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AUTHOR(S): Holger Nielsen, Niels Bohr Institute
Hans Rugh, Universty of Warwick
Svend Rugh, T-6

MASTER

Los Alamos   Los Alamos National Laboratory
                  Los Alamos, New Mexico 87545

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CHAOS, SCALING AND EXISTENCE OF A CONTINUUM LIMIT
IN CLASSICAL NON-ABELIAN LATTICE GAUGE THEORY

Holger Bech Nielsen

The Niels Bohr Institute, Blegdamsvej 17, 2100 København Ø, Denmark

Hans Henrik Rugh

Department of Mathematics, University of Warwick, Coventry, CV4 7AL, England

Svend Erik Rugh

Theoretical Division, T-6, MS B 288, University of California, Los Alamos National Laboratory, Los Alamos, New Mexico, NM 87545, U.S.A.

We discuss space-time chaos and scaling properties for classical non-Abelian gauge fields discretized on a spatial lattice. We emphasize that there is a "no go" for simulating the original continuum classical gauge fields over a long time span since there is a never ending dynamical cascading towards the ultraviolet. We note that the temporal chaotic properties of the original continuum gauge fields and the lattice gauge system have entirely different scaling properties thereby emphasizing that they are entirely different dynamical systems which have only very little in common. Considered as a statistical system in its own right the lattice gauge system in a situation where it has reached equilibrium comes closest to what could be termed a "continuum limit" in the limit of very small energies (weak non-linearities). We discuss the lattice system both in the limit for small energies and in the limit of high energies where we show that there is a saturation of the temporal chaos as a pure lattice artifact. Our discussion focuses not only on the temporal correlations but to a large extent also on the spatial correlations in the lattice system. We argue that various conclusions of physics have been based on monitoring the non-Abelian lattice system in regimes where the fields are correlated over few lattice units only. This is further evidenced by comparison with results for Abelian lattice gauge theory. How the real time simulations of the classical lattice gauge theory may reach contact with the real time evolution of (semi-classical aspects of) the quantum gauge theory (e.g. Q.C.D.) is left as an important question to be further examined.

There are some indications - and it would be a beautiful principle if it was true - that we have the "approximative laws and regularities" which we know, because they are infrared stable against modifications of "short distance physics" in the ultraviolet [1]. These "approximative laws", which at present accessible scales to the best of our knowledge comprise a sector consisting of the Standard Model of quantum Yang-Mills fields based on the group $S(U_2 \times U_3) \sim U(1) \otimes SU(2) \otimes SU(3)$ and a sector consisting of the gravitational interactions, may (thus) be robust and stable in the sense of Ref.[1], but they are full of unstable and chaotic solutions! Indeed, both the gravitational field (the Einstein equations) and the Yang-Mills fields exhibit dynamical chaos [2,3] for generic solutions (solutions without too much symmetry) in regions where (semi)classical treatments are justified and non-linearities of the interactions are non-negligible. In the present contribution we shall report on some observations [4,5,6] concerning the lattice implementation of space-time chaos of classical Yang-Mills fields, but let us first note some few physical motivations for studying dynamical chaos in classical Yang-Mills fields:

(1) Despite the word "chaos" at first suggests something structureless it is really a study of ways in which "structure" may be generated (from various initial configurations) during the evolution of the governing equations of motion. It is of interest - but so far only little is known - to know how structures form and evolve (e.g. the formation and evolution of embedded topological structures) during the time evolution of the Yang-Mills equations. In the semi-classical regimes of the Standard Model, the quantum solutions build around the classical solutions (and it is not unlikely that quantum fluctuations on top of classical chaos only enhance chaos [a]).

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[a]The remark that the full quantum theory for a system with bounded configuration space (and a discrete spectrum of quantum states) has no chaos due to quasiperiodic evo-

(2) For example, the possibility of baryon non-conservation (via the famous ABJ-anomaly) within the electroweak theory has achieved quite some attention recently, and it is related to a detailed understanding of the *dynamics* of the electroweak fields (as the Universe cools down). It is quite natural to speculate about a relationship between dynamical chaos and an activity of formation and destruction of embedded topologically interesting field configurations (such a relationship is, for example, well known for the complex Landau Ginzburg equation, see also e.g. discussion in Ref.[8] which contemplates the relevance of a concept of *topological turbulence* of the gauge fields). We note that field configurations for which the rate of production of baryon number

$$\dot{B} \sim \int_{\Omega \subset R^3} d^3x \, Tr(F \, F^*) \qquad (1)$$

vanishes will span a surface of co-dimension one. Thus, in fact, most field configurations (in the hot electroweak plasma) will contribute to the right hand side of equation (1). For a particular class of field configurations which contributes to the baryon non-conservation see also Ref.[9].

(3) Fast equilibration processes which take place in heavy-ion collisions[10] are very likely connected to non-linear chaotic dynamics; an idea which in principle dates back e.g. to Fermi, Pasta and Ulam, Ref.[11].

As a further motivation of the study of *classical* Yang-Mills chaos, we should also note that not many non-perturbative tools are available to study the time evolution of quantum Yang-Mills fields, so the study of classical Yang-Mills fields is a natural starting point for semi-classical understanding of the dynamics.

In order to facilitate a numerical study of spatio-temporal chaos (formation of space-time structure) in inhomogeneous Yang-Mills fields one has in some way to discretize the space-time continuum on which the Yang-Mills fields are defined. This can be achieved in a way which breaks gauge invariance (see e.g. [14]) or in a gauge invariant way which is called lattice gauge theory[15].

We shall refer to[4,6] for more details in the discussion which will follow. The study of chaos in toy-models where the gauge fields are spatially homogeneous was initiated by Sergei Matinyan and George Savvidy[12]. An important new qualitative dynamical feature comes into play, however, when one considers the spatially inhomogeneous classical Yang-Mills equations: There is a never ending cascading of the dynamical degrees of freedom towards the ultraviolet, generated by the time evolution of the Yang-Mills equations! (In spite of this "ultraviolet catastrophe", the solutions are well behaved in the sense that there are no "finite time blow up of singularities"). The non-linear self-coupling terms which open up the possibility for a chaotic behavior in the classical evolution lead in the non-homogeneous case to the infinite cascade of energy from the long wavelength modes towards the ultraviolet. (Note, there is a priori no concept of temperature and in the case of Abelian (electromagnetic) fields this cascading would <u>not</u> show up dynamically, unless one couples the fields to charged particles). This tendency of the mode frequencies cascading towards the ultraviolet will completely dominate the qualitative behavior of the *classical* Yang-Mills equations, and the "ultraviolet catastrophe" has for some time been emphasized by us (cf. e.g. discussion in Ref.[8]) as a major obstacle to simulate the classical continuum Yang-Mills fields in a numerical experiment over a long time span. [b] There is no mechanism, within the classical equations, which prevents this never ending cascading of the modes towards the ultraviolet. Nature needs $\hbar$, the Planck constant, as an ultraviolet regulator. Indeed, both Abelian and non-Abelian gauge fields are implemented as quantum theories in Nature.

Here we shall discuss the possibility of using a lattice cutoff in a purely classical treatment to regularize the equations. There will still be a cascading of modes towards the ultraviolet, i.e. towards the lattice cut-off, and this ultraviolet cascade will still dominate the dynamical evolution of smooth initial field configurations. However, in a lattice formulation of the Yang-Mills fields on a large but finite lattice, the phase space is compact for any given energy and thus the system can

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lution of the wave function is of little importance to us since the number of degrees of freedom involved here are so large that the time it will take to evolve quasi-periodically through the states will be substantially larger than the lifetime of the Universe.

[b]This is an obstacle which, in our opinion, has received insufficient attention in the various studies attempting at discussing and modeling chaotic properties of spatially inhomogeneous classical Yang-Mills fields.

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reach an equilibrium state among the modes (a 'thermodynamic equilibrium'). The lattice regularization of the theory opens up for the definition of dynamic and thermodynamic properties, which are not defined in the classical Yang-Mills field theory without regularization. It could e.g. be ergodic (modulo constraints) with respect to the Liouville measure, in which case it makes sense to talk about its micro-canonical distribution and approximating this by looking at 'typical' classical trajectories. The fundamental assumption of thermodynamics asserts that on average the two approaches give the same result if we have a large system, and we may then naturally introduce correlation functions and possibly a correlation length ξ (measured in lattice units) of the system.

One hopes to define a continuum theory if, by judicious choice of the parameters in the system, one obtains a physical correlation length in the limit when the lattice constant goes to zero, i.e.

$$\xi(a, E(a), ...) \times a \;\; \rightarrow \;\; \ell \neq 0 \; , \;\; \text{as } a \rightarrow 0. \quad (2)$$

In equation (2) the correlation length ξ in the lattice system is a function of lattice model parameters such as lattice spacing a, average energy density $E(a)$, etc. Condition (2) implies that the correlation length diverges when measured in lattice units and only if this is the case do we expect the lattice system to lose its memory of the underlying lattice structure.

We shall restrict attention to the lattice gauge theory in 3+1 dimensions based on the gauge group $SU(2)$. We consider a finite size 3 dimensional hypercubic lattice having N^3 points where nearest neighbor points are separated by a distance $a > 0$. The phase space is a fibered space where the tangent manifold of the Lie-group $SU(2)$ is assigned to each of the links, $i \in \Lambda$, connecting nearest neighbor lattice points: To each $i \in \Lambda$ is associated a link variable $U_i \in SU(2)$ as well as its canonical momentum[c] $P_i \in T_{U_i}SU(2)$. A point in the entire phase space will be denoted

$$\mathbf{X} = \{U_i, P_i\}_{i \in \Lambda} \; \in \; M = \prod_\Lambda T\,SU(2) \; . \quad (3)$$

A Hamiltonian is constructed so as to correspond to the continuum, classical Yang-Mills Hamiltonian in the limit $a \rightarrow 0$. The construction of such

[c] In principle P_i is a cotangent vector, but the Lie algebra inner product gives a natural identification of the cotangent space with the tangent space.

a Hamiltonian is of course ambiguous in the sense that extra terms of order $O(a)$ may be added to the Hamiltonian. Alluding to some sort of "universality" (especially in the limit $a \rightarrow 0$), we expect that the precise choice of Hamiltonian is not so important in what follows. A Hamiltonian which is often employed to generate the time evolution of the orbit $\mathbf{X}(t)$ is the Kogut-Susskind Hamiltonian, which can be written in the following way[d]:

$$H(a, \mathbf{X}^{(a)}) =$$
$$\frac{1}{a} \sum_{i \in \Lambda} \frac{1}{2} \text{tr}(P_i P_i^\dagger) + \frac{1}{a} \sum_\square (1 - \frac{1}{2} \text{tr} U_\square) \; . \quad (4)$$

Here the last sum is over elementary plaquettes bounded by 4 links, and $U_\square$ denotes the path-ordered product of the 4 gauge elements along the boundary of the plaquette $\square$. The last term, the potential term, is automatically bounded and, for a given finite total energy, the same is the case for the first term, the kinetic term. Thus the phase space corresponding to a given energy-surface is compact.

As is standard we shall discuss the temporal correlations (temporal chaos) of the lattice gauge system in terms of its spectrum of Lyapunov exponents. The compactness of the phase space implies[4] that the spectrum of Lyapunov exponents (which we overall will assume to be well defined quantities for the lattice system) is independent of the choice of norm on the space of field configurations. In fact, it follows[3,4] from the scaling properties of the equations of motion generated by the lattice Hamiltonian (4), that in order to study the dependence of the maximal Lyapunov exponent with the energy density of the system, it is sufficient to consider the equations of motion for a fixed value of the lattice constant a, e.g. $a = 1$, as a function of energy density ($\propto$ energy/plaquette) and then rescale the results back afterwards.

We have several different forms of lattice artifacts in the lattice simulation of real-time dynamical behavior of the continuum classical Yang-Mills fields:

(1) Lattice artifacts due to the compactness of the group. The magnetic term (the second term) in the Kogut-Susskind Hamiltonian (4) is

[d] For a derivation we refer to [15] and [3,16] from which we adapt our notation. For simplicity we omit the coupling constant factor $2/g^2$ which anyway is arbitrary in a classical theory.

uniformly bounded, $0 \leq 1 - \frac{1}{2}TrU_\square \leq 2$, due to the $SU(2)$ compactification. Thinking in terms of statistical mechanics for our classical lattice system, we expect that after some time the typical field configuration has equally much energy in all modes of vibration - independent of the frequency[e] - and the total amplitude of the classical field, and the energy per lattice plaquette, is thus small for a fixed low energy. For low energy, when the average energy per plaquette is small, the lattice artifacts due to the compactness of the gauge group are thus negligible.

(2) For small energy per plaquette, we thus expect that the dominant form for lattice artifacts is due to the fact that an appreciable amount of the activity (for example the energy) is in the field modes with wavelengths comparable to the lattice constant a. This short wavelength activity at lattice cut-off scales is unavoidable in the limit of long time simulation of an initially smooth field configuration (relative to the lattice spacing), or already after a short time if we initially have an irregular field configuration.

In which way can the classical lattice gauge theory approach a *continuum limit* ? This is a difficult question to which we shall only be able to provide a very partial answer here.

For the study of the time evolution of Yang-Mills fields which initially are *far from an equilibrium situation*, the (classical) field modes will exhibit a never ending dynamical cascade towards the ultraviolet and after a certain transient time, the cut-off provided by the spatial lattice will prevent the lattice gauge theory from simulating this cascade. It is therefore immediately clear that the lattice regularized, classical fields will not approach a "continuum limit" in the sense of simulating the dynamical behavior of the classical *continuum* fields in the $t \to \infty$ limit. For the simulation of classical continuum gauge fields far from equilibrium, we conclude that we will have the best "continuum limit" if we simulate, for a short period of time, an initial smooth ansatz[f] for the fields

in the region of low energy per plaquette.[g]

Gauge fields exist, however, as quantum theories in Nature and the interesting definition - as concerns applications in *physics* - of a "continuum limit" of the classical lattice gauge theory, is thus to identify regions in the parameter space for the classical lattice gauge theory which probe the behavior of the time evolution of semi-classical initial configurations (with many quanta) of the quantum theory for a shorter or longer interval of time or in an equilibrium situation. (See also e.g. discussion in Ref.[22]). Since the classical lattice gauge theory does not contain a relationship like $E = \hbar\omega$ (implying a damping of the high frequency modes relative to the soft modes), we must expect that the simulation of quantum gauge theory will be distorted by this fact (even if implemented with effective lattice Hamiltonians as e.g. devised by Ref.[22]).

We shall in the following restrict attention to real-time simulations of the classical lattice gauge theory which have reached an *equilibrium situation* on the lattice. Such simulations are hoped [3] to yield insights into the dynamical behavior of (long wavelength modes in) the high temperature Q.C.D. fields in situations where an equilibrium situation has been reached.

Before we attempt an analysis of aspects of chaos (in time and space) of the lattice gauge theory let us first note that it is far from obvious how to establish contact between an *effective lattice temperature* of the *classical* lattice gauge theory (which has reached an equilibrium situation due to the presence of the lattice cutoff $\Lambda = 1/a$) and the *physical* temperature T of the Q.C.D. theory which is a *quantum* theory and which can reach an equilibrium situation due to a cut off of quantum

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[e]Note, this situation is very different in the quantum case. Planck's constant $\hbar$ introduces a relation $E = \hbar\omega$ between the energy of a mode of vibration and its frequency, implying that a mode with a high frequency also has a high energy. With a given available finite total energy, modes with high frequencies will therefore be suppressed. Quantum mechanically, we thus have that at low energy only excitations of the longest wavelengths appear.

[f]By "smooth" configurations we mean lattice config-

urations which are 'good' approximations to continuum configurations.

[g]I.e. we are here imagining a situation where the spatial correlations in the monitored field variables are so large that they lose memory of the underlying lattice structure (including the lattice spacing a). In the extreme opposite limit, one could imagine situations with randomly fluctuating fields on the scale of the lattice constant, i.e. with (almost) no spatial correlations from link to link. If the field variables fluctuate independently of each other (independent of their neighbors), one could imagine the model to be invariant (with respect to the monitoring of many variables) under changes of the lattice spacing, a. Thus, it appears that lattice cut-off independence of numerical results can <u>not</u> be a sufficient criterion for the results to report "continuum physics".

mechanical origin (i.e. ultimately due to the existence of a Planck constant $\hbar$). Some discrepancy in the literature [20,25,26] illustrates that this is not an easy question.

In the classical lattice gauge theory we have an "effective lattice temperature" $T_L = 1/\beta$ where β has been determined by looking at the probability distribution (Gibbs distribution) of the kinetic energy (or the magnetic energy in a plaquette), $p(E_k) \propto \exp(-\beta E_k)$ (after equilibration). Cf. e.g. Ref. [3] (p. 206-207) and Ref. [26].

In Ref.[3,18,26] it is asserted that the physical temperature T (of the quantum Q.C.D. fields in an equilibrium situation) and the average energy $E_\square$ per plaquette in the classical lattice gauge study in equilibrium are related by

$$E_\square \approx \frac{2}{3}\left(n^2 - 1\right) T \quad \text{(for } SU(n)\text{).} \quad (5)$$

It is not clear to us how serious one should take this relationship (one objection being that a scale has not been fixed in the continuum limit of the classical theory) but we shall leave a discussion of this issue aside in the following.

Let us now attempt an understanding of chaotic aspects of the lattice gauge theory - i.e. the temporal chaos (as monitored by the spectrum of Lyapunov exponents) and its relationship with spatial chaos (as monitored by the spatial correlations, e.g. a spatial correlation length) in the dynamical system.

A sequence of articles[16,17,18] and a recent book by Biró, Matinyan and Müller[3] present numerical results for the classical $SU(2)$ lattice gauge model and provide evidence that the maximal Lyapunov exponent is a monotonically increasing continuous function of the scale free energy/plaquette with the value zero at zero energy.

Ref.[3,16,18] reports a particularly interesting interpretation of numerical results for the dynamics on the lattice, namely that there is a *linear* scaling relation between the scale free maximal Lyapunov exponent, $\lambda_{max}(a = 1)$ and the average energy per plaquette $E_\square(a = 1)$. The possible physical relevance of this result follows from the observation[16] that when we rescale back to a variable lattice spacing a we note that the observed relationship is in fact a graph of $a\lambda_{max}(a)$ as a function of $a E_\square(a)$. Thus being linear,

$$a\,\lambda_{max} = \text{const} \times a\,E_\square(a) \quad (6)$$

cancellation of a factor a implies that $\lambda_{max}(a) = \text{const} \times E_\square(a)$ and thus there is a continuum limit $a \to 0$, either of both sides simultaneously or of none of them. In the particular case where the energy per mode ($\propto$ energy per plaquette) is interpreted to be a fixed temperature T (cf. equation (5) above), one deduces that the maximal Lyapunov exponent has a continuum limit in real time, proportional to the temperature of the gauge field.

Note, also, the suggested relationship[3,16,24] between the maximal Lyapunov exponent λ of the gauge fields on the lattice and the "gluon damping rate" $\gamma(0)$ for a thermal gluon at rest, arrived at in re-summed perturbation theory in finite temperature quantum field theory, see also Refs.[25,26]. For the $SU(2)$ gauge theory, this suggested relation reads

$$\lambda = 2\gamma(0) = 2 \times 6.635\frac{2}{24\pi}g^2T \sim 0.34\,g^2T \quad (7)$$

It is also our understanding that $\gamma(0)$ is a quantity of semiclassical origin.[17] A relation like (7) is nevertheless remarkable in chaos theory since it suggests that a complicated dynamical quantity like a temporal Lyapunov exponent (which is usually only possible to extract after a considerable numerical effort) is analytically calculable by summing up some diagrams in finite temperature quantum field theory.

In Ref.[4] we argued (in view of the numerical evidence presented in e.g. [3,16,18]) that the apparent linear scaling relation (6) is a transient phenomena residing in a region extending at most a decade between two scaling regions, namely for small energies where the Lyapunov exponent scales with an exponent which could be close to $k \sim 1/4$, and a high energy region where the scaling exponent is at most zero:

$$\eta = \frac{d\log\lambda}{d\log E} = \begin{cases} = k \sim 1/4 & \text{for } E \to 0 \\ \leq 0 & \text{for } E \to \infty \end{cases} \quad (8)$$

The proposed scaling relation $k \sim 1/4$ was, for reasons we shall give below, suggested to hold in the limit $E \to 0$ from general scaling arguments [8] of the continuum classical Yang-Mills equations in accordance with simulations on homogeneous models and consistent with the figures in [16] and [18]. However, a more recent numerical analysis[5] argues rather convincingly that the data points presented in e.g. [16,18,3] were subject to "finite time"

artifacts, and that long time simulations with a correct procedure for extracting the principal Lyapunov exponent could support that $k \equiv 1$ even in the limit as $E \to 0$.

Before we discuss the limit of low energies ($E \to 0$) let us note that regarding the limit for high energies per plaquette a rigorous result[7] (also reported in [4,6]) shows that the $SU(2)$ scale free ($a = 1$) lattice Hamiltonian in d spatial dimensions has an upper bound for the maximal Lyapunov exponent

$$\lambda_B = \sqrt{(d-1)(4+\sqrt{17})} \qquad (9)$$

which for $d = 3$ becomes $\lambda_B = 4.03...$ This result is arrived at by constructing an appropriate norm on the phase space and showing that the time derivative of this norm can be bounded by a constant times the norm itself, hence giving us an upper bound as to how exponentially fast the norm can grow in time. The upper bound (9) shows that a linear scaling region, i.e. $\eta = d \log \lambda / d \log E = 1$, cannot extend further than around $E \sim 10$ on the figure 1. Beyond that point the maximal Lyapunov exponent either saturates and scales with energy with an exponent which approaches zero or it may even decrease over a region of high energies, yielding a negative exponent thus justifying the $E \to \infty$ limit of equation (8).

It should be noted that the upper bound (9) is independent of the lattice size and the energy (but scales with $1/a$). There is quite a simple intuitive explanation for the saturation of the maximal Lyapunov exponent in the regime for high energy per plaquette of the $SU(2)$ lattice gauge model, since the potential (magnetic) term in the Hamiltonian (4) is uniformly bounded, $0 \leq 1 - \frac{1}{2}TrU_\square \leq 2$, due to the $SU(2)$ compactification. For high energies, almost all the energy is thus put in the (integrable) kinetic energy term in the lattice Hamiltonian (4), and the spectrum of Lyapunov exponents will saturate as the energy per plaquette increases for the model. (The chaos generated by the non-linear potential energy term does not increase, only the energy in the kinetic energy (the electric fields) increases).

As we shall argue (a discussion which is rather suppressed in [16,17,18,3]), it is however in the opposite limit, i.e. in the limit where the average energy per plaquette goes to zero, i.e. $E \to 0$, that the equilibrium lattice gauge theory has the possibil-

ity to reach contact with "continuum physics" - not in the sense (as we have seen) that the lattice system simulates the original continuum classical gauge fields (over a long time span), but in the sense that the lattice gauge theory - considered as a statistical system in its own right - will develop spatial correlations in the fields (as monitored by a correlation length ξ, say) which may approach a formal "continuum limit",

$$\xi \to \infty \text{ as } E \to 0 \qquad (10)$$

There is a well known analogy[19] between a quantum field theory in the Euclidean formulation with a compactified (periodic) imaginary time axis $0 \leq \tau \leq \beta$ and finite temperature statistical mechanics of the quantum field theory at a temperature $T = 1/\beta$ which is inversely proportional to the above-mentioned time extension.

Calculations in finite temperature Euclidean quantum field theory suggest that a characteristic correlation length of static magnetic fields in the thermal quantum gauge theory is of the order $\xi \sim (g^2 T)^{-1}$. If we (cf. discussion above, equation (5)) make use of the relationship, $E_\square = 2T$ (for $SU(2)$ Yang-Mills theory) between temperature T and average energy per plaquette $E_\square$, this could suggest that the (magnetic sector) of field configurations on the lattice will have characteristic correlation lengths of the order

$$\xi \sim (g^2 T)^{-1} \sim (\frac{1}{2}g^2 E_\square)^{-1} \to \infty \text{ as } E_\square \to 0 . \qquad (11)$$

In systems with space-time chaos (and local propagation of disturbances with a fixed speed $\sim c$) one often has a relationship between a correlation length (coherence length) ξ and the maximal Lyapunov exponent λ (see also e.g. discussions in Ref.[23]) of the form $\xi \sim c/\lambda$. This suggests, via equation (11),

$$\lambda \sim c/\xi \sim g^2 E_\square \qquad (12)$$

i.e. a Lyapunov exponent which scales linearly with the average energy per plaquette $E_\square$. Clearly, it deserves further investigation to establish a precise relationship between λ_{max} and the spatial correlation length ξ (in lattice units). However, we emphasize that it is in the limit as the energy per plaquette goes to zero (and where the lattice Lyapunov exponent λ_{max} goes to zero) that we expect

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the spatial correlation length ξ (in lattice units) will diverge. The finite size of the lattice makes it difficult to analyze the behavior of the gauge fields and the principal Lyapunov exponent on the lattice in the limit $E \to 0$.

Let us reiterate the arguments which led us to the conclusion (in Ref.[4]) that the scaling exponent of the Lyapunov exponent likely will be closer to $\sim 1/4$ in the limit $E \to 0$ than the scaling exponent ~ 1 observed in the intermediate energy region $1/2 \leq E \leq 4$.

In figure 1 we display a log-log plot of the results obtained in Ref.[3,16,18]. It appears that for $E \sim 1/2$ there is a cross-over to another scaling region. Although the data points in this region are determined with some numerical uncertainty (finite time artifacts, as argued in Ref.[5]) we did note that they are consistent with a scaling with exponent $\sim 1/4$. Moreover, it is well known that the homogeneous Yang-Mills equations have a non-zero Lyapunov exponent which by elementary scaling arguments scales with the fourth root of the energy density and has the approximate form [13]:

$$\lambda_{\max} \approx 1/3 \, E^{1/4} \ . \tag{13}$$

On the lattice the above scaling relation is valid for spatially homogeneous fields, i.e. the maximal Lyapunov exponent scales with the fourth root of the energy per plaquette. By continuity, fields which are almost homogeneous [h] on the lattice will, in their transient, initial dynamical behavior, exhibit a scaling exponent close to 1/4. In fact, the same scaling exponent would also hold for the inhomogeneous Yang-Mills equations [8] had the fields been smooth relative to the lattice scale, so derivatives, ∂_μ, are well approximated by their lattice equivalent and we are allowed to scale lengths as well. This is seen from scaling arguments for the continuum Yang-Mills equations $D_\mu F^{\mu\nu} = 0$ where $D_\mu = \partial_\mu - ig[A_\mu, \ ]$ and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$. Not taking boundary conditions into account, these equations are invariant when ∂_μ and A_μ are scaled with the same factor α. That is, if $A(x,t)$ is a solution to the equations, then $\frac{1}{\alpha}A(\alpha x, \alpha t)$ is also a solution. The energy density E, which is quadratic in the Yang-Mills field curvature tensor $F^{\mu\nu}$, then scales

with α^4. This indicates [4] that if we perform a measurement of the maximal Lyapunov exponent λ_{max} over a time short enough for the solutions to stay smooth, then λ_{max} scales with $E^{1/4}$. Such a scaling was also observed for "smooth" configurations by Müller and Trayanov p. 3389 in Ref.[16]. These scaling arguments do not carry over to infinite time averages since solutions on the lattice tend to be irregular.

In any case we expect that the lattice gauge theory does not probe field configurations which are substantially correlated beyond one lattice unit when the lattice gauge theory is monitored for energies per plaquette in the regime $E \sim 1/2 - 4$. That is, we expect $\xi/a \sim 1$ for $E \sim 1/2 - 4$ and thus the system is indeed very far from reaching contact with "continuum physics" when monitored in that regime of energies. For example, the numerical results for the Lyapunov exponents as presented e.g. in Ref. [3,16,18] (c.f. e.g. fig. 8.4 in Ref.[3]) report on lattice size $N = 20$ for energies $E \sim 1/2 - 4$ and coupling strength $g \approx 2$, which suggest (cf. also the equation (11)) that the lattice gauge theory in this regime of parameters monitor lattice field configurations which have a correlation length up to a few lattice units only. There is also numerical evidence [i] that the correlation length for energies $E \sim 1$ is of the order of a few lattice units. This observation applies also to the numerical studies of the $SU(3)$ lattice gauge theory reported in [17,18] in the range of energies per plaquette $E_\Box \sim 4 - 6$.

These observations are further substantiated by the numerical simulations of Ref.[18] (figure 12) for the lattice gauge theory with a $U(1)$ group showing a steep increase of the maximal Lyapunov exponent with energy/plaquette in the interval $1 \leq E \leq 4$. The continuum theory here corresponds to the classical electromagnetic fields which have no self-interaction, and thus the Lyapunov exponent in this limit should vanish. The discrepancies in this case were attributed in [18] to a combined effect of the discreteness of the lattice and the compactness of the gauge group $U(1)$ and were not connected with finite size effects. This suggests strongly that we, in the case of numerical studies of a $SU(2)$ Kogut-Susskind Hamilto-

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[h] Such field configurations are (ungeneric) examples of field configurations which exhibit correlation lengths much larger than the lattice spacing.

[i] J. Ambjørn, priv.communication. See also e.g. Ref. [21] which reports investigations of finite temperature QCD on the lattice with measurements of a magnetic mass (inverse correlation length) of the order $m_{mag} \sim 0.5 \, g^2 \, T$.
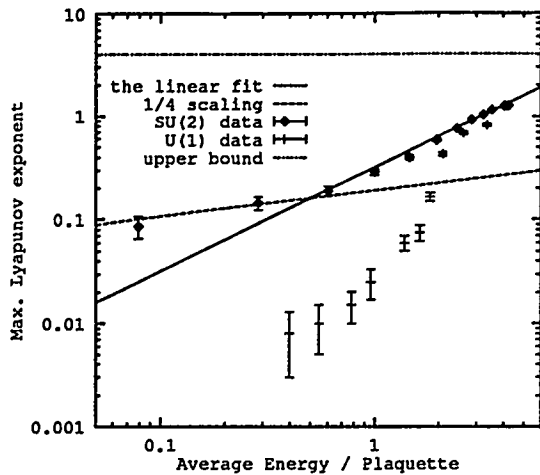
Figure 1: The maximal Lyapunov exponent as a function of the average energy per plaquette for the $SU(2)$ and the $U(1)$ lattice gauge theory. The data points (diamonds for $SU(2)$, ticks for $U(1)$) are adapted from Müller et al. [16] and Biró et al. [3], p. 192. The solid line is a linear fit through the origin, the dashed line is the function $\frac{1}{2} \times 1/3 \times E^{1/4}$ (half of the homogeneous case result (13)). The dot-dashed line shows the rigorous upper bound for the $SU(2)$ lattice model (saturation of temporal chaos). As is seen, the linear scaling region for the $SU(2)$ data is positioned where the $U(1)$ data display strong lattice artifacts.

nian system, cannot base continuum physics on results from simulations in the same interval of energies where the $U(1)$ simulations fail to display continuum physics. On the contrary, we suspect that what could reasonably be called "continuum physics" (for the lattice gauge theory considered as a statistical system in its own right) has to be extracted from investigations of the Kogut-Susskind lattice simulations for energies per plaquette which are at least much smaller than $E \sim 1/2$.

We would like to conclude with some final points of discussion.

Exactly in which way we may speak about a "continuum limit" in a simulation of a classical gauge theory in real time on a spatial lattice (a theory which does not have a continuum limit without lattice regularization) appears to be a question of some fundamental as well as practical interest, since not many non-perturbative tools are available to study the time evolution of quantized Yang-Mills fields.

Since by intrinsic scaling arguments (cf. Refs, [3,4]) one has a functional relation between

$a\lambda_{max}(a)$ and $aE_\square(a)$, a scaling (at small energies $E \to 0$) with exponent k, according to the scaling relation (8), would imply that

$$\lambda(a) \propto a^{k-1} E_\square(a)^k , \qquad (14)$$

in which case one cannot achieve a continuum limit simultaneously for the maximal Lyapunov exponent and the temperature (assuming that it is proportional to the average energy per plaquette $E(a)$, as given by equation (5)), except in the special case where $k \equiv 1$. In particular, if $k < 1$, the former would be divergent if the temperature is kept fixed. There is no particular contradiction in this statement, however, as there is - a priori - no reason for having a finite Lyapunov exponent in the continuum limit. The erratic and fluctuating behavior of the fields one expects in time as well as in space (for numerical evidence, cf. also [14]) on very small scales could suggest that a Lyapunov exponent would not be well defined in the "continuum limit" (as $a \to 0$ and $E \to 0$). Clearly, this question deserves further investigation.

If the scaling-relation $k \equiv 1$ according to equation (8) holds in the limit $E \to 0$ for the lattice gauge theory, it will in a most striking way illustrate the point that the continuum gauge theory (with scaling $k \sim 1/4$) and the lattice gauge theory (probed in a situation where it has reached equilibrium) are two *entirely different* dynamical systems - despite the lattice theory (4) is at first set up to be an approximation to the continuum theory. (Thus the lattice theory does <u>not</u> simulate the continuum theory; it is an entirely new statistical theory in its own right and - important for applications in physics - the relationship with quantum non-Abelian gauge theory remains to be established on a more rigorous basis). Understanding of this crucial difference could, perhaps, be obtained from renormalization group analysis: We may say that the limit $E \to 0$ is a critical point for the lattice Hamiltonian (4), i.e. that the lattice correlation length diverges in that limit. As is common for field theories studied in a neighborhood of a critical point, one expects classical scaling arguments to break down, or rather, that scaling relations are subjected to renormalization which gives rise to anomalies in the scaling exponents. Often such anomalous scaling exponents seem to be 'ugly' irrational numbers, perhaps with the 2-d Ising model as a notable exception. It would therefore be quite miraculous

if the 1/4 classical-scaling of the Lyapunov exponent emphasized above renormalizes to an exponent which equals unity. Even if this did happen it is not clear that such a behavior should be independent of the regularization procedure employed, since Lyapunov exponents are measures of local instabilities, i.e. short range rather than long range structures.

As regards the numerical evidence for the relationship (7) between the maximal Lyapunov exponent and the "gluon damping rate" [24] we note that in the hierarchy of scales $g^2T \ll gT \ll T$ (this separation of scales is assumed in hot perturbative gauge theory) the "gluon damping rate" is connected to the decay of the "soft" modes $\sim g^2T$. In order for the lattice gauge theory to probe decays of "soft" gauge modes, this requires the existence of some "soft" modes on the lattice (in a background of "hard" modes). Thus we must monitor the lattice gauge system in a region where there are fields (in equilibrium with "hard" modes $\Lambda \sim 1/a$) which have spatial correlations substantially larger than the lattice unit a. As we have seen this is not the case in the regime where the lattice gauge theory was studied ($N = 20, g \approx 2, E \sim 1 - 4$) in Ref. [3,16,18] providing the numerical support for equation (7). It appears, that the numerical support of the relation (7) is somewhat of an accident if the relation (7) is to be interpreted as a "continuum result". Studies of the lattice gauge theory have to be conducted for much smaller energies per plaquette which are, however, also difficult, since finite size effects then will become of appreciable size. [j]

Acknowledgements

[j]Finite size effects are of course - in the context of our discussion - a "good sign" which witnesses that the lattice monitors field configurations which have correlations of the order of the lattice size $\sim Na$, i.e. much beyond a single lattice unit.

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