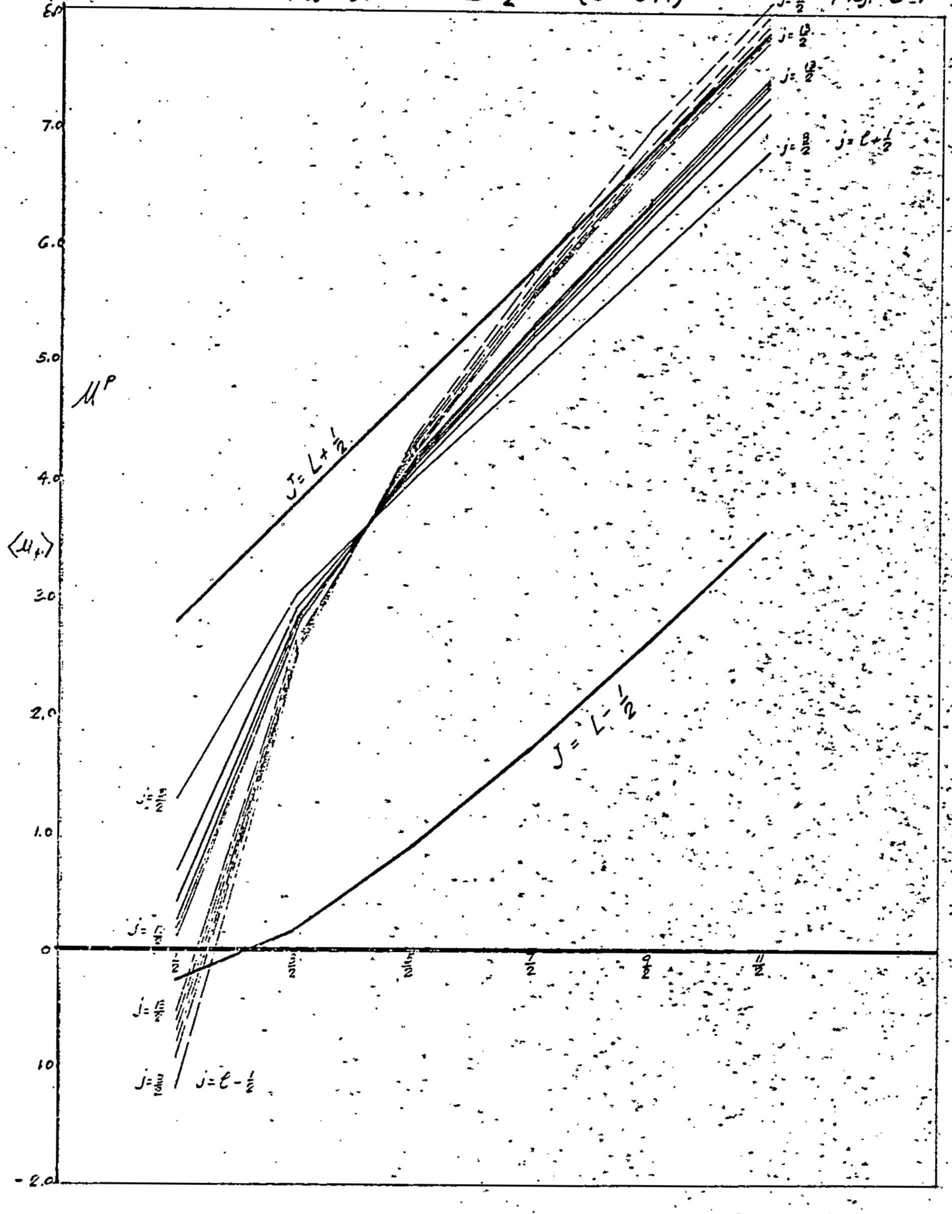
$\mathcal{M}^P(2jj'':J)$ $J = L + \frac{1}{2}$ $(j'' = J+1)$ Fig. 5.P



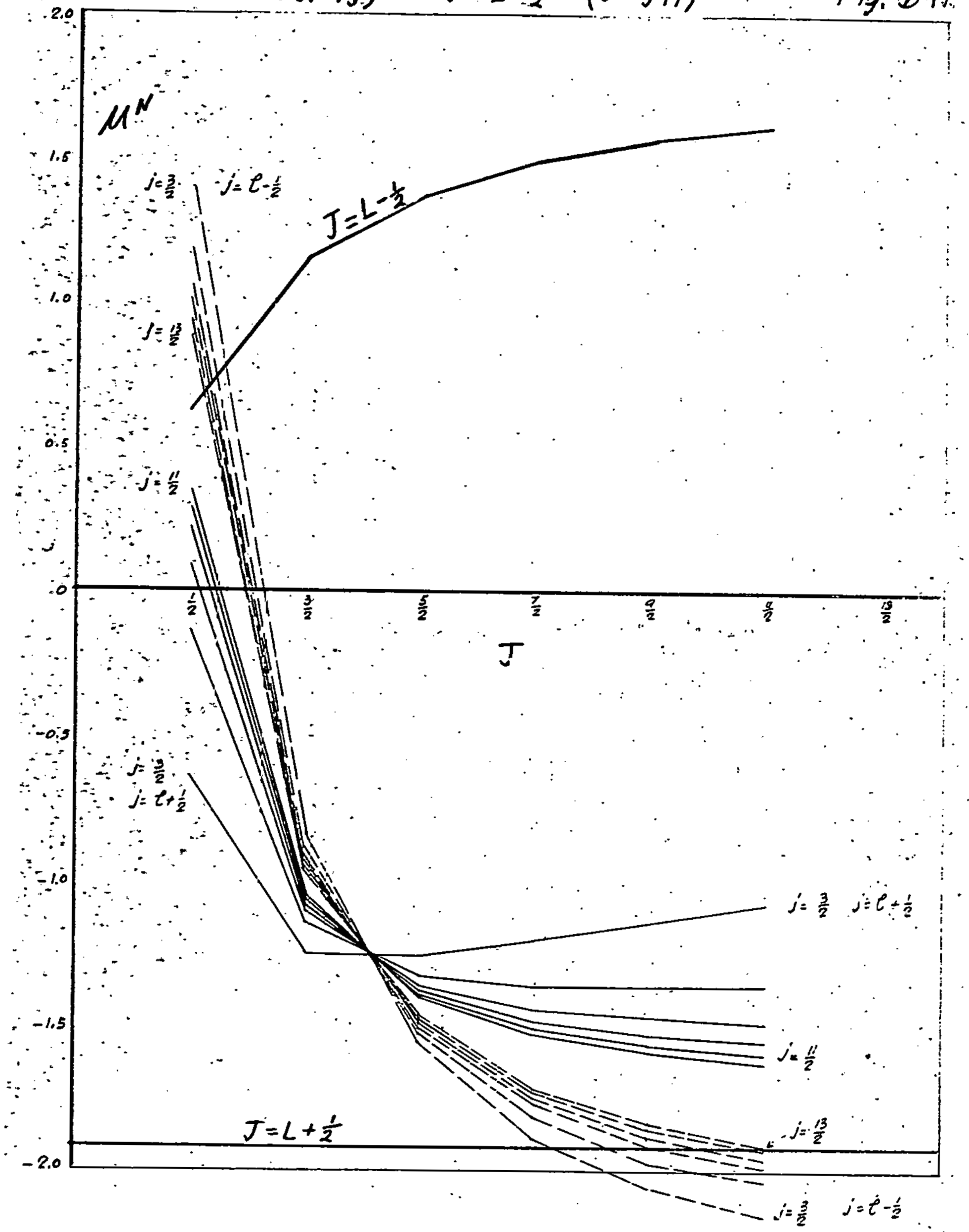
$\mathcal{M}^N(2jj'':J)$ $J = L + \frac{1}{2}$ $(J'' = J+1)$ Fig. 5.N



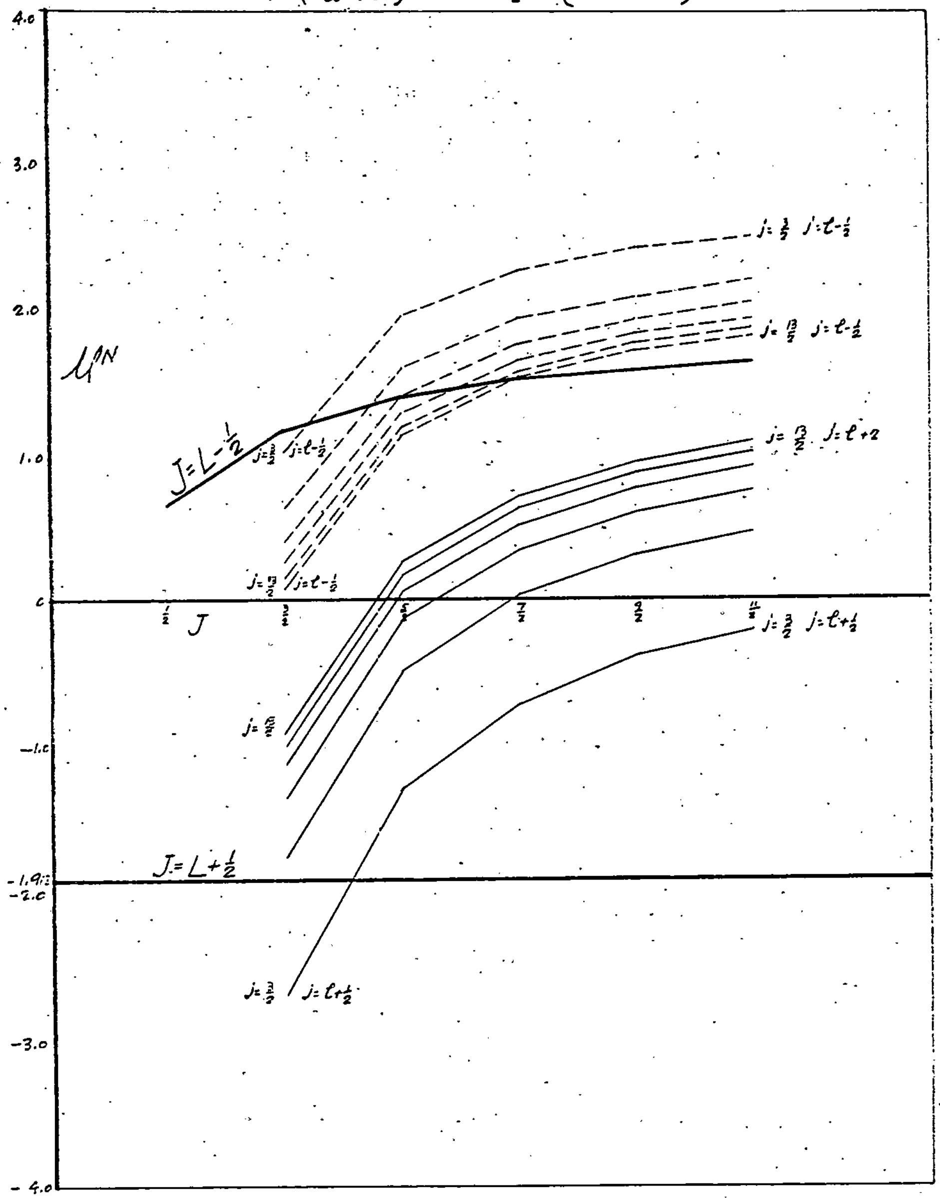
$\mu^P(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J+1)$ Fig. 6 P



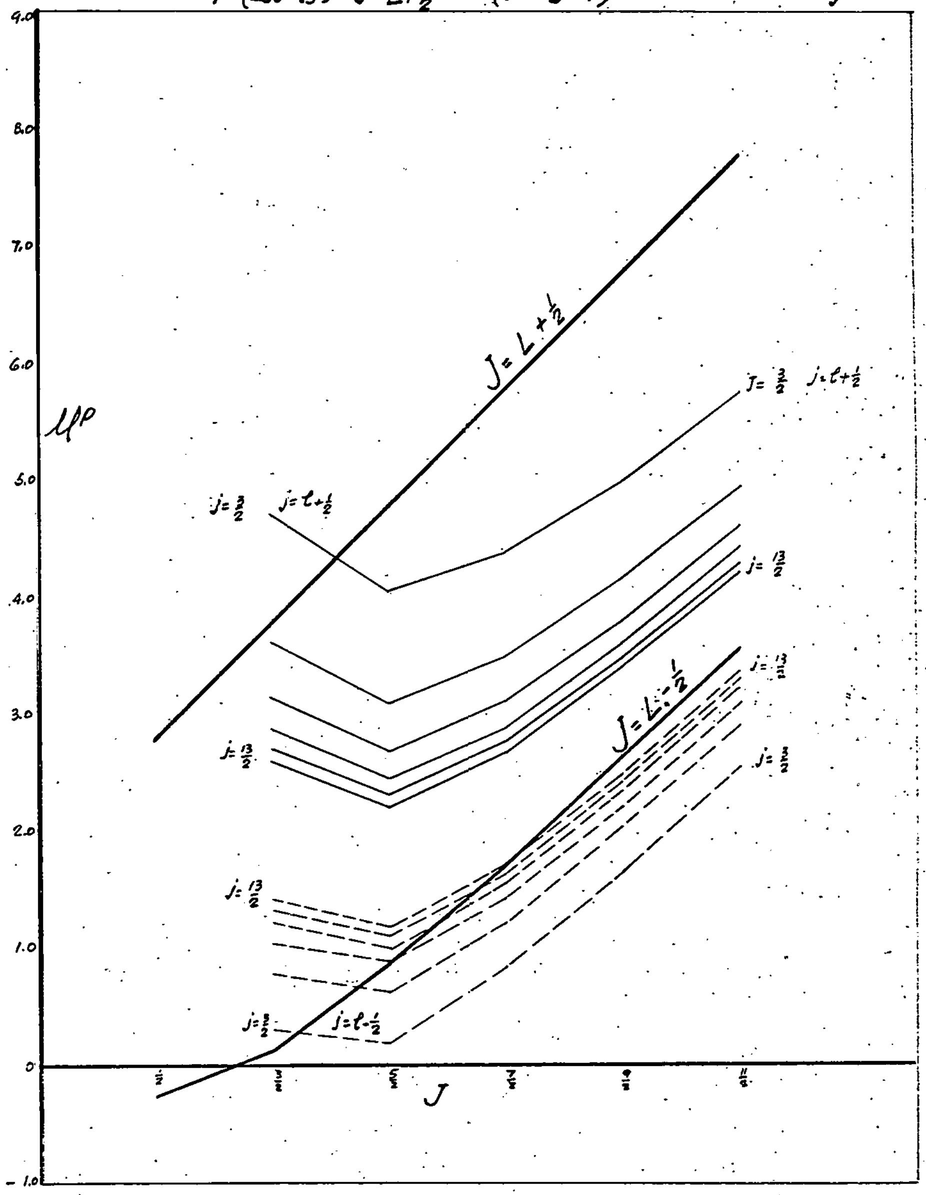
$\mu^N(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J+1)$ Fig. 6 N

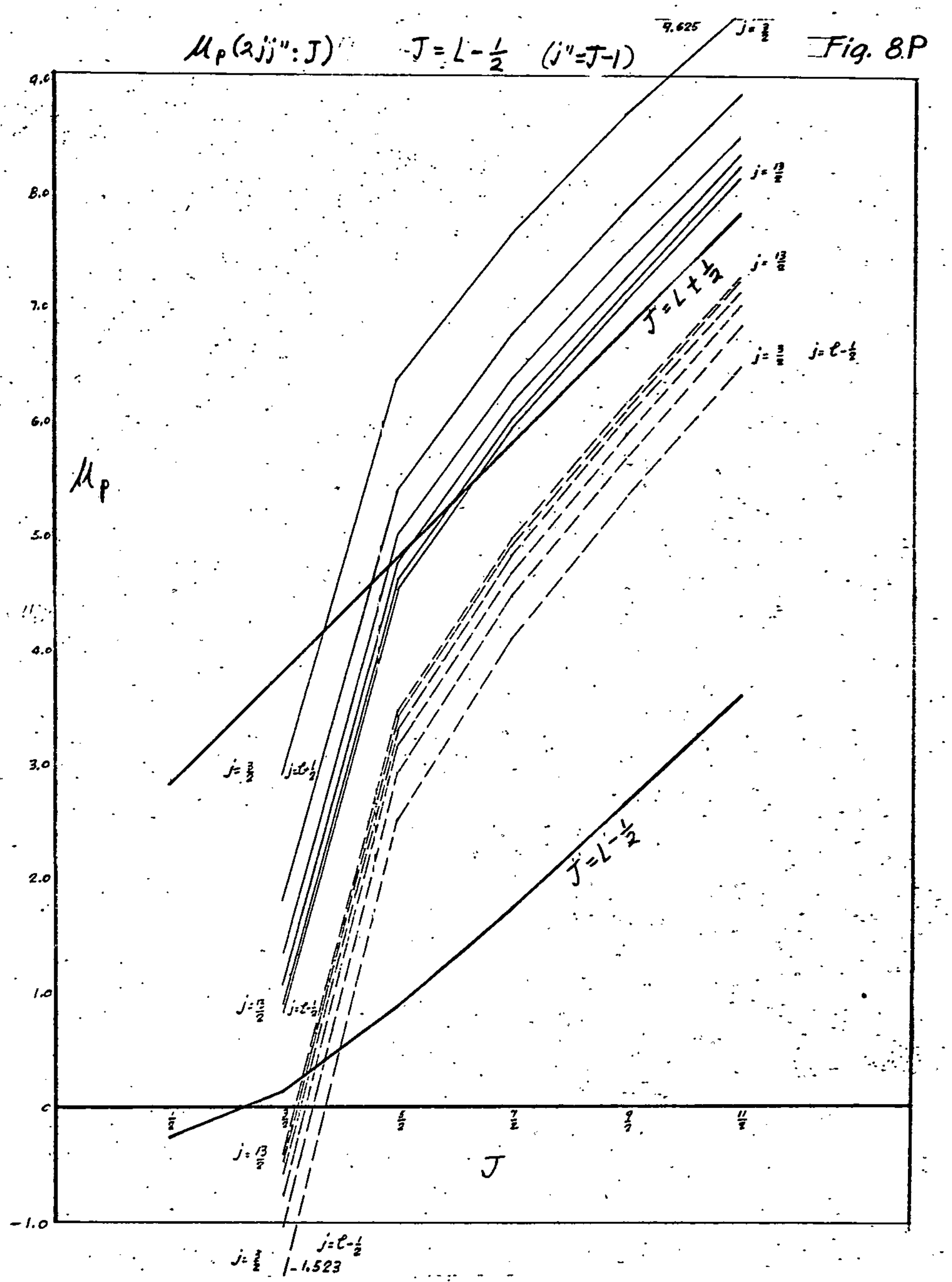


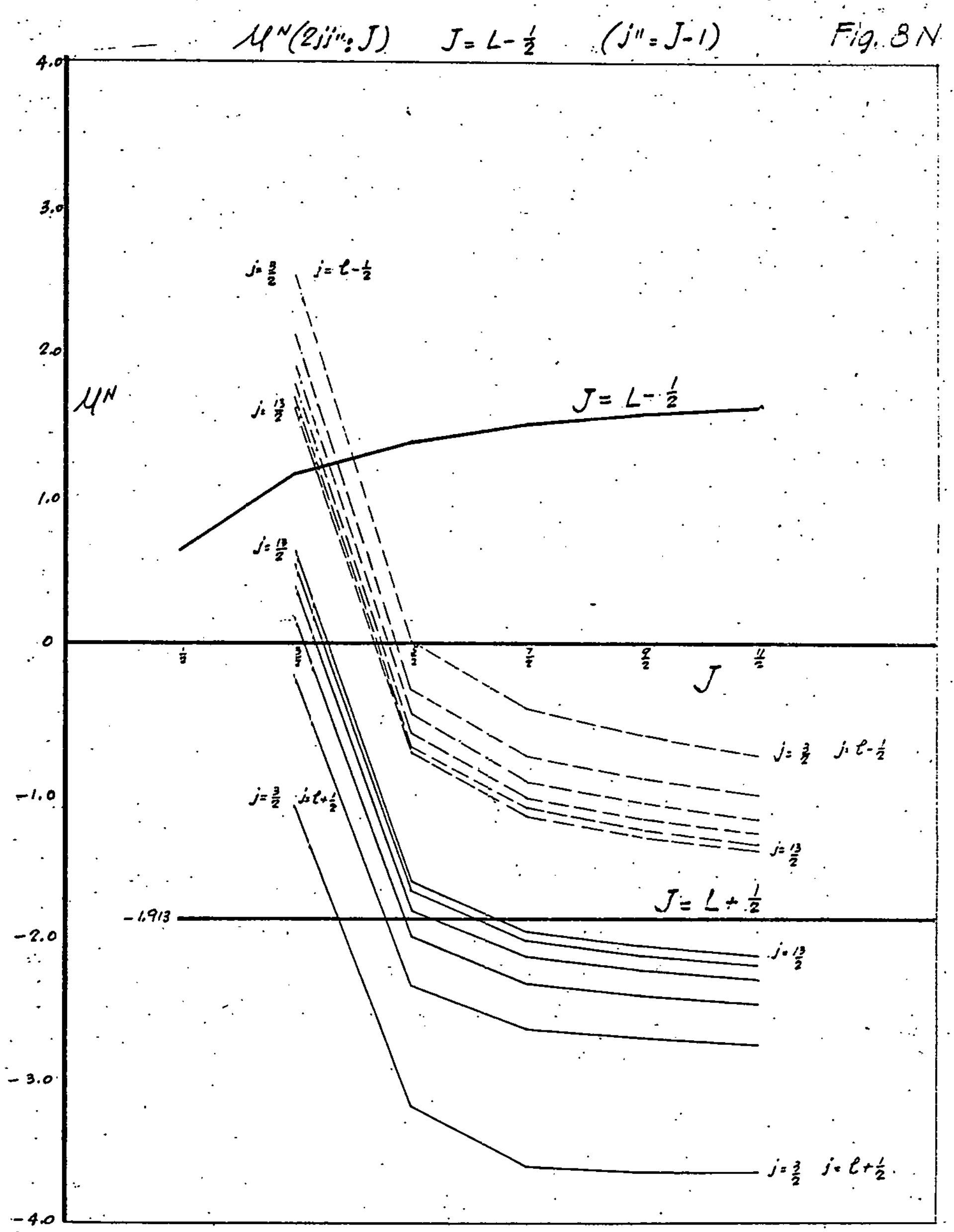
$\mathcal{M}^N(2jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J-1)$ Fig. 7N



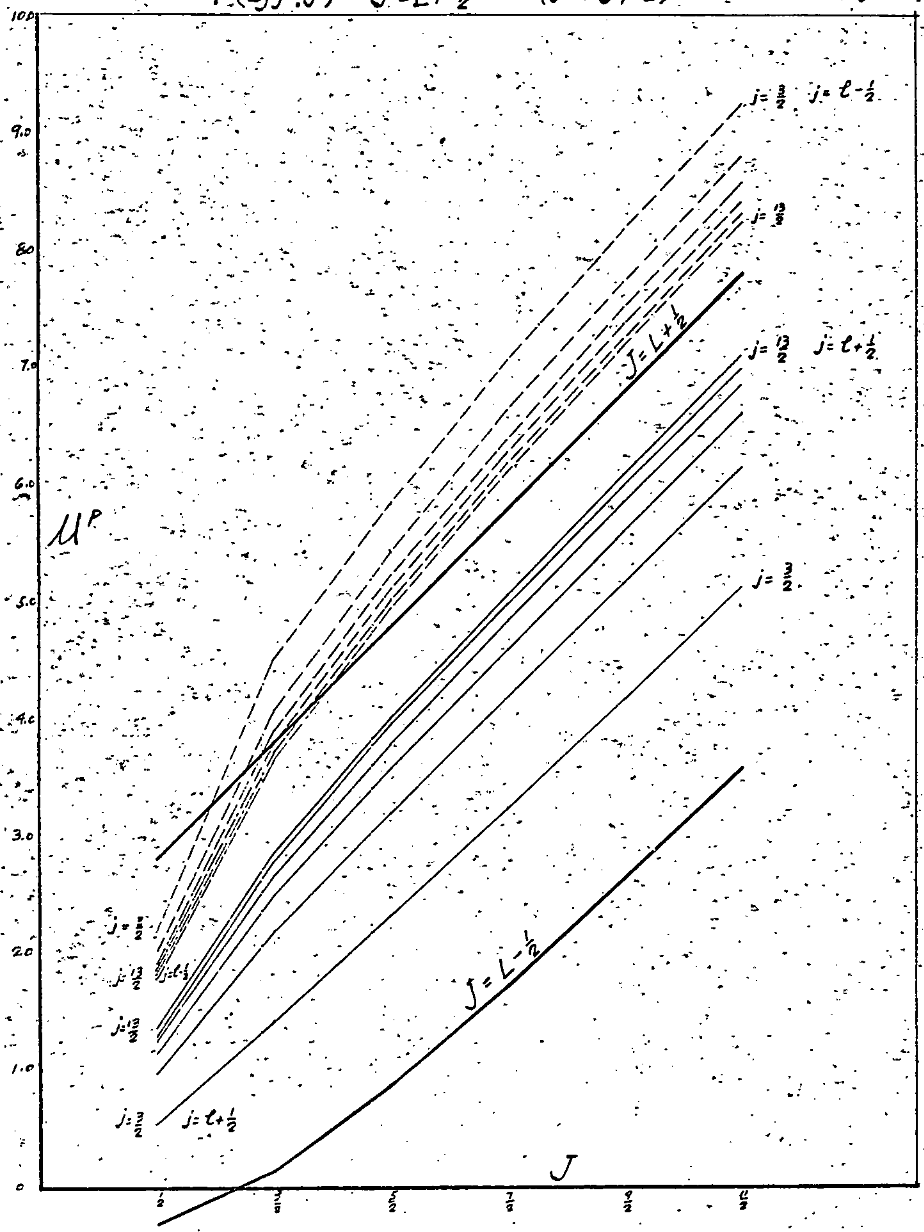
$\mathcal{M}^P(2jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J-1)$ Fig. 7P



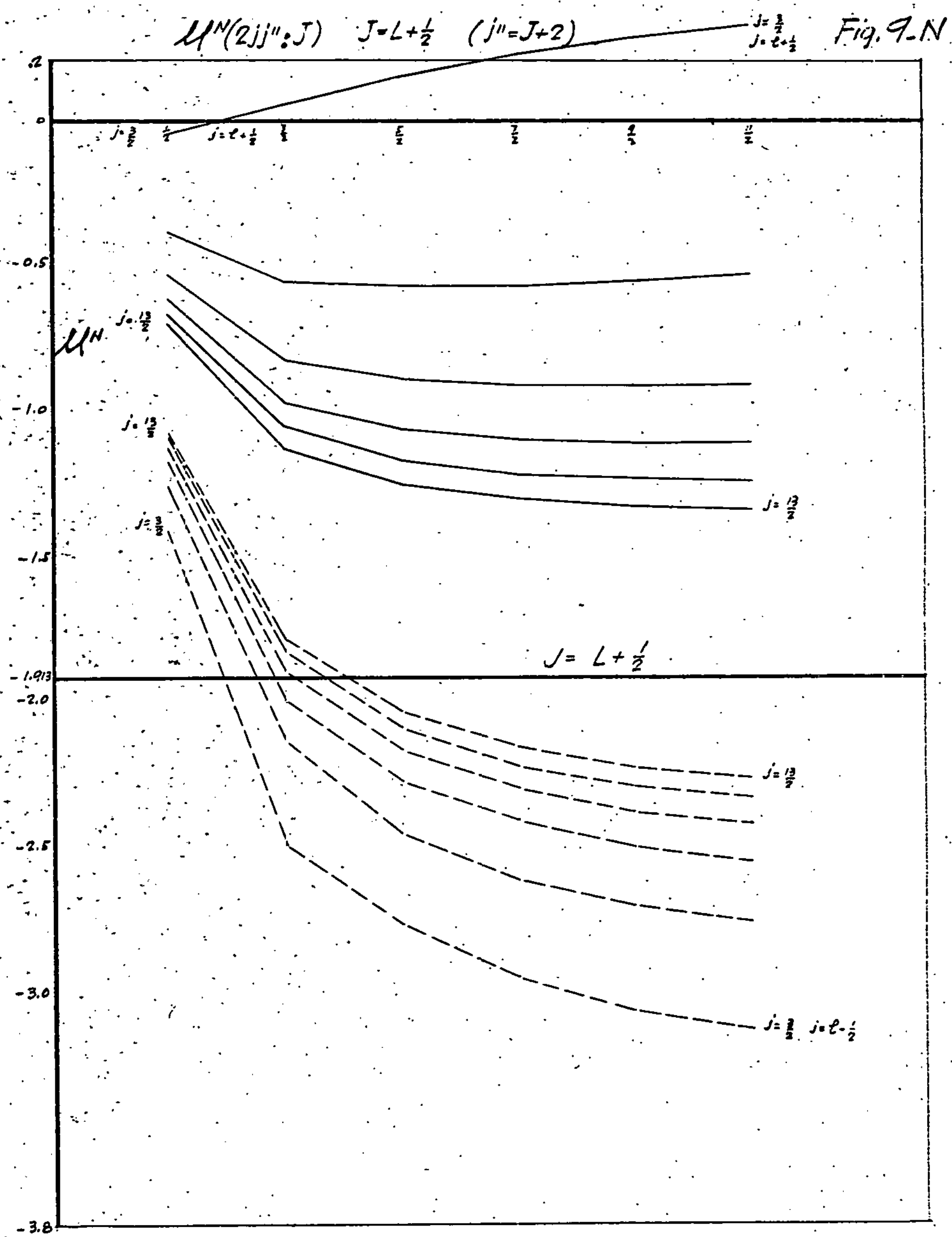

$\mu_p(2jj'':J)$ $J=L-\frac{1}{2}$ $(j''=J-1)$
Fig. 8P
μ_p
J
$J=L+\frac{1}{2}$
$J=L-\frac{1}{2}$
9.625
-1.523

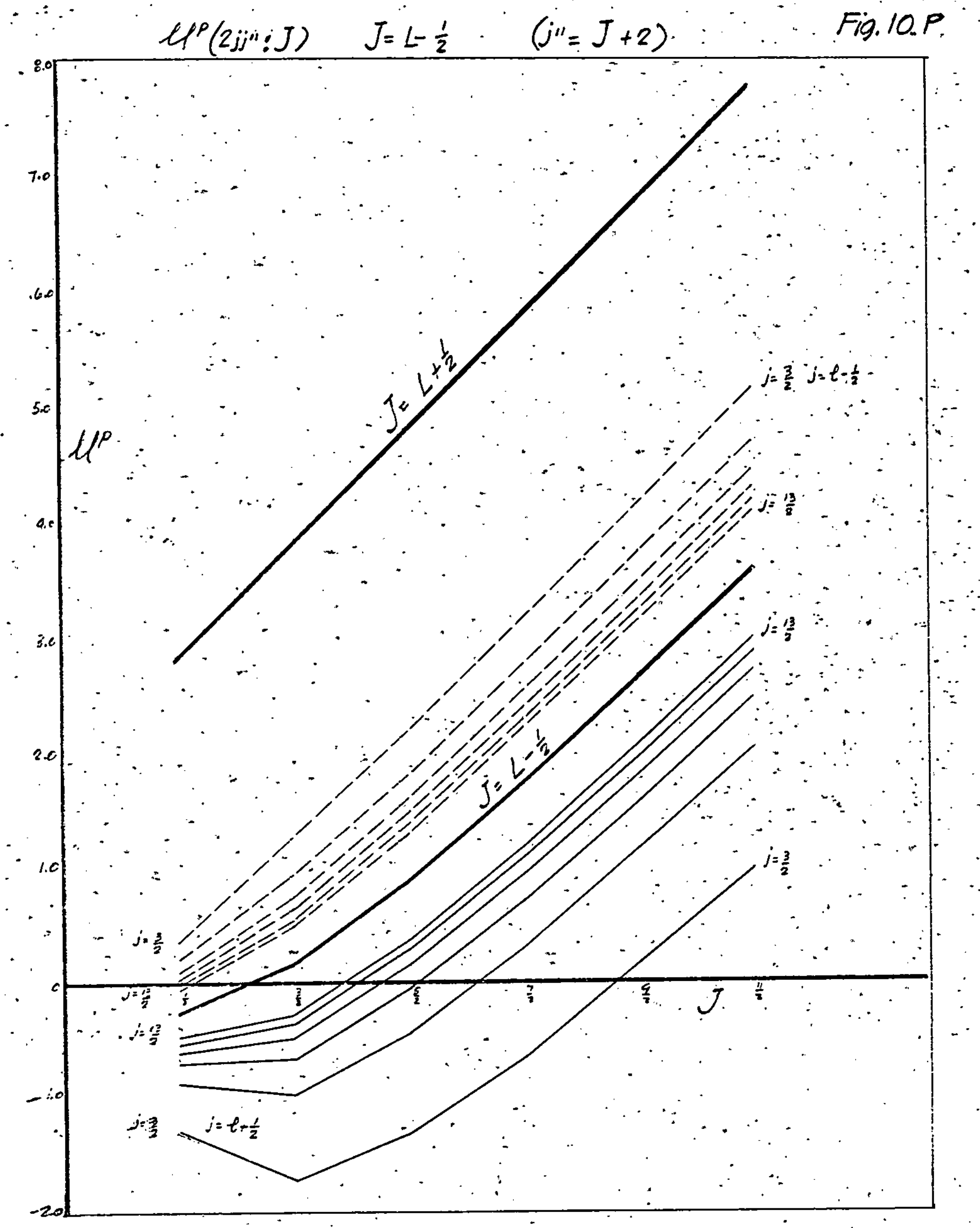

$\mu^N(2jj'':J)$ $J=L-\frac{1}{2}$ $(j''=J-1)$
Fig. 8N
μ^N
J
$J=L-\frac{1}{2}$
$J=L+\frac{1}{2}$
-1.913

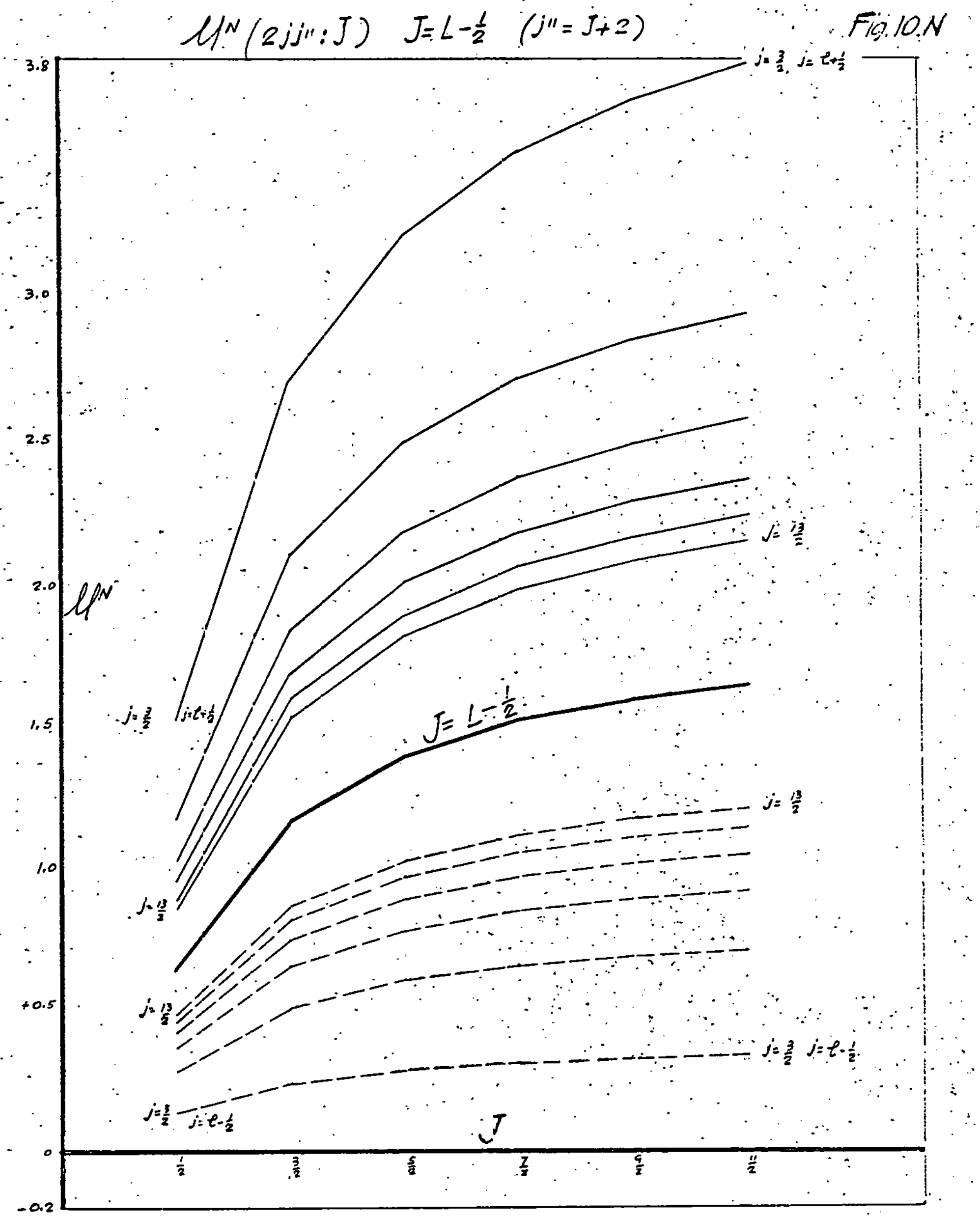
$\mathcal{U}^P(2jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J+2)$ Fig. 9P



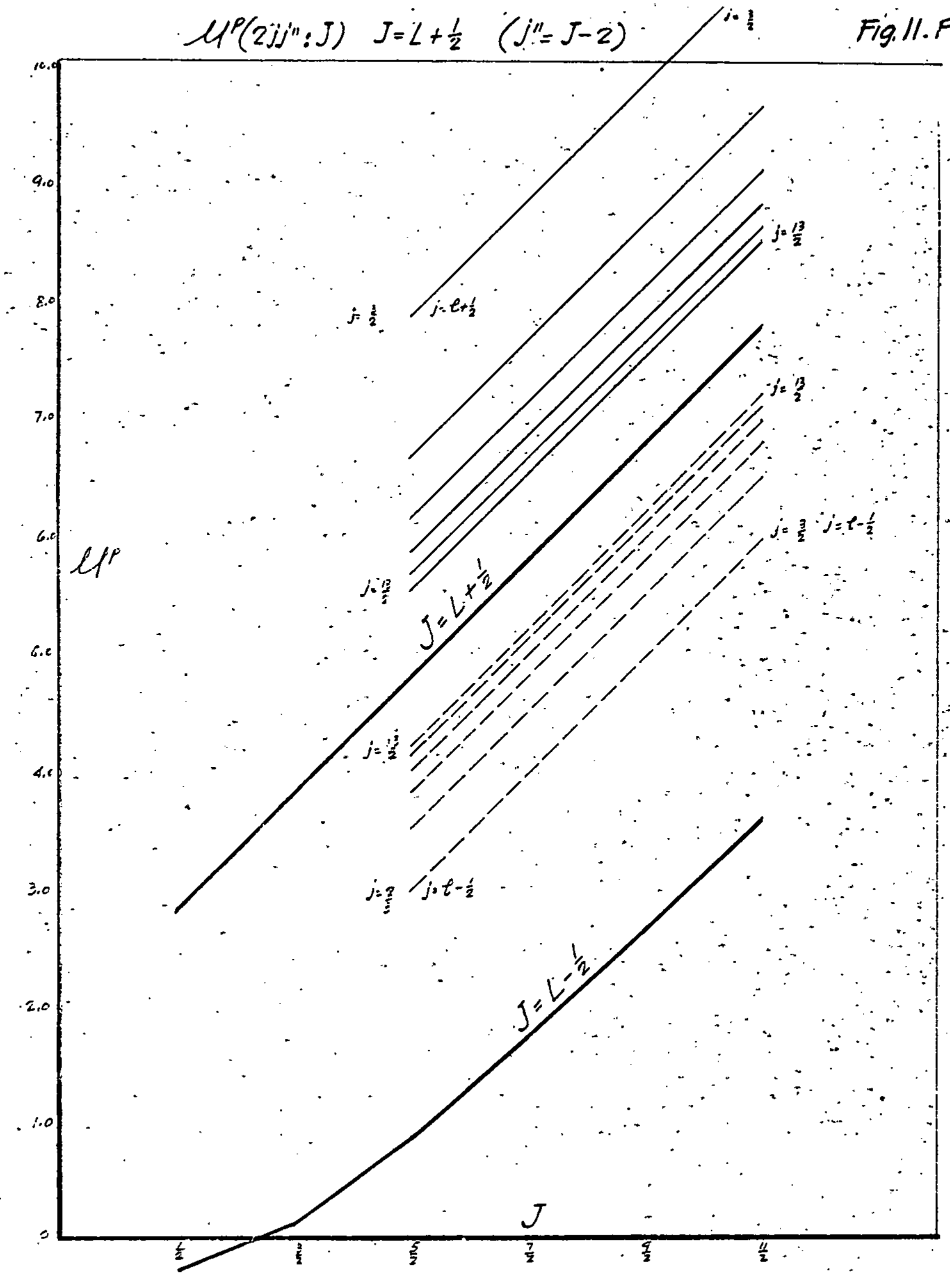
$\mathcal{U}^N(2jj'':J)$ $J=L+\frac{1}{2}$ $(j''=J+2)$ Fig. 9-N



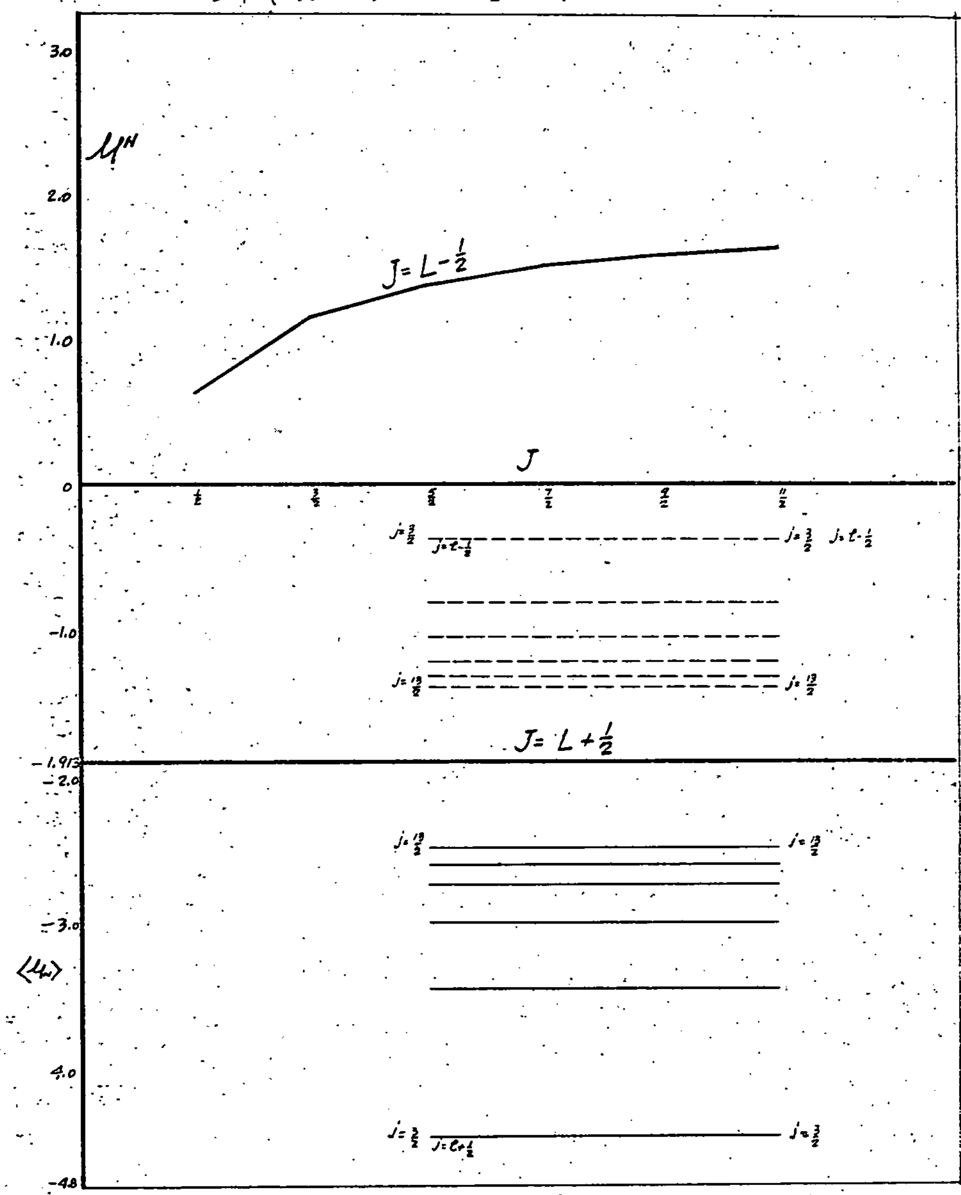

$U^P(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J+2)$
Fig. 10.P
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
-1.0
-2.0
U^P
$J = L + \frac{1}{2}$
$J = L - \frac{1}{2}$
$j = \frac{3}{2}$ $j = \ell - \frac{1}{2}$
$j = \frac{13}{2}$
$j = \frac{3}{2}$ $j = \ell + \frac{1}{2}$
J


$U^N(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J+2)$
Fig. 10.N
3.8
3.0
2.5
2.0
1.5
1.0
+0.5
0
-0.2
U^N
$J = L - \frac{1}{2}$
$j = \frac{3}{2}$ $j = \ell + \frac{1}{2}$
$j = \frac{13}{2}$
$j = \frac{3}{2}$ $j = \ell - \frac{1}{2}$
J

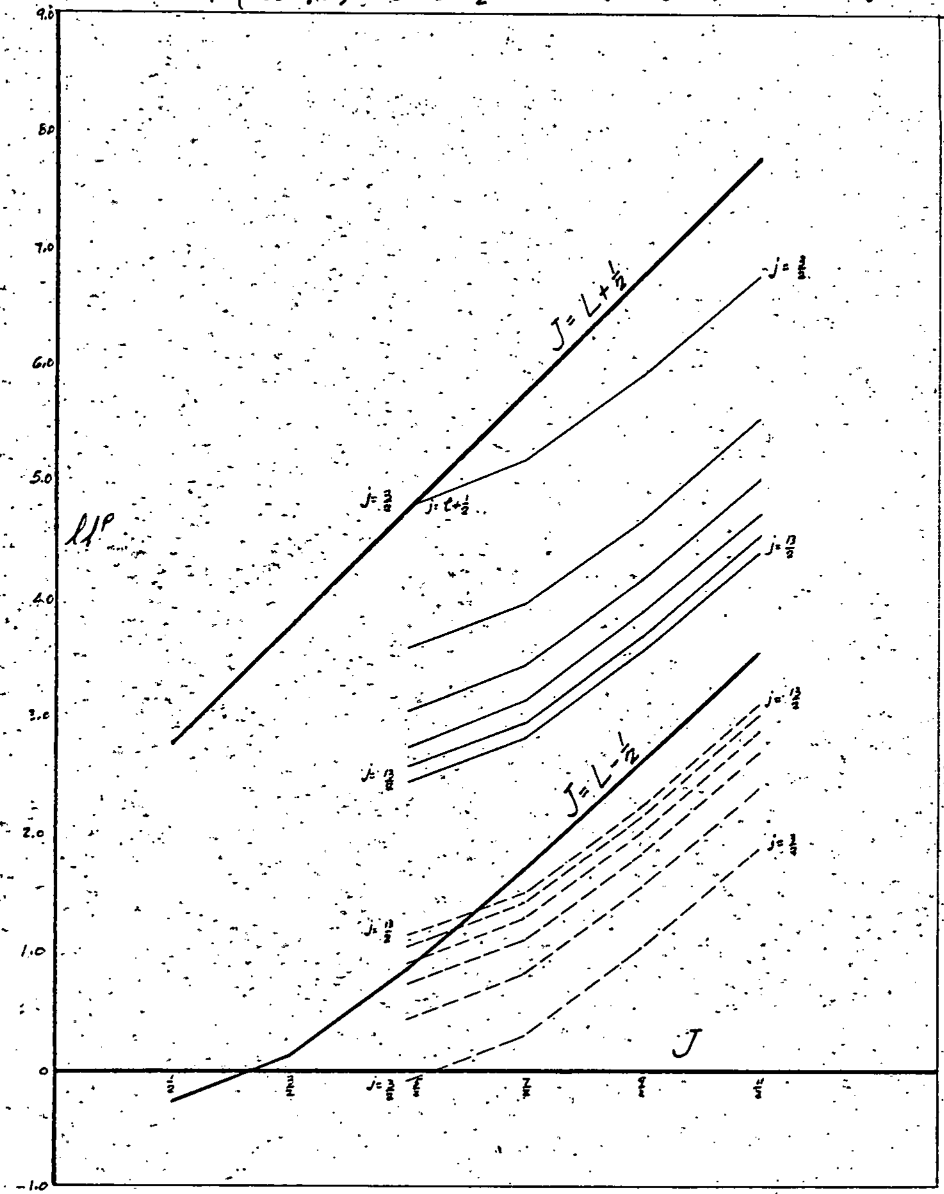
$\mathcal{U}^P(2jj'':J)$ $J = L + \frac{1}{2}$ $(j'' = J-2)$ Fig. 11. P



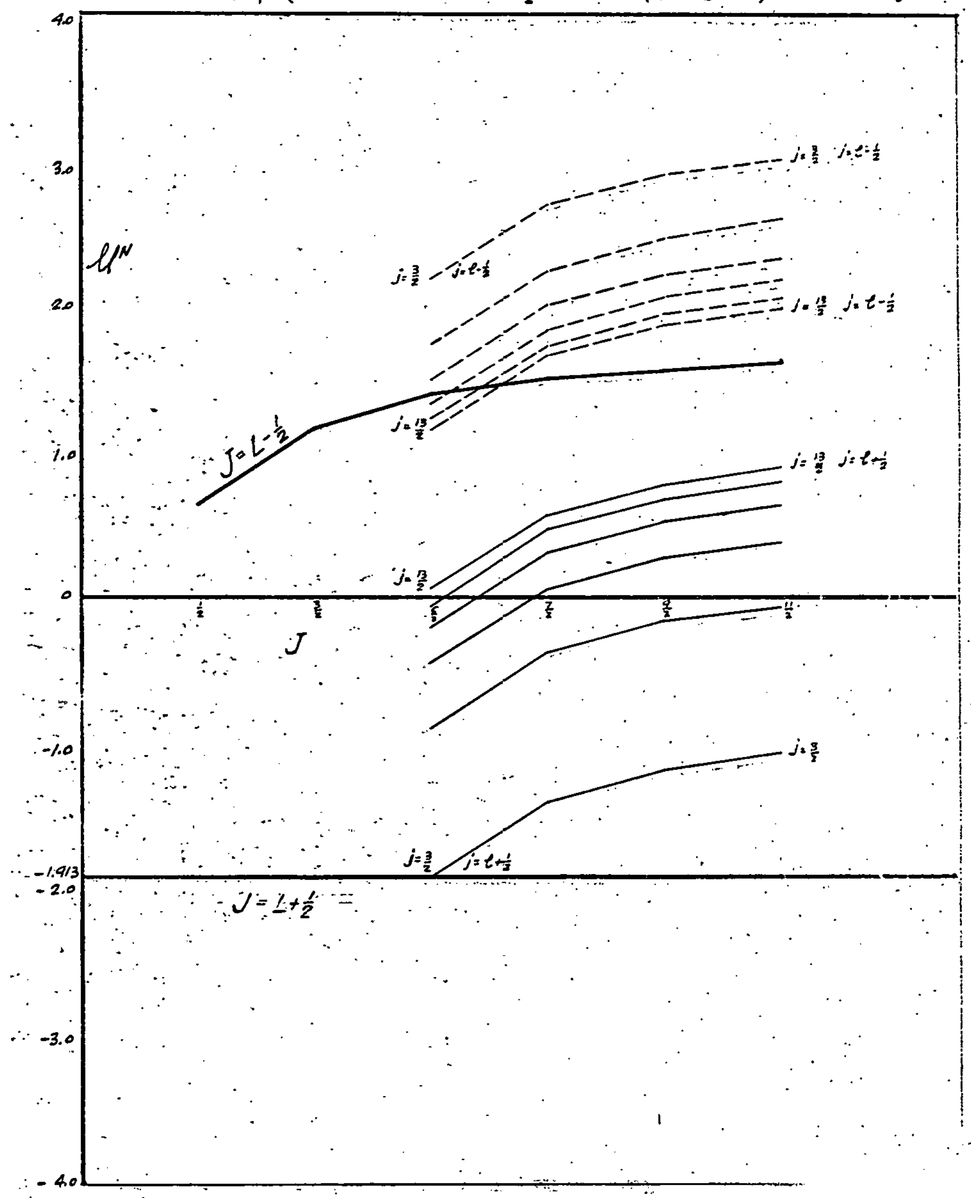
$\mathcal{U}^N(2jj'':J)$ $J = L + \frac{1}{2}$ $(j'' = J-2)$ Fig 11.N



$\mathcal{U}^P(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J-2)$ Fig. 12 P



$\mathcal{U}^N(2jj'':J)$ $J = L - \frac{1}{2}$ $(j'' = J-2)$ Fig. 12 N



plots for $\mu^n(2jj'':J)$. Each possible value of j (of the two particles combined to $j' = 2$) gives a line on every plot. In order to obtain all magnetic moment values that are of interest six values of $j = l + 1/2$ must be plotted ($j = 3/2, 5/2$ ---- $13/2$) and five values of $j = l - 1/2$ ($j = 3/2,$ ------ $11/2$). Thus each plot contains eleven lines each giving a magnetic moment value at the half integer values of j'.

The appropriate Schmidt lines are included on each graph of $\mu^p(2jj'':J)$ and $\mu^n(2jj'':J)$. The systematic features of these graphs will now be indicated. There is a symmetry between the deviations from the Schmidt lines of a given $\mu^p(2jj'':J)$ plot and the equivalent $\mu^n(2jj'':J)$. Later it will be shown that the appropriately defined deviations are actually equal to each other. These equivalent plots have the same figure number and are only differentiated by letter, e.g. Fig. 3p and Fig. 3n. Therefore, in the following discussion it is necessary to consider only the systematics of the $\mu^p(2jj'':J)$ plots. The deviations from the Schmidt lines of equivalent $\mu^n(2jj'':J)$ have nearly the same gross features.

In Fig. 3p the plot indicates the deviation from the $J = L + 1/2$ Schmidt line of an odd proton nucleus for the coupling case $j'' = J$. These characteristics are indicated at the top of the figure where the notation $\mu^p(2jj'':J)$, $J = L + 1/2$ and ($j'' = J$) appear. The Schmidt lines are indicated by extra heavy lines. The $j = l + 1/2$ lines are given by fine solid lines, while the $j = l - 1/2$ lines are given by fine broken lines. The above notation will be used in all the following plots. The $j = l + 1/2$ lines always tend to fall in a group which is ordered by ascending or descending values of j. The gap between lines of given j decreases as j increases. The $j = l - 1/2$ lines display the same properties.

The most apparent feature of Fig. 3p is that most values of $\mu^p(2jj'':J)$ fall <u>inside</u> the $J = L + 1/2$ Schmidt line. For $J = 3/2$ no $\mu^p(2jj'':J)$ falls outside the $J = L + 1/2$ Schmidt line (one value falls exactly on the Schmidt line). For each increase in J, one additional value of $\mu^p(2jj'':J)$ falls outside the Schmidt value, but even for $J = 9/2$ the center of gravity of all the $\mu^p(2jj'':J)$ values falls inside the Schmidt line. The same qualitative features hold for Fig. 4p where $J = L - 1/2$ and $j'' = J$. As previously mentioned it is not possible to construct a wave function $\Psi_J^J(2jj'')$ for $J = 1/2$ when $j'' = J$.

In anticipation of a detailed discussion of the experimentally observed magnetic moment deviations from the Schmidt lines, it should be mentioned here that the observed deviations for $J \geq 7/2$ cannot be explained by the magnetic moments plotted in Fig. 3p, Fig. 3n, Fig. 4p and Fig. 4n. The observed deviations are usually greater than the greatest deviation of any $\mu^p(2jj'':J)$ or $\mu^n(2jj'':J)$ for $J \geq 7/2$. Thus the coupling $j'' = J$ gives values of $\mu^p(2jj'':J)$ and $\mu^n(2jj'':J)$ which have the correct qualitative features to explain the observed deviations but which are not adequate to explain a number of observed magnetic moment deviations.

The effect on $\mu^p(2jj'':J)$ of the coupling $j'' = J + 1$ is illustrated in Fig. 5p and Fig. 6p for $J = L + 1/2$ and $J = L - 1/2$ respectively. In Fig. 5p ($J = L + 1/2$) all the $\mu^p(2jj'':J)$ fall <u>inside</u> the $J = L + 1/2$ Schmidt line. The deviations of these $\mu^p(2jj'':J)$ are large, especially for $J \geq 3/2$. The same general characteristics are true for the $\mu^p(2jj'':J)$ in Fig. 6p ($J = L - 1/2$) with the exception of those $\mu^p(2jj'':J)$ for $J = 1/2$ and $j = l - 1/2$. These values of $\mu^p(2jj'':J)$ fall <u>outside</u> the $J = L - 1/2$ Schmidt line. However, the center of gravity of the values of $\mu^p(2jj'':J)$ still falls slightly inside the Schmidt line. This behavior of $\mu^p(2jj'':J)$ for $J = L - 1/2 = 1/2$ and $j = l - 1/2$ gives a possible explanation for the experimentally observed magnetic moment deviations of those odd-even nuclei with $J = L - 1/2 = 1/2$. The experimentally observed magnetic moment deviations for these nuclei are all much smaller than average and in three instances have deviations which actually fall slightly outside the Schmidt line. This behavior is clearly consistent with the $\mu^p(2jj'':J)$ values plotted in Fig. 6p.

The values of $\mu^p(2jj'':J)$ plotted in Fig. 7p ($J = L + 1/2$) and Fig. 8p ($J = L - 1/2$) for the coupling $j'' = J - 1$ have the same general properties as in the previous coupling ($j'' = J + 1$). However, the $J = 1/2$ state is not possible for $j'' = J - 1$ and the values of $\mu^p(2jj'':J)$ which fall outside the $J = L - 1/2$ Schmidt line in Fig. 8p occur for $J = L - 1/2 = 3/2$. These results are also compatible with the observed magnetic moment deviations of odd-even nuclei with $J = L - 1/2 = 3/2$.

Fig. 9p and Fig. 10 p show the values of $\mu^{p}(2jj'':J)$ for the coupling $j'' = J + 2$ for $J = L + 1/2$ and $J = L - 1/2$ respectively. The general tendency of these values is for the $j = l + 1/2$ values to fall <u>inside</u> the $J = L - 1/2$ Schmidt line in Fig. 9p and to fall <u>outside</u> the $J = L - 1/2$ Schmidt line in Fig. 10p. The $j = l - 1/2$ values tend to fall <u>outside</u> the $J = L + 1/2$ Schmidt line in Fig. 9p (except at $J = 1/2$ and $J = 3/2$) and to fall <u>inside</u> the $J = L - 1/2$ Schmidt line in Fig. 10p. The center of gravity of the $\mu^{p}(2jj'':J)$ values is inside the appropriate Schmidt line in Fig. 9p and slightly outside the appropriate Schmidt line in Fig. 10p. Fig. 10 represents the only possible values of $\mu^{p}(2jj'':J)$ for $J = 1/2$ besides those already discussed in connection with Fig. 6p. These two sets of values have a center of gravity very close to the Schmidt line which may explain the small experimental magnetic moment deviations for $J = L - 1/2 = 1/2$ already noted in connection with Fig. 6p.

Fig. 11p and Fig. 12p show the values of $\mu^{p}(2jj'':J)$ for the coupling $j'' = J - 2$ for $J = L + 1/2$ and $J = L - 1/2$ respectively. For this coupling J is restricted to the values $J \geqslant 5/2$. In Fig. 11p the $j = l + 1/2$ values of $\mu^{p}(2jj'':J)$ fall <u>outside</u> the $J = L + 1/2$ Schmidt line while the $j = l - 1/2$ values of $\mu^{p}(2jj'':J)$ fall <u>inside</u> the

$J = L + 1/2$ Schmidt line. The center of gravity of these lines falls slightly outside the Schmidt line. In Fig. 12p the general picture is similar to that of Fig. 11p with the roles of $j = l + 1/2$ and $j = l - 1/2$ reversed. However, the center of gravity of the $\mu^{p}(2jj'':J)$ values in this case falls slightly inside the $J = L - 1/2$ Schmidt line.

The overall tendency of all the couplings considered is to give magnetic moment values which fall inside the Schmidt lines. If only a single type of coupling is considered (such as $j'' = J$), many of the observed magnetic moment deviations cannot be explained. However, all experimental magnetic moment deviations can be explained by using appropriate combinations of all the couplings considered above. The correlation of the calculated $\mu^{p}(2jj'':J)$ and $\mu^{n}(2jj'':J)$ values with the experimentally observed magnetic moment deviations will be considered in greater detail in the section on experimental magnetic moment deviations which follows.

Before discussing the experimental magnetic moment deviations, there is still one type of wave function modification, mentioned in the introduction, to be considered. Two of the three particles sharing the angular momentum J may be even particles (both particles are assumed to have the same j and l) while the third is an odd particle. The two even particles are coupled to give an angular momentum $j' = 2$, and the third particle with an angular momentum $j'' = J$ is coupled to the two even particles so as to give a total angular momentum J for the system of three particles.

As mentioned in the introduction, the only j" that will be considered when even particles share the angular momentum is j" = J.

When two even neutrons combine with an odd proton, the magnetic moment is

$$\mu^{p}(2j(n)J(p):J) = J - \frac{3}{J+1} + \varepsilon(J)2(\mu_p - \tfrac{1}{2})\left[\frac{J}{2L+1} - \frac{3}{(J+1)(2L+1)}\right] + \frac{\varepsilon(j)6\mu_N}{(2\ell+1)(J+1)} \qquad (\text{II}-11)$$

When two even protons combine with an odd neutron, the magnetic moment is

$$\mu^{N}(2j(p)J(n):J) = \varepsilon(J)\frac{2J\mu_N}{2L+1} + \left(1 + \frac{2\varepsilon(j)(\mu_p - \frac{1}{2})}{2\ell+1} - \frac{2\varepsilon(J)\mu_N}{2L+1}\right)\frac{3}{J+1} \qquad (\text{II}-12)$$

The magnetic moments calculated from II-11 and II-12 are plotted in Fig. 13p, Fig. 14p and Fig. 13m, Fig. 14m respectively. Again the general characteristics of the equivalent (same j, ł, J and L) $\mu^{p}(2J(n)J(p):J)$ and $\mu^{n}(2j(p)J(n):J)$ values are similar. However, the deviations of II-11 and II-12 are not truly symmetric with respect to the Schmidt lines as are the magnetic moment deviations

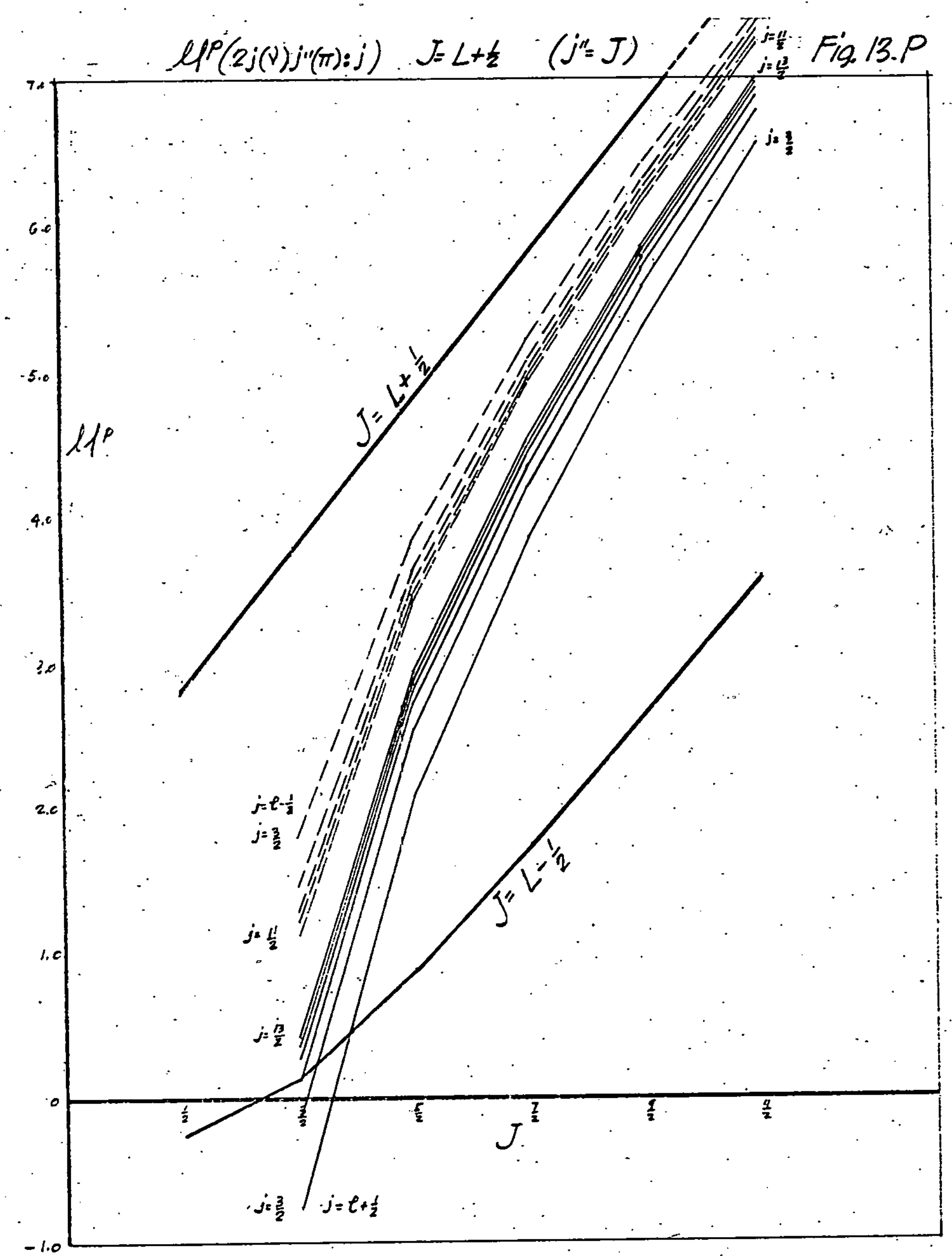


Fig. 13.P
J= L+½
J= L-½
J

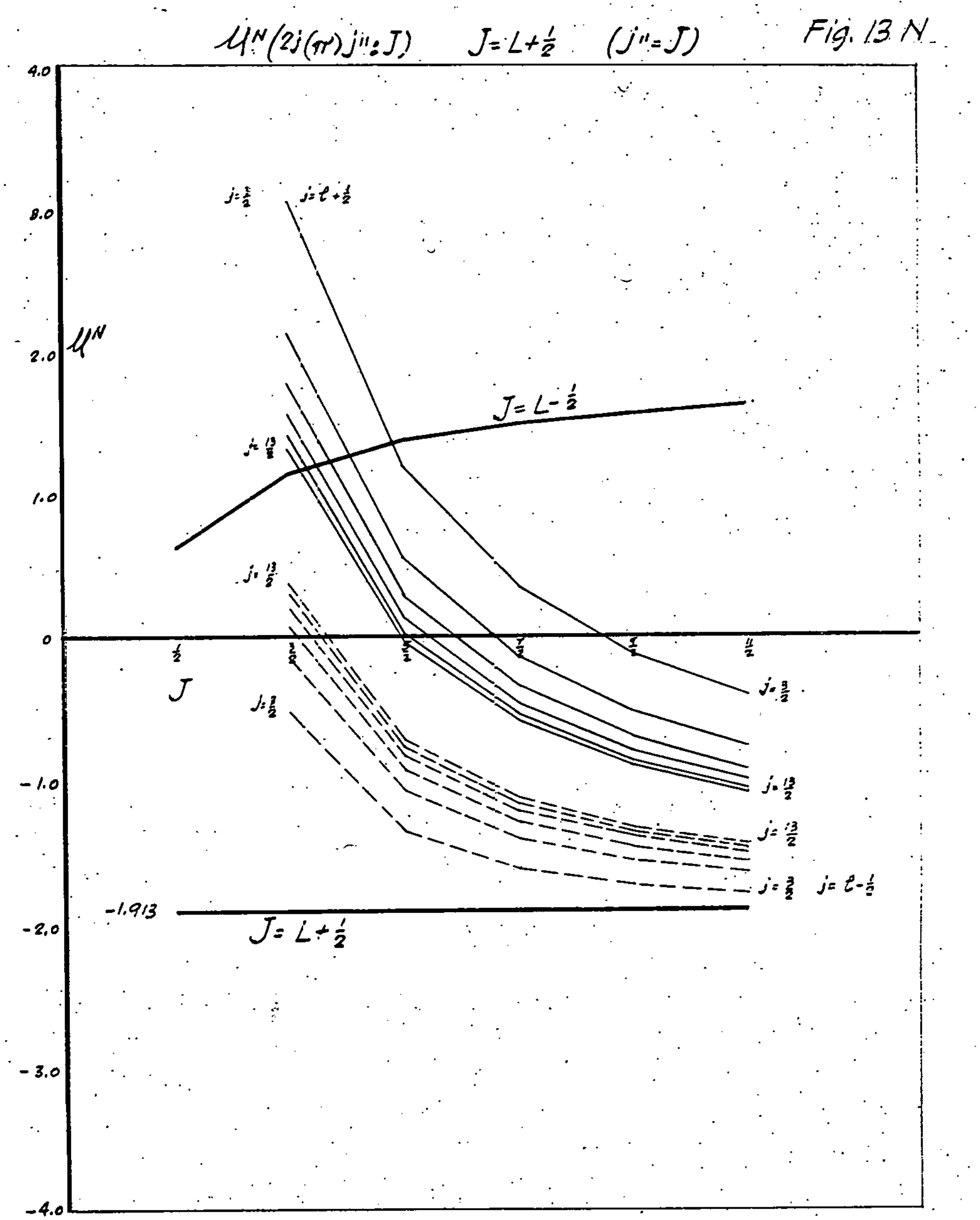
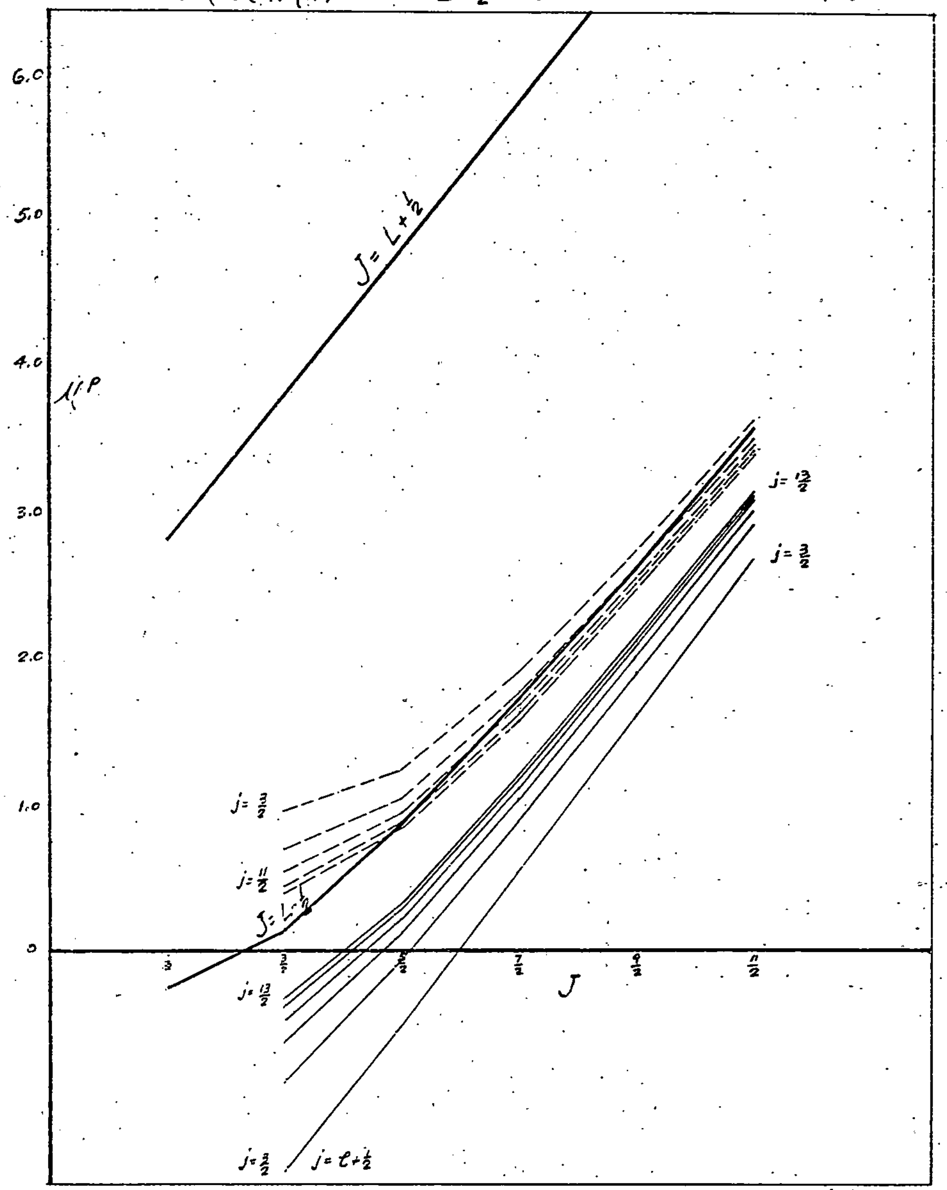
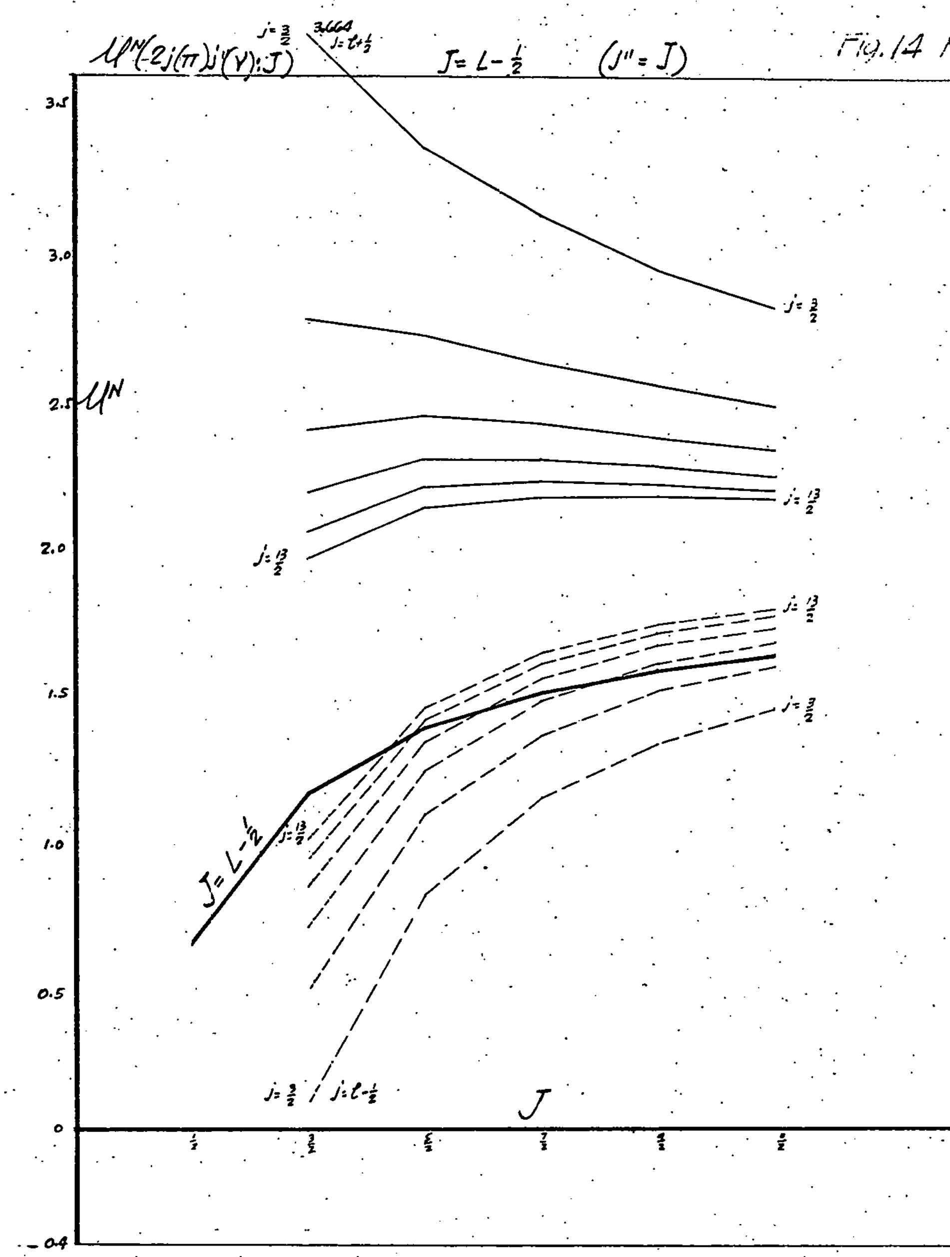


Fig. 13 N
J= L-½
J= L+½
-1.913
J

$\mu^P(2j(\nu)j''(\pi):J \quad J = L - \frac{1}{2} \quad (j'' = J)$ Fig. 14 P.



$\mu^N(2j(\pi)j'(\nu):J) \quad J = L - \frac{1}{2} \quad (j'' = J)$ Fig. 14 N.



given by II-9a and II-9b. The question of symmetry with respect to the Schmidt lines of the odd proton and odd neutron magnetic moments calculated from modified shell model wave functions will be discussed in some detail in the next section.

In Fig. 13p and Fig. 13n ($J = L + 1/2$), the magnetic moments all fall inside the $J = L + 1/2$ Schmidt line. The deviations inside the $J = L + 1/2$ Schmidt line are quite large for $J = 3/2$ and become considerably smaller as J increases. In Fig. 14p and Fig. 14n ($J = L - 1/2$) most of the magnetic moment values fall outside the $J = L - 1/2$ Schmidt line. When the $j = l + 1/2$ even particles share the angular momentum with a $J = L - 1/2$ odd particle, the resulting magnetic moment is considerably outside the $J = L - 1/2$ Schmidt line. When the even particles have $j = l - 1/2$, the magnetic moment deviations (some of which are outside and some of which are inside the $J = L - 1/2$ Schmidt line) are considerably smaller than when $j = l + 1/2$.

Thus, for odd-even nuclei which are assigned $J = L + 1/2$ by the shell model, the sharing of the angular momentum J with even particles is consistent with the observed deviations of the magnetic moments from the $J = L + 1/2$ Schmidt line. For odd-even nuclei which are assigned $J = L - 1/2$, the sharing of the angular momentum among the even particles would tend to make the magnetic moments fall outside the $J = L - 1/2$ Schmidt line.

III. EXPERIMENTAL MAGNETIC MOMENT DEVIATIONS OF ODD-EVEN NUCLEI.

When comparing the experimentally observed magnetic moments of odd-even nuclei to the Schmidt lines, it is important to provide a scale in terms of which deviations can be measured. The following quantities provide an especially convenient measure of the deviations from the Schmidt lines:

$$\text{a)} \quad \Delta^P = \frac{\mu_s^P - \mu_{ex}^P}{(\mu_P - \frac{1}{2})}$$

$$\text{b)} \quad \Delta^N = \frac{\mu_s^N - \mu_{ex}^N}{\mu_N} \qquad \text{(III-1)}$$

where Δ^p and Δ^n refer to odd proton nuclei and odd neutron nuclei respectively, μ_s^p is the Schmidt value for the magnetic moment of the odd proton nucleus, μ_s^n is the Schmidt value for the magnetic moment of the odd neutron nucleus, and μ_{ex}^p and μ_{ex}^n are the experimental magnetic moments of the corresponding nuclei. The quantities Δ^p and Δ^n will be referred to simply as "deviations" in the following discussion.

The experimental magnetic moment deviations, as determined by III-1a and III-1b, are plotted[18] in Fig. 15 as a

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[18] The experimental magnetic moments used to obtain this plot are from an unpublished compilation of nuclear moment data by Harold E. Walchli.

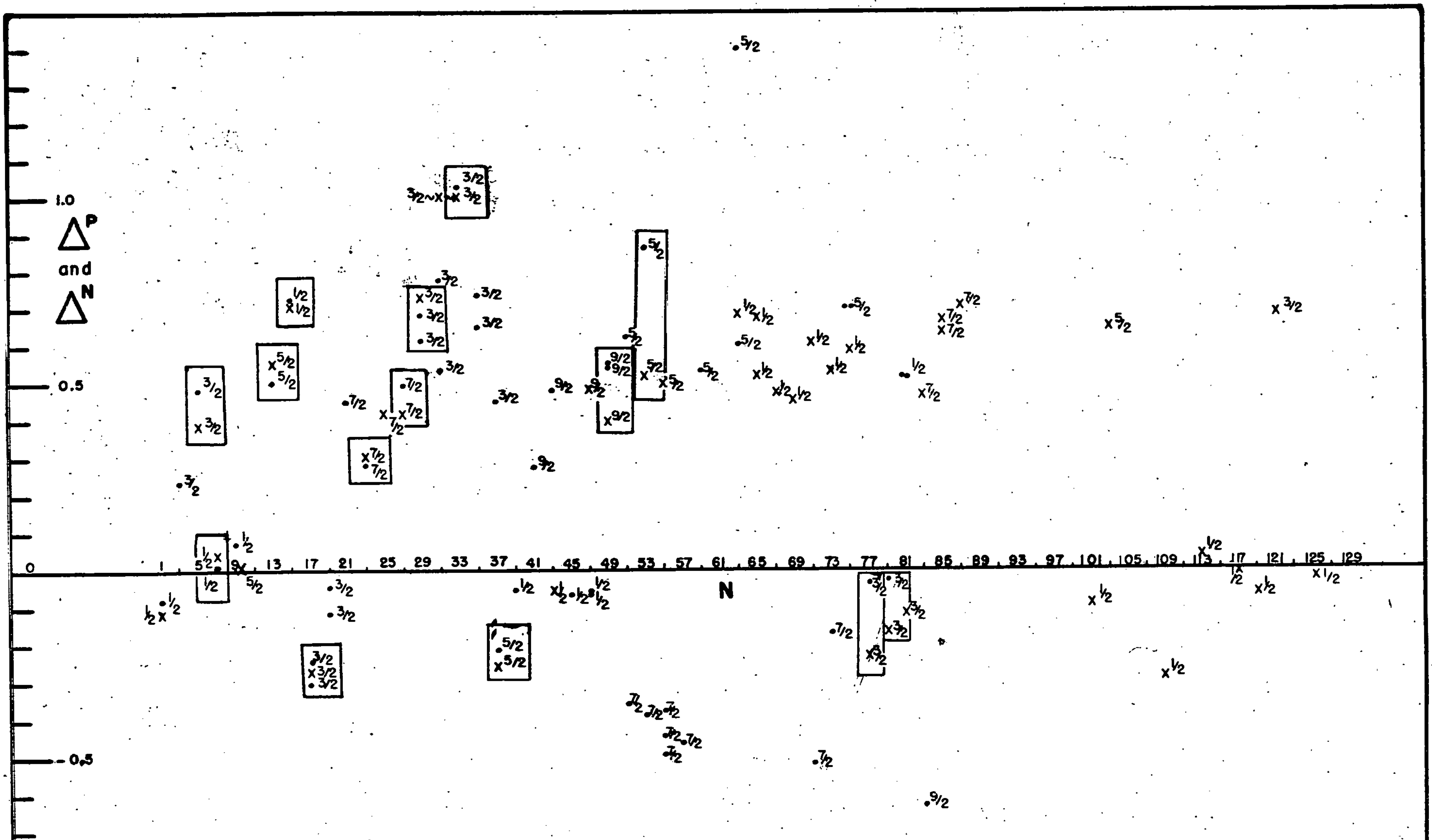

FIG. 15 Magnetic moment deviations.

function of the number of odd particles in the odd-even nucleus. Odd proton magnetic moment deviations are indicated by dots while odd neutron magnetic moments are indicated by crosses. The total angular momentum J is written beside the indicated deviation. The Schmidt values are such that deviations inside the J = L + 1/2 Schmidt line are positive (for both odd proton and odd neutron nuclei), while deviations inside the J = L - 1/2 Schmidt lines are negative. Except for those nuclei with n = 1, 7, and 113 odd nucleons, all the deviations plotted in Fig. 15 represent deviations inside the appropriate Schmidt line.

A number of systematic features of the magnetic moments are suggested by Fig. 15. The negative deviations (inside the J = L - 1/2 Schmidt line) are, on the whole, much smaller than the positive deviations (inside the J = L + 1/2 Schmidt line). The deviations for n smaller than 53 seem to vary in a systematic manner which may be related to the shell model[19], since they assume definite minimum values in the region of the closed shells at n = 2, 8 and 20 (the systematic variations would become more apparent if only the magnitudes of the deviations were plotted). A fairly definite minimum also occurs in the region of the closed subshell at n = 40. A less definite minimum occurs at the closed n = 50 shell. At the n = 28 subshell the deviations show no minimum

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[19] This behavior of the magnetic moment deviations was noted by J. P. Davidson, Phys. Rev. 85, 432 (1952).

but instead a general rising trend to a maximum at n = 33.

The positive deviations seem to cluster around an average value Δ^p and $\Delta^n \sim 0.58$ which does not depend in any significant way on the value of J (except all J = 9/2 deviations are below this average). The fluctuations from this average are greatest for n < 53. The positive deviations in the region of closed shells or subshells on the whole do not become appreciably smaller than the average except for n = 9.

The negative deviations are not only smaller than the positive deviations, but the magnitudes of the negative deviations appear to depend on the value of J. The average (for a specific J) of the negative deviations is smallest for J = 1/2 and increases with J reaching a maximum value with the highest experimentally observed value J = 9/2. The magnitude of the negative deviations for J = 7/2 and J = 9/2 is nearly the same as the average for the positive deviations. The behavior of the negative deviations in the region of the closed shells at n = 50 and n = 82 is quite interesting. The deviations are small when J is small and large when J is large, which suggests that the small negative deviations are primarily a consequence of a small J = L - 1/2 rather than any closed shell property.

The deviations of the magnetic moments of isotopes, which are indicated by two dots with the same n (odd proton isotopes) or by two or more crosses with adjacent n values (even proton isotopes), are usually close to each other. There are fifteen values of n (excluding n = 1 which is anomalous) for which experimental deviations are recorded for both odd proton and odd neutron nuclei having the same n and the same angular momentum J (conjugate pairs). The pairs (or in some cases, triplets) are boxed off in Fig. 15[20]. It is seen that the deviations for the odd proton and odd neutron nuclei with the same n and J are generally close to each other. The importance of this fact, which was first noted by Schawlow and Townes, has already been discussed in the Introduction.

The close correlation of the majority of the conjugate pairs (or triplets) seems to indicate some mirror property with respect to odd proton and odd neutron states. Other, but less direct, evidence for such a mirror property is given by the near equality of the magnetic moment deviations for odd proton and odd neutron nuclei with the same J whose n values differ by two, four, or six. Many such cases are found distributed throughout Fig. 15. The addition of two, four, or six odd particles does not appear, in these cases, to affect seriously the magnetic moment deviations.[21] These additional particles can be considered as filling subshells

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[20]The measured odd neutron magnetic moments for n = 31 ($_{26}Fe^{57}$) and n = 33 ($_{28}Ni^{61}$) are quoted as $\mu \approx 0$ (P.F.A. Klinkenberg, Rev. of Mod. Phys., 24, 63 (1952).) with a fairly large error possible. The conjugate pair with n = 33 is boxed off because of the good correlation of the deviations, while the conjugate triplet with n = 31 is not indicated because of the poor correlation. This procedure is highly arbitrary.

[21]In many instances there is also an additional increase in the number of even particles.

in such a way as not to change the admixture of odd particle states. The odd proton states may thus be essentially the same as the odd neutron states even though the number of odd protons is different from the number of odd neutrons. This is not inconceivable, since four particles (in general any possible even number of particles) in equivalent states (same j and l) coupled to J = 2 cannot be distinguished, with respect to any effect on the magnetic moments, from two particles in equivalent states coupled to J = 2.

The experimentally observed correlations of the conjugate pairs can be understood in terms of two conditions which are consistent with, but much more general than, the shell model. One of these is that <u>only the odd nucleons</u> carry the total angular momentum of the nucleons, the other, that the odd nucleon wave function satisfies a mirror condition. When the angular momentum J is assumed to be distributed only among the odd particles the magnetic moment deviations as defined by III-1a and III-1b are

$$\text{a)} \quad \Delta^P = \varepsilon(J)\frac{2J}{2L+1} - \left\langle \Psi_J^J(P), \sum_P \sigma_P^z\, \Psi_J^J(P) \right\rangle$$

(III-2)

$$\text{b)} \quad \Delta^N = \varepsilon(J)\frac{2J}{2L+1} - \left\langle \Psi_J^J(N), \sum_N \sigma_N^z\, \Psi_J^J(N) \right\rangle$$

where $\Psi_J^J(p)$ and $\Psi_J^J(n)$ are the wave functions of the odd protons of the odd proton nucleus and of the odd neutrons of the odd neutron nucleus respectively. If the wave functions

$\Psi_J^J(p)$ and $\Psi_J^J(n)$ are the same when the number of protons is equal to the number of neutrons, then

$$\langle \Psi_J^J(P), \sum_P \sigma_P^z \Psi_J^J(P) \rangle = \langle \Psi_J^J(N), \sum_N \sigma_N^z \Psi_J^J(N) \rangle \qquad \text{(III-3)}$$

This leads to the result

$$\Delta^P = \Delta^N \qquad \text{(III-4)}$$

Therefore, the experimentally observed correlations of the magnetic moment deviations can be understood by assuming that the even particles do not share the angular momentum J and that the odd particle states are described by the same wave function.

However, the experimental deviations of the conjugate pairs are not really equal, especially[22] for larger n. Differences in the deviations of conjugate pairs may arise either from a lack of perfect mirroring of the odd particle wave functions or to a sharing of the angular momentum with the even particles.

The interpretation of the conjugate pairs is further complicated by the fact that the nature of the admixtures (number of admixed states and their relative importance) contributing to the magnetic moment deviations is also im-

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[22]The poor correlation for the conjugate pair with n = 53 may be associated in part with the anomalous behavior of the $_{53}I^{127}$ and $_{53}I^{129}$ isotopes in having different total angular momenta ($_{53}I^{127}$ has J = 5/2 while $_{53}I^{129}$ has J = 7/2).

portant when trying to determine what properties of the wave function are responsible for the correlation (or lack of correlation) of conjugate pairs. If only a few states contribute to the nuclear wave function (the probability amplitudes of these states are not necessarily small), then the experimental correlations are a good indication that the admixtures involve only odd particle states, which are very nearly the same for the paired nuclei. For, if the number of states in the admixture is small, the magnetic moment deviation would depend quite sensitively on the probability amplitudes of these states and on possible interference effects[23]. Under these circumstances even particle contributions would tend to destroy the correlation in conjugate pairs in two ways, both due to the large difference in the number of even particles in the odd proton nucleus as compared to the odd neutron nucleus. First, the even particle contributions would generally come from different shells in the two nuclei, so equal contributions to the magnetic moment deviations for the two nuclei would be unlikely. Second, the admixtures of the even particle states from different shells would be expected to occur with different probability amplitudes. This would, of necessity, destroy any mirroring of the odd particle states since the probability amplitudes of the odd particle admixtures could no longer be

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[23] Interference effects can only occur, with the states being considered, between two states $\Psi_J^J(2jj'')$ and $\Psi_J^J(2\underline{j}\underline{l}'')$ where $j = \underline{j}$, $l = \underline{l}$ and $l'' = \underline{l}''$. The Mayer Jensen shell model state can give no interference contributions.

the same (because of normalization conditions) for both nuclei. Thus if the magnetic moment deviations are due to admixtures of only a few states, experimental evidence indicates that only odd particles are responsible for most of the angular momentum and the odd particle states display a mirror property between neutrons and protons.

If many states contribute to the admixture describing an odd-even nucleus, then the experimental evidence does not clearly exclude sharing of the angular momentum among the even particles. The deviations would not be expected to depend sensitively on the probability amplitude of any single state (the probability of any single state being comparatively small). Interference effects would also tend to be of less importance in the deviations since interference terms would occur with arbitrary signs and would be expected to average to zero if the number of admixed states were very large.

The experimental correlations would require that the general admixture of states be more or less the same for the two nuclei. The correlations would also require that the average contribution to the magnetic moment deviations due to the admixture of even particle states be nearly the same for the two nuclei. In Section II it was noted that the magnetic moment deviations for two even particles coupled to a third odd particle ($j'' = J$) were roughly symmetric for odd proton-even neutron and odd neutron-even proton nuclei (see Fig. 13p, Fig. 13n and Fig. 14p, Fig. 14n), i.e. the deviations from the Schmidt lines are almost the same for the two cases, though not exactly equal, as they are when all the particles are odd. Thus, if the even particles share the angular momentum in the manner described in the previous section, then a large number of contributing states could easily tend to give about the same contribution to the magnetic moment deviation even though the contributing states come from different shells.

The interpretation of the correlations observed by Schawlow and Townes is thus seen to be quite difficult. For the lighter nuclei ($n < 29$) where shell effects seem important and where the number of possible admixtures is comparatively small, the correlations are probably due to mirroring of the odd particle states with little or no sharing of the angular momentum among the even particles. For heavier nuclei it is not possible to exclude a sharing of the angular momentum by the even particles. Such a sharing may indeed account for the poorer correlations of the conjugate pairs found for larger n.

On the basis of the experimental correlation of conjugate pairs, odd particle admixtures will be assumed to be primarily responsible for magnetic moment deviations even when the number of admixed states is assumed to be large. However, even particle contributions will also be considered, but with the value of j'' restricted to $j'' = J$. Such a restriction on the value of j'' limits the number of possible even particle states available for admixture as compared to the number of possible odd particle states. Thus, even if all possible states were admixed with equal probability, the odd particle contributions would be of most importance in determining the magnetic moment deviations in agreement with the evidence of the conjugate pairs.

An attempt will now be made to understand the magnetic moment deviations in terms of admixtures of those states considered in Section II. The Mayer-Jensen shell model is used as a guide to determine which states are to be admixed to form the nuclear ground state wave function. Of primary importance is the energy level system proposed by Mayer[1] and Jensen[2] to account for the magic shells at 2, 8, 20, 50, 82, and 126 protons or neutrons.

The level scheme[24] used in this thesis is shown in Fig. 16. The level ordering of protons and neutrons are very similar with the slight differences being attributed to coulomb effects among the protons. Every state shown in Fig. 16 has a $2J + 1$ fold degeneracy of the magnetic quantum number M ($M = J, J-1, \ldots -J$). Klinkenberg indicates 28 as a closed shell but this is usually considered as a subshell and shall be so treated in this work.

The odd-even nuclei are assumed to have their nucleons in those configurations specified by the shell scheme given in Fig. 16. In those cases where the levels are fairly close to each other so that there is competition between the states, the shell model configuration of a given odd-even

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[24] From the Tables of Nuclear Shell Structure by P. F. A. Klinkenberg, Rev. Mod. Phys., 24, 63, (1952).

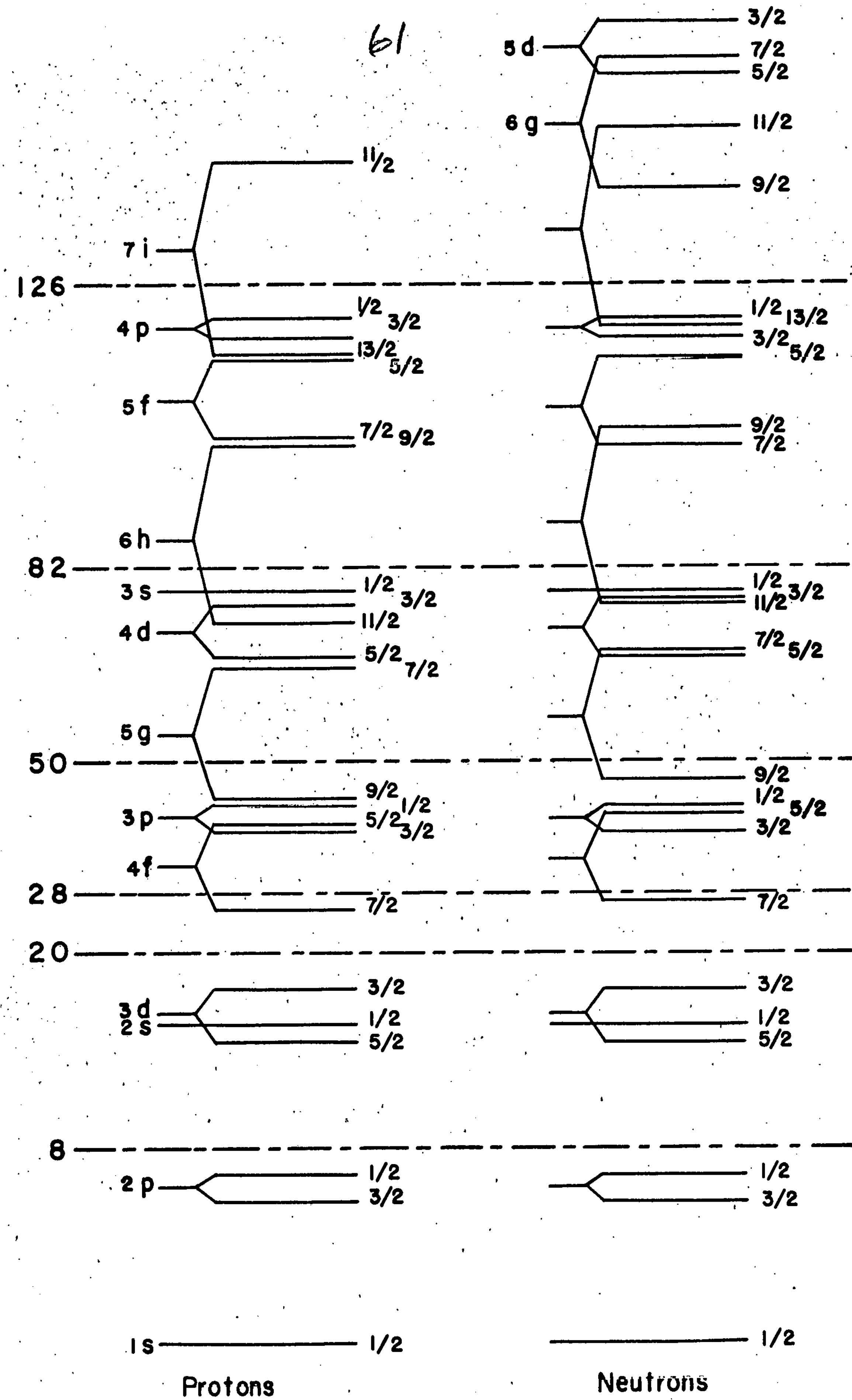

FIG. 16. Nuclear shell schemes.

nucleus must be deduced from the experimental data, as has been done, for example, by Klinkenberg[24].

Although the point of view here is that the wave function may differ quite appreciably from the shell model function, it will be assumed that the energy levels and the nuclear configurations given by Klinkenberg may be used to determine the "most probable" states which are admixed in the nuclear wave functions. Considerable freedom will be assumed in mixing configurations as well as in mixing wave functions of a given configuration. However, it seems reasonable (for the sake of simplicity if nothing else) to limit any mixing of configurations, i.e. any new distribution of the particles among the energy levels, to those levels within a given shell. Of course the different configurations must form wave functions with the correct angular momentum and parity in order to be part of the nuclear wave function.

Thus, for a given odd-even nucleus the three particle states[25] considered for any admixture are constructed from the various possible configurations having two particles in equivalent states within a given shell. When the three particles are all odd, all possible odd particle configurations with two particles in equivalent states within the unfilled

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[25]It should be emphasized again that in all considerations of the three particle states that actually many more than three particles may be sharing the angular momentum in the wave functions specified in Section II. The wave function of any even number of particles in equivalent states combined to give $j' = 2$, will give the same magnetic moment as two particles in equivalent states combined to give $j' = 2$.

odd shell are considered. The even shell is assumed to be coupled to angular momentum $j' = 0$. When two particles of the three particle state are even, the odd particle configuration is assumed to be the same as in the shell model, while all possible even particle configurations within the unfilled even shell (with the two particles in equivalent states) are considered.[26] A typical nucleus will be discussed here in order to illustrate the method for obtaining possible states for admixture.

$_{37}Rb^{85}$ is an odd proton nucleus with a measured magnetic moment equal to + 1.353 n.m. (nuclear magnetons). It has the following shell model configuration for the odd particles, $(20)(4f\ 7/2)^8(3p\ 3/2)^4(4f\ 5/2)^5$, where the (20) indicates that the 20 shell is filled and where $(nL_J)^x$ indicates that x particles are in the state specified by the radial quantum number n, the orbital angular momentum L, and the total angular momentum J. For the nucleus being considered, the shell model specifies that the 4f 7/2 and 3p 3/2 subshells are completely filled (and of necessity coupled to angular momentum $j' = 0$) while there are five out of a possible six particles in the 4f 5/2 subshell. Four of these five particles are, according to the shell model, coupled to $j' = 0$.

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[26]When the shell is closed or one less or one more than closed, there is some ambiguity as to which configurations should be used in admixtures. In this work two particles are assumed to be promoted out of the closed or near closed shell (into equivalent states and usually coupled to $j' = 0$), and the configurations of the now opened shell are usually used for admixture. In some cases, when there is one particle more than a closed shell, the closed shell is treated as still closed even though two particles have been promoted to what is then treated as the unfilled shell. Again this procedure is completely arbitrary.

The fifth particle has the total angular momentum $J = 5/2$ and determines the parity (odd) of the nucleus. The states, 4f 5/2, 3p 1/2, and 5g 9/2 of the (20)-(50) shell have energies close to those that are filled and the nucleons may find it advantageous to distribute themselves among these states rather than those indicated by the strict shell model. The configurations of three odd particles (having the correct parity and capable of giving the correct angular momentum), allowed by the prescription of the previous paragraph are indicated in Table 1 along with the appropriate magnetic moment as calculated in Section II.

In Table 1 the radial quantum numbers have been omitted for convenience and only the three particles sharing the angular momentum in the configuration are indicated, where $\Psi_J^J(2jj'')$ is defined by II-8. Some of these configurations represent considerable modifications of the shell model. Thus the three particle configuration $l_j = f_{7/2}$ and $l''_{j''} = f_{5/2}$ can only be formed by taking two particles out of the filled $f_{7/2}$ subshell and placing them, coupled to $j' = 0$, in an unfilled subshell. Four of the remaining $f_{7/2}$ particles are coupled to $j' = 0$ while the other two $f_{7/2}$ particles are coupled to $j' = 2$ which in turn is coupled to the $f_{5/2}$ particle to give the final total angular momentum $J = 5/2$. Other three particle configurations represent similar shifting of particles, appropriately coupled to give the final angular momentum correctly.

TABLE 1

POSSIBLE THREE PARTICLE CONFIGURATION (2jj") (in $\psi_J^J(2jj'')$)		MAGNETIC MOMENT
ℓ_j	$\ell''_{j''}$	
--	f 5/2	0.862
f 7/2	f 5/2	2.506
p 3/2	f 5/2	2.734
g 9/2	f 5/2	2.395
f 7/2	f 7/2	4.135
p 3/2	f 7/2	4.010
f 5/2	f 7/2	4.323
g 9/2	f 7/2	4.156
f 7/2	p 3/2	4.949
f 5/2	p 3/2	2.888
g 9/2	p 3/2	4.721
f 7/2	p 1/2	3.045
p 3/2	p 1/2	4.793
f 5/2	p 1/2	0.427
g 9/2	p 1/2	3.143

The even neutrons of the $_{37}Rb^{85}$ nucleus have the shell model configuration $(20)(4f_{7/2})^8(3p_{3/2})^4(4f_{5/2})^6(3p_{1/2})^2(5g_{9/2})^8$, so just two $5g_{9/2}$ particles are needed to close the 50 shell. The three nucleon configurations which will lead to possible admixed states in this case are given in Table 2. The two particles in equivalent states are neutrons and the third particle is a proton. Only the one proton state $f_{5/2}$ is considered for the reasons stated earlier.

Altogether there are eighteen possible configurations, corresponding to eighteen wave functions. The magnetic moments of these states vary considerably with the great majority falling inside the J = L - 1/2 Schmidt line for protons, i.e. the great majority of magnetic moments are greater than 0.862 n.m., the Schmidt value for an $f_{5/2}$ odd proton nucleus. Thus the experimental magnetic moment of 1.353 n.m. can be obtained by a simple admixture of the shell model state with almost any of the eighteen states being considered.

Within the scope of the model being used here, the maximum probability for the shell model state that is consistent with the observed magnetic moment is 87.2%. In this case the only admixed configuration is $I_j = f_{7/2}$ and $I''_{j''} = p_{3/2}$ (18.2%). For most nuclei the maximum shell model probability would be considerably less, since the magnetic moment deviation of $_{37}Rb^{85}$ is smaller than average ($\Delta_{Rb}^{p} \sim -0.2$). In general, if the admixed wave functions are of the type considered in Section II and restricted by the prescription

TABLE 2

THREE PARTICLE (TWO EVEN-ONE ODD) CONFIGURATION (2jJ)		MAGNETIC MOMENT
ℓ_j	L_J	
f 7/2	f 5/2	0.100
p 3/2	f 5/2	-0.525
f 5/2	f 5/2	1.036
g 9/2	f 5/2	0.204

for the admixed wave functions given in this Section, then the magnetic moment deviations indicate that at best the shell model states are not much more probable than the admixed states. If interference effects are considered, then the admixed states can possibly become even more important relative to the shell model state.

In many nuclei the magnetic moment deviation can be explained by assuming that the shell model state has little or no probability amplitude compared to a single possible admixed state. However, this is not a general rule and there is no reason to believe that such agreement between the observed magnetic moment and the magnetic moment given by a single admixed state is not fortuitous.

If more than one of the possible states is admixed with the shell model state, the correlation of the magnetic moment deviations with the wave functions becomes impossible without further assumptions. The observed magnetic moment can only give a functional relation among the probability amplitudes of the various admixed states if there are more than two states. Furthermore interference effects can occur in many cases when more than one admixed state is considered, e.g. states formed from the configurations $l_j = f_{5/2}$, $l''_{j''} = p_{3/2}$ and $l_j = f_{5/2}$, $l''_{j''} = p_{1/2}$ will give a non-vanishing matrix element for the magnetic moment operator. The contributions of interference effects occur with an unknown sign since the probability amplitudes of the admixed states are undetermined with respect to sign. In many possible admixtures involving several states, the interference contributions to the calculated magnetic moment may be of considerable importance. Thus any specific assignment of more than one admixed state on the basis of magnetic moment deviations is highly arbitrary.

Since there is little basis for an a priori selection of one admixed state over another in constructing a nuclear model, it seems reasonable to consider a statistical model, i.e., to assume that, within the classes of the admixed wave functions being considered, all possible states (including the shell model state) are equally probable. This represents, in effect, a sharing of the angular momentum among most of the "unfilled" shell states of both the odd and even particles. However, the angular momentum is not shared uniformly among the states, but instead is subject to a decided influence of the shell model energy levels as entailed in the particular specification of the admixed states being used.

Such a model of complex nuclei is fairly reasonable on the basis of the rather contradictory evidence of the two body problem and complex nuclei. The nuclear forces deduced from two body data, e.g. the deuteron, proton-proton scattering and proton-neutron scattering are strong and short ranged. Data on complex nuclei, e.g. nuclear spins, magnetic moments, isomeric states, etc., indicate that a given nucleon moves in an average central potential, i.e. the independent particle shell model. Strong short range nuclear forces would appear to be inconsistent with the idea of an average potential, necessary for the independent particle model. The interactions between nucleons due to the short range forces would be expected to cause continual interchanges of energy and momenta between particles. Such interchanges would also involve the angular momentum of the nucleus, and on the average for very strong and short range interactions it is reasonable to assume that the total angular momentum of the nucleus, which of course is a constant of the motion, is equally shared among all the nucleons of the nucleus. This assumption carried to the extreme leads to the Margenau-Wigner values for the magnetic moments of complex nuclei. The Margenau-Wigner model takes no account of the shell model but the statistical model suggested here takes account of that model by introducing it as a basis for the selection of states. In that sense the present model is a synthesis of the basic principles of the shell model and those of the Margenau-Wigner scheme.

A further assumption must be made before magnetic moments can be calculated on the basis of what may now be called the "statistical shell model". Namely, interference effects are assumed to average to zero. If the number of admixed states is very large, this assumption is probably justified. However, the number of states usually considered in the statistical shell model is not very large and the number of interference terms is still smaller. In these cases the interference terms are assumed to be equal to zero just for the sake of simplicity.

When using the statistical shell model to calculate theoretical magnetic moments, every odd-even nucleus is classified by the usual shell model state (by total angular momentum and parity), the unfilled odd particle shell[26], and by the unfilled even particle shell[26]. The allowed configurations of three particles are determined by the unfilled shells of the odd particles and of the even particles. The allowed admixed states (constructed from these configurations) are then determined by the usually assigned shell model state. All states are assumed to occur with equal probability. Since the interference terms are assumed to be equal to zero, the theoretical magnetic moment is obtained by taking a simple average of the magnetic moments obtained in Section II for each of the admixed states (the usual shell model state being included as one of the states). It then follows that all odd-even nuclei in the same class, i.e. having the same shell model state and the same unfilled shells, would have the same magnetic moment according to the statistical shell model. Such a behavior is noted experimentally as can be seen in Fig. 15 which shows that the magnetic moment deviations for nuclei with the same J, parity, and unfilled shells are nearly the same. A striking example of this behavior is offered by the negative deviations for $J = 7/2$ between $n = 51$ and $n = 57$ (the $J = 7/2$ negative deviations at $n = 71$ and $n = 73$ are in a different class since the even shell is different).

Eighty-five odd-even nuclei with measured magnetic moments have been combined into 15 odd proton-even neutron classes and 15 odd neutron-even proton classes. These classes are shown in Table 3 and Table 4 respectively. The total angular momentum J is indicated in the first column while the usually assigned shell model state is given in the second column. The third and fourth columns indicate the unfilled neutron shell and proton shell respectively (the unfilled shell is indicated by the two closed shell numbers which constitute its upper and lower bounds). The fifth column gives the number of nuclei in the particular class being considered. The sixth column gives the average experimental magnetic moment, $\mu^{p}_{ex}(ave)$ or $\mu^{n}_{ex}(ave)$, of the nuclei in the class. The seventh column gives the magnetic moment, μ^{p}_{1} or μ^{n}_{1}, obtained from considering admixtures of only odd particle configurations in the statistical shell model, while the eighth column gives the magnetic moment, μ^{p}_{2} or μ^{n}_{2}, obtained from considering all possible admixtures in the statistical shell model. The ninth column gives the average of the absolute value of the magnetic moment deviations Δ^{p} or Δ^{n} for the nuclei in a given class which are to be compared to the deviations from the statistical shell model magnetic moments. The next two columns give the absolute deviations of the average experimental magnetic moment, $\mu^{p}_{ex}(ave)$ or $\mu^{n}_{ex}(ave)$, from μ^{p}_{1}, μ^{n}_{1} and μ^{p}_{2}, μ^{n}_{2} where these absolute deviations are defined as

a) $$\delta_1^{p} = \frac{|\mu_1^{p} - \mu_{ex}^{p}(ave)|}{\mu_p - \frac{1}{2}} \qquad \text{(III-5)}$$

$$\delta_1^{N} = \frac{|\mu_1^{N} - \mu_{ex}^{N}(ave)|}{\mu_N}$$

TABLE 3 - ODD P NUCLEI

J	SHELL MODEL STATE	N SHELL	P SHELL	NO. OF NUCLEI	μ^P(AVE)	μ_1^p	μ_2^p	AVERAGE Δ^p	δ_1^p	δ_2^p
1/2	p 1/2	50-82	20-50	4	-0.123	-0.320	-0.370	0.062	0.108	0.108
1/2	s 1/2	82-126	50-82	2	1.620	1.482	1.482	0.512	0.060	0.060
3/2	p 3/2	2-8	2-8	1	2.689	2.723	2.224	0.481	0.015	0.203
3/2	d 3/2	8-20	8-20	2	1.520	1.677	1.354	0.608	0.068	0.072
3/2	d 3/2	20-50	8-20	3	0.430	1.677	1.098	0.133	0.543	0.291
3/2	p 3/2	20-50	20-50	8	2.219	2.213	1.820	0.686	0.003	0.174
3/2	d 3/2	82-126	50-82	2	0.165	1.03	0.728	0.017	0.377	0.246
5/2	d 5/2	8-20	8-20	1	3.641	3.515	3.856	0.502	0.055	0.094
5/2	f 5/2	20-50	20-50	2	2.411	3.357	2.693	0.676	0.413	0.123
5/2	d 5/2	82-126	50-82	7	3.058	2.976	3.013	0.757	0.036	0.019
7/2	f 7/2	20-50	20-50	3	4.851	4.790	4.693	0.411	0.027	0.069
7/2	g 7/2	50-82	50-82	6	2.681	3.469	2.952	0.421	0.344	0.118
7/2	g 7/2	82-126	50-82	3	2.667	3.469	2.809	0.415	0.350	0.062
9/2	g 9/2	50-82	20-50	4	5.613	5.638	5.648	0.515	0.011	0.015
9/2	h 9/2	82-126	82-126	1	4.082	4.288	3.796	0.633	0.090	0.124

TABLE 4 - ODD N NUCLEI

J	SHELL MODEL STATE	N SHELL	P SHELL	NO. OF NUCLEI	μ^n(AVE)	μ_1^n	μ_2^n	AVERAGE Δ^n	δ_1^n	δ_2^n
1/2	p 1/2	2-8	2-8	1	0.702	0.637	0.637	0.034	0.034	0.034
1/2	s 1/2	8-20	8-20	1	-0.555	-1.212	-1.212	0.710	0.343	0.343
1/2	s 1/2	50-82	50-82	8	-0.700	-1.050	-1.050	0.634	0.183	0.183
1/2	p 1/2	82-126	50-82	6	0.421	0.774	0.774	0.113	0.185	0.185
3/2	p 3/2	2-8	2-8	1	-1.177	-1.425	-0.679	0.385	0.130	0.286
3/2	d 3/2	8-20	8-20	1	0.643	0.626	0.856	0.264	0.009	0.111
3/2	p 3/2	20-50	20-50	1	-0.5	-0.621	-0.437	0.739	0.063	0.033
3/2	d 3/2	50-82	50-82	3	0.825	0.392	0.593	0.169	0.226	0.121
3/2	p 3/2	82-126	50-82	1	-0.613	-0.424	-0.171	0.630	0.099	0.231
5/2	f 5/2	28-50*	20-50	1	0.876	-0.170	0.669	0.257	0.547	0.108
5/2	d 5/2	50-82	20-50	3	-0.828	-0.532	-0.399	0.567	0.155	0.224
5/2	f 5/2	82-126	50-82	1	-0.66	-0.549	-0.169	1.060	0.058	0.257
7/2	f 7/2	20-50	20-50	2	-1.104	-1.076	-0.901	0.423	0.015	0.106
7/2	f 7/2	82-126	50-82	4	-0.713	-0.266	-0.377	0.627	0.234	0.176
9/2	g 9/2	20-50	20-50	2	-1.035	-0.928	-0.903	0.459	0.056	0.069

* The f 7/2 shell model level has been treated as a closed shell in this case in order to obtain satisfactory agreement.

$$\delta_2^P = \frac{|\mu_2^P - \mu_{ex}^P(ave)|}{\mu_P - \frac{1}{2}}$$

b) (III-5)

$$\delta_2^N = \frac{|\mu_2^N - \mu_{ex}^N(ave)|}{\mu_N}$$

The two deviations δ_1 and δ_2 are considered separately, since, as has already been noted, the correlations of conjugate pairs of nuclei seem to indicate that only odd particles share the angular momentum in lighter nuclei, while for heavier nuclei the evidence of conjugate pairs does not exclude even particle contributions. As will be indicated in the following discussion, the conclusions obtained from the conjugate pair data seem to be verified by the results tabulated in Table 3 and Table 4.

In Table 5 the weighted averages of the last three columns of Table 3 and Table 4 are given. In the averaging process the weight for each class is given by the number of nuclei in the class. This procedure seems called for by the statistical nature of the theory. Thus the average magnetic moment of a class which contains a large number of nuclei would be expected to agree with the theoretical magnetic moment on the statistical model better than a class composed of one or two nuclei. Individual differences in the ground states of nuclei within a class would be expected to average to zero if there is a sufficiently large number of nuclei in the

TABLE 5

	Ave Ave Δ^p	Ave δ_1^p	Ave δ_2^p	Ave Ave Δ^n	Ave δ_1^n	Ave δ_2^n
All Nuclei	0.485	0.167	0.121	0.454	0.169	0.169
$J = L\ \ \frac{1}{2}$	0.555	0.020	0.076	0.590	0.158	0.178
$J = L - \frac{1}{2}$	0.383	0.348	0.159	0.215	0.187	0.152

class. A weighted average emphasizes this expectation.

Besides the simple weighted averages, Table 5 also gives the weighted averages of the J = L + 1/2 and J = L - 1/2 classes separately. When the average δ_1^p, δ_2^p and δ_1^n, δ_2^n are compared to the appropriate averaged deviations from the Schmidt lines Δ^p and Δ^n, it is seen that the statistical shell model gives a much better fit to the magnetic moment data than the Mayer-Jensen shell model. Thus in the best case, involving 29 nuclei, the J = L + 1/2 odd proton nuclei have an average $\delta_1^p = 0.020$, i.e. an average deviation of the theoretical magnetic moments from the experimental magnetic moments of approximately 0.04 nuclear magnetons. This is to be compared to the average deviation from the Schmidt lines of $\Delta^p = 0.555$, i.e. over 1 nuclear magneton. The surprisingly good agreement in this case is probably fortuitous to some extent.

The J = L + 1/2 nuclei are seen to be in generally good agreement with the statistical shell model, where the improvement over the deviations from the Schmidt value Δ^p and Δ^n is by at least a factor of 4. The improvement for J = L - 1/2 nuclei is seen to be negligible in comparison with that for the J = L + 1/2 nuclei. The comparatively poor agreement for the J = L - 1/2 nuclei may be attributed in part to the fact that most J = L - 1/2 odd-even nuclei occur near the closing of a shell. This may lead to either the ambiguous prescription for mixing discussed in footnote 26, or may actually result in a much more restricted form of mixing. In any event, there is still some improvement for the $J = L - \frac{1}{2}$ nuclei, especially for δ_2 where both even and odd particle admixtures are considered. The average deviations for the δ_2's are equivalent to a difference of approximately 0.33 nuclear magnetons between the experimental magnetic moments and the magnetic moments predicted by the statistical shell model. The average Schmidt value deviations are not much larger than this as can be seen in Table 5.

In connection with this discussion of the $J = L - \frac{1}{2}$ nuclei, it is interesting to note that most of the deviations predicted by the statistical shell model for the $J = L - \frac{1}{2}$ classes are too far <u>inside</u> the corresponding Schmidt limit, compared to the experimental magnetic moments. Agreement can be immediately improved by assuming that for $J = L - \frac{1}{2}$ nuclei the shell model state has a somewhat greater weight than any other single admixed state.

These averages, while quite significant, obscure some interesting features found in the body of Table 3 and Table 4. First, there is the fact that the variation of the average magnetic moments for classes with the same J and parity, but different unfilled shells, is in large measure predicted by the statistical shell model magnetic moments, e.g. the $p_{3/2}$ and $d_{5/2}$ classes in Table 3 and the $p_{3/2}$ classes in Table 4. Furthermore the δ_1's generally represent a better fit for the smaller unfilled shells (lighter nuclei), while the δ_2's represent a better fit for the larger unfilled shells (heavier nuclei), i.e., odd particle admixtures alone give better agreement for lighter nuclei, while both odd particle admixtures and even particle admixtures give better agreement for heavier nuclei.

Agreement for nuclei in many classes could be improved by eliminating a few admixed states or by a judicious weighting of the admixed states, a weighting based on considerations of the shell model. However, such detailed treatment does not seem warranted, since the theory is already rather arbitrary. The improved agreement by use of the statistical shell model with the experimental magnetic moments seems to indicate that a fairly large number of admixed states is necessary and that the usual shell model may be a rather inadequate description of complex nuclei.

IV. "FORBIDDEN" MAGNETIC DIPOLE TRANSITIONS.

Graham and Bell[7] have measured the lifetimes of several isomeric magnetic dipole (M1) radiative transitions which should be forbidden on the basis of the Mayer-Jensen shell model assignment for the excited state and the ground state. The measured lifetimes of these "forbidden" transitions were found to be comparable to the lifetimes of allowed M1 transitions. The experimental lifetimes of the "forbidden" transitions indicate that either the ground state and excited state are not adequately described by the Mayer-Jensen shell model or the magnetic moment operator has terms in addition to the ordinary magnetic moment operator, e.g. exchange magnetic moment operators, which have different selection rules than the ordinary magnetic moment operator. The latter point of view was that considered by Sachs and Ross[6].

The investigation in Section III of the magnetic moment deviations indicated that the Mayer-Jensen shell model may be quite inadequate in describing the ground states of complex nuclei. If this is the case, then the excited states are probably also composed of admixtures somewhat similar to those of the ground states. The statistical shell model, used in the last section to account for the static magnetic moment deviations, will be investigated in this section as a possible explanation for the "forbidden" M1 radiative transitions as opposed to the use of additional magnetic moment operators. However, the actual cause of the "forbidden" M1 transitions is probably a rather complicated combination of exchange moments and admixtures of states rather than a single one of the two.

The M1 radiative transition probability is given by the expression

$$w_\gamma = \frac{1}{(2J'+1)}\left(\frac{e^2}{\hbar c}\right)\left(\frac{m}{M_n}\right)^2\left(\frac{mc^2}{\hbar}\right) E_\gamma^3 \sum_{M,M'} \left|\langle \Psi_{J'}^{M'}, \vec{n}\cdot\vec{\mu}\,\Psi_J^M \rangle\right|^2 \qquad \text{(IV-1)}$$

$$= \frac{1.68\times 10^{12}}{2J'+1} E_\gamma^3 |\mu|^2 \text{ sec}^{-1}$$

where E_γ is the photon energy of the radiative transition in units of mc^2, J' and M' are the total angular momentum and z component of the total angular momentum of the initial (excited) state of the nucleus in the transition, J and M are the total angular momentum and z component of the total angular momentum of the final (ground) state of the nucleus, $\vec{n}$ is an arbitrary unit vector which will be taken as the z axis in further calculations, $\vec{\mu}$ is the appropriate magnetic moment operator which will be assumed to be the ordinary magnetic moment operator II-1 in all further work in this section, and $|\mu|^2 = \sum_{M,M'} \left|\langle \Psi_{J'}^{M'}, \vec{n}\cdot\vec{\mu}\,\Psi_J^M \rangle\right|^2$.

The symmetry properties of the emitted M1 radiation requires that the following selection rules hold for the initial and final states of the nucleus

$$|\Delta J| = 1 \text{ ; no change of parity} \qquad \text{(IV-2)}$$

The assumption that $\vec{\mu}$ is an ordinary moment operator (sum of the nucleon moments) results in the further selection rule

$$\Delta L = 0 \qquad \text{(IV-3)}$$

More complicated magnetic moment operators would in general not have such a stringent selection rule on L.

The "forbidden" M1 transitions investigated by Graham and Bell were interpreted, on the basis of the usual shell model, as single particle transitions between either d $3/2$ and s $1/2$ states or d $5/2$ and g $7/2$ states. These transitions involve $|\Delta L| = 2$ and are therefore incompatible with the selection rules of the ordinary magnetic moment operator as given by IV-3, hence the term "forbidden" transition.

The statistical shell model will be used to calculate $|\mu|^2 = \sum_{M,M'} \left| \langle \Psi_{J'}^{M'}, \vec{m}\cdot\vec{\mu}\ \Psi_J^M \rangle \right|^2$ where only the ordinary magnetic operator II-1 will be considered. These calculated values must then be compared with the values obtained from the experimental lifetimes determined by Graham and Bell using the relation IV-1. The experimental data and the experimental values for $|\mu|^2$ are given in Table 6. In determining the experimental values of $|\mu|^2$ from measured half lives ($T_{\frac{1}{2}}$ in the third column of Table 6), the following relation between the transition probability and half life must be used

$$w_\gamma (1 + \alpha) = \frac{\ln 2}{T_{1/2}} \qquad \text{(IV-4)}$$

TABLE 6

NUCLEUS	E (kev.)	$T_{\frac{1}{2}}$ (sec.)	Total α	Spins J'	Spins J	$\|M\|^2$
Li^7	478	(5.2 1.7) x 10^{-14}	0	p 3/2	p 1/2	53.308
Fe^{57}	14	1.1 x 10^{-7}	10	p 1/2	p 3/2	0.031
Te^{123}	159	(1.9 0.3) x 10^{-10}	0.21	d 3/2	s 1/2	0.223
Te^{125}	35.4	(1.58 0.15) x 10^{-9}	18	d 3/2	s 1/2	0.157
Xe^{131}	80	(4.8 2.0) x 10^{-10}	2.0	s 1/2	d 3/2	0.137
Cs^{133}	81	(6.0 0.4) x 10^{-9}	2.1	d 5/2	g 7/2	0.031
Cs^{135}	248	(2.8 0.8) x 10^{-10}	0.1	d 5/2	g 7/2	0.072
Pm^{147}	91.5	(2.44 0.08) x 10^{-9}	2.2	-	d 5/2	0.083

where α is the total conversion coefficient.

All the nuclei listed in Table 6, with the exception of Li^7 and Fe^{57}, provide examples of the "forbidden" transitions. The value of $|\mu|^2$ varies from ~ 0.22 to ~ 0.03 for the forbidden transitions. These values are smaller than the predicted value for $|\mu|^2$ for allowed transitions by a factor of approximately 50 to 150 ($|\mu|$ differs by a factor of 7 to 13). The experimental values of $|\mu|^2$ for the allowed transitions in Li^7 and Fe^{57} are larger than predicted in the first instance and smaller in the second. It is interesting to note that the experimental $|\mu|^2$ for Fe^{57}, a supposedly allowed transition, is equal to the smallest experimental value of $|\mu|^2$ for the "forbidden" transitions.[27] It should also be noted that the magnetic moment deviation of Fe^{57} ($\mu_{ex} \sim 0$) is very large on the basis of the assumed $p_{3/2}$ ground state.

In calculating the value of $|\mu|^2$ a slight generalization of the statistical shell model will be used in order to obtain a more general theoretical expression for $|\mu|^2$. The prescription for choosing states for admixture is the same, but for the present it will no longer be assumed that the probability of each admixed state is the same. The ground state Ψ_J^M will be written as

[27]It is quite possible that the assignment for Fe^{57} given by Graham and Bell ($p_{\frac{1}{2}}$ $p_{3/2}$) is incorrect. On the basis of the Mayer-Jensen shell model and the observed magnetic moments the transition may actually be $f_{5/2}$ $p_{3/2}$, i.e. a "forbidden" M1 transition.

$$\Psi_J^M = a_s \phi_J^M + \sum_{j,j''} a_{jj''} \Psi_J^M(2jj'') + \sum_j b_j \Psi_J^M(2jJ) \qquad \text{(IV-6)}$$

where ϕ_J^M is the shell model wave function and $\Psi_J^M(2jj'')$ is the state of two equivalent particles coupled to j' = 2 which are in turn coupled to a third particle of angular momentum j" to give a total angular momentum J, i.e.

$$\Psi_J^M(2jj'') = \sum_m \langle j'', 2, J, M \mid j'', 2, M-m, m \rangle \; J_2^m(2j)\,\phi_{j''}^{M-m} \qquad \text{(IV-7)}$$

The second term in IV-6 refers to states of three odd nucleons and the third term to states of two even nucleons (j) and one odd one. The summation indices j and j" include <u>only those states prescribed by the statistical shell model as described in Section III</u>. The probability amplitudes are subject to the usual normalization condition

$$|a_s|^2 + \sum_{j,j''} |a_{jj''}|^2 + \sum_j |b_j|^2 = 1 \qquad \text{(IV-8)}$$

It can be shown that the probability amplitudes a_s, $a_{jj''}$, and b_j are all real, but they are still arbitrary with respect to their sign.

In a similar manner the wave function of the excited state is written as

$$\Psi_{J'}^{M'} = a_s' \phi_{J'}^{M'} + \sum_{j,j''} a'_{jj''} \Psi_{J'}^{M'}(2jj'') + \sum_j b_j' \times$$

$$\times \Psi_{J'}^{M'}(2jJ) \qquad \text{(IV-9)}$$

The probability amplitudes a_s', $a'_{jj''}$, b_j' are assumed to have no simple correlation with the ground state amplitudes.

It is now possible to calculate the matrix element $\langle \Psi_{J'}^{M'}, \mu^z \Psi_J^M \rangle$. Several important simplifications are possible in this calculation. First, because of the selection rule IV-3 it follows that

$$\text{a)} \quad \langle \phi_{J'}^{M'}, \mu^z \phi_J^M \rangle = 0$$

(IV-10)

$$\text{b)} \quad \langle \sum_j b'_j \Psi_{J'}^{M'}(2jJ'), \mu^z \sum_j b_j \Psi_J^M(2jJ) \rangle = 0$$

since $|L' - L| = 2$ according to the shell model assignments for the transitions under consideration. Now, because the functions are orthogonal,

$$\langle J_0^0 (j^2), J_2^m(j^2) \rangle = 0$$

and because $|\Delta J| = 2$ is forbidden for magnetic dipole transitions

$$\langle J_0^0 (j^2), (\mu_1^z + \mu_2^z) J_2^0(j^2) \rangle = 0$$

Here J_0^0 is the usual shell model function of two particles (labeled 1 and 2) in equivalent states coupled to total angular momentum $j' = 0$, J_2^m is the function for the same two particles coupled to total angular momentum $j' = 2$, and μ_1^z and μ_2^z are the z components of the single particle magnetic moment operators for the two particles. It follows that

a) $\langle \phi_{J'}^{M'}, \mu^z \sum_{jj''} a_{jj''} \Psi_J^M(2jj'') \rangle = \langle \sum_{jj''} a'_{jj''} \Psi_{J'}^{M'}(2jj''), \mu^z \phi_J^M \rangle = 0$

b) $\langle \phi_{J'}^{M'}, \mu^z \sum_j b_j \Psi_J^M(2jJ) \rangle = \langle \sum_j b'_j \Psi_{J'}^{M'}(2jJ), \mu^z \phi_J^M \rangle$ (IV-11)

c) $\langle \sum_{jj''} a'_{jj''} \Psi_{J'}^{M'}(2jj''), \mu^z \sum_j b_j \Psi_J^M(2jJ) \rangle =$

$$= \langle \sum_J b'_j \Psi_{J'}^{M'}(2jJ), \mu^z \sum_{jj''} a_{jj''} \Psi_J^M(2jj'') \rangle = 0$$

With these results in mind the calculation of the matrix $\langle \Psi_{J'}^{M'}, \mu^z \Psi_J^M \rangle$ is reduced to

$$\langle \Psi_{J'}^{M'}, \mu^z \Psi_J^M \rangle = \langle \sum_{jj''} a'_{jj''} \Psi_{J'}^{M'}(2jj''), \mu^z \sum_{jj''} a_{jj''} \Psi_J^M(2jj'') \rangle$$

$$= \sum_j \langle \sum_{j''} a'_{jj''} \Psi_{J'}^{M'}(2jj'', \mu^z \sum_{j''} a_{jj''} \Psi_J^M(2jj'') \rangle \qquad \text{(IV-12)}$$

In IV-12 the value of j must be the same in $\Psi_{J'}^{M'}(2jj'')$ and $\Psi_J^M(2jj'')$ because μ^z, being a one particle operator, can lead to a change in the state of only one nucleon. The values of j" can be different in $\Psi_{J'}^{M'}(2jj'')$ and $\Psi_J^M(2jj'')$, but the l'' values must be the same in order to satisfy the selection rule IV-3. Hence Δ j" = 0, $\pm$ 1 and the matrix element may finally be written as

$$\langle \Psi_{J'}^{M'}, \mu^z \Psi_J^M \rangle = \delta_{MM'} \Big\{ \sum_{jj''} a'_{jj''} a_{jj''} \langle \Psi_{J'}^M(2jj''), \mu^z \Psi_J^M(2jj'') \rangle$$

$$+ \sum_{j,j''} a'_{jj''} a_{jj''-1} \langle \Psi_{J'}^M(2jj''), \mu^z \Psi_J^M(2jj''-1) \rangle \qquad \text{(IV-13)}$$

$$+ \sum_{j,j''} a'_{jj''-1} a_{jj''} \langle \Psi_{J'}^M(2jj''-1), \mu^z \Psi_J^M(2jj'') \rangle \Big\}$$

In the last two sums the states specified by j'' and $j'' - 1$ must have the same l'' value, i.e. $j'' = l'' + 1/2$ and $j'' - 1 = l'' - 1/2$.

By using the same techniques as in Section II the matrix elements on the right side of IV-13 are found to be

a) $$\left\langle \Psi_{J'}^{M}(2jj''), \mu^{z}\, \Psi_{J}^{M}(2jj'') \right\rangle =$$
$$= 2\left(\frac{\mathcal{E}(j)}{2l+1} - \frac{\mathcal{E}(j'')}{2l''+1}\right)\left(\sum_{m} m \langle j'',2,J',M \,|\, j'',2,M-m,m\rangle \times \langle j'',2,J,M \,|\, j'',2,M-m,m\rangle\right)$$

b) $$\left\langle \Psi_{J'}^{M}(2jj''), \mu^{z}\, \Psi_{J}^{M}(2jj''_{-1}) \right\rangle = \qquad \text{(IV-14)}$$
$$= -\frac{2}{2l''+1}\sum_{m}\sqrt{(j''+M-m)(j''-M+m)}\,\langle j'',2,J',M \,|\, j'',2,M-m,m\rangle\langle j''-1,2,J,M \,|\, j''-1,2,M-m,m\rangle$$

c) $$\left\langle \Psi_{J'}^{M}(2jj''_{-1}), \mu^{z}\, \Psi_{J}^{M}(2jj'') \right\rangle =$$
$$= \frac{-2}{2l''+1}\sum_{m}\sqrt{(j''+M-m)(j''-M+m)}\,\langle j''-1,2,J',M \,|\, j''-1,2,M-m,m\rangle\langle j'',2,J,M \,|\, j'',2,M-m,m\rangle$$

The matrix elements given by IV-14 are easily evaluated and thus the theoretical determination of the transition probability depends on the values assumed for $a^{1}_{jj''}$ and $a_{jj''}$. The indeterminateness of the sign of these probability amplitudes requires some simplifying assumption when evaluating the absolute square of the matrix element given by IV-13.

The assumption is made that the cross terms average to zero (or more strictly to a number small compared to the sum of the squared terms). This is a reasonable assumption as long as there are a fairly large number of terms ~~by~~ contributing to the sums in IV-13.

With the use of this assumption

$$\begin{aligned} \left|\left\langle \psi_{J'}^{M'}, \mu^z \psi_J^M \right\rangle\right|^2 = \delta_{MM'} \Big\{ & \sum_{j,j''} a'^2_{jj''} a^2_{jj''} \left|\left\langle \psi_{J'}^{M}(2jj''), \mu^z \psi_J^M(2jj'') \right\rangle\right|^2 \\ & + \sum_{j,j''} a'^2_{jj''} a^2_{jj''-1} \left|\left\langle \psi_{J'}^{M}(2jj''), \mu^z \psi_J^M(2jj''-1) \right\rangle\right|^2 \\ & + \sum_{j,j''} a'^2_{jj''-1} a^2_{jj''} \left|\left\langle \psi_{J'}^{M}(2jj''-1), \mu^z \psi_J^M(2jj'') \right\rangle\right|^2 \Big\}. \end{aligned} \qquad \text{(IV-15)}$$

The nuclei involved in the measured "forbidden" magnetic dipole radiative transitions are of two types. The odd neutron nuclei $_{52}Te^{123}$, $_{52}Te^{125}$, and $_{54}Xe^{131}$ involve transitions between states having total angular momentum 3/2 and 1/2 (d 3/2 and s 1/2 states according to the Mayer-Jensen shell model) while the odd proton nuclei[28] $_{55}Cs^{133}$ and $_{55}Cs^{135}$ involve transitions between states having total angular momentum 5/2 and 7/2 (d 5/2 and g 7/2 states according to the Mayer-Jensen shell model). All the nuclei are in the statistical shell model classes of 50-82 even particles and 50-82 odd particles.

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[28] $_{61}Pm^{147}$ will not be considered in detail because there has been no definite assignment of Mayer-Jensen shell model states for this nucleus. However, the results for this case are probably very similar to those obtained for the Cs isotopes.

$|\mu|^2$ will now be calculated on the basis of the statistical shell model used in Section III. Thus

$$\text{a)} \quad a_s = a_{jj''} = b_j = \frac{1}{\sqrt{N}}$$
$$\text{b)} \quad a'_s = a'_{jj''} = b'_j = \frac{1}{\sqrt{N'}} \qquad \text{(IV-16)}$$

for all values of j and j'' allowed by the statistical shell model, where N is the total number of states (as given by IV-7) admixed to give the ground state, and N' is the total number of states (as given by IV-9) admixed to give the excited state. Thus every state in the admixture is assumed to occur with equal probability.

The value for $|\mu|^2$ becomes

$$|\mu|^2 = \sum_{M,M'} \left|\langle \psi_{J'}^{M'}, \mu^z \psi_J^M \rangle\right|^2$$
$$= \frac{1}{NN'} \Big\{ \sum_{j,j''} \sum_M \left|\langle \psi_{J'}^M(2jj''), \mu^z \psi_J^M(2jj'')\rangle\right|^2 \qquad \text{(IV-17)}$$
$$+ \sum_{j,j''} \sum_M \left|\langle \psi_{J'}^M(2jj''), \mu^z \psi_J^M(2jj''-1)\rangle\right|^2$$
$$+ \sum_{j,j''} \sum_M \left|\langle \psi_{J'}^M(2jj''), \mu^z \psi_J^M(2jj''-1)\rangle\right|^2 \Big\}$$

Since the operator $\vec{\mu}$ is hermitian, and the wave functions with the same J of the Te isotopes and Xe isotope are the same according to the statistical shell model, the value of $|\mu|^2$ will be the same for all these isotopes even though the role of the ground state and excited state is interchanged between the Te isotopes and the Xe isotope.

Table 7 gives the value of $|\mu|^2$ calculated on the basis of the statistical shell model ($|\mu|^2_{ssm}$) as well as the experimental value of $|\mu|^2$ ($|\mu|^2_{ex}$). Also indicated in the table is μ_s the Schmidt value of the magnetic moment, μ_{ex} the experimental value of the magnetic moment, μ_{ssm} the statistical shell model magnetic moment, and N and N' the total number of states (including the shell model states) admixed in the ground state and excited state respectively.

It is seen that the values of μ_{ssm} represent a better fit to the experimental magnetic moments μ_{ex} than do the Schmidt magnetic moments μ_s. The theoretical values of $|\mu|^2_{ssm}$ are larger than the corresponding experimental values $|\mu|^2_{ex}$. The agreement is quite good for the odd neutron isotopes while the agreement is poor for the odd proton isotopes. The poorest agreement (for Cs^{133}) gives $|\mu|^2_{ssm} / |\mu|^2_{ex} \sim 10$ or $|\mu|_{ssm} / |\mu|_{ex} \sim 3$.

However, the fact that the theoretical values of $|\mu|^2_{ssm}$ are larger than $|\mu|^2_{ex}$ is satisfying since almost any reasonable modification of the statistical shell model would tend to make $|\mu|^2_{ssm}$ smaller. Thus any additional mixing of even particle states would make N and N' larger while not increasing (IV-10b) the square of the transition matrix element IV-15. Also if the shell model state occurs with a larger probability amplitude than assumed by the statistical shell model, then $a_{jj''}$ and $a'_{jj''}$ must become

TABLE 7.

Nucleus	μ_s	μ_{ex}	μ_{ssm}	N	N'	$\|\mu\|^2_{ex}$	$\|\mu\|^2_{ssm}$
$_{52}Te^{123}$	-1.913	-0.736	-1.050	7	20	0.223	0.263
$_{52}Te^{125}$	-1.913	-0.887	-1.050	7	20	0.157	0.263
$_{54}Xe^{131}$	1.148	0.708	0.593	20	7	0.137	0.263
$_{55}Cs^{133}$	1.715	2.577	2.952	16	20	0.031	0.403
$_{55}Cs^{135}$	1.715	2.727	2.952	16	20	0.072	0.403

smaller (assuming all states other than the usual shell model state to have equal probability) because of the normalization condition IV-8. Since the shell model states cannot contribute to the transition matrix element (IV-10a), $|\mu|^2_{ssm}$ would decrease as a_s increases.

In line with these considerations it is noted that the statistical shell model magnetic moment μ_{ssm} represents too great a deviation of the magnetic moment inside of the Schmidt line when compared to the experimental moment of the two Cs isotopes. If the shell model probability amplitude is increased sufficiently to give the correct magnetic moment for these isotopes, then $|\mu|^2_{ssm}$ is decreased by approximately a factor of two (assuming the shell model state in the excited state has the same increase in its probability amplitude).

It is seen that the value of $|\mu|^2$ can vary as the statistical shell model is modified. However, if the admixture of states, necessary to fit the observed magnetic moment, involves several different wave functions and the excited state wave functions are formed from the same configurations as the ground state wave functions, then there is no violent fluctuation in the value of $|\mu|^2$. Thus, a very considerable modification of the statistical shell model, such as eliminating all the wave functions formed from an arbitrary configuration of the unfilled shell, does not result in too great a change in $|\mu|^2$. This seems to indicate that the calculated values of $|\mu|^2$ are not just fortuitous, and that the occurrence of the "forbidden" M1 transitions may be due in part to ground state and excited state admixtures of the type considered in the last two sections.

It is interesting to note that on the basis of the statistical shell model the value of $|\mu|^2$ for supposedly allowed magnetic dipole transitions would be expected to be of the same order of magnitude as for the "forbidden" transitions. This would explain the rather small experimental value[28] of $|\mu|^2$ found for Fe^{57}. The large value of $|\mu|^2$ found for Li^7 can probably be attributed to the fact that the statistical shell model would not be expected to apply to so light a nucleus.

Thus the statistical shell model appears to offer at least a partial explanation of two anomalous features of magnetic dipole phenomena as interpreted by the Mayer-Jensen shell model. First, the deviations of the static magnetic moments of odd-even nuclei are in large measure predicted by means of the statistical shell model. Then, the supposedly forbidden magnetic dipole radiative transitions are found to be allowed by the statistical shell model and the theoretical values of the transition probability are of the same order of magnitude as the experimentally determined values. The statistical shell model is undoubtedly an oversimplification of the true physical situation, but the success of this comparatively simple model indicates that the Mayer-Jensen shell model probably gives a rather poor description of the ground state wave function of complex nuclei. The great success of the Mayer-Jensen model in predicting spins and parities does not justify the assumption that it also gives an adequate representation of the dynamical features of complex nuclei.

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