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THEORY OF LOW POWER KINETICS OF CIRCULATING FUEL REACTORS WITH SEVERAL GROUPS OF DELAYED NEUTRONS

J. A. Fleck, Jr.

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ABSTRACT

A method is described for computing reactor periods of circulating fuel reactors by finding the eigenvalues of a system of differential equations. The method accommodates two energy groups of neutrons and five groups of delayed neutrons. Approximate inhour equations are also derived for flat and sinusoidal fluxes and compared with the eigenvalue method. Discrepancies are largely attributable to differences in the delayed neutron spatial distribution.

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THEORY OF LOW POWER KINETICS OF CIRCULATING FUEL REACTORS WITH SEVERAL GROUPS OF DELAYED NEUTRONS

A method has been previously described by the author* for obtaining the low power kinetic behavior of closed loop, circulating fuel reactors at constant reactivity by solving a system of linear partial differential equations. The equations consisted of a one-group, kinetic neutron diffusion equation coupled with the hydrodynamic equation for the density of a single group of delayed neutron emitters. The latter equation was simplified by the assumption of slug flow. If the time dependence of the neutron and emitter densities is assumed to be exponential, the determination of the reactor period or e-folding time reduces to an eigenvalue problem. This method has now been generalized to include five groups of delayed neutrons and two neutron energy groups. The reason for the latter refinement is this: the distribution of delayed neutron emitters can be highly distorted by flow from the flux distribution which it would assume if there were no circulation; the kinetic behavior of the reactor is sensitive to distortions in the delayed neutron distribution, but it is somewhat smoothed out by the slowing down of the delayed neutrons. The inclusion of two neutron energy groups helps, if perhaps inadequately, to take into account this smoothing out because of slowing down.

Calculation of reactor periods by the eigenvalue method, when several groups of delayed neutrons are considered, is a highly involved operation and requires the use of an automatic computing machine. Some sort of generalized inhour equation would therefore be highly desirable. Two approximate inhour equations are derived in this report and are used to calculate periods which are compared with those obtained by the eigenvalue method. The eigenvalue results were obtained from a Univac code developed by Herbert Goertzel and Herbert Keller of the AEC Computing Center at New York University.

The Eigenvalue Method

It will be assumed that the reactor core is a right circular cylinder of length a. The circulating fuel passes through the core once before entering the external loop, which has a length h and the same cross-sectional area as the core. No generality is lost in the last assumption, since only the transit time through the external loop is important to the calculation.

Under the assumption of uniform (slug) flow, the kinetic equations of the flux-emitter system can be written

$$\tau\nabla^2\phi_f - \phi_f + k(1-\bar{\beta})\frac{\Sigma_s}{\Sigma_f}\phi_s + \frac{1}{\Sigma_f}\sum_{i=1}^{5}\lambda_i c_i = \ell_f \frac{\partial\phi_f}{\partial t} \tag{1a}$$

$$L^2\nabla^2\phi_s - \phi_s + \frac{\Sigma_f}{\Sigma_s}\phi_f = \ell_s \frac{\partial\phi_s}{\partial t} \tag{1b}$$

$$\frac{\partial c_i}{\partial t} + q\frac{\partial c_i}{\partial z} = -\lambda_i c_i + \bar{\beta}_i k\Sigma_s\phi_s; \quad i = 1, 2, \ldots 5$$

$$0 \leqslant z \leqslant a \tag{1c}$$

*J.A. Fleck, Jr., "Kinetics of Circulating Reactors at Low Power," Nucleonics 12, 11 (1954).

$$\frac{\partial c_i}{\partial t} + q\frac{\partial c_i}{\partial z} = -\lambda_i c_i \; ; \quad i = 1, 2, \ldots 5$$

$$a \leqslant z \leqslant h \; . \tag{1d}$$

Here z is the displacement along the core axis, ℓ_s and ℓ_f are the infinite medium mean lifetimes for slow and fast neutrons, respectively, and the $\bar{\beta}_i$ terms are the delayed neutron fractions with $\bar{\beta} = \sum_{i=1}^{5} \bar{\beta}_i$. The remaining notation is the conventional one for two-group theory with the subscripts s and f referring to the slow and fast groups, respectively.

Since equations (1) are linear with constant coefficients, we look for solutions of the form:

$$\phi_f = F\, e^{\alpha z + \omega t} f(r) \tag{2a}$$

$$\phi_s = S\, e^{\alpha z + \omega t} f(r) \tag{2b}$$

$$c_i = c_i\, e^{\alpha z + \omega t} f(r) \; ; \quad i = 1, 2, \ldots 5 \; , \tag{2c}$$

where

$$\nabla^2 f(r) = -B_r^2 f(r) \; . \tag{3}$$

By substituting the solutions (2) in equations (1), canceling exponentials and equating to zero the determinant of the resulting homogeneous equations in the coefficients F, S, and the c_i, we obtain the following relation which α and ω must satisfy:

$$\left\{\tau(B_r^2 - \alpha^2) + 1 + \omega\ell_f\right\}\left\{L^2(B_r^2 - \alpha^2) + 1 + \omega\ell_s\right\} = k\left\{(1 - \bar{\beta}) + \sum_{i=1}^{5} \frac{\beta_i\lambda_i}{\omega + \lambda_i + \alpha q}\right\} . \tag{4}$$

If this equation is solved for α in terms of ω, there will be seven real and two complex roots, which are written in the form

$$r_1 = \gamma + i\beta, \quad r_2 = \gamma - i\beta, \quad r_3 = \alpha_3, \ldots r_9 = \alpha_9 \; . \tag{5}$$

The expression ϕ_s then has the form

$$\phi_s = A_1\, e^{\gamma z} \sin \beta z + A_2\, e^{\gamma z} \cos \beta z + \sum_{j=3}^{9} A_j\, e^{\alpha_j z} \; . \tag{6}$$

(For convenience the factors f(r), $e^{\omega t}$ are omitted.) The remaining functions are given by

$$\phi_f = (u_{61}A_1 + u_{62}A_2)\, e^{\gamma z} \sin \beta z + (v_{61}A_1 + v_{62}A_2)\, e^{\gamma z} \cos \beta z + \sum_{i=3}^{9} w_{6j}A_j\, e^{\alpha_j z}$$

$$c_i = (u_{i1}A_1 + u_{i2}A_2)\, e^{\gamma z} \sin \beta z + (v_{i1}A_1 + v_{i2}A_2)\, e^{\gamma z} \cos \beta z + \sum_{j=3}^{9} w_{ij}A_j e^{\alpha_j z} \; ;$$

$$i = 1, 2, \ldots 5 \; . \tag{7}$$

Here, the subscript 6 labels quantities which pertain to ϕ_f. The quantities u_{ij}, v_{ij}, w_{ij} are given as follows:

$$u_{i1} = \frac{\bar{\beta}_i k \Sigma_s (\omega + \lambda_i + q\gamma)}{(\omega + \lambda_i + q\gamma)^2 + (q\beta)^2} ,$$

$$u_{i2} = \frac{\bar{\beta}_i k \Sigma_s (q\beta)}{(\omega + \lambda_i + q\gamma)^2 + (q\beta)^2} ,$$

$$v_{i1} = \frac{-\bar{\beta}_i k \Sigma_s (q\beta)}{(\omega + \lambda_i + q\gamma)^2 + (q\beta)^2} ,$$

$$v_{i2} = \frac{\bar{\beta}_i k \Sigma_s (\omega + \lambda_i + q\gamma)}{(\omega + \lambda_i + q\gamma)^2 + (q\beta)^2} ,$$

$$w_{ij} = \frac{\bar{\beta}_i k \Sigma_s}{\omega + \lambda_i + q\alpha j} ; \quad i = 1, 2, \ldots 5 ; \quad j = 3, 4, \ldots 9 .$$

$$u_{61} = \frac{\Sigma_s}{\Sigma_f} \left\{ L^2 (B_r^2 + \beta^2 - \gamma^2) + 1 + \ell_s \omega \right\}$$

$$u_{62} = 2L^2 \frac{\Sigma_s}{\Sigma_f} \gamma\beta ,$$

$$v_{61} = -2L^2 \frac{\Sigma_s}{\Sigma_f} \gamma\beta ,$$

$$v_{62} = \frac{\Sigma_s}{\Sigma_f} \left\{ L^2 (B_r^2 + \beta^2 - \gamma)^2 + 1 + \ell_s \omega \right\} ,$$

$$w_{6j} = \frac{\Sigma_s}{\Sigma_f} \left\{ L^2 (B_r^2 - a_j^2) + 1 + \ell_s \omega \right\} ; \quad j = 3, 4, \ldots 9 . \qquad (8)$$

A set of boundary conditions on the fluxes suitable for heavy element fuels like U-Bi is that the fluxes vanish at suitable extrapolated boundaries:

$$\phi_f (z = 0) = \phi_f (z = a) = 0$$
$$\phi_s (z = 0) = \phi_f (z = a) = 0 . \qquad (9a)$$

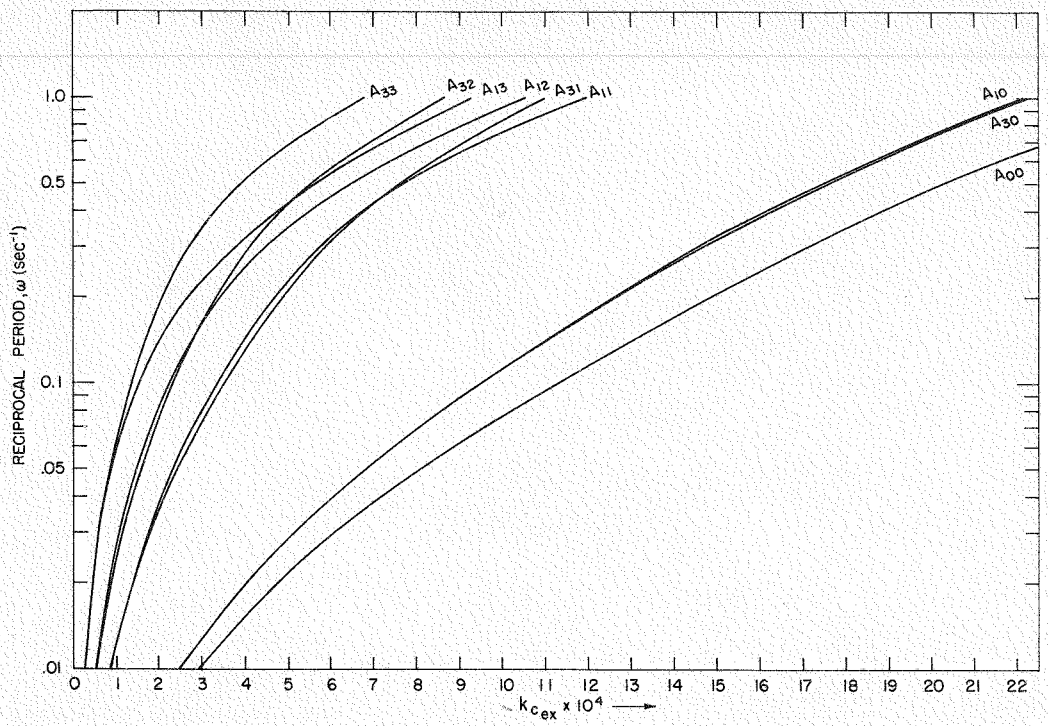

Figure 1. Eigenvalue curves for U^{233} for core transit time t_o = 5, .5, corresponding to first index equal to 1 and 3, respectively. Second indices 0, 1, 2, 3 indicate ratio a/(a + h) or fraction of time spent in core equal to 0, 1/3, 1/5, 1/11, respectively.

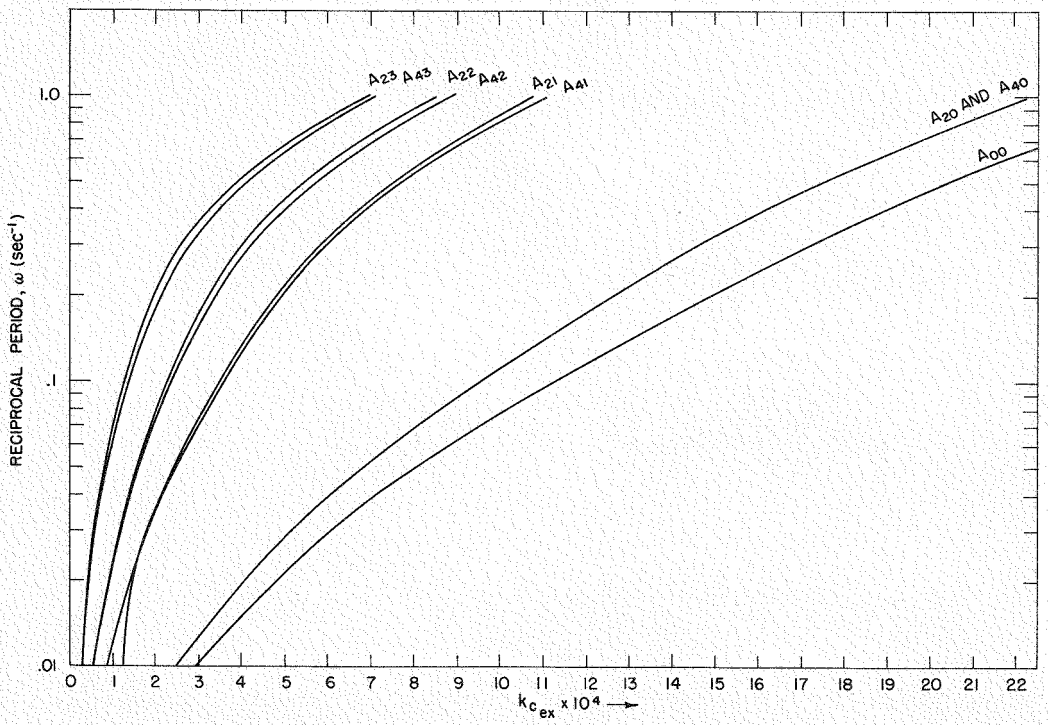

Figure 2. Eigenvalue curves for U^{233} (cont'd) for t_o = 1, .1, or first index equal to 2 and 4, respectively.

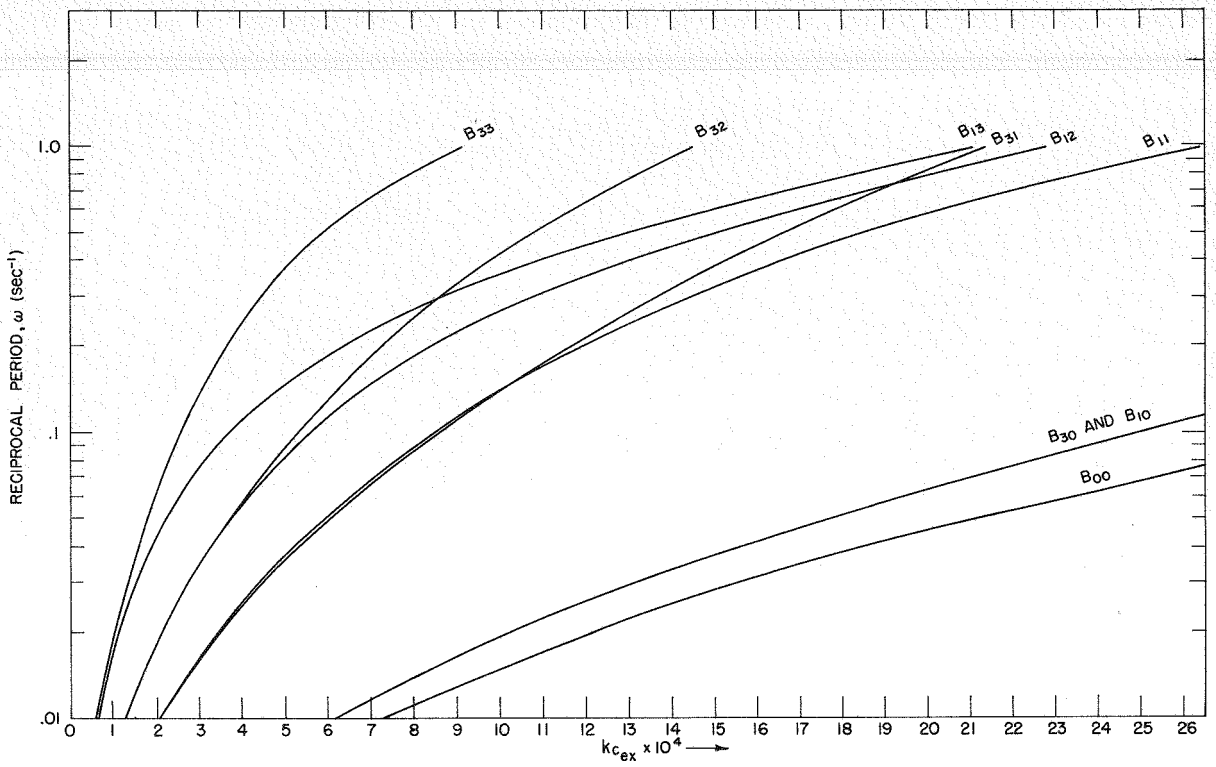

Figure 3. Eigenvalue curves for U^{235}, t_o = 5, .5.

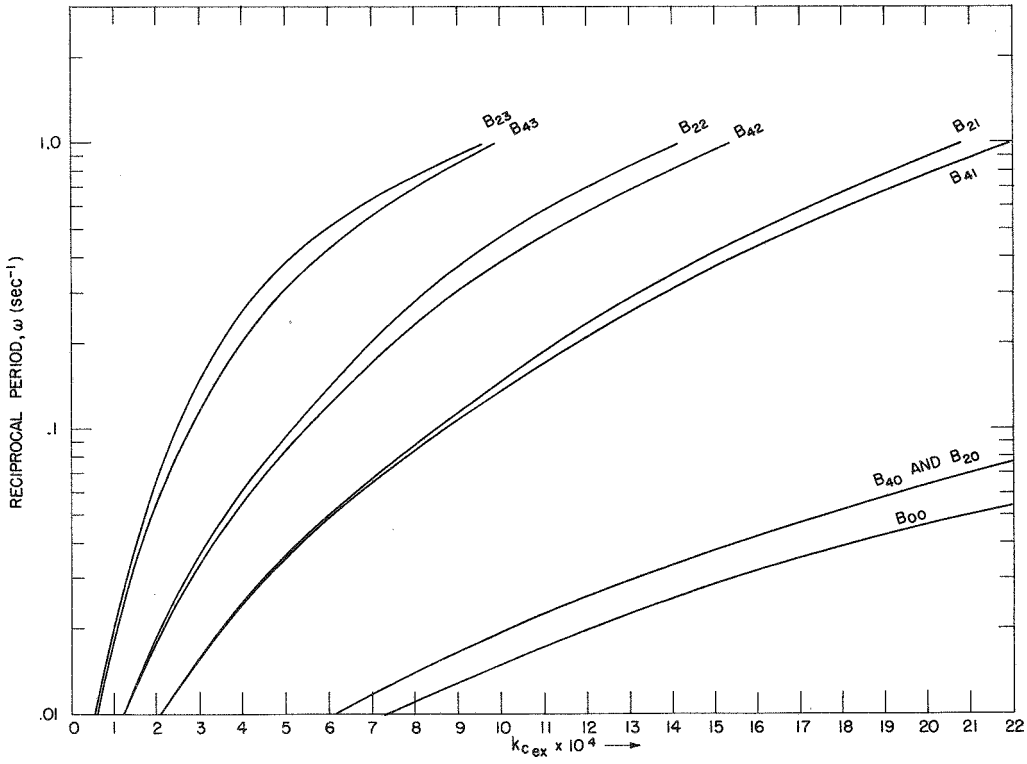

Figure 4. Eigenvalue curves for U^{235} (cont'd), t_o = 1, .1.

The boundary conditions on the c_i are obtained by requiring that they be continuous for each ω at the core inlet and outlet boundaries. Since equation (1d) has a solution of the form $c_i = c_i\, e^{\omega t - (\omega + \lambda_i)\, z/q}$, this boundary condition is expressed by the relation

$$c_i\,(z = 0) = c_i\,(z = a)\, e^{-(\omega + \lambda_i)\, h/q};\quad i = 1, 2, \ldots 5. \tag{9b}$$

Substitution of equations (6) and (7) into the boundary conditions (9) yields the following nine homogeneous equations:

$$A_2 + \sum_{j=3}^{9} A_j = 0$$

$$e^{\gamma a} \sin \beta a A_1 + e^{\gamma a} \cos \beta a A_2 + \sum_{j=3}^{9} e^{\alpha_j a} A_j = 0$$

$$v_{61} A_1 + v_{62} A_2 + \sum_{j=3}^{9} w_{6j} A_j = 0$$

$$(u_{61} e^{\gamma a} \sin \beta a + v_{61} e^{\gamma a} \cos \beta a) A_1 + (u_{62} e^{\gamma a} \sin \beta a + v_{62} e^{\gamma a} \cos \beta a) A_2 + \sum_{j=3}^{9} w_{6j} e^{\alpha_j a} A_j = 0$$

$$\vdots$$

$$\left\{v_{i1} - e^{\gamma a - s_i h}(u_{i1} \sin \beta a + v_{i1} \cos \beta a)\right\} A_1 + \left\{v_{i2} - e^{\gamma a - s_i h}(u_{i2} \sin \beta a + v_{i2} \cos \beta a)\right\} A_2$$

$$+ \sum_{j=3}^{9} w_{ij}(1 - e^{\alpha_j a - s_i h}) A_j = 0;\quad i = 1, 2, \ldots 5. \tag{10}$$

Here $s_i = (\omega + \lambda_i)/q$. The determinant of the resulting equations must be 0:

$$\begin{vmatrix} 0 & 1 \ldots\ldots\ldots\ldots\ldots 1 \\ e^{\gamma a} \sin \beta a & e^{\gamma a} \cos \beta a \ldots\ldots\ldots e^{\alpha_9 a} \\ v_{61} & v_{62} \ldots\ldots\ldots\ldots w_{69} \\ e^{\gamma a}(u_{61} \sin \beta a + v_{61} \cos \beta a) & e^{\gamma a}(u_{62} \sin \beta a + v_{62} \cos \beta a) \ldots w_{69} e^{\alpha_9 a} \\ \left\{v_{51} - e^{\gamma a - s_i h}(u_{61} \sin \beta a + v_{61} \cos \beta a)\right\} & \left\{u_{52} - e^{\gamma a - s_i h}(u_{62} \sin \beta a + v_{62} \cos \beta a)\right\} \\ & \ldots\ldots w_{69}(1 - e^{\alpha_9 a - s_i h}) \end{vmatrix} = 0 \tag{11}$$

Equations (4) and (11) define an infinite set of discrete eigenvalues ω in terms of the reactor parameters. For small reactivities, one of these values will be positive and the remaining values negative. It is possible in principle to satisfy any set of initial conditions by use of a suitable linear combination of the eigenfunctions (6) and (7) associated with the various values of ω. The eigenfunctions with negative exponentials, however, will damp out, so that the fluxes and emitters will behave asymptotically like $e^{\omega t}$, where ω is a positive number, the reactor's reciprocal period.

The procedure for computation is the following: ω is first set equal to zero; then with $k = k_o + \epsilon_o$, where k_o is the infinite multiplication necessary to make the reactor critical without circulation, equations (4) and (11) are solved simultaneously for ϵ_o. Once ϵ_o has been determined, we put $k = k_o + \epsilon_o + \epsilon$, and solve (4) and (11) simultaneously for ω for each value of ϵ_1. Results of computations made with the Goetzel-Keller Univac code are given in Figures 1 to 4 for a reactor having the fixed parameters listed in the Appendix. The plots are of ω vs. $k_{c_{ex}}$, where $k_{c_{ex}}$, the "circulating" reactivity, is defined in terms of reactivities for the stationary reactor system:

$$k_{c_{ex}} = \frac{k_o + \epsilon_o + \epsilon_1}{(1 + L^2B^2)(1 + \tau B^2)} - \frac{k_o + \epsilon_o}{(1 + L^2B^2)(1 + \tau B^2)} . \tag{12}$$

The stationary reactivity of a circulating fuel reactor we define as that reactivity which the system would have if the fuel were suddenly made stationary. The circulating reactivity, $k_{c_{ex}}$, is thus the stationary reactivity of the system less the critical stationary reactivity, i.e., the reactivity which must be added to the stationary fuel reactor to maintain criticality when its fuel is caused to circulate.

The dependence of L^2 on k was introduced into the Univac code. The relationship chosen was the following:

$$L^2 = L_o^2 \left(1 - \frac{\epsilon + \epsilon_o}{\eta - k_o}\right). \tag{13}$$

Here, L_o^2 is the diffusion area for the critical stationary reactor. However, the dependence of ω on $k_{c_{ex}}$ should be fairly insensitive to the form of L^2 chosen.

Generalized Inhour Equations

Calculation by the eigenvalue method is obviously a complex operation, hence a simple functional relationship between the reactor period and reactivity would be highly useful. We shall now derive two such functional relationships, one for a sinusoidal flux distribution, the other for a flat flux distribution.

If displacement is measured in units of q, the solution to equation (1c) in terms of $\phi_s(z,t)$, valid for large times and satisfying the condition of continuity with the solution of (1d) at the reactor boundaries, is

$$c_i = \bar{\beta}_i k \Sigma_s \left\{ \sum_{m=i}^{\infty} e^{-m(t_o + t_1)\lambda_i} \int_0^{t_o} \phi_s \left(z', t - (z - z') - m(t_o + t_1)\right) e^{\lambda_i(z' - z)}\, dz' \right.$$

$$\left. + \int_0^{z} \phi_s \left(z', t - (z - z')\right) e^{\lambda_i(z' - z)}\, dz' \right\}. \tag{14}$$

Here $t_o = a/q$, 0 and $t_1 = h/q$ are the transit times through the core and external loop respectively. Equation (14) can be derived from equations (1c) and (1d) by the application of Laplace transform or simply by summing the contributions to the emitter density from behind and in front of, respectively, the point in question for all passes from time 0 to the time in question, which is assumed to be large. The contribution from initial conditions decays exponentially with time and can therefore be ignored for large times.*

*For short times, c_i has a different form according to whether $N(t_o+t_1)<t<N(t_o+t_1)+z$; $N(t_o+t_1)<t<N(t_o+t_1)+t_o$; $N(t_o+t_1)+t_o<t<N(t_o+t_1)+t_1$; $N(t_o+t_1)+t_1<t<(N+1)(t_o+t_1)$; N being an integer. However, for large times these differences become insignificant, and c_i is accurately represented by the asymptotic form (14).

With the change of variable, $u = z - z'$, equation (14) becomes

$$c_i = \bar{\beta}_i k \Sigma_s \left\{ \sum_{m=1}^{\infty} e^{-m(t_o + t_1)\lambda_i} \int_{z-t_o}^{z} \phi_s \left(z-u,\ t-u-m(t_o + t_1)\right) e^{-\lambda_i u}\, du \right.$$

$$\left. + \int_0^z \phi_s(z-u,\ t-u)\, e^{-\lambda_i u}\, du \right\} . \tag{15}$$

For convenience, let equations (1a) and (1b) be written in the following forms:

$$\tau \left\{ \frac{\partial^2 \phi_f}{\partial z^2} + \left(\frac{\pi}{t_o}\right)^2 \phi_f \right\} - (\tau B^2 + 1)\phi_f + k(1 - \bar{\beta}) \frac{\Sigma_s}{\Sigma_f} \phi_s + \frac{1}{\Sigma_f} \sum_{i=1}^{5} \lambda_i c_i = \ell_f \frac{\partial \phi_f}{\partial t} \tag{16a}$$

$$L^2 \left\{ \frac{\partial^2 \phi_s}{\partial z^2} + \left(\frac{\pi}{t_o}\right)^2 \phi_s \right\} - (L^2 B^2 + 1)\phi_s + \frac{\Sigma_f}{\Sigma_s} \phi_f = \ell_s \frac{\partial \phi_s}{\partial t} . \tag{16b}$$

Here we have added and subtracted $\tau(\pi/t_o)^2$ and $L^2(\pi/t_o)^2$ from the left-hand members of equations (16a) and (16b), respectively; $B^2 = B_r^2 + (\pi/t_o)^2$ is the geometrical buckling of the reactor.

One way to solve equations (16a) and (16b) for ϕ_s and ϕ_f would be to expand them in Fourier series:

$$\phi_s = \sum_{j=1}^{\infty} \phi_{s_j}(t) \sin \frac{j\pi}{t_o} z \tag{17a}$$

$$\phi_f = \sum_{j=1}^{\infty} \phi_{f_j}(t) \sin \frac{j\pi}{t_o} z . \tag{17b}$$

(For convenience we omit the dependence on r.) Equation (17a) would then be substituted into equation (15), and c_i, as given by equation (15), would in turn be substituted into equation (16a). Upon multiplying equations (16a) and (16b) by $\sin j\pi z/t_o$ for all values of j and integrating from 0 to t_o, we would then obtain a set of integro-differential equations for the Fourier coefficients ϕ_{s_j} and ϕ_{f_j}, with time the only dependent variable. Each equation would contain all the Fourier coefficients. For purposes of calculation, we should have to truncate the flux expansions. We shall be concerned here only with the approximation in which the first harmonic is retained.

In this case c_i becomes

$$c_i = \bar{\beta}_i k \Sigma_s \left\{ \sum_{m=1}^{\infty} e^{-m(t_o + t_1)\lambda_i} \int_{z-t_o}^{z} \phi_{s_1} \left(t-u-m(t_o + t_1)\right) \sin \frac{\pi}{t_o}(z - u)\, e^{-\lambda_i u}\, du \right.$$

$$\left. + \int_0^z \phi_{s_1}(t - u) \sin \frac{\pi}{t_o}(z - u)\, e^{-\lambda_i u}\, du \right\} . \tag{18}$$

Equation (18) is next substituted into (16a), and the expansions (17a) and (17b) are truncated at the first term into (16a) and (16b). We then multiply (16a) and (16b) by

$\sin(\pi/t_o)z$ and integrate from 0 to t_o. Making use of the following reversals of integration order

$$\int_0^{t_o} \sin\frac{\pi z}{t_o}\,dz \int_0^{z} \phi\left(t-u-m(t_o+t_1)\right) \sin\frac{\pi}{t_o}(z-u)\,du$$

$$=\int_0^{t_o} \phi\left(t-u-m(t_o+t_1)\right) du \int_u^{t_o} \sin\frac{\pi z}{t_o} \sin\frac{\pi}{t_o}(z-u)\,du\,,$$

$$\int_0^{t_o} \sin\frac{\pi z}{t_o}\,dz \int_{z-t_o}^{0} \phi\left(t-u-m(t_o+t_1)\right) \sin\frac{\pi}{t_o}(z-u)\,du$$

$$=\int_0^{t_o} \phi\left(t+u-m(t_o+t_1)\right) du \int_0^{t_o-u} \sin\frac{\pi z}{t_o} \sin\frac{\pi}{t_o}(z+u)\,du \qquad (19)$$

and eliminating ϕ_{1_f} in terms of ϕ_{1_s}, we obtain

$$\ell\frac{d\phi_{s_1}}{dt} = \left(k_{eff}\ (1-\bar{\beta}) - 1\right)\phi_{s_1} + k_{eff}\sum_{i=1}^{5}\lambda_i\bar{\beta}_i\left\{\sum_{m=0}^{\infty} e^{-m(t_o+t_1)}\right.$$

$$\int_0^{t_o} \phi_{s_1}\left(t-u-m(t_o+t_1)\right) K(u)\, e^{-\lambda_i u}du$$

$$\left. + \sum_{m=1}^{\infty} e^{-m(t_o+t_1)}\int_0^{t_o} \phi_{s_1}\left(t+u-m(t_o+t_1)\right) K(u)\, e^{\lambda_i u}\,du\right\}. \qquad (20)$$

Here $\ell = \ell_s/(1+L^2B^2) + \ell_f/(1+\tau B^2)$ is the sum of the effective slow and fast neutron lifetimes, $k_{eff} = k/(1+L^2B^2)\,(1+\tau B^2)$ is the effective multiplication of the stationary reactor, and the kernel K(u) is given by

$$K(u) = \cos\frac{\pi u}{t_o}\ \frac{(t_o-u)}{t_o} + \frac{1}{\pi}\sin\frac{\pi u}{t_o}. \qquad (21)$$

In deriving equation (20), we have neglected a term $(d^2\phi_{s_1}/dt^2)\cdot \ell_s\ell_f/(1+\tau B^2)\,(1+L^2B^2)$. This term should be negligible for any practical range of reactivity. Equation (20) is generally valid for any number of groups so long as ℓ is the sum of the effective lifetimes for all groups and only terms to the first order in ℓ are retained.

The derivation of equation (20) can be generalized nonrigorously to include a flux of arbitrary shape. The flux distributions ϕ_s and ϕ_f in the first harmonic approximation are determined by the curly bracket terms in equations (16) and satisfy

$$\left\{\frac{\partial^2}{\partial z^2} + (\pi/t_o)^2\right\}\phi_i = 0,\quad i = f,\ s\ . \qquad (22)$$

Let us consider replacing the sinusoidal flux distribution operator in equation (22) by some sort of generalized flux distribution operator $\underset{\sim}{L}$, such that the generalized flux distribution ψ satisfies

$$\underset{\sim}{L}\psi = 0 \ . \tag{23}$$

Then equation (20) can be derived by substitution of $\psi(z)$ for $\sin \pi z/t_o$ in the previous derivation. Only the kernel K(u) will be changed and will be determined by replacing $\sin \pi z/t_o$ by $\psi(z)$ in equation (19). For example, for a flat flux K(u) takes the form

$$K(u) = \frac{1}{t_o}\int_u^{t_o} dz = \frac{1}{t_o}\int_0^{t_o-u} dz = \frac{t_o - u}{t_o}. \tag{24}$$

Equation (20) is satisfied by a solution of the form

$$\phi_{1_s} = S\, e^{\omega t} \ . \tag{25}$$

By substitution, it can be verified that ω must satisfy

$$k_{eff} - 1 = \ell\omega + k_{eff}\,\bar{\beta} - k_{eff}\sum_{i=1}^{5}\lambda_i\bar{\beta}_i$$

$$\left[\frac{(\lambda_i + \omega)}{(\lambda_i + \omega)^2 + (\pi/t_o)^2} + \frac{2}{t_o}\,\frac{\left(1 + e^{-t_1(\lambda_i + \omega)}\right)\left(1 + e^{-(t_o + t_1)(\lambda_i + \omega)}\right)}{\left[(\lambda_i + \omega)^2 + (\pi/t_o)^2\right]^2\ \left(1 - e^{-(t_o + t_1)(\lambda_i + \omega)}\right)}\right] \tag{26}$$

for a sinusoidal flux and

$$k_{eff} - 1 = \ell\omega + k_{eff}\,\bar{\beta} - k_{eff}\sum_{i=1}^{5}\lambda_i\bar{\beta}_i\left[\frac{1}{(\lambda_i + \omega)} - \frac{\left(1 - e^{-(\lambda_i + \omega)t_o}\right)\left(1 - e^{-(\lambda_i + \omega)t_1}\right)}{t_o(\lambda_i + \omega)^2\left(1 - e^{-(t_o + t_1)(\lambda_i + \omega)}\right)}\right]$$

for a flat flux.

Equations (26) and (27) express $k_{eff} - 1 = k_{ex}$, the stationary reactivity of the system, in terms of ω. To obtain the dependence of $k_{c_{ex}}$, the circulating reactivity, on ω we must subtract from the left-hand sides of (26) and (27) the amount of reactivity (k_{ex}) necessary to keep the circulating system just critical. This is obtained by setting $\omega = 0$ in (26) and (27) and computing k_{ex}.

The following limiting cases are of interest:

$$t_1,\ t_o \longrightarrow \infty$$

and

$$t_1,\ t_o \longrightarrow 0$$

in such a way that t_o/t_1 remains a constant, namely a/h. The first is the case of no circulation, and it is easily verified that (26) and (27) reduce to the conventional inhour equation*

$$k_{ex} = \ell\omega + k_{eff}\sum_{i=1}^{5}\frac{\omega\beta_i}{\omega + \lambda_i} \ . \tag{28}$$

*See for example: Glasstone and Edlund, The Elements of Nuclear Reactor Theory, p. 298, Van Nostrand Co., Inc., New York, 1952.

The second is the case of infinite circulation velocity, whereby (26) and (27) reduce to the form

$$k_{c_{ex}} = \ell\omega + k_{eff}\,\bar{\gamma} \sum_{i=1}^{5} \frac{\omega\beta_i}{\omega + \lambda_i} . \tag{29}$$

Here

$$\bar{\gamma} = \frac{8}{\pi^2} \frac{a}{a + h}$$

for a sinusoidal flux, and for a flat flux distribution

$$\bar{\gamma} = a/a + h .$$

In other words $k_{c_{ex}}$ and ω are related by an inhour equation in which the delayed neutron fractions are multiplied by a number, γ, roughly or exactly equal to the fraction of time spent in the core by the fuel for a sinusoidal and flat flux, respectively. At intermediate velocities, the interpretation is not quite so simple.

Let us examine the flat flux inhour equation (27) more closely. If we substitute

$$\phi_s = Se^{\omega t}$$

in equation (15), we obtain for c_i in the core and external loop, respectively,

$$c_i = \frac{S\bar{\beta}_i k\,\Sigma_s}{(\lambda_i + \omega)\left(1 - e^{-(t_o + t_1)(\lambda_i + \omega)}\right)} \left\{ \left(1 - e^{-(\lambda_i + \omega)z}\right) + \left(e^{-(t_o + t_1)(\lambda_i + \omega)}\right)\left(e^{-(\lambda_i + \omega)(z - t_o)} - 1\right) \right\} \tag{30a}$$

$$c_i = \frac{S\bar{\beta}_i k\,\Sigma_s\, e^{-(\lambda_i + \omega)z}\left(1 - e^{-(\lambda_i + \omega)a}\right)}{(\lambda_i + \omega)\left(1 - e^{-(t_o + t_1)(\lambda_i + \omega)}\right)} . \tag{30b}$$

The number of delayed neutron emitters in the core is proportional to the integral of (30a) from 0 to t_o, and the number in the external loop is proportional to the integral of (30b) from t_o to $t_o + t_1$. From these numbers the fraction of delayed neutron emitters of the i^{th} kind in the core is readily obtained:

$$\bar{\gamma}_i(\omega) = \frac{c_i\,(\text{core})}{c_i\,(\text{core}) + c_i\,(\text{loop})} = 1 - \frac{\left(1 - e^{-t_1(\lambda_i + \omega)}\right)\left(1 - e^{-t_o(\lambda_i + \omega)}\right)}{t_o(\lambda_i + \omega)\left(1 - e^{-(t_o + t_1)(\lambda_i + a)}\right)} . \tag{31}$$

If we first subtract the critical stationary reactivity from the right-hand side, equation (27) can now be written in the form

$$k_{c_{ex}} = \ell\omega + k_{eff}\,\bar{\beta}\left[1 - \sum_{i=1}^{5} \frac{\left(1 - e^{-t_1\lambda_i}\right)\left(1 - e^{-t_o\lambda_i}\right)}{t_o\lambda_i\left(1 - e^{-(t_o + t_1)\lambda_i}\right)}\right]$$

$$- k_{eff} \sum_{i=1}^{5} \frac{\lambda_i \beta_i}{\lambda_i + \omega} \left[1 - \frac{\left(1 - e^{-t_1(\lambda_i + \omega)}\right)\left(1 - e^{-t_o(\lambda_i + \omega)}\right)}{t_o(\lambda_i + \omega)\left(1 - e^{-(t_o + t_1)(\lambda_i + \omega)}\right)} \right] \tag{32}$$

If $\omega << \lambda_i$, then equation (32) reduces to the form

$$k_{c_{ex}} = \ell\omega + k_{eff} \sum_{i=1}^{5} \frac{\omega \bar{\beta}_i \bar{\gamma}_i(0)}{\omega + \lambda_i} . \tag{33}$$

Thus the system behaves exactly like a stationary system, except that the delayed neutron fractions $\bar{\beta}_i$ have been reduced to $\bar{\beta}_i \bar{\gamma}_i(0)$, $\bar{\gamma}_i(0)$ being the steady state fraction of delayed neutron emitters in the core. As soon as ω becomes commensurate with the λ_i of most delayed neutron emitters, equation (33) must be replaced by

$$k_{c_{ex}} = \ell\omega + k_{eff} \sum_{i=1}^{5} \frac{\omega \bar{\beta}_i \bar{\gamma}_i(0)}{\omega + \lambda_i}$$

$$- k_{eff} \sum_{i=1}^{5} \frac{\lambda_i \bar{\beta}_i}{\lambda_i + \omega} \left(\bar{\gamma}_i(\omega) - \bar{\gamma}_i(0) \right) . \tag{34}$$

Since $\bar{\gamma}_i(\omega)$ increases monotonically with ω, the third term on the right-hand side of (34) is always negative, hence equation (33) overestimates the value of $k_{c_{ex}}$ associated with a given positive ω. The physical situation is this. While the flux is increasing, there are more delayed neutrons in the system than would be predicted by the fraction $\bar{\beta}_i \bar{\gamma}_i(0) k_{eff}$ because the mean lifetime for the emission of delayed neutrons has been reduced from $1/\lambda_i$ to $1/(\lambda_i + \omega)$. The number of additional delayed neutrons in the core per neutron absorbed is

$$k_{eff} \sum_{i=1}^{5} \frac{\lambda_i \bar{\beta}_i}{\lambda_i + \omega} \left(\bar{\gamma}_i(\omega) - \bar{\gamma}_i(0) \right) .$$

As the flux increases, the circulating fuel reactor simply loses less reactivity to the external loop than when it is just critical. The system thus behaves as though it contained an additional latent reactivity whenever the power rises.

We can conclude, in summary, that the use of effective delayed neutron fractions $\gamma_i(0)\bar{\beta}_i$ in a conventional inhour expression is permissible only when periods are large (small ω) or when the flow is so rapid that the $\bar{\gamma}_i$ are insensitive to delayed neutron emitter lifetimes. In this case all values of $\bar{\gamma}_i = \bar{\gamma}$.

In Figure 6, $k_{c_{ex}}$ is plotted (dotted lines) as a function of ω using equation (34) for $t_o = 1$ (sec) and $t_o = 0$. (infinite circulation velocity) with $a/(a + h) = .25$. The curves exhibit the interesting feature that ω rises more rapidly with increasing $k_{c_{ex}}$ for a finite circulation velocity than for an infinite one. While the value of $\bar{\gamma}_i(0)\bar{\beta}_i$ is more favorable for the finite velocity case than for the infinite, the latent delayed neutrons tip the balance in favor of a steeper curve for the former.

The variation of statistical weight with position in a sinusoidal flux reactor further complicates the dependence of ω on $k_{c_{ex}}$, which must now depend on the exact shape of the delayed neutron emitter distribution. The factor $8/\pi^2$ in the value of γ for a sinusoidal flux is the result of a sizable proportion of the delayed neutrons being emitted in regions of low statistical weight.

Numerical Results

The results of the Univac eigenvalue calculations are given in Figures 1 through 4. Each curve is labeled A_{mn} or B_{mn} according to whether the fuel is U^{233} or U^{235}. The subscript m defines the value of t_o according to

$$\begin{aligned} t_o &= \infty, \quad m = 0 \\ t_o &= 5, \quad m = 1 \\ t_o &= 1, \quad m = 2 \\ t_o &= .5, \quad m = 3 \\ t_o &= .1, \quad m = 4 \end{aligned}$$

and the subscript n defines the ratio $t_1/t_o = h/a$ in accordance with:

$$\begin{aligned} t_1/t_o &= 0, \quad n = 0 \\ t_1/t_o &= 2, \quad n = 1 \\ t_1/t_o &= 4, \quad n = 2 \\ t_1/t_o &= 10, \quad n = 3. \end{aligned}$$

It is necessary in carrying through a numerical calculation to assume a complete set of reactor parameters*: k_o, τ, L^2, B^2, a, B_r^2, etc., so that ω is a unique function of reactivity and transit times for any given reactor, although this function is probably not oversensitive to small changes in the reactor parameters. The values of the parameters selected for the calculations in this report are taken from the Brookhaven graphite moderated, U-Bi design (LMFR) and are listed in the Appendix.

The increments in reactivity necessary to maintain criticality, the critical stationary reactivities, are listed in Table 1 for the various cases.

Some of the curves, for example those in Figures 2 and 4, appear to be inverted, i.e., the lower velocity curve rises more steeply than does its higher velocity counterpart. These inversions can be explained by the same arguments applied in the previous section to the flat flux inhour equation. Namely, although the lower velocity reactor has the greater steady state delayed reactivity, which might be thought of as determining the neutron generation time, it also receives more latent delayed neutrons during a transient than does the higher velocity reactor. The latter effect outweighs the former, so that the curve for lower velocity rises more rapidly. As soon as the differences in steady state delayed reactivity become more important, the position of the curves is reversed as in Figure 1.

Figures 5 and 6 compare plots of the generalized inhour equations (26) and (27) (dotted lines) with eigenvalue plots (solid lines) for the cases B_{22} and B_{42}. In Figure 5 the B_{42} curves fit reasonably well, but the B_{22} curves do not. It was pointed out in the

*The importance of the various reactor parameters is that they determine the distribution of the thermalized delayed neutrons and, consequently, their relative effectiveness. More will be said on this subject shortly.

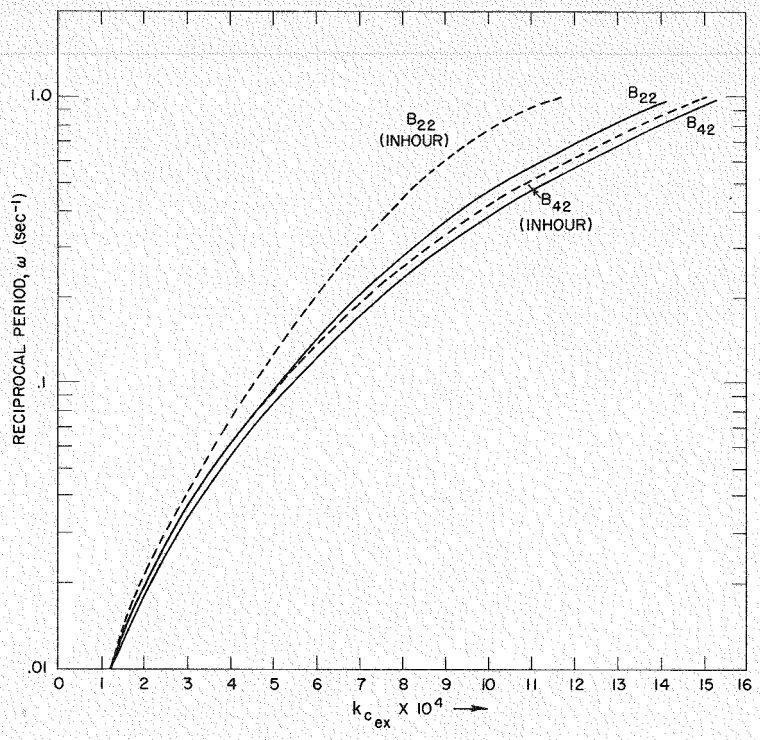

Figure 5. Comparison between sinusoidal flux inhour equation (dotted lines) and eigenvalue method (solid lines) for cases B_{22} and B_{42} or t_o = 1, .1, a/(a + h)=1/5.

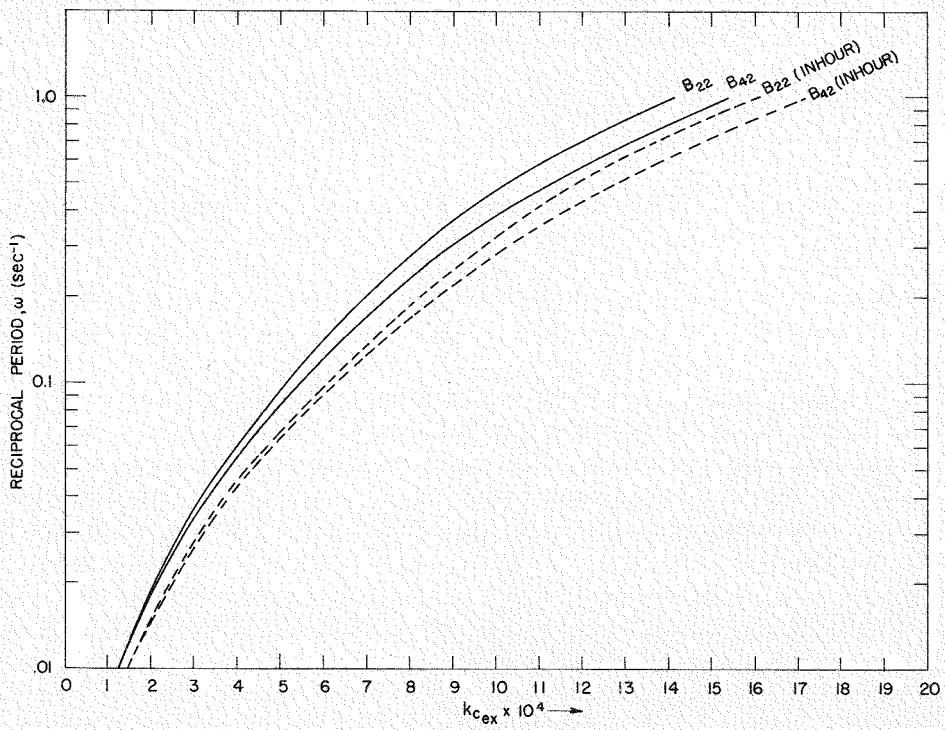

Figure 6. Comparison between flat flux inhour equation (dotted lines) and eigenvalue method (solid lines) for cases B_{22} and B_{42} or t_o = 1, .1, a/(a + h) = 1/5.

previous section that the kinetic equation derived in the first harmonic approximation is independent of the number of groups and does not depend on the reactor parameters other than reactivity and neutron generation time. This approximation can thus take no account of the smoothing out of the delayed neutron distribution which occurs in slowing down. In the $t_o = 1$ case the emitter distribution is concentrated near the core exit, a region of low statistical weight. In the first harmonic approximation, delayed neutrons are absorbed at the point of emission. Hence the average effectiveness of the delayed neutrons is less than for the $t_o = 1$ case, in which the emitter distribution is

Table 1

Values of k_{ex} to maintain criticality

Case A				
m	n = 0	n = 1	n = 2	n = 3
1	3.70441×10^{-4}	1.40363×10^{-3}	1.53983×10^{-3}	1.61661×10^{-3}
2	3.91478×10^{-4}	1.69866×10^{-3}	1.93462×10^{-3}	2.10125×10^{-3}
3	3.92244×10^{-4}	1.73494×10^{-3}	1.99152×10^{-3}	2.18597×10^{-3}
4	3.92429×10^{-4}	1.75077×10^{-3}	2.02186×10^{-3}	2.24190×10^{-3}
Case B				
m	n = 0	n = 1	n = 2	n = 3
1	1.05407×10^{-3}	3.91964×10^{-3}	4.27604×10^{-3}	4.47562×10^{-3}
2	1.16455×10^{-3}	5.16883×10^{-3}	5.85229×10^{-3}	6.30632×10^{-3}
3	1.17144×10^{-3}	5.34187×10^{-3}	6.12018×10^{-3}	6.68556×10^{-3}
4	1.17392×10^{-3}	5.41756×10^{-3}	6.26414×10^{-3}	6.94859×10^{-3}

uniform. The spread between the inverted curves, therefore, is accentuated. The two-group eigenvalue method will yield for $t_o = 1$ a more uniform distribution of thermalized delayed neutrons, perhaps not very different from the uniform distribution for $t_o = .1$. Hence the spread between the eigenvalue curves is similar to the one between the curves in Figure 6 calculated for a flat flux.

Naturally the flat flux curves will always rise more slowly than their eigenvalue counterparts, since the former take no account of the positional variation of the effectiveness of delayed neutrons. Interestingly, equation (32) can be "fudged" to give results in very close agreement with the eigenvalue method. The fudging consists of expressing $\bar{\gamma}$ in equation (27) as $\bar{\gamma} = xa/(a + h)$ and determining the value of x which gives the closest fit with the eigenvalue curve for infinite velocity ($t_o = .1$ will do). The $\bar{\beta}_i$ terms in equation (32) are then each multiplied by x. The resulting curves of ω vs. $k_{c_{ex}}$ for finite circulation velocities are in very close agreement with their eigenvalue counterparts. This fudge would be useful in a determination of the kinetics for a reactivity varying with time, in which case the eigenvalue method would be of no avail and the integro-differential equation (20) would have to be used.

Acknowledgement

The author wishes to express appreciation to Herbert Goertzel and Herbert Keller of the AEC Computing Center at New York University for their fine work in coding the eigenvalue method for Univac, and to Jack Chernick for illuminating discussions.

APPENDIX OF PARAMETERS

Parameters which are fixed throughout the computation are given below:

$a = 160$ cm

$B_r^2 = 0.0009$ cm^2

$k_o = 2.1459$

$\tau = 550$ cm^2

$L^2 = 200$ cm^2

$\Sigma_f = 0.00128$ cm^{-1}

$\Sigma_s = 0.01$ cm^{-1}

$\ell_s = 4.03 \times 10^{-4}$ sec

$\ell_f = 4.1 \times 10^{-4}$ sec

The values of $\bar{\beta}_i$, $\bar{\beta}$, λ_i are given below for U^{233} and U^{235}, respectively:

(a) $\bar{\beta}_1 = 1.83 \times 10^{-4}$ $\lambda_1 = 0.01247$

$\bar{\beta}_2 = 5.82 \times 10^{-4}$ $\lambda_2 = 0.03151$

$\bar{\beta}_3 = 8.57 \times 10^{-4}$ $\lambda_3 = 0.1536$

$\bar{\beta}_4 = 6.22 \times 10^{-4}$ $\lambda_4 = 0.4566$

$\bar{\beta}_5 = 1.82 \times 10^{-4}$ $\lambda_5 = 1.639$

$\Sigma_i \bar{\beta}_i = \bar{\beta} = 0.00243$

(b) $\bar{\beta}_1 = 2.60 \times 10^{-4}$ $\lambda_1 = 0.01253$

$\bar{\beta}_2 = 17.26 \times 10^{-4}$ $\lambda_2 = 0.03115$

$\bar{\beta}_3 = 22.81 \times 10^{-4}$ $\lambda_3 = 0.1548$

$\bar{\beta}_4 = 24.45 \times 10^{-4}$ $\lambda_4 = 0.5263$

$\bar{\beta}_5 = 8.50 \times 10^{-4}$ $\lambda_5 = 1.639$

$\Sigma_i \bar{\beta}_i = \bar{\beta} = .00756$

List of Symbols

a = core length

B^2 = geometrical buckling

B_r^2 = radial geometrical buckling

c_i = density of delayed neutron emitters of the i^{th} group

h = length of external loop

k = infinite multiplication

k_o = value of k which makes stationary reactor critical

k_{eff} = effective multiplication of stationary system

k_{ex} = stationary reactivity

$k_{c_{ex}}$ = circulating reactivity

L^2 = thermal diffusion length

L_o^2 = value of L^2 for critical stationary reactor

ℓ_s = infinite medium slow neutron lifetime

ℓ_f = infinite medium fast neutron lifetime

ℓ = effective neutron lifetime equal to sum of fast and slow neutron lifetimes

q = circulation velocity

$s_i = (\omega + \lambda_i)/q$

t = time

t_o = transit time through core

t_1 = transit time through external loop

z = displacement along core axis

α_j = real roots to equation (4)

$\gamma \pm i\beta$ = complex roots to equation (4)

ϵ = increment in k

ϵ_o = increment in k necessary to make reactor critical under circulation conditions

$\bar{\gamma}_i$ = fraction of delayed neutrons present in core

$\bar{\gamma}$ = effectiveness of delayed neutrons at infinite circulation velocity

η = fast neutrons produced per thermal capture

Σ_f = fast neutron slowing down cross section

Σ_s = thermal absorption cross section

τ = neutron age

ω = reciprocal period