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# SOME PRACTICAL METHODS OF USING CHARACTERISTICS IN THE CALCULATION OF NON-STEADY COMPRESSIBLE FLOW

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Summary

Three methods of using characteristics for the evaluation of solutions of equations of non-steady isentropic compressible flow in one space variable are considered and compared.

One involves the use of a grid of characteristics, in terms of Eulerian variables (§3.1), two others involve use of the properties of characteristics to relate the flow at the beginning and end of a given time-interval, but do not involve the use of a grid of characteristics. One of these uses Eulerian variables (§4), and the other Lagrangian variables (§7).

The arithmetical process of solution is appreciably simpler for the equations in terms of Eulerian variables rather than in terms of Lagrangian variables. Also use of specified time intervals seems to have several advantages over the use of a grid of characteristics, in simplifying the numerical process, in providing results in the form most likely to be required, namely the flow field at different times, and in other ways (§4.1). Hence the use of specified time intervals, in Eulerian variables, is chosen as the method to develop further.

The extension to anisentropic flow is found to be numerically simple and straightforward (§5) and the treatment of a shock appears surprisingly simple (§6).

Some tentative suggestions are made of methods for evaluating unsteady flow in two space dimensions, retaining as much as possible of the advantages of the use of characteristics in unsteady flow in one space dimension (§8).

## §1. Introduction.

Two branches of fluid dynamics in which hyperbolic partial differential equations occur are steady supersonic flow in two space variables (flow in two dimensions or axially symmetrical flow in three dimensions) and non-steady flow of a compressible fluid in one space variable (flow in one dimension, radial flow in two or three dimensions).

In numerical work on the evaluation of steady flows, much use has been made of the curves known as "characteristics" of such equations [2, 4, 5, 7, 11, 12]. In the evaluation of non-steady flows, it seems that much of the work has been done by direct numerical integration of the equations in Lagrangian form, though proposals for using characteristics in this context have also been made and some work done on these lines (see, for example [13, 14, 15]).

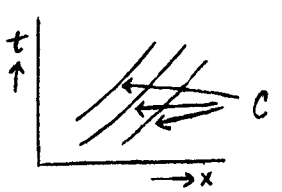
In much of the work using characteristics, a grid of characteristics has been used, to such an extent that use of such a grid is often referred to as "the method of characteristics" as if there were no other. But there are other methods of using characteristics, and it is the main purpose of the present report to propose another method which seems to have several advantages, at least for non-steady flow in one space variable. It gives a simple numerical treatment both of isentropic and anisentropic flow, and also promises to give a simple treatment of the propagation of shocks, though this has not been examined numerically. It may also be applied to the evaluation of steady flow in two space variables, though its advantage in this context may not be so marked, and to other examples of hyperbolic equations.

This report is written primarily for those who are immediately concerned with carrying out calculations of non-steady flow, either

by hand or by means of automatic digital machines. Its main purpose is to give practical numerical processes rather than formal theory; for this reason some attention is given to details of procedure which would be out of place in a presentation of general theory. The methods discussed are suitable for hand calculation, and are simple enough to be used on a large scale; exploratory hand calculations have been made on an example of isentropic flow.

The methods should also be quite practicable for use with automatic digital machines, though the details of the organization for such a machine would depend on such machine properties as storage capacity, nature of auxiliary store (if any), and means of transfer to and from it. Although the best way of carrying out a calculation by means of an automatic machine is often not the programmed form of the method best suited to hand calculation, it seems likely that in this case the best procedure for hand calculation may well form the basis for a good method for machine calculation. This is another reason for giving some details of the procedure found to be most convenient for hand calculation. However, no detailed program in terms of the facilities and order of a particular machine has yet been drawn up.

In most methods for the numerical integration of partial differential equations, integration is carried out along a set of curves C in the space of the independent variables; for example, along particle paths, along characteristic curves, or along lines x = constant. The quantities which are integrated are rates of change along the curves C along which integration is being carried, and the evaluation of these integrands may involve the evaluation of derivatives in other directions, "across" the curves



C; for example, if the Eulerian form of the equations for one-dimen sional flow

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0$$

are integrated with respect to t along a set of curves x = constant, the values of the time-derivatives, which are the integrands, depend on the values of x-derivatives, which are derivatives "across the direction in which the integration is being carried; such deriva tives will be called "crosswise derivatives", and the process of determining them will be called "crosswise differentiation"[1].

[1]The term "cross-derivative" is sometimes used for a derivative $\partial^2 u/\partial x \partial y$ obtained by an operation of differentiation $(\partial/\partial x)$ "across" the direction of another operation of differentiation $(\partial/\partial y)$. "Crosswise differentiation" is differentiation across a direction in which an operation of integration is being carried.

The extent to which crosswise differentiation is involved is an important point in the consideration of numerical methods for the integration of partial differential equations, for the following reason. The form of these equations suggests that some crosswise differentiation will be involved in their solution; but in numerical work differentiation is a notoriously unsatisfactory process, whereas integration is a satisfactory one, and it is usually desirable to organize numerical calculations so as to avoid differentiation as far as possible.

In the numerical integration of partial differential equations it may not be possible to avoid crosswise differentiation entirely. However, for hyperbolic equations in two variables, one property of the characteristics is that they are curves C such that integration along them involves no crosswise differentiation; in the numerical integration of such equations this is the important property of

the set of characteristics as a whole.

As will be seen in §2.1, this property of characteristics is only another aspect of a property which, expressed in different terms, forms part of the general theory of characteristics of hyperbolic differential equations. But since the formal theory is normally developed without reference to practical numerical procedures, the importance of this property in this context is not usually emphasized. It will, however, be recognized by those who have had practical experience in the evaluation of solutions of partial differential equations by methods involving the use of characteristics and by other methods. It is hoped that the explicit expression of this property in the above terms may suggest practical methods for work in two space dimensions; a tentative first essay in this direction is considered in §8.

§1.1. Characteristic variables.

In the exposition of the general theory of characteristics (for example, [4, 5]) it is customary to introduce "characteristic variables" ("characteristic parameters" or "characteristic coordinates"), $\alpha$ and $\beta$, say, which are supposed defined in some way such that the characteristics of one set are curves $\alpha$ = constant and those of the other set are curves $\beta$ = constant.

In the application of characteristics to specific problems, two distinct steps may be involved, one involving use of the properties of characteristics without reference to characteristic variables, and the other involving the introduction of characteristic variables, which, in numerical work, means the assignment of a definite numerical value to each characteristic of each set. The first step does not imply the second; but in the application of characteristics, the formulation in terms of characteristic variables is sometimes adopted

as the starting point, without considering whether it is necessary or even helpful.

The introduction of characteristic variables is sometimes referred to as a "natural" step in the theory (see, for example, [4, p. 42]), and so, from the point of view of the formal presentation of the general theory, it may be. But any numerical work is concerned with one or more specific cases, and from the point of view of numerical work the introduction of characteristic variables seems not natural but highly artificial. This is emphasized by the high degree of arbitrariness in the choice of numerical values to be assigned to the individual characteristics of a set, and by the fact that if such numbers are assigned, they play no part in the calculation. They are quite irrelevant to the method considered in §§4-6, and even in work on a grid of characteristics they play no part except for indexing purposes. They seem to complicate rather than to help the argument, and use of them has deliberately been avoided in this report.

## §1.2. Some points of terminology and notation.

Following the usage of Courant-Friedrichs [4], "adiabatic" will be used to mean that the entropy per unit mass of each element of fluid is constant in time, and "isentropic" to mean that this constant entropy is the same for all fluid elements in a region of the field of integration considered. It will be assumed that the motion is adiabatic, except at shocks, but not necessarily that it is isentropic.

The time-rate of change of a quantity $\phi$ following the motion of a particle will be written $d\phi/dt$ (except in §7 which is concerned with the Lagrangian form of the equations); the time rate of change of $\phi$ along a curve $C$ will be written $(d\phi/dt)_C$.

The form of the terms involving the derivatives of highest order is the same for flow in one dimension, radial flow in two dimensions, and spherically symmetrical flow in three dimensions. These will be classed together as "flows in one space variable" in $n = 1$, 2 and 3 dimensions respectively.

## §2. Eulerian form of equations.

The equations to be solved, and the numerical procedure for evaluating a solution, are appreciably simpler for isentropic flow. The equations for anisentropic flow will first be obtained, then procedures for the calculation of isentropic flow will be considered to illustrate in a simple context the essential features of the methods to be discussed. The generalizations to anisentropic flow and to flow with shocks will be considered later (§5 and §6 respectively).

In Eulerian form, the basic equations are the equation of motion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \quad , \tag{2.1}$$

the equation of conservation of mass,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} = - (n-1) \rho u / r \quad , \tag{2.2}$$

and the equation of state of the material.

For anisentropic flow, we must take the general equation of state, and since we are concerned with notions in which the behavior of each element of the fluid is adiabatic (apart from shocks), it is convenient to take the equation of state in the form

$$p = p(\rho, s) \quad , \tag{2.3}$$

where $s$ is either the entropy per unit mass or some function of this quantity (for example the value, for a perfect gas, of $p/\rho^{\gamma}$, which is constant on an adiabatic and has different values on differ-

ent adiabatics). Then if $s$ varies from particle to particle,

$$\left(\frac{\partial p}{\partial r}\right)_t = \left(\frac{\partial p}{\partial \rho}\right)_s \left(\frac{\partial \rho}{\partial r}\right)_t + \left(\frac{\partial p}{\partial s}\right)_\rho \left(\frac{\partial s}{\partial r}\right)_t .$$

Now $(\partial p/\partial \rho)_s$ is just the square of the adiabatic velocity of sound $a$; hence (2.1) can be written

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial r} + \frac{1}{\rho} \left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial s}{\partial r} = 0 . \tag{2.4}$$

For isentropic flow the last term on the left-hand side is zero since $s$ is uniform over the field of flow. For anisentropic flow, however, in which the behavior of each particle is adiabatic, $s$ is constant along a particle path, so

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial r} = 0 . \tag{2.5}$$

§2.1. Characteristics from the Eulerian form of the equations.

Equations (2.2), (2.4), (2.5) are three linear first-order equations for three dependent variables. The procedure for obtaining the equations for the characteristics and the variation of $u$, $\rho$, and $s$ along them is standard (see, for example, [4, §§22, 31]) and its application to the equations of isentropic flow can be found in various references (see, for example [4, §23]). We require its application, however, to anisentropic flow.

We form a linear combination of these equations, and determine the coefficients of the linear combination in such a way that the differential operators operating on $u$, $\rho$ and $s$ represent rates of change in the same direction in the $(r,t)$ plane. This is the step which ensures that the resulting equations for the rates of change along the characteristics involve no crosswise derivatives, and it is this property which makes this equation so attractive for numerical work.

The result of forming the linear combination of (2.4), (2.2) and (2.5) with coefficients 1, $\lambda$, $\mu$ respectively is

$$(2.6)\quad \left[\frac{\partial}{\partial t} + (u+d\rho)\frac{\partial}{\partial r}\right]u + \left[\lambda\frac{\partial}{\partial t} + \left\{\lambda u + (a^2/\rho)\right\}\frac{\partial}{\partial r}\right]\rho + \left[\mu\frac{\partial}{\partial t} + \left\{\mu u + \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\right\}\frac{\partial}{\partial r}\right]s = -(n-1)\lambda\rho u/r .$$

The condition that these three differential operators shall represent rates of change in the same direction is

$$(u + \lambda\rho)/1 = \left\{\lambda u + (a^2/\rho)\right\}/\lambda = \left\{\mu u + \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\right\}/\mu ,$$

whence

$$(2.7)\qquad \lambda = \pm a/\rho , \qquad \mu = (\lambda/a^2)(\partial p/\partial s)_\rho .$$

Further, if C is a curve in the (r,t) plane on which $(dr/dt)_C = u + \lambda\rho$, then for any function $\phi(r, t)$

$$\left[\frac{\partial}{\partial t} + (u+\lambda\rho)\frac{\partial}{\partial r}\right]\phi = \left(\frac{d\phi}{dt}\right)_C .$$

Hence the root $\lambda = + a/\rho$ of (2.7) gives

$$(2.8)\qquad \left(\frac{du}{dt}\right)_I + \frac{a}{\rho}\left(\frac{d\rho}{dt}\right)_I + \frac{1}{a\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\left(\frac{ds}{dt}\right)_I = -(n-1)au/r$$

along curves I such that

$$(2.9)\qquad (dr/dt)_I = u + a ,$$

and the root $\lambda = -a/\rho$ gives

$$(2.10)\qquad -\left(\frac{du}{dt}\right)_{II} + \frac{a}{\rho}\left(\frac{d\rho}{dt}\right)_{II} + \frac{1}{a\rho}\left(\frac{\partial p}{\partial s}\right)\left(\frac{ds}{dt}\right)_{II} = -(n-1)au/r$$

along curves II such that

$$(2.11)\qquad (dr/dt)_{II} = u - a .$$

For isentropic flow, the third term on the left-hand side of each of equations (2.8), (2.10) is zero, and equations (2.8)-(2.11)

are sufficient to determine the motion. Further, $ad\rho/\rho$ is an exact differential, say

$$ad\rho/\rho = dW(\rho)$$

so that equations (2.8), (2.10) become

$$[\frac{d}{dt}\{W(\rho) + u\}]_I = -(n-1)au/r \ , \tag{2.12}$$

$$[\frac{d}{dt}\{W(\rho) - u\}]_{II} = -(n-1)au/r \tag{2.13}$$

(the quantities $\{W(\rho) \pm u\}$ are sometimes called the "Riemann invariants", though they are not invariant except for the plane case $n = 1$).

If the adiabatic equation of state is

$$p = A\rho^{\gamma} + B \ , \tag{2.14}$$

where $\gamma$, A and B are constants, then (and only then) $W(\rho)$ is a constant multiple of $a(\rho)$; in fact

$$W(\rho) = [2/(\gamma-1)]a(\rho) \ . \tag{2.15}$$

Since $[W(\rho) \pm u]$ are the quantities obtained by integration of equations (2.12), (2.13), whereas $a(\rho)$ is required for the evaluation of the integrands in equations (2.9), (2.11), (2.12) and (2.13). the simple relation (2.15) is very convenient for numerical work. This is a practical argument for the addition of an equation of state of the form (2.14) if such a formula gives an adequate approximation to the properties of the fluid over the relevant pressure range.

For anisentropic flow, equations (2.8) to (2.11) give only two equations for the three dependent variables $u, \rho, s$. However, there is a third equation, namely

$$(ds/dt)_{III} = 0 \tag{2.16}$$

on particle paths, that is, on curves III such that

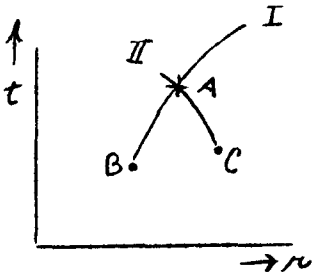
$$(dr/dt)_{III} = u \ ; \tag{2.17}$$

(this is not included in (2.6), since a non-zero coefficient of equation (2.4) has been assumed in forming this linear combination).

Equations (2.8)-(2.13) and (2.15) hold in any consistent set of units. It may be convenient to consider the units so chosen that the density $\rho_o$ of the fluid, and the velocity of sound $a_o$ in it, in some standard state, both have value unity. If the fluid starts from rest at a uniform pressure and density, this is a convenient state to adopt as a standard.

## §3. Isentropic flow. Finite-difference approximation.

Consider first the treatment of isentropic flow.




The finite-difference approximation which will be used here is that expressed by the trapezium-rule formula

$$(3.1) \qquad \int_{x_o}^{x_1} f(x)dx = \tfrac{1}{2}(f_o + f_1)(x_1 - x_o) .$$

For brevity, it is convenient to write

$$(3.2) \qquad v = a + u, \quad w = a - u,$$

$$(3.3) \qquad P = W(\rho) + u, \quad Q = W(\rho) - u .$$

Then if A is the intersection of the characteristic of I through B and the characteristic of set II through C, we have:-

(3.4) from equation (2.12) $P_A - P_B = -\frac{1}{2}(n-1)[(au/r)_A + (au/r)_B](t_A - t_B)$

(3.5) " " (2.9) $r_A - r_B = \frac{1}{2}(v_A + v_B)(t_A - t_B)$

(3.6) " " (2.13) $Q_A - Q_C = -\frac{1}{2}(n-1)[(au/r)_A + (au/r)_C](t_A - t_C)$

(3.7) " " (2.11) $r_A - r_C = -\frac{1}{2}(w_A + w_C)(t_A - t_C)$ .

$u_A$ is given by

$$(3.8) \qquad 2u_A = P_A - Q_A ,$$

and if the adiabatic equation of state is (2.14), so that $W(\rho)$ and $a(\rho)$ are related by the simple formula (2.15), $a_A$ is given by

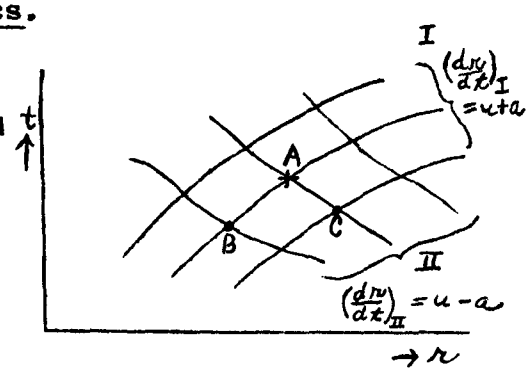
$$[4/(\gamma-1)]a_A = P_A + Q_A \; ; \tag{3.9}$$

for a more general equation of state, $a_A$ has to be deduced from $W_A = \frac{1}{2}(P_A + Q_A)$ and the equation of state adopted.

There are several ways of using the set of equations (3.4) - (3.9) to obtain an approximate numerical solution of the original partial differential equations. One is to use a grid of characteristics; another is to use specified time-intervals, relating the values of a and u at selected points A at the end of the time interval to values at the beginning of the time interval by means of the characteristic relations (3.4)-(3.7). The latter procedure was suggested some years ago as a possible process for use on analogue equipment, and a schematic set-up worked out for a differential analyzer [6, §3.8]. It was not at that time regarded as a practicable method for digital calculation, but I have recently found it to be quite practicable, and indeed it seems preferable in some cases to the use of a grid of characteristics. Both these methods will be considered, to give a basis for a comparison of them.

## §3.1. Use of a grid of characteristics.

In this method, for each pair of points B and C located on the grid of characteristics as shown in the figure, equations (2.3)-(3.9) are regarded as equations for the four unknowns $r_A$, $t_A$, $u_A$, $a_A$. They form a set of non-linear simultaneous equations for these quantities, and the problem is to organize the work of solving them. The following is the most convenient procedure I have found; it is similar to a



method used by A. J. R. Schneider [13], but Schneider used two drafting machines and graphical constructions for some stages of the calculation, whereas a purely numerical process seems easier, more accurate, more reliable, and easier to check.

First estimate $a_A$ and $u_A$. If the characteristics for which the calculations are being carried out are chosen in a systematic way, and tables of the results are kept as the calculation proceeds, it should be possible to make these estimates quite accurately (in a trial calculation, carried out with $u$ and $a$ expressed in units such that the values of $\rho_o$ and $a_o$ were unity, I found it practicable to estimate both $u_A$ and $a_A$ to $\pm$ 1 in the third decimal place).

Then from (3.5) and (3.7)

$$(3.9)\quad \begin{cases} t_A - t_B = [2(r_C-r_B)-(w_A+w_C)(t_B-t_C)]/(v_A+v_B+w_A+w_C) \\ t_A - t_C = [2(r_C-r_B)+(v_A+v_B)(t_B-t_C)]/(v_A+v_B+w_A+w_C) \end{cases}$$

These quantities should <u>both</u> be calculated; the agreement between the values of $t_A$ derived from them provides a useful check, which should be used since these time-differences are used later in the calculation.

Then $r_A - r_B$, $r_A - r_C$ are calculated from (3.5), (3.7) and the values of $(t_A - t_B)$, $(t_A - t_C)$ just obtained, and the agreement between the values of $r_A$ deduced from them is a further check that the arithmetic has been correctly carried out so far. The value of $r_A$ obtained here is used in the subsequent calculation for this point.

The following seems the most convenient way to use formulas (3.4) and (3.6). For brevity, write

$$(3.10)\qquad \begin{cases} P_B^* = P_B - \frac{1}{2}(n-1)(au/r)_B(t_A - t_B) \\ Q_C^* = Q_C - \frac{1}{2}(n-1)(qu/r)_C(t_A - t_C) \end{cases}$$

Then the result of subtracting (3.6) from (3.4) is

$$2u_A = P_A - Q_A = P_B^* - Q_C^* - \tfrac{1}{2}(n-1)(au/r)_A(t_B - t_C) ,$$

so that

$$(3.11)\qquad u_A = [P_B^* - Q^*_C]/[2 + \tfrac{1}{2}(n-1)(a/r)_A(t_B - t_C)] ;$$

the right-hand side of this formula involves only the estimate of $a_A$ and quantities already calculated.

Then the sum of (3.4) and (3.6) gives

$$(3.12)\qquad 2W_A = P^*_B + Q^*_C + \tfrac{1}{2}(n-1)(au/r)_A[2t_A - t_B - t_C] .$$

If the equation of state is such that the relation between $W(\rho)$ and $a(\rho)$ is the simple relation (2.15), then (3.12) is a linear equation for $a_A$ and gives

$$(3.13)\qquad a_A = [P^*_B + Q^*_C]/[\{4/(\gamma-1)\} + \tfrac{1}{2}(n-1)(u/r)_A(2t_A - t_B - t_C)] .$$

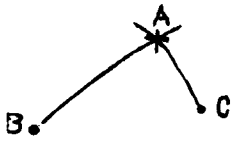
The values of $u_A$, $a_A$ calculated from (3.11), (3.13) will be called "final" values.

The procedure has been given in some detail since details of procedure may have considerable effect on the case with which a calculation is carried out. The one suggested has been reached as the result of several exploratory calculations, and is quite practicable for hand computation. A suggested computing schedule is given in Table I.

An alternative procedure, after evaluating $r_A$, is to use the estimates of both $a_A$ and $u_A$ to evaluate $(au/r)_A$ from (3.4) and (3.6), hence evaluate $P_A$ and $Q_A$, and then "final" values of $u_A$ and $a_A$ from $P_A \pm Q_A$. But, in some cases at any rate, the main

## TABLE I

### Suggested computing schedule for use when working on a grid of characteristics




| | | Notes |
|---|---|---|
| $\boxed{v_B}$ | $\boxed{w_C}$ | |
| $\boxed{t_C - t_B}$ | | |
| Est $a_A$ | | |
| Est $u_A$ | | |
| $v_A = a_A + u_A$ | $w_A = a_A - u_A$ | |
| $v_A + v_B$ | $w_A + w_C$ | |
| Sum = S | | |
| $\boxed{2(r_C - r_B)}$ | $\boxed{2(r_C - r_B)}$ | |
| $(w_A + w_C)(t_C - t_B)$ | $-(v_A + v_B)(t_C - t_B)$ | |
| Sum = $S_1$ | Sum = $S_2$ | |
| $S_1/S = t_A - t_B$ | $S_2/S = t_A - t_C$ | |
| $\boxed{t_B}$ | $\boxed{t_C}$ | |
| Sum = $t_A$ | Sum = $t_A$ | Check |
| $(v_A + v_B)(t_A - t_B)$ | $-(w_A + w_C)(t_A - t_C)$ | |
| $\boxed{r_B}$ | $\boxed{r_C}$ | |
| Sum = $r_A$ | Sum = $r_A$ | Check |
| $\boxed{P_B}$ | $\boxed{Q_C}$ | |
| $-\frac{1}{2}(n-1)(au/r)_B(t_A - t_B)$ | $-\frac{1}{2}(n-1)(au/r)_C(t_A - t_C)$ | |
| Sum = $P^*_B$ | Sum = $Q^*_C$ | |
| $P^*_B - Q^*_C$ | | |
| $P^*_B + Q^*_C$ | | |
| $S_3 = 2 - \frac{1}{2}(n-1)(a/r)_A(t_C - t_B)$ | | |
| $u_A = (P^*_B - Q^*_C)/S_3$ | | |
| $S_4 = [4/(\gamma-1)] - \frac{1}{2}(n-1)(u/r)_A(t_C - t_B)$ | | |
| $a_A = (P^*_B + Q^*_C)/S_4$ | | |
| $[2/(\gamma-1)]a_A$ | | Only calculated when adequate agreement between initial and final values of $u_A$, $a_A$ has been attained. |
| $P_A = [2/(\gamma-1)]a_A + u_A$ | $Q_A = [2/(\gamma-1)]a_A - u_A$ | |
| $v_A = a_A + u_A$ | $w_A = a_A - u_A$ | |

Note: Quantities in "boxes" $\boxed{\quad}$ are copied from the initial data or results of previous calculations.

influence of the estimates on the "final" values of $u_A$ and $a_A$ evaluated by this process arises through the dependence of $(au/r)_A$ on the estimate of $u_A$; this is avoided if the form (3.11), (3.13) is used. Further, $t_B - t_C$ is often fairly small, and then $u_A$ calculated from (3.11) is not sensitive to the $a_A$ in the denominator.

The procedure used to improve the estimates of $u_A$, $a_A$ from which the calculation for a point A starts may depend on the circumstances (accuracy required, intervals between characteristics, etc.) of the particular calculation being carried out. One possible procedure is direct iteration, the "final" values of one iteration being taken as estimates for the next. Alternatively, the calculation for each point A may be carried out with three estimates, say $(u_A^{(0)}, a_A^{(0)})$, $(u_A^{(0)} + \Delta u_A, a_A^{(0)})$, $(u_A^{(0)}, a_A^{(0)} + \Delta a_A)$, and the results used to determine what fractions of $\Delta u_A$, $\Delta a_A$ should be taken to obtain agreement between estimated and "final" values of $u_A$, $a_A$; it would be advisable to make a check calculation with the resulting values of $u_A$, $a_A$ as estimates. This procedure is unduly lengthy if fewer than four iterations, on the average, are required when a direct iterative process is used. In the exploratory calculations which I have made, I found that for a calculation to 4 significant figures, the first estimate could usually be made closely enough for further estimates to be unnecessary.

An alternative procedure has been suggested to me by Dr. R. F. Clippinger, who has had extensive experience of the use of a grid of characteristics in the calculation of steady supersonic flow. One can correct for the main part of the truncation error of the integration formula by carrying out two (or more) independent calculations with grids of different mesh size, and using the results to extrapolate to zero mesh size on the lines of Richardson's

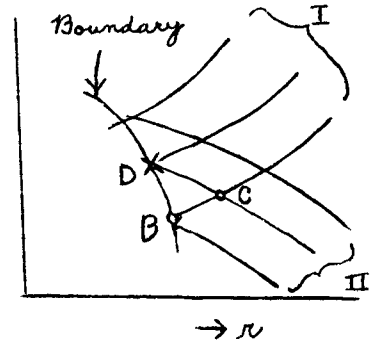
process of "$h^2$-extrapolation" [9]. Clippinger has found that if this procedure is used, and values of the dependent variables at A are found by an iterative process based on a consistent use of the crude integration formula

$$\int_{x_o}^{x_1} f(x)\,dx = (x_1 - x_o) f_o$$

to give the first estimates, then there is no ultimate advantage in carrying the iteration more than one stage further. For although the results on each grid separately may be affected appreciably by carrying the iteration further, the final results obtained by extrapolation to zero mesh size are not. The same would probably be true of equations (3.4) to (3.7).

§3.2. Boundaries.

If the fluid has a physical boundary, the regionof the (r,t) plane in which the solution of equations (2.2), (2.4) is required will be bounded by a curve in the (r,t) plane which in general is not a characteristic.




An internal boundary will be considered here; the treatment of an outer boundary is similar with the roles of characteristics of sets I and II interchanged.

In working on a grid of characteristics, the points calculated on the boundary in the (r,t) plane will be the points where the successive characteristics of set II cut it. Let B be the last boundary point to have been determined, C the point of intersection of the characteristic of set I through B and the next characteristic of set II, and D the intersection of this characteristic with the boundary curve.

Then for CD we have relations (3.6) and (3.7) with A re-

placed by D, and need two more relations at D to determine $r_D$, $t_D$, $u_D$ and $a_D$. The nature of these relations, and their numerical treatment, will depend on the conditions at the boundary.

If the boundary is a solid wall whose motion is given, we have

$$r_D = f(t_D), \quad u_D = \phi(t_D) ,$$

where $f(t)$ and $\phi(t)$ are given functions of $t$; in general these give an implicit equation for $t_D$, which may have to be solved by iteration (for each estimate of $u_A$, $a_A$).

If the boundary is a boundary between the fluid and an enclosed gas (e.g., expanding or contracting bubble) in which pressure gradients can be neglected, then (to the accuracy of the trapezoidal integration formula)

$$r_D - r_B = \tfrac{1}{2}(u_D + u_B)(t_D - t_B) \tag{3.14}$$

for the motion of the boundary, and

$$a_D = F(r_D), \tag{3.15}$$

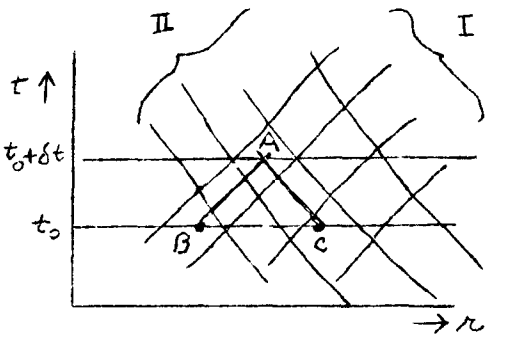
determined from the pressure-volume relation for the enclosed gas and the pressure-(velocity of sound) relation for the fluid.

If the boundary is a shock, the motion of the fluid on the other side of the boundary is relevant, and the relations are more involved; further, the motion is in general anisentropic. For this reason, consideration of the treatment of shocks is postponed till later (§6).

The only difference between equations (3.14) and (3.5) is that $v$ in (3.5) is replaced by $u$ in (3.14). Hence as far as the determination of $r_A$ (now $r_D$) the procedure of §3.1 stands, with $v$ replaced by $u$. From $r_D$, $a_D$ is given by (3.15), then $u_D$ is given by (3.6) (with A replaced by D). $a_D$ then follows directly from (3.15), and $Q_A$ then follows from (3.6), (3.7), in which $r_A$ is now known.

§4. Use of specified time intervals.

In this process we consider $u$ and $a$ as known functions of $r$ at time $t$, either as given initial conditions or as results of a previous stage of the calculation, and we use equations (3.4) to (3.7) to determine $u_A$, $a_A$ at a set of specified values of $r_A$ at a later time $t_o + \delta t$. The equations are the same as for the process of §3.1, but which variables are known, and which are to be determined, are different; $t_A$, $r_A$ are now known, and both $u$ and $a$ as functions of $r$ at time $t_o$; the unknowns are $u_A$, $a_A$, $r_B$, $r_C$.




The best procedure seems to be as follows. As in the method of §3.1, start with the calculation for each point A with estimates of $u_A$, $a_A$. From them form $r_A - \frac{1}{2} v_A \delta t$, $r_A + \frac{1}{2} w_A \delta t$. To determine $r_B$, write equation (3.5) in the form

$$r_A - \tfrac{1}{2} v_A \delta t = r_B + \tfrac{1}{2} v_B \delta t \ . \tag{4.1}$$

The right-hand side is a known function of $r_B$, and the left-hand side has been calculated, so that $r_B$ can be determined; similarly $r_C$ can be determined from equation (3.7) in the form

$$r_A + \tfrac{1}{2} w_A \delta t = r_C - \tfrac{1}{2} w_C \delta t \ . \tag{4.2}$$

The best way of performing these interpolations for $r_B$, $r_C$ is likely to be different for hand calculations than for work with an automatic digital calculating machine. If $r + \frac{1}{2} v\delta t$, $r - \frac{1}{2} w\delta t$ at time $t_o$ are given at equal intervals of $r$, the determination of $r_B$ for a value of $(r_B + \frac{1}{2} v_B \delta t)$ obtained from (4.1) involves inverse interpolation. For hand calculation it seems best to use the data at $t = t_o$ in the form of a table of

(4.3) $u$ $a$ $v$ $r + \frac{1}{2} v\delta t$ $w$ $r - \frac{1}{2} w\delta t$

as functions of $r$, at close enough intervals for linear interpolation of $v$ and $w$ as functions of $(r + \frac{1}{2} v\delta t)$, $(r - \frac{1}{2} w\delta t)$; then $r_B$ is formed as

$$r_B = (r_B + \tfrac{1}{2} v_B \delta t) - \tfrac{1}{2} v_B \delta t \ , \tag{4.4}$$

the quantity in brackets being evaluated from (4.1), and similarly for $r_C$. The advantage of this procedure is that the interpolation is performed on $v$, which is multiplied by the time interval $\delta t$ and usually gives a relatively small contribution to $r$, rather than on $r$ itself.

For an automatic machine with a sub-routine for non-linear interpolation with unequal values of the argument, a table of

$$u \qquad a \qquad r + \tfrac{1}{2} v\delta t \qquad r - \tfrac{1}{2} w\delta t$$

as functions of $r$, and requiring quadratic or perhaps cubic interpolation, would probably be more convenient, and should not make embarrassing demands on storage capacity. Relative to the method of §3.1, some space is saved since it is not now necessary to store a separate value of $t$ for each point.

In equations (3.4) and (3.6) the time intervals $(t_A - t_B)$ and $(t_A - t_C)$ are <u>given</u>, and not dependent on the estimates of $u_A$, $a_A$ as they are in the method of §3.1; also they are <u>equal</u>, so that formulas (3.10) become

$$\begin{cases} P^*_B = P_B - (au/r)_B \delta t \\ Q^*_C = Q_C - (au/r)_C \delta t \end{cases} , \tag{4.5}$$

and formulas (3.11), (3.13) simplify to

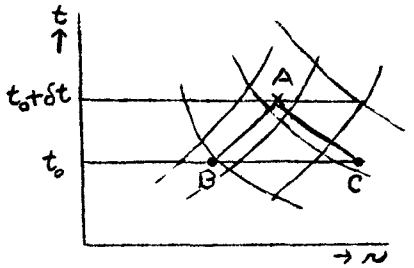
$$u_A = \tfrac{1}{2} (P^*_B - Q^*_C) \tag{4.6}$$

$$a_A = (P^*_B + Q^*_C)/[\{4/(\gamma-1)\} + (n-1)(u/r)_A \delta t] \ , \tag{4.7}$$

of which the former does not involve the estimate of $a_A$ explicitly (as (3.11) does).

## TABLE II

Suggested computing scheme
for use with specified time intervals




| | |
|---|---|
| $r_A$ | |
| Est $a_A$ | |
| Est $u_A$ | |
| $a_A + u_A = v_a$ | $a_A - u_A = w_A$ |
| $- \frac{1}{2} v_A \delta t$ | $+ \frac{1}{2} w_A \delta t$ |
| $r_A - \frac{1}{2} v_A \delta t$ | $r_A + \frac{1}{2} w_A \delta t$ |
| $= r_B + \frac{1}{2} v_B \delta t$ | $= r_C - \frac{1}{2} w_C \delta t$ |
| $v_B$ | $w_C$ |
| $- \frac{1}{2} v_B \delta t$ | $+ \frac{1}{2} w_C \delta t$ |
| $r_B$ | $r_C$ |
| $u_B$ | $u_C$ |
| $a_B$ | $a_C$ |
| $\frac{2}{\gamma-1} a_B$ | $\frac{2}{\gamma-1} a_C$ |
| $u_B$ | $-u_C$ |
| $- \frac{1}{2} (n-1)(\frac{au}{r})_B \delta t$ | $+ \frac{1}{2} (n-1)(\frac{au}{r})_C \delta t$ |
| Sum = $P^*_B$ | Sum = $Q^*_C$ |

$$u_A = \frac{1}{2} (P^*_B - Q^*_C)$$

$$P^*_A + Q^*_C$$

$$S_1 = \left\{4/(\gamma-1)\right\} + (n-1)(u/r)_A \delta t$$

$$a_A = (P^*_B + Q^*_C)/S_1$$

A suggested computing schedule is given in Table II. Experience with exploratory calculations with both methods is that the present method [including the interpolation in the table of quantities (4.3) is simpler to carry out in practice than that of §3.1. The computing schedule involves fewer steps of computation, as can be seen by comparing Table I and Table II, though it must be remembered that for the present method Table II does not include the preparation of the table of the input data; this, however, is very simple and can be done separately from the calculations for the various points A, and this division of the calculation into two parts which can be done separately is probably an advantage rather than the reverse, both for hand computation and for work with an automatic machine, at least with one which has an auxiliary store.

The process for revision of the estimates of $u_A$ and $a_A$, should it be necessary, follows that of §3.1, but my experience with some exploratory calculations has been that for work of moderate accuracy the first estimates can usually be made closely enough for such revision to be unnecessary.

For hand calculations it is convenient, as already mentioned, to have the input data (4.3) at close enough intervals for linear interpolation of $v$ and $w$ as functions of $(r + \frac{1}{2} v\delta t)$ and $(r - \frac{1}{2} w\delta t)$, and perhaps of $u$ and $a$ as functions of $r$. But it is not necessary to carry out the calculations using formulas (4.1), (4.2), (4.5) and (4.6) for values of $r_A$ at such close spacing. It might, for example, be adequate to carry out these calculations, for part of the range of $r$, for values of $r_A$ at two, four or five times the r-interval required for the input table for the next time interval, and to fill in the data for the intermediate values of $r$ by subtabulation[2].

[2] The "end-figure" method of subtabulation [3] is recommended.

§ 1. Comparison of method using a grid of characteristics (§3.1) with method using specified intervals in time (§4).

One reason why the method of §4 is simpler to carry out numerically than that of §3.1 is that now $r_A$ and $t_A$ are given and known exactly, and only two quantities at A have to be determined by integration, namely $u_A$ and $a_A$, whereas in the method of §3.1, all four of these quantities have to be determined by integration. Another reason is that, as already mentioned, the time-differences $t_A - t_B$ and $t_A - t_C$ on the two characteristics through A are not only known but equal.

Further, for the calculation of unsteady flow the present method has the considerable advantage that it produces results directly in the form most likely to be needed, namely the velocity distribution in space at different times. Also, the value of $\delta t$, and the values of $r_A$ at which $u_A$ and $a_A$ are calculated, are entirely under the current control of the individual who is doing the work (or superintending the machine, in an automatic calculation); if at any time it appears advisable to shorten the time interval $\delta t$, it can be halved (for example) without any previous notice of the change being taken in the work; and small intervals in r can be taken, for example near boundaries, shocks, or incipient shocks, without difficulty.

This method in terms of Eulerian variables is appreciably simpler than in terms of Lagrangian variables (§7). It has therefore been chosen for further development; to anisentropic flow (§5) and to the treatment of shocks (§6).

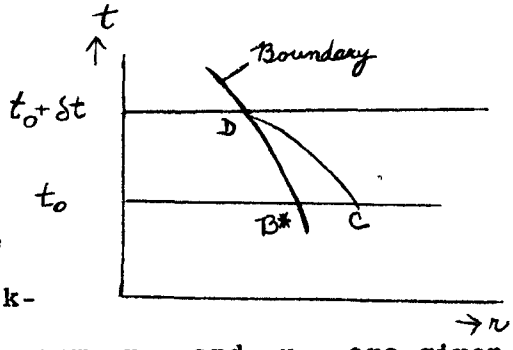
Some, but not all, of the advantages of use of specified time intervals in non-steady flow would also apply to the use of a similar method for calculating steady supersonic flow in two space variables. However, as Dr. Clippinger has pointed out to me, there

are two advantages of the use of a grid of characteristics in this context. One is that consistent use of a grid of characteristics in different solutions (for example, flows round cone-cylinders at zero for different cone angles) may simplify interpolation between those solutions. The other is that a characteristic is a possible line of discontinuity, and if such a discontinuity does occur, it is convenient to determine the corresponding characteristic (this could however, be done when using specified intervals of one independent variable).

For $n = 1$, the calculation is simplified by the absence of the terms on the right-hand side of equations (3.4), (3.6). In this case the solution at A depends strictly (and not only to the approximation of the integration formula) on that at B and C only. This corresponds to the "existence of a Huyghens Principle" in one dimension. But the procedure $n = 2$ is just the same as for $n = 3$ whether the calculation is carried out on a grid of characteristics or by use of specified intervals of time, there is no distinction corresponding to the "existence of a Huyghens Principle" in three dimensions but not in two.

§4.1. Boundaries.

Let B*, D be the points in the (r,t) plane representing the boundary at times $t_o$, $t_o + \delta t$ respectively.




The treatment of a boundary whose motion is <u>given</u> is simpler than in working on a grid of characteristics, since now $r_D$ and $u_D$ are <u>given</u>, and it is only necessary to determine $r_C$ and $a_D$ by integration along the characteristic of set II (for an interval boundary) through D.

For a boundary at which there is a given relation between $r$ and $p$, and hence between $r_D$ and $a_D$, the following procedure, which only requires an estimate of $u_D$, seems convenient.

Estimate $u_D$ and from it calculate

$$r_D = r_{B*} + \tfrac{1}{2}(u_{B*} + u_D)\delta t, \tag{4.8}$$

use the relation between $r_D$ and $a_D$ to determine $a_D$, from this determine $r_C$ from (4.2) (with D substituted for A). Then

$$u_D + \frac{2}{\gamma-1} a_D = Q^*_C - \tfrac{1}{2}(n-1)[(au/r)_C + (au/r)_D]\delta t$$

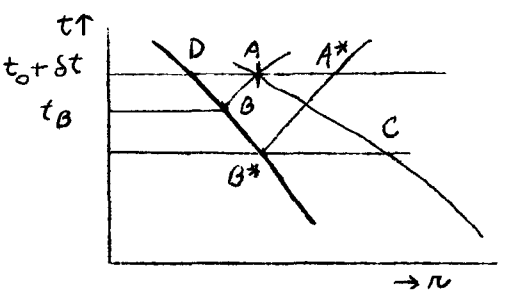
and $a_D$ has been determined, so $u_D$ is calculated from

$$u_D = [Q^*_C - \tfrac{1}{2}(n-1)(au/r)_C\delta t - \frac{2}{\gamma-1} a_D]/[1 + \tfrac{1}{2}(n-1)(a/r)_D\delta t] \tag{4.9}$$

If $r_B$ or $u_B$ departs appreciably from linearity in $t$ during the time interval $\delta t$, it may be advisable to evaluate some intermediate points between B and D on the curve in the $(r,t)$ plane representing the motion of the boundary. This can be done by replacing $\delta t$ in (4.8), (4.2), (4.9) by the value of $(t_D - t_B)$ for each intermediate point calculated.

## §4.2. Points near boundary.

Let B*,D be the boundary points in the $(r,t)$ plane at $t = t_o$, $t + \delta t$ respectively, as in §4.1, and let A* be the point at which the characteristic through B* cuts the line $t = t_o + \delta t$. The method of §4 can only be used for points A for which $r_A > r_{A*}$ (for an internal boundary; $r_A < r_{A*}$ for an external boundary); for points A between D and A*, the appropriate equations are (3.4) to (3.7) where B is the point at which the characteristic of set I through A cuts the boundary curve.



A* itself is given by

$$(4.10) \qquad r_{A*} - r_{B*} = \tfrac{1}{2}(v_{A*} + v_{B*})\delta t \ ;$$

from estimates of $u_{A*}$, $a_{A*}$, the value of $r_{A*}$ is calculated from (4.10) and the rest of the calculation follows the procedure of §4. The determination of A* should be the first calculation carried out for each time interval, to avoid attempts to carry out the standard procedure of §4 for points A for which $r_A < r_{A*}$.

Suppose the motion of the boundary between B* and D has already been calculated. Then, given A, we have to solve

$$(4.11) \qquad r_A - r_B = \tfrac{1}{2}(v_A + v_B)(t_A - t_B)$$

for $t_B$, where $r_B$ and $v_B$ are functions of $t_B$ already known in numerical form from the calculation of the motion of the boundary. This equation involves the unknown $t_B$ in an awkwardly implicit way. But we have also, for the motion of the boundary between B and D,

$$(4.12) \qquad r_D - r_B = \tfrac{1}{2}(u_D + u_B)(t_A - t_B) \qquad (\text{since } t_D = t_A),$$

and can eliminate $r_B$ from (4.11) and (4.12), giving

$$(4.13) \qquad t_A - t_B = 2(r_A - r_D)/(v_A + a_B - u_D) \ .$$

The right-hand side only involves the point B through the term $a_B$ in the denominator, and in most cases it should be possible to estimate this to sufficient accuracy so that no iteration is required. The value of $t_A - t_B$ can be checked by interpolating $r_B$, $v_B$ for the corresponding value of $t_B$, and verifying that the values satisfy equations (4.11). $r_C$ is determined as in §4.

Then $u_A$, $a_A$ are given by (3.11), (3.12).

§5. Anisentropic flow of a perfect gas.

For a perfect gas the equations of anisentropic flow simplify considerably. It seems most convenient to take for $s$ a constant multiple of the entropy per unit mass, that is

$$s = (1/\beta) \log (p/\rho^\gamma) \tag{5.1}$$

where $\beta$ is a constant to be determined later. Then

$$p = \rho^\gamma e^{\beta s} \tag{5.2}$$

(a multiplying constant in $p$ does not affect the final formulas), and

$$a^2 = (\partial p/\partial \rho)_s = \gamma \rho^{\gamma-1} e^{\beta s} = \gamma p/\rho$$

$$(\partial (a^2)/\partial \rho)_s = (\gamma - 1)a^2/\rho , \quad (\partial (a^2)/\partial s)_\rho = \beta a^2 \tag{5.3}$$

and

$$(\partial p/\partial s)_\rho = \beta p = \beta a^2 \rho /\gamma . \tag{5.4}$$

Then along any curve $C$ in the $(r,t)$ plane

$$2a\left(\frac{da}{dt}\right)_C = \left(\frac{\partial (a^2)}{\partial \rho}\right)_s \left(\frac{d\rho}{dt}\right)_C + \left(\frac{\partial (a^2)}{\partial s}\right)_\rho \left(\frac{ds}{dt}\right)_C$$

$$= (\gamma - 1)/(a^2/\rho)(d\rho/dt)_C + \beta a^2 (ds/dt)_C$$

from (5.3), so

$$(\gamma - 1)(a/\rho)(d\rho/dt)_C = 2(da/dt)_C - \beta a (ds/dt)_C . \tag{5.5}$$

Now in both equations (2.8) and (2.10) there occurs the combination

$$(a/\rho)(d\rho/dt)_C + (1/a\rho)(\partial p/\partial s)_\rho (ds/dt)_C .$$

Substitution in the first term from (5.5) and in the second from (5.4) gives

$$(\gamma-1)[(a/\rho)(d\rho/dt)_C + (1/a\rho)(\partial p/\partial s)_\rho (ds/dt)_C]$$

$$= 2(da/dt)_C - (\beta/\gamma)a(ds/dt)_C .$$

It now appears most convenient to put

$$\beta = \gamma(\gamma - 1), \tag{5.6}$$

and equations (2.8), (2.10) become[3]

[3] Sauer [11, §23.4] quotes equations for anisentropic flow, in u, a, s as dependent variables, and obtains equations similar to these; the derivation of the basic equations is in Sauer [10, §5.16, formulas (55)].

$$\left(\frac{dP}{dt}\right)_I - a\left(\frac{ds}{dt}\right)_I = -(n-1)au/r \tag{5.7}$$

$$\left(\frac{dQ}{dt}\right)_{II} - a\left(\frac{ds}{dt}\right)_{II} = -(n-1)au/r \tag{5.8}$$

where

$$P = [2/(\gamma-1)]a + u, \quad Q = [2/(\gamma-1)]a - u$$

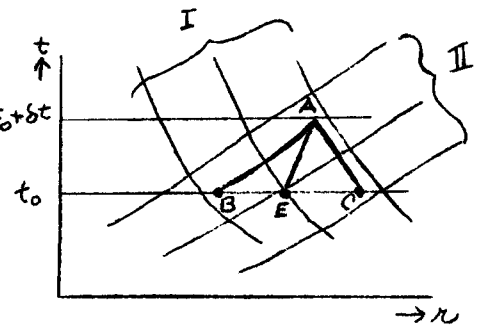
as for the isentropic case, and

$$s = [1/\gamma(\gamma-1)] \log (p/\rho^{\gamma}). \tag{5.9}$$

The other equations are (2.9), (2.11) as for the isentropic case, and the additional equations (2.16), (2.17).

## §5.1. Integration using specified time intervals.

The equations for the anisentropic flow of a perfect gas are so nearly similar to those for isentropic flow that much the same procedure can be used for them.




We start with estimates of $u_A$, $a_A$. The equations for the characteristics are the same as for isentropic flow, so exactly the same procedure can be used as far as the determination of $r_B$, $r_C$. We have now, in addition, to find $r_E$, the point at $t = t_o$ on the particle path through A; this is given by integration of equation (2.17):-

$$r_A - r_E = \frac{1}{2}(u_A + u_E)\delta t ,$$

whence

$$(5.10) \qquad r_A - \tfrac{1}{2} u_A \delta t = r_E + \tfrac{1}{2} u_E \delta t$$

from which $r_E$ is found by the same procedure as used for finding $r_B$ and $r_C$. $s(r)$ at $t = t_o$ has to be included in the table of input data, and since $s$ is constant along a particle path, interpolation for $s$ at $r = r_E$ gives $s_A$. This value is then used in the further calculation.

The integrated forms of (5.7), (5.8), corresponding to equations (3.4), (3.6) for the isentropic case, with $t_A - t_B = t_A - t_C = \delta t$, are

$$(5.11) \quad P_A - P_B = -\tfrac{1}{2}(n-1)[(au/r)_A + (au/r)_B]\delta t + \tfrac{1}{2}(a_A + a_B)(s_A - s_B)$$

$$(5.12) \quad Q_A - Q_C = -\tfrac{1}{2}(n-1)[(au/r)_A + (au/r)_C]\delta t + \tfrac{1}{2}(a_A + a_C)(s_A - s_C)$$

and the most convenient way of dealing with these relations seems to be one which is an extension of that used for the isentropic case. Subtraction of (5.12) from (5.11) gives

$$(5.13) \quad 2u_A = P_A - Q_A = P^*_B - Q^*_C + \tfrac{1}{2}[a_A(s_C - s_B) + a_B(s_A - s_B) - a_C(s_A - s_C)];$$

where $P^*_B$, $Q^*_C$ are given by (4.6) as before; the estimated value of $a_A$ is used in the first term in the square brackets, the other terms involve quantities already calculated.

Addition of (5.11), (5.12) gives

$$[4/(\gamma-1)]a_A = P^*_B + Q^*_C - (n-1)(au/r)_A + \tfrac{1}{2} a_A(2s_A - s_B - s_C) + \tfrac{1}{2} a_B(s_A - s_B) + \tfrac{1}{2} a_C(s_A - s_C)$$

whence

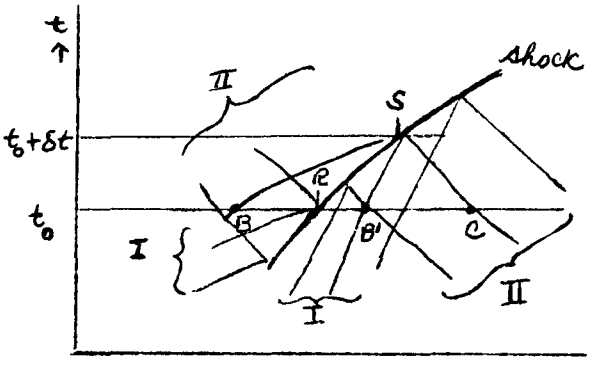
$$(5.14) \quad a_A = [P^*_B + Q^*_C + \tfrac{1}{2}\{a_B(s_A - s_B) + \tfrac{1}{2} a_C(s_A - s_C)\}]/$$
$$[\{4/(\gamma-1)\} + (n-1)(u/r)_A + \tfrac{1}{2}(2s_A - s_B - s_C)] .$$

Equations (5.13), (5.14) take the place of (4.6), (4.7) for isentropic flow.

The evaluation of formula (5.9) only has to be carried out for the initial conditions and at a shock; in the absence of shocks, no reference has to be made to this formula in the course of the integration.

## §6. Shocks.

The treatment of shocks looks involved at first sight, but turns out to be fairly simple when using specified time intervals, and is probably simpler than in work on a grid of characteristics.




Let R and S refer to the shock at times $t_0$ and $t_0 + \delta t$, and let suffixes + and - refer to conditions just ahead of the shock and just behind it, respectively; also, let U be the velocity of the shock. $U_R$ and $r_R$, as well as a, u and s as functions of r at time $t = t_0$, are supposed known, either as given initial conditions or as results of calculation for previous time intervals.

If the fluid ahead of the shock has not been traversed by previous shocks, its motion is isentropic and $s_{S+}$ is known. Then for the point S we have ten unknowns, namely:-

$$r_S,\ U_S,\ a_{S-},\ u_{S-},\ s_{S-},\ a_{S+},\ u_{S+},\ r_B,\ r_{B'},\ r_C\ ,$$

and need ten equations relating them. These are given by the following considerations.

The velocity of the shock relative to the fluid ahead of it is greater than the velocity of sound in that fluid, so that the motion of a particle of this fluid is independent of the presence of the shock until the shock reaches it; whereas the velocity of the shock relative to the fluid behind it is less than the velocity of sound

in that fluid. It follows that there are three characteristics through S and intersecting the line $t = t_0$, namely:-

BS, a characteristic of set I in the fluid behind the shock;

B'S, a characteristic of set I in the fluid ahead of the shock;

CS, a characteristic of set II in the fluid ahead of the shock.

Each of these gives two relations between the unknowns, making six altogether.

Then there are three relations, given by the conservation of mass, momentum, and of energy, between the shock velocity U and the values of a, u and s on the two sides of the shock. Finally there is the relation between the velocity and position of the shock itself,

$$(6.1) \qquad r_S - r_R = \frac{1}{2}(U_S + U_R)\delta t \ .$$

These give the required ten equations, several of them non-linear.

However, it turns out that if they are taken in the right order, it is necessary to estimate only <u>one</u> of the unknowns in order to determine the corresponding values of all the others. It seems surprising that it is necessary to estimate only one quantity in order to deal with the equations for the shock whereas it is necessary (as far as I have found) to estimate two quantities for dealing with the fewer and simpler equations involved in calculating the continuous region of the flow.

The quantity to start from is an estimate of the shock velocity $U_S$. Then (6.1) gives $r_S$ and, since the flow ahead of the shock is independent of the presence of the shock, $r_B$, $r_C$, $a_{S+}$, $u_{S+}$, can be found by the procedure of §4.1 without further reference to the shock (if the flow ahead of the shock is anisentropic, the method of §5.1 is used and gives $s_{S+}$ also). Then from $a_{S+}$, $u_{S+}$, $s_{S+}$, and $U_S$, the equations of conservation of mass, momentum, and

energy through the shock give $a_{S-}$, $u_{S-}$ and $s_{S-}$. For a perfect gas, the Rankine-Hugoniot relations, which give the density ratio and the velocity ratio through the shock in terms of the pressure ratio, are not the most convenient relations to use for this purpose; it seems more convenient to proceed as follows (see Courant-Friedrichs [4, §68, A]).

Let $u^*_1$, $u^*_2$ be the velocities, relative to the shock, of the fluid in front of the shock and behind it respectively, and $a_1 = a_{S+}$, $a_2 = a_{S-}$ the corresponding velocities of sound. Then from the equation of energy, the quantity $\frac{1}{2}(u^*)^2 + [1/(\gamma-1)]a^2$ has the same value on both sides of the shock, and if we write $[(\gamma+1)/2(\gamma-1)](a^*)^2$ for this common value, so that

$$(6.2) \quad (\gamma-1)(u^*_1)^2 + 2a_1^2 = (\gamma-1)(u^*_2)^2 + 2a_2^2 = (\gamma+1)(a^*)^2$$

we have the result[4]

[4] See Liepmann-Puckett [7], formula (4.3); Courant-Friedrichs [4], formula (66.02). $u^*_1$, $u^*_2$ are the $u_1$, $u_2$ of Liepmann-Puckett, whose notation is otherwise followed here. The $\mu^2$ of Courant-Friedrichs is $(\gamma-1)/(\gamma+1)$, and their $c^*$ is the $a^*$ of Liepmann-Puckett.

$$(6.3) \quad u_1 u_2 = (a^*)^2 .$$

Now $a_1 = a_{S+}$, and $u^*_1 = U_S - u_{S+}$ can be determined from the estimates $U_s$ and quantities already calculated, so $(a^*)^2$ can be calculated from (6.2), then $u^*_2$ from (6.3), then $a_2^2$ from (6.2). Also $\gamma p/\rho^\gamma = a^2/\rho^{\gamma-1}$, and $\rho u^*$ is conserved through the shock, so the change in $s$ can be calculated from

$$(6.4) \quad S_2 - S_1 = [\log (a_2^2/a_1^2) + (\gamma-1) \log (u_2/u_1)]/\gamma(\gamma-1).$$

Also $u_{S-} = U_S - u^*_2$. Thus all quantities just behind the shock are calculated in terms of the estimated $U_S$, and no further estimates or iteration has been needed in this stage of the work.

Finally we obtain two values of $P_{S-}$, one directly from

$$P_{S-} = [2/(\gamma-1)]a_{S-} - u_{S-}$$

and the other by determining $r_B$ from (4.1) and integrating along the characteristic BS by (5.11). If the estimate of $U_S$ is correct these values of $P_{S-}$ will agree. There are clearly several ways of arranging the numerical work of adjusting $U_S$ so that this criterion is satisfied, and some exploratory numerical work would be necessary before the best process could be determined.

This procedure for dealing with shocks when using specified time intervals seems simpler than that required when working on a grid of characteristics. The only special procedure required is that for dealing with the relations at the shock itself, and as far as the use of characteristics is involved, the interpolation required is no more than if S were a point in the region of the (r,t) plane in which the flow is continuous; whereas when working on a grid of characteristics, some special procedure is required for departing from the mesh points of this grid in the neighborhood of shocks (see, for example, [15]).

If $u_A$ and $a_A$ are required at points A between S and the characteristic of set II through R, they can be found by a procedure similar to that of §4.2.

## §6.1. Contact discontinuities.

A surface of separation of two different fluids, or two volumes of the same fluid but with different values of the entropy ("contact discontinuities", see [4, §56]) can probably be treated in a similar manner to shocks, but no details of procedure have been worked out. The conditions at the boundary are continuity of pressure and of fluid velocity.

## §7. Lagrangian form of the equations.

The use of characteristics with the Lagrangian form of the equations will only be considered briefly, as it seems to offer no advantages over the use of characteristics with the Eulerian form. The main purpose of this section and the following is to show why this is.

Let $q$ be a Lagrangian variable, having a constant value for each particle, and different values for different particles such that $r$ is a monotonic function of $q$ and $\partial r/\partial q$ exists.

The equation of motion is now

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\right) . \tag{7.1}$$

[$(\partial/\partial t)$ now means $(\partial/\partial t)_q$, a time rate of change for a point moving with the fluid.] If at $t = t_o$ the density at the point $r = r_o(q)$ is $\rho_o$, the equation of conservation of mass is

$$\rho r^{n-1}\partial r/\partial q = \rho_o r_o^{n-1}\partial r_o/\partial q , \tag{7.2}$$

and finally

$$\partial r/\partial t = u . \tag{7.3}$$

A particular choice of the variable $q$ which is often convenient is $q = r_o$; another is

$$q = m \equiv \int \rho_o r_o^{n-1} dr_o ; \tag{7.4}$$

if the latter is adopted, (7.2) becomes

$$\partial r/\partial m = 1/\rho r^{n-1} \tag{7.5}$$

and (7.1) can be written

$$\frac{\partial u}{\partial t} = -r^{n-1}\left(\frac{\partial p}{\partial m}\right)_t . \tag{7.6}$$

## §7.1. Characteristics from Lagrangian form of the equations[5].

[5] This derivation of the characteristics from the Lagrangian form of the equations is based on one shown me by Mr. Hugh Flynn; it may not be new, but I had not seen it before.

Only isentropic flow will be considered; the time-variation of $\rho$, u and s along characteristics must be independent of the variables in the plane of which the characteristics are drawn, so the extension to anisentropic flow can be taken over directly from §5.1.

For isentropic flow, $\partial p/\partial m = a^2 \partial\rho/\partial m$, so (7.6) is

$$\frac{\partial u}{\partial t} = - a^2 r^{n-1} \frac{\partial \rho}{\partial m} . \tag{7.7}$$

Also from (7.3), (7.5)

$$\frac{\partial u}{\partial m} = \frac{\partial}{\partial t}\left(\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(1/\rho r^{n-1}\right)$$

$$= - \frac{1}{\rho^2 r^{n-1}} \frac{\partial \rho}{\partial t} - \frac{n-1}{\rho r^n} \frac{\partial n}{\partial r} ,$$

so that

$$\frac{1}{\rho} \frac{\partial \rho}{\partial p} + \rho r^{n-1} \frac{\partial u}{\partial m} = - \frac{(n-1)u}{r} . \tag{7.8}$$

The linear combination of (7.7), (7.8) with coefficients 1, $\lambda$ respectively is

$$\left(\frac{\partial}{\partial t} + \lambda\rho r^{n-1} \frac{\partial}{\partial m}\right)u + \left[\left(\gamma/\rho\right) \frac{\partial}{\partial \rho} + a^2 r^{n-1} \frac{\partial}{\partial m}\right] = -(n-1)\lambda u/r.$$

The condition that the differential operators represent differentiation in the same direction is

$$\lambda\rho r^{n-1}/1 = a^2 r^{n-1}/(\lambda/\rho)$$

whence $\lambda = \pm a$. The characteristics are curves I and II in the (m,t) plane such that

$$\frac{dm}{dt} = \pm a\rho r^{n-1} \tag{7.9}$$

and the equations for the variation of u and $\rho$ along them are as before.

The main reason why these equations are not so convenient for numerical work as those for the Eulerian form of the equations is that the evaluation of the integrand in equations (7.9) involves appreciably more work than for the corresponding equations (2.9), (2.11) of the Eulerian form. First, for a fluid with an equation of state (2.14),

$$a\rho = \text{const} \times a^{(\gamma+1)/(\gamma-1)},$$

and however the calculation is arranged, this power or its inverse has to be evaluated at some stage of the work, for each point A; secondly, each value of $a\rho$ has to be multiplied by $r^{n-1}$; and thirdly, r has to be evaluated by integration of (7.3), whether the flow is isentropic or not, whereas with the Eulerian form r is given and equation (2.17), which corresponds to (7.3), does <u>not</u> have to be integrated except for anisentropic flow. The first of these points is the main one, and is quite substantial; one could hardly expect simpler integrands to evaluate than the $u \pm a$ of the Eulerian form. The extra integration is a simple one, but it is an additional step which has to be carried out for each point A with the Lagrange form. The only advantage of the Lagrange form is that in isentropic flow no interpolation of s as a function of m is required (except at shocks), but this does not seem a sufficient compensation for the disadvantages.

## §8. <u>Motion in two space variables</u>.

One possible method for calculating non-steady flow in two spac variables is to integrate with respect to time the equations in Lagrangian form. These are

$$\frac{d^2x}{dt^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{d^2y}{dx^2} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{8.1}$$

and, if (x,y) are the current coordinates of the particle initially

at $(x_o, y_o)$, and $\rho_o(x_o)$ is the initial density,

$$\rho_o/\rho = \partial(x, y)/\partial(x_o, y_o) . \tag{8.2}$$

This involves no reference to characteristics, but it involves six numerical differentiations for each point of the (x,y,t) grid, four for the four derivatives required in the evaluation of the density and two for the pressure gradients; further, the latter involve further differentiation of quantities already obtained by earlier differentiations. Any simple way of avoiding some of these crosswise differentiations is likely to be an advantage, and particularly if it avoids repeated differentiation.

In the numerical calculation of non-steady motion in one dimension, the advantage of the use of characteristics, in avoiding crosswise differentiation, is so marked that for non-steady flow in two space variables it seems worth exploring the use of integration along selected curves in (x, y, t) space with the intention of reducing the number of crosswise differentiations to be carried out.

It can be shown that in general there are no curves with the properties of the characteristics in one space variable, that the rate of change along any one such curve involves no crosswise differentiation, though of course there may be such curves in special cases (for example, for motion with circular symmetry). Also the entities in (x, y, t) space most closely corresponding to the characteristics in (r, t) space are characteristic surfaces, not curves; it is possible that these could be used as the basis for a practical numerical method (see [16, §3], [1] and [10], and other papers referred to in [10]), but such a process appears likely to be complicated and not amenable to the direct production of results as functions of t; the derivation of results as functions of (x,y,t) from results on a grid of intersections of characteristic surfaces

would be a substantial piece of interpolation!

The case of one space variable suggests that for two space variables the Eulerian form of the equations is the more promising. Consider therefore these equations, which for isentropic flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \tag{8.3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial y} = 0 \tag{8.4}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,, \tag{8.5}$$

and, for irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,. \tag{8.6}$$

On a curve C on which

$$\left(\frac{dx}{dt}\right)_C = u + \alpha a, \quad \left(\frac{dy}{dt}\right)_C = v + \beta a, \tag{8.7}$$

we have

$$\left(\frac{du}{dt}\right)_C = a\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} - \frac{a}{\rho}\frac{\partial \rho}{\partial x}\right) \tag{8.8}$$

$$\left(\frac{dv}{dt}\right)_C = a\left(\alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} - \frac{a}{\rho}\frac{\partial \rho}{\partial y}\right) \tag{8.9}$$

$$\left(\frac{d\rho}{dt}\right)_C = \alpha a\frac{\partial \rho}{\partial x} + \beta a\frac{\partial \rho}{\partial y} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\,. \tag{8.10}$$

The linear combination of (8.8), (8.9), (8.10) and (8.6), with coefficients $\lambda, \mu$, $(a/\rho)$ and $\kappa$ respectively, is

$$\begin{aligned}\lambda\left(\frac{du}{dt}\right)_C + \mu\left(\frac{du}{dt}\right)_C + \frac{a}{\rho}\left(\frac{d\rho}{dt}\right)_C = a\Big[&(\lambda\alpha - 1)\frac{\partial u}{\partial x} + (\lambda\beta - \kappa)\frac{\partial u}{\partial y} \\ &+ (\mu\alpha + \kappa)\frac{\partial v}{\partial x} + (\mu\beta - 1)\frac{\partial v}{\partial y} + \frac{1}{\rho}\left\{(\alpha - \lambda)\frac{\partial \rho}{\partial x} + (\beta - \mu)\frac{\partial \rho}{\partial y}\right\}\end{aligned} \tag{8.11}$$

We want to choose $\alpha, \beta, \kappa, \lambda, \mu$ so as to give equations involving as few crosswise derivatives as possible.

First, it is clear that we cannot remove all the derivatives of velocity components, for to make the coefficients of $\partial u/\partial x$ and $\partial v/\partial y$ zero, we require $\lambda\alpha = 1$, $\mu\beta = 1$ and hence $\lambda\mu\alpha\beta = 1$, whereas to make the coefficients of $\partial u/\partial y$ and $\partial v/\partial x$ zero, we

require $\lambda\mu\alpha\beta = -\kappa^2$.

We can, however, eliminate <u>all but one</u> of the six terms on the right-hand side of (8.11), in several ways. Four possibilities are

$$(8.12)\qquad \begin{cases} \mu = \beta = 0, \quad \lambda = \alpha = \pm 1, \quad \kappa = 0 \\ \lambda = \alpha = 0, \quad \mu = \beta = \pm 1, \quad \kappa = 0 . \end{cases}$$

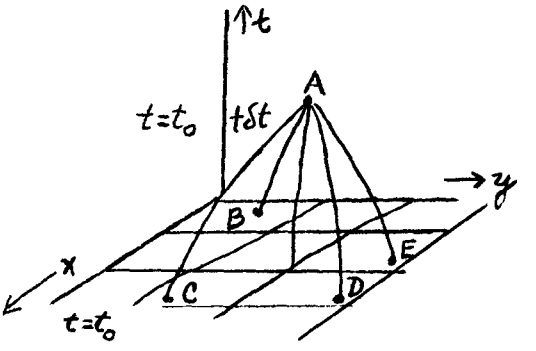
$\mu = \beta = \kappa = 0$, $\quad \lambda = \alpha = +1$ gives

$$(8.13)\qquad \left[\frac{d}{dt}\left\{u + W(\rho)\right\}\right]_I = -a\,\frac{\partial u}{\partial y},$$

on curves C such that

$$(8.14)\qquad \left(\frac{dx}{dt}\right)_C = u + a, \quad \left(\frac{dy}{dt}\right)_C = v,$$

and similarly for the other alternatives. These give altogether four equations involving altogether only two crosswise derivatives; three of these four equations like (8.13) would suffice to determine u, v and $W(\rho)$ at $(x, y, t_o + \delta t)$ given u, v and $W(\rho)$ [or $a(\rho)$] as functions of (x,y) at time t; the fourth would serve as a check. It should be possible to extend the procedure of §4.1 to deal with this.




Alternatively, for irrotational flow it might be possible to use only two of equations (8.13), involving the same crosswise derivative (say those given by $\lambda = \kappa = \pm 1$) and equation (8.6) expressing the irrotational character of the flow. It is not clear, however, how this would best be done; it might involve a further cross-differentiation, and then nothing would be gained.

From equations (8.3) - (8.5), it follows that flow initially irrotational remains so. But rounding errors in numerical work will

unavoidably introduce a certain amount of spurious circulation, in a more or less random fashion. This might simulate turbulence; but it is a purely numerical phenomena and would not appear in a correct solution of equations (8.3) to (8.5), supposing that this could be evaluated. Whether or not this "numerical turbulence" is serious might depend on the circumstances of the particular solution being evaluated; if it were serious, it might be necessary to derive procedures for preventing it building up as the solution proceeds.

Another possible choice of coefficients in (8.11) is

$$\alpha = \lambda = \cos\chi, \quad \beta = \mu = \sin\chi, \quad \kappa = 0 . \tag{8.15}$$

Then the right-hand side of (8.11) becomes

$$-a[\sin^2\chi\frac{\partial u}{\partial x} - \sin\chi\cos\chi(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \cos^2\chi\frac{\partial v}{\partial y}] . \tag{8.16}$$

Now if $\psi$ is the angle which the velocity vector V at (x,y,t) makes with the x axis, the quantity in square brackets in (8.16) is

$$\sin(\chi - \psi)(\sin\chi\frac{\partial V}{\partial x} - \cos\chi\frac{\partial V}{\partial y}] - V\cos(\chi - \psi)[\sin\chi\frac{\partial\psi}{\partial x} - \cos\chi\frac{\partial\psi}{\partial y}].$$

For $\chi = \psi$, $\chi = \psi + \pi$ this gives two equations involving the crosswise derivatives $\partial\psi/\partial x$, $\partial\psi/\partial y$ only, and for $\chi = \psi + \frac{1}{2}\pi$, $\chi = \psi + \frac{3}{2}\pi$ it gives two equations involving $\partial V/\partial x$, $\partial V/\partial y$ only. For irrotational flow it might be possible to use one of these pairs of equations with (8.6).

In all these cases, reduction in the number of crosswise derivatives to be evaluated is achieved at the cost of a considerable amount of interpolation in two independent variables. Whether this is avoidable, and if not, whether the price is too heavy a one, are questions for future exploratory work.

The treatment of anisentropic flow and shocks should wait until it has been established whether it is practicable to treat isentropic flow by a method of this kind, and if so, what is the best method.

## §9. Acknowledgements.

I wish to express my warm thanks to Professor Garrett Birkhoff for the invitation which has enabled me to pay a visit to Harvard University during which the work covered in this report was carried out, and also to thank Professor Birkhoff and Dr. R. F. Clippinger for several valuable discussions.

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