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PHYSICS

GAS DISCHARGES

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I. INTRODUCTION

Most books on the subject of gas discharges describe the multiple banded structure which appears in the glow between two electrodes. This structure is actually due to the difficulty in getting electrons out of the cathode to support the discharge. These electrons are obtained by having a large enough potential drop near the cathode so that the positive ions may strike it hard enough to produce secondary electrons. If one substitutes a hot cathode to produce a copious quantity of electrons, the banded structure will be radically changed. Hence the banding is not a pure property of the discharge, but is a complication brought on by the presence of electrodes.

For these reasons, gas discharges may be more easily studied without the presence of electrodes. Such discharges are produced in resonant cavities excited by radio frequency fields, or in a torus threaded by a changing magnetic field.

These lectures will be limited to a) discharges in a torus, produced by a dB/dt through the center, and b) discharges produced in hydrogen gas.

II. MATHEMATICAL TREATMENT

In describing the torroidal discharge mathematically we will make the modification to an infinitely long cylinder, radius r_1, with an electric field E uniform from $-\infty$ to $+\infty$. With these assumptions

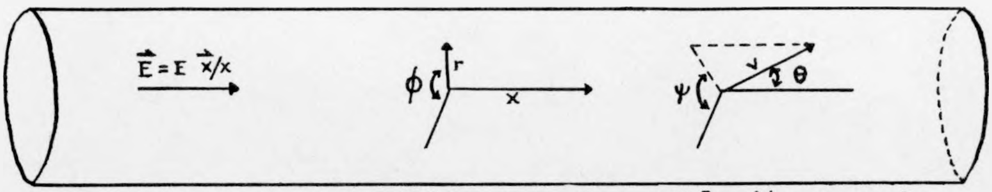

Fig. 1

the current down the tube and the distribution functions should be independent of x and ϕ .

The electrons in a discharge are about a thousand times more mobile than the positive ions, and although the two are connected together by coulomb forces, the properties of the discharges are largely determined by the electron distribution function $F(\vec{v}, r, t)$.

We may now write down the Boltzman equation which says that the number of electrons in space and velocity volumes are conserved. $\vec{v}F$ would be the number of electrons flowing through a given space area per unit time, and $\vec{a}F$ would be the number flowing through a given area in velocity space per unit time. The net loss to the system, $\partial F/\partial t$, equals the divergences of the two flows. Or:

$$\frac{\partial F}{\partial t} = -\nabla_r \cdot \vec{v}F - \nabla_v \cdot \vec{a}F - \text{collision terms} \qquad (2\text{-}1)$$

The collision terms represent electrons that are lost (or gained) from the volume element in velocity space because of encounters with other particles.

We now assume F independent of ϕ and ψ, almost independent of θ, and expand F in spherical harmonics. (We ignore the third and higher terms.)

$$F = F^0(r, x, v) + F^1(r, x, v)\cos\theta = F^0 + \frac{\vec{F}^1 \cdot \vec{v}}{v} \qquad (2\text{-}2)$$

the expansion (2-2) in (2-1) gives

$$\underset{(1)}{\frac{\partial F^0}{\partial t}} + \underset{(2)}{\frac{\vec{v}}{v} \cdot \frac{\partial \vec{F}^1}{\partial t}} = \underset{(3)}{-\vec{v} \cdot \nabla_r F^0} - \underset{(4)}{\frac{\vec{v}}{v} \cdot \nabla_r(\vec{v} \cdot \vec{F}^1)} - \underset{(5)}{\frac{\vec{v} \cdot \vec{a}}{v}\frac{\partial F^0}{\partial v}}$$

$$\underset{(6)}{-\vec{a} \cdot \nabla_v \left(\frac{\vec{v} \cdot \vec{F}^1}{v}\right)} - \underset{(7)}{\text{collision terms}} \qquad (2\text{-}3)$$

Terms in $\frac{\partial v}{\partial t}$, $\nabla_r \cdot \vec{v}$, $\nabla_v \cdot \vec{a} = 0$ because of the independence of variables. Also $\nabla_v F^0 = \frac{\vec{v}}{v}\frac{\partial F^0}{\partial v}$. We will separate Equation (2-3) into parts 1) dependent on, and 2) independent of, $\cos\theta$. However it will be convenient first to expand several terms. Term (4) of Equation (2-3) may be expanded:

$$\frac{\vec{v}}{v} \cdot \nabla_r(\vec{v} \cdot \vec{F}^1) = \frac{1}{v}\left[(\vec{v} \cdot \vec{v})(\nabla_r \cdot \vec{F}^1) + (\vec{v} \cdot \vec{F}^1)(\nabla_r \cdot \vec{v})\right]$$

$$= \frac{1}{v}\left[\frac{v^2}{3}(\nabla_r \cdot \vec{F}^1)\right] = \frac{v}{3}\nabla_r \cdot \vec{F}^1 \qquad (2\text{-}3a)$$

The reason for the appearance of the factor 1/3 may be shown by taking the x axis (of a new set of cartesian co-ordinates in velocity space) along $\vec{F}^1$.

Now

$$\vec{v} \cdot \nabla_r(\vec{v} \cdot \vec{F}^1) = (v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z})(v_x F^1)$$

$$= v_x^2 \frac{\partial F^1}{\partial x} + v_x v_y \frac{\partial F^1}{\partial y} + v_x v_z \frac{\partial F^1}{\partial z}$$

the average over angles θ and ψ of $v_x v_y = v_y v_z = 0$

while $\overline{v_x^2} = v^2/3$. Term (6) of Equation (2-3) may be expanded also:

$$\vec{a} \cdot \nabla_v (\frac{\vec{v} \cdot \vec{F}^1}{v}) = \frac{\vec{a} \cdot \vec{F}^1}{v} + (\vec{a} \cdot \vec{v})(\nabla_v \cdot \frac{\vec{F}^1}{v})$$

$$= \frac{\vec{a} \cdot \vec{F}^1}{v} + \frac{\vec{a} \cdot \vec{v}}{3v} \frac{\partial F^1}{\partial v} - \frac{1}{3} \frac{\vec{a} \cdot \vec{F}^1}{v} \qquad (2\text{-}3b)$$

$$= \frac{1}{v^2} \frac{\partial}{\partial v} \frac{v^2}{3} \vec{a} \cdot \vec{F}^1$$

where the factor 1/3 appears in the same manner as in term (4) above.

Now Equation (2-3) may be split into two parts:

a, a scalar equation of terms 1, 4, 6, and 7, and b, a vector equation of terms 2, 3, 5, and 7, which involve $\vec{v}$.

$$\frac{\partial F^0}{\partial t} = - \frac{v}{3} \nabla_r \cdot \vec{F}^1 - \frac{1}{v^2} \frac{\partial}{\partial v} \frac{v^2}{3} \vec{a} \cdot \vec{F}^1 - \text{collisions} \qquad (2\text{-}4a)$$

$$\frac{\partial F^1}{\partial t} = - v \nabla_r F^0 - \vec{a} \frac{\partial F^0}{\partial v} - \text{collisions} \qquad (2\text{-}4b)$$

All terms in B have been divided by $\frac{\vec{v}}{v}$.

Collision terms: We define an elastic collision frequency

$$\nu_c(v) = \oint v \, \sigma_c \, n_g (1 - \cos \theta) \sin \theta \, d\theta \, d\psi$$

where v = velocity of electron

n_g = number gas particles per unit volume

σ_c = total collision cross section.

We use a collision frequency instead of the mean free paths, since ν for different v are directly additive, and ν_c is more nearly independent of v than is a mean free path.

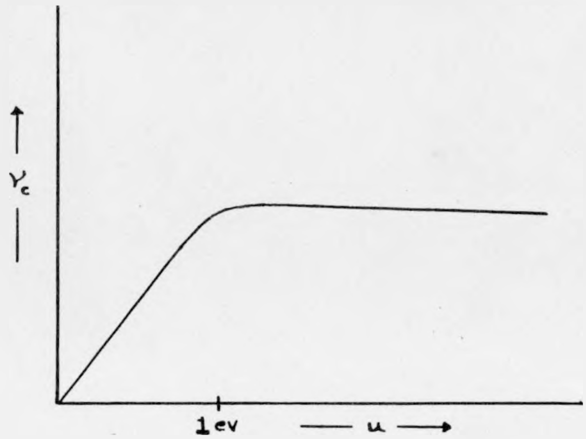

Fig. 2. A plot of ν_c (for hydrogen) vs energy $u = mv^2/2e$

Inelastic collisions are less frequent than elastic, and give rise either to ionizing collisions of frequency ν_i or exciting collisions of frequency ν_x. Known shapes of these functions are

Fig. 3. Plot of excitation and ionizing collision frequencies vs electron energy.

The ionization is $H_2 \longrightarrow H_2^+ + e^-$.

Between Equations (2-4a) and (2-4b) we may eliminate F^1 if we also use

$$\left(\frac{\partial F^1}{\partial t}\right)_{\text{collisions}} = \nu_c F^1 \qquad (2\text{-}5a)$$

which is valid because direction is destroyed in each collision. From (2-4b)

$$F^1 = -\frac{v \nabla_r F^0}{\nu_c} - \frac{\vec{a}}{\nu_c} \frac{\partial F^0}{\partial v} \qquad (2\text{-}5b)$$

Since inelastic collisions remove fast electrons at the rate $(\nu_i + \nu_x)F^0$ and create slow electrons at the rate $(\nu_x + 2\nu_i)F^0$, in the average we will have

$$\left(\frac{\partial F^0}{\partial t}\right)_{\text{collisions}} = -\left[\nu_i + \nu_x - (\nu_x + 2\nu_i)\right] F^0 = \nu_i F^0 \qquad (2\text{-}6)$$

Substituting (2-5) and (2-6) into (2-4a) and (2-4b) and eliminating F^1:

$$\overset{(1)}{\nu_i F^0} = -\overset{(2)}{\frac{v^2}{3\nu_c} \nabla^2 F^0} - \overset{(3)}{\frac{v}{3\nu_c} \nabla_r \cdot \vec{a} \frac{\partial F^0}{\partial v}}$$

$$-\overset{(4)}{\frac{1}{v^2}\frac{\partial}{\partial v}\frac{v^3}{3\nu_c} \vec{a} \cdot \nabla_r F^0} - \overset{(5)}{\frac{1}{v^2}\frac{\partial}{\partial v}\frac{v^2}{3\nu_c} a^2 \frac{\partial F^0}{\partial v}} \qquad (2\text{-}7)$$

(6)
- coulomb interactions

The third and fourth terms are zero because $\vec{a}$ is along the tube and the gradients are radial. We now integrate over v space to get a pure space distribution.

Here

$$n = \int_0^\infty F \; 4\pi v^2 dv = \int_0^\infty F^0 \; 4\pi v^2 dv \qquad (2\text{-}8)$$

since F^1 contributes only to direction.

Multiplying (2-7) by $4\pi v^2 dv$ and integrating, terms 3, 4, and 5 go to zero, and the collision terms drop out (because the gas and the electrons are in equilibrium); leaving

$$\overline{\nu_i} n = -\nabla^2 Dn$$

or $\qquad (2\text{-}9)$

$$\nabla^2 n = -\overline{\nu_i} n/D$$

where $\quad D \equiv \frac{4\pi}{3n} \oint F^0 \; v^4 dv.$

Now a solution must be found such that $n = o$ at the walls of the tube. This is possible only for a set of characteristic values of D/ν_i and only for the lowest of these is the solution $n(r)$ everywhere positive. This value of $\sqrt{D/\nu_i}$ is called the diffusion length Λ of the tube. For an infinite cylindrical tube $\Lambda = r/2.4$.

Coulomb interactions are of two types, long and short range. The dividing line is the Debye length $\ell_D^2 = \dfrac{\epsilon_o kT_-}{e^2 n_-}$.

Long Range:*

1. Space charge $\overline{E_s}$ due to charge separation
2. Plasma oscillations $\omega_p^2 = ne^2/\epsilon_o m$

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* Bohm and Pines, Phys. Rev., January 1952.

Short Range:

3. "Frictional" effect -- or resistivity of ionized plasma

4. Random coulomb field in plasma seen by moving electron

5. Bremsstrahlung

The effect of random field is to produce a Maxwellian distribution, and if the coulomb interactions are large one may assume the distribution $F^o = Ae^{-u/T}$.

III. LANGMUIR AND TONKS EQUATIONS

The scalar and vector equations lead to five Langmuir and Tonks equations:

1. Ion generation
2. Energy balance
3. Wall current
4. Plasma balance
5. Arc current

1. Ion generation equation.

The average ionizing collision frequency:

$$\overline{\nu_i} = \frac{\int \nu_i(u)F^o\, 4\pi v^2 dv}{\int F^o\, 4\pi v^2 dv} = \frac{4\pi}{n}\int \nu_i F^o v^2 dv = \frac{\text{const}}{n} P\sqrt{\frac{T}{v_i}}\, e^{-v_i/T} \quad (3\text{-}1)$$

where T is the temperature of electrons in electron volts, v_i = ionizing potential and P = pressure.

The form of Equation (3-1) comes from the assumption that F^o is Maxwellian, i.e., $F = Ae^{-u/T}$ where u is the energy of the electrons.

2. Energy balance equation.

Per unit length of discharge (in cylinder), the power in equals the power out, or

$$EI = 2\pi r \Gamma_r \left[2e(T_- + T_+) + N\nu_i\, e\, v_i + N\nu_x\, e\, v_x\right] \qquad (3\text{-}2a)$$

Here E = voltage drop per unit length

I = current

Γ_r = radial flow of particles

$2eT_-$ = energy removed from discharge by each electron which reaches the wall

(This would normally be $3eT/2$, but becomes $2eT$ when one averages over the electron velocity distribution, since fast electrons get to the wall faster and carry more energy each.)

$2eT_+$ = energy removed by positive ions, by the same mechanism. T_+ is usually so small as to be ignored. Here we assume $\Gamma_{r+} = \Gamma_{r-} = \Gamma_r$.

$N\nu_i$ = number of ionizing collisions per electron moving to wall

$N\nu_x$ = ditto for exciting collisions

$e\,v_i$ = energy loss per ionizing collision

$e\,v_x$ = energy loss per exciting collision

Energy losses due to radiation and thermal conduction are small and ignored. If we now add up the sums of energies lost in the ionization of H_2 to $2p^+ + 2e^-$ we obtain 32 volts or two times the first ionization potential v_i = 16 volts. This allows us to rewrite (3-2a) as

$$2\pi r\, \Gamma_r\, [2e]\, [T_- + T_+ + v_i] \qquad (3\text{-}2b)$$

3. Wall current equation.

If $u = mv^2/2e$ then $du = \frac{mv}{e} dv$ so

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial u}\frac{du}{dv} = \frac{mv}{e}\frac{\partial F}{\partial u}$$

and the vector equation (2-4b), broken into r and x components, becomes, with Equation (2-5a),

$$\nu_c F_x^1 = -\frac{eE_x}{m} \cdot \frac{mv}{e}\frac{\partial F^0}{\partial u} = -vE_x \frac{\partial F^0}{\partial u} \tag{3-3a}$$

and

$$\nu_c F_r^1 = -v\nabla_r F^0 - vE_r \frac{\partial F^0}{\partial u} \tag{3-3b}$$

Multiply vF_x^1 by $4\pi v^2 dv$ and integrate to get Γ_x, or

$$\Gamma_x = \int vF_x^1 4\pi v^2 dv = -\frac{4\pi eE_x}{3m\nu_c}\int v^3 \frac{\partial F^0}{\partial v} dv \equiv -n_- \mu_- E_x$$

The factor of 3 arises because of the product vF averaged over θ and ψ as on page 4. The mobility μ_- is defined by

$$\mu = \frac{1}{n}\left(\frac{4\pi e}{3m\nu_c}\right)\int \frac{\partial F^0}{\partial v} v^3 dv$$

Similarly in the radial direction

$$\Gamma_r = -\frac{4\pi}{3}\int \left(E_r \frac{e}{mv}\frac{\partial F^0}{\partial v} + \nabla F^0\right) v^4 dv$$

$$= -\frac{4\pi eE_r}{3m\nu_c}\int v^3 \frac{\partial F^0}{\partial v} dv - \frac{4\pi}{3}\int \nabla_r F^0 v^4 dv$$

or

$$\Gamma_{r-} = -n_-\mu_- E_r - \nabla(D_- n_-) \tag{3-4}$$

where the electron diffusion coefficient $D = \frac{4\pi}{3n}\int F^o v^4 dv$. Now, because electrons are a thousand times more mobile than the protons, one will obtain space charges. These are negligible however in a discharge which is going under conditions where the current is only slightly dependent on E. In the plot of Fig. 4 this is true in the region B, and one may make the assumptions that $n_+ = n_-$ and $\Gamma_{r_-} = \Gamma_{r_+}$.

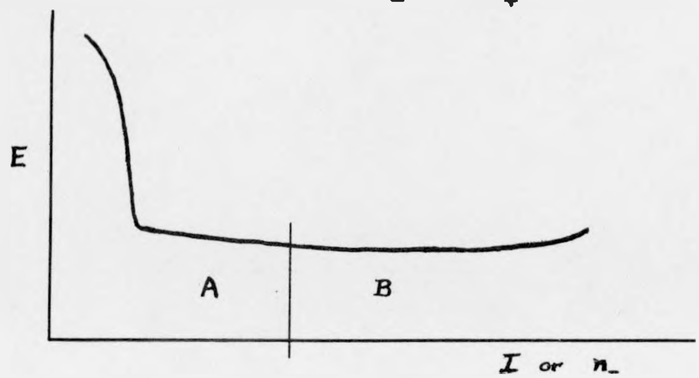

Fig. 4. Potential across a discharge vs current through the discharge.

The positive radial current will be given by

$$\Gamma_{r_+} = n_+ \mu_+ E_r - \nabla(D_+ n_+) \qquad (3\text{-}5)$$

and equations (3-4) and (3-5) will then lead (with assumption $n_+ = n_- = n$ and $\Gamma_{r_-} = \Gamma_{r_+} = \Gamma_r$) to the wall current equation:

$$\Gamma_r = -\frac{\nabla(D_- \mu_+ + D_+ \mu_-)n}{\mu_+ + \mu_-} \equiv -\nabla D_a n \qquad (3\text{-}6)$$

where D_a is the so-called ambipolar diffusion constant. This approximation is obviously invalid unless $(n_+ - n_-) << n_-$. D_a is defined by

$$D_a \equiv \frac{D_+ \mu_- + D_- \mu_+}{\mu_+ + \mu_-} \qquad (3\text{-}7)$$

If F is a Maxwell distribution, the Einstein relation holds:

$$D = T\mu$$

so that

$$D_a = \mu_+ \mu_- \frac{T_+ + T_-}{\mu_+ + \mu_-} \qquad (3\text{-}8)$$

4. <u>The plasma balance equation.</u>

Because $\vec{a}$ has no component along r we can use Equations (2-1) and (2-6) to obtain

$$\frac{\partial F}{\partial t} = - \nabla_r \cdot \vec{v}F = - \nu_i F^0 \qquad (3\text{-}9)$$

which when integrated over $4\pi v^2 dv$ gives

$$\nabla \Gamma_r = n \overline{\nu_i} \qquad (3\text{-}10)$$

This equation, with (3-6), gives the differential equation

$$\nabla^2 D_a n = \overline{\nu_i} n \qquad (3\text{-}11)$$

which has the Bessel function $n_- = n_-^o J_o\left(\sqrt{\frac{\nu_i}{D_a}}\, r\right)$ as a solution. This solution, with the boundary condition $n_- = o$ for $r = r_1$, gives the plasma balance equation

$$\frac{\overline{\nu_i}}{D_a}\left(\frac{r_1}{2.4}\right)^2 = 1 \qquad (3\text{-}12)$$

5. The arc current equation.

The number of particles transported along the direction x is

$$\Gamma_x = \Gamma_{x_-} + \Gamma_{x_+} = E_x(n_-\mu_- + n_+\mu_+). \qquad (3\text{-}13)$$

This, multiplied by the charge per particle, gives the arc current equation:

$$J_x = eE_x(\mu_- n_- + \mu_+ n_+) \qquad (3\text{-}14)$$

Hence the five Langmuir and Tonks equations are:

Ion generation: $\overline{\nu_i}(T) = \int \nu_i F_0 \, 4\pi v^2 dv$ 1.

Energy balance: $EI = 2\pi r \, \Gamma_r [2e][T_- + T_+ + v_i]$ 2.

Wall current: $\Gamma_r = -\nabla D_a n$ 3.

Plasma balance: $\nabla \Gamma_r = n\overline{\nu_i}$ or $\overline{\nu_i} r_i^2 = (2.4)^2 D_a$ 4.

Arc current: $J = ne(\mu_+ + \mu_-) E_x$ 5.

IV. DISCUSSION OF LANGMUIR EQUATIONS AND THE GENERAL PHENOMENA

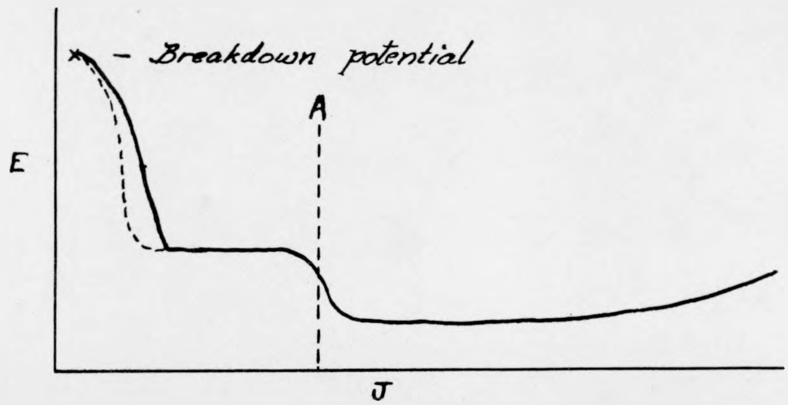

Fig. 5

After breakdown the potential falls rapidly. Since the load line of experimental apparatus is to the right of the theoretical behavior (shown by the dotted line in Fig. 5), the shape of the rapid drop after breakdown is determined by the load line. For this reason we speak of the arc as "striking." We obtain further drops as the current increases, such as at A. These may be due to a variety of reasons. One reason might be that if one has an a-c field then such a drop occurs when the frequency ω of the a-c field coincides with the plasma oscillation frequency $\omega_p = \sqrt{ne/\epsilon_o m}$. Resonance helps the discharge because in the region in the tube where resonance occurs the field E is increased each half cycle over its dc value. This leads to an increase in ν_i and increases the electron density. This might be termed a self-constriction since the current tends to localize itself.

A transition from a uniform discharge (to the left of A) to a self-constricted discharge (to the right of A) is usually called a transition from a glow discharge to an arc discharge. In addition to the special case of plasma oscillations such transitions might be due to 1) a hot spot on an electrode emitting electrons (i.e., a surface constriction), 2) cumulative or stepwise ionization due to the presence of a metastable state in the gas supporting the discharge (e.g., mercury vapor may be ionized by collision with two 6 volt electrons in succession due to a 6 volt metastable state, while it requires a 10 volt electron to produce ionization in one collision); this might be termed a body constriction. This process produces a self-constriction because the discharge finds it advantageous to increase the density of metastable ions and the density of electrons to hit them. Another body constriction occurs when the gas gets hot in a localized volume. If the rest of the gas stays cool, the density will drop in the hot region, the mean free path will increase, the energy the electrons gain between collisions will increase, so ν_i will go up and the arc will constrict. 3) Pinch effect (see below).

Eventually the potential will have to go up as the current J gets very large, if for no other reason than because the gas becomes totally ionized. It is to be noted that the constricting effects occur only because a change in area of the discharge produces a change in the percentage ionization. The rise in potential will occur only for very large current densities: 10,000 to 100,000 amp/cm².

V. THEORY OF THE PINCH EFFECT

1. Energy Distribution and Ion Generation

In discharges at low pressure and high current densities the interaction between the electrons far outweighs all other energy interchanges, except for electrons so fast that they may reach the walls directly. For these conditions it is then not necessary to solve the Boltzmann equation for the energy distribution as this distribution must be Maxwellian, and be given by the expression

$$F^o = n(m/2\pi e T_-)^{3/2}\, e^{-u/T_-} \tag{5-1}$$

which will be correct except for a depletion at the high energy end which may generally be overlooked. Knowing the excitation function for hydrogen one then obtains the ion generation equation by integration

$$n\overline{\nu_i} = \int_0^\infty \nu_i(u) F^o\, 4\pi v^2 dv \tag{5-2}$$

The ionization function for hydrogen may be approximated near the ionization potential by

$$\nu_i = 2.18 \times 10^8 \,(v - 16.2)\, p/\sqrt{v} \ \ \sec^{-1} \tag{5-3a}$$

and at high energies by

$$\nu_i = 0.95 \times 10^{10}\, p/v^{1/4} \ \ \sec^{-1} \tag{5-3b}$$

where p is pressure and v is the energy of the electron in volts. Using these approximations one obtains

for $T_- < 150$ volts; $\overline{\nu_i} = 10^9\, p \sqrt{T_-/v_i}\ \ e^{-v_i/T_-}\ \ \sec^{-1}$

for $T_- > 150$ volts; $\nu_i = 10^{10}\, p\, T_-^{-1/4} \left[1 - 28.7/T_-^{5/4}\right]\ \ \sec^{-1}$ (5-4)

In these expressions p is the actual pressure of neutral gas, or

$$p = p_o(1 - i) \tag{5-5}$$

where p_o is the pressure of hydrogen originally introduced and i the fraction ionized. v_i is the ionization potential of hydrogen.

2. Power Density Balance

The power input, $eE \cdot v_+$, to an ion is balanced, in the stationary state, by the losses in collisions. (Here v_+ is the velocity of a + ion.) When two particles collide their energies tend on the average to equalize; the energy transfer being proportional to the original energy difference and the mass factor $3Mm/(M + m)^2$ which makes the transfer inefficient when the masses differ greatly. If ν_{pe} and ν_{pg} are the frequencies of proton-electron and proton-gas collision, the energy loss due to collisions will be given by:

$$\frac{3mM}{(M + m)^2} \nu_{pe} \, e(T_+ - T_-) + \frac{2}{3} \nu_{pg} \, e(T_+ - T_g)$$

$$= \frac{3ne^2}{M\sigma} (T_+ - T_-) + \frac{e}{M\mu_+} (T_+ - T_g)$$

where substitution has been made for the collision frequencies in terms of the conductivity σ of a completely ionized plasma and the mobility μ_+ of the ions.

In a given volume element there are also losses through the walls of the element. If ions are created at the rate $\overline{\nu_i}$, and an equal number leave the volume, they will carry with them the thermal energy $2eT_+\overline{\nu_i}$; the factor 2 entering instead of 3/2 because fast ions are somewhat more likely to leave than slow ones. There will also be a thermal conductivity loss $-eK_+\nabla^2 T_+$. Adding these we have

$$E \cdot v_+ = 2T_+\overline{\nu_i} + \frac{3ne}{M\sigma} (T_+ - T_-) + \frac{1}{M\mu_+} (T_+ - T_g) + K_+\nabla^2 T_+ \qquad (5\text{-}6)$$

For electrons the loss to the gas is negligible but there are two additional terms. For each complete ionization the electrons must supply 16.2 volts to ionize H_2, 1.7 volts to dissociate H_2^+, and 13.6 volts to ionize H. Then it must supply 8.9 volts or more for every excitation of H_2 and 5.2 for every excitation of H. If there were two excitations per ionization this would add up to 60 volts for each double ionization, or 30 volts per single ionization. We shall use the figure $2v_i = 32.4$ volts which is convenient though probably excessive. There will also be a bremsstrahlung loss βne where $\beta = 0.66\sqrt{T_-}$ and n = number of ions per cubic meter. Thus we have

$$E \cdot v_- = 2(T_- + v_i)\nu_i + \frac{3ne}{M\sigma}(T_- - T_+) + K_-\nabla^2 T_- + \beta ne \qquad (5\text{-}7)$$

Adding Equations (5-6) and (5-7), neglecting the loss to the gas, and multiplying by ne, we obtain the power density equation:

$$EJ = ne\left[2(T_+ + T_- + v_i)\nu_i + \nabla^2(K_+T_+ + K_-T_-) + \beta ne\right] \qquad (5\text{-}8)$$

The heat conduction terms are small in ordinary glow discharges because temperature gradients in the discharge are small and the heat transfer to the walls is essentially zero. We shall neglect them at present mainly because they lead to intractable differential equations. The bremsstrahlung term is proportional to n^2 and will therefore increase towards the axis whereas the ionization term should dominate near the walls.

3. Temperature Ratio

One can also eliminate the field E between Equations (5-6) and (5-7) to obtain the ratio T_+/T_- of ion to electron temperatures. Neglecting bremsstrahlung and conductivity in this calculation we find

$$\frac{T_+}{T_-} = \frac{1 + \frac{2}{3}\frac{m}{e}\frac{\sigma \overline{\nu_i}}{ne}\left(\frac{T_- + v_i}{T_-}\right)}{1 + \left(\frac{2M\overline{\nu_i}}{e} + \frac{1}{\mu_+}\right)\frac{\sigma}{3ne}} \qquad (5\text{-}9)$$

Near the walls n is small and this ratio will be of the order of $m/M = 1/1837$, whereas near the center the ratio may be near 1. The ions lose most of their energy to the gas while diffusing out whereas the electrons carry much of their energy directly to the walls.

4. Arc Current Density

The vector components of the Boltzmann equation for the electrons, in the presence of a magnetic field B, are

$$(\nu_{eg} + \nu_{ep})F_x + \frac{eB}{m}F_r = vE_x\frac{\partial F^o}{\partial u} \qquad (5\text{-}10)$$

$$\nu_{eg}F_r - \frac{eB}{m}F_x = v\left(E_r\frac{\partial F^o}{\partial u} - \nabla_r F^o\right) \qquad (5\text{-}11)$$

On the right hand side the gradient is absent from the first equation because we have assumed no longitudinal gradients. On the left we have the retarding effect of collisions and the deflection effect of the magnetic field. In moving along the tube the electrons collide with both neutral gas atoms and with protons, but in their radial flow they are moving with the protons and hence collisions between radially moving electrons and protons must not be included.

In the first equation (5-10) we hope that the radial flow will be sufficiently small, compared to the axial flow, for the term in F_r to be neglected. The remaining terms will then yield the arc current density equation

$$E = \frac{J}{ne(\mu_+ + \mu_-)} + \frac{J}{\sigma} \tag{5-12}$$

where $\sigma \left(= \sigma_1 T_-^{3/2} = 3620\ T_-^{3/2} \text{ mho/meter}\right)$ (5-13)

is the conductivity of a completely ionized gas;

$$\mu_- = 30/p \quad \text{and} \quad \mu_+ = 1.07/p \tag{5-14}$$

and the current due to the positive ions has been added arbitrarily.

From (5-12) and (5-8) one finds that

$$\frac{ne(\mu_+ + \mu_-)}{\sigma} = \frac{E^2(\mu_+ + \mu_-)}{2(T_+ + T_- + v_i)\overline{\nu_i} + \beta ne} - 1 \tag{5-15}$$

As the denominator on the right increases towards the axis it is seen that the mobility decreases relative to the conductivity towards the axis, and therefore, by (5-12), the current is limited on the axis by the neutral gas, while on the outside it is limited by the protons, which is just the opposite of what one might expect.

5. Wall Current

Multiplying the radial equation (5-11) by $4\pi v^2 dv/3\nu_c$ and integrating yields the radial flow

$$\Gamma_{r_-} = -\nabla D_- n_- - E_r \mu_- n_- + B\mu_- \Gamma_{x_-} \quad (5\text{-}16)$$

In this equation slightly different quantities have been represented by μ_-. It occurs first as the true mobility, and then as the Hall mobility. Both these quantities are averages of $e/m\nu_c$ but cover somewhat different distributions. It is customary to neglect the difference between these two mobilities in first theories.

Equation (5-16) must be combined with the similar one for positive ions:

$$\Gamma_{r_+} = -\nabla D_+ n_+ + E_r \mu_+ n_+ - B\mu_+ \Gamma_{x_+} \quad (5\text{-}17)$$

in order to eliminate the unknown space-charge field E_r. Assuming $n_+ = n_- = n$, which is substantially true if the Debye length

$$\ell_D = \sqrt{\epsilon_o T_- / ne} \quad (5\text{-}18)$$

is less than the diffusion length for the tube, we obtain

$$\Gamma_r = -\nabla D_a n - B \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \left(\Gamma_{x_+} - \Gamma_{x_-} \right) \quad (5\text{-}19)$$

where the ambipolar diffusion coefficient

$$D_a = \mu_+ \mu_- \frac{T_+ + T_-}{\mu_+ + \mu_-} \quad (5\text{-}20)$$

Also, by Ampere's law

$$B = \mu_0 I/2\pi r \tag{5-21}$$

where I(r) is the current contained in the tube out to the radius r, and

$$\Gamma_{x_+} - \Gamma_{x_-} = \frac{J}{e} = \frac{1}{2\pi r}\,\frac{d\,J/e}{dr} \tag{5-22}$$

Inserting these in (5-19) we obtain the wall current equation

$$J_r = -\frac{\mu_+ \mu_-}{\mu_+ + \mu_-}\left[\nabla ne(T_+ + T_-) + \frac{\mu_0}{8\pi^2 r^2}\nabla^2 I^2\right] \tag{5-23}$$

If I is selected as the dependent variable one should now use (5-8) and (5-12) to express $ne(T_+ + T_-)$ in terms of J. This is complicated, but it turns out that $ne(T_+ + T_-)$ is never far from proportional to J. Accordingly we shall write

$$ne(T_+ + T_-) = cJ = \frac{c}{\pi}\,\frac{dI}{d(r^2)} \tag{5-24}$$

but remember that some error is introduced in treating c as a constant. The wall current equation may now be written

$$2\pi r\, J_r = -\frac{4\mu_+\mu_-}{\mu_+ + \mu_-}\,\frac{d}{d(r^2)}\left(cr^2\frac{dI}{d(r^2)} - cI + \frac{\mu_0 I^2}{8\pi}\right) \tag{5-25}$$

6. Bennett Relations

The wall current J_r is expected to be small and useful relations were obtained by Bennett for the limit $J_r = 0$. Now from (5-25), with $J_r = 0$,

$$cr^2\frac{dI}{d(r^2)} = cI - \frac{\mu_0 I^2}{8\pi} \tag{5-26}$$

And at the boundaries $\pi J = \frac{dI}{d(r^2)} = 0$, so

$$c = \mu_0 I_1/8\pi \tag{5-27}$$

where $I_1 = I(r_1)$ is the total current in the stream. Hence we find the Bennett relation from Equations (5-24) and (5-27)

$$\frac{\mu_0 I_1^2}{8\pi} = \int cJ\ 2\pi r dr = Ne(T_+ + T_-)_a \tag{5-28}$$

where N is the number of electrons or ions per meter length of stream and $(T_+ + T_-)_a$ the average temperatures. If c were not constant it would be a weighted average, but the Bennett relation would still hold as it represents a balance between the magnetic and kinetic pressures. Integrating (5-25) we find

$$I = \frac{\pi J_0 r^2}{1 + br^2} \qquad \text{where} \qquad b = \frac{\pi J_0}{I_1} \tag{5-29}$$

whence the Bennett distribution

$$J = J_0/(1 + br^2)^2 \tag{5-30}$$

This expression is more sensitive to the assumption regarding c and it is quite possible that the exponent in the denominator differs somewhat from 2. However, it is seen that $J_0 = 0$ only at $r_1 = \infty$, so that the Bennett distribution, and the assumption of zero wall current, can hold strictly only in free space.

The magnetic energy within a radius r_1 of the current given by (5-30), and including the thermal energy given by (5-28), is given by $LI_1^2/2$ where $L = \frac{\mu_0}{4\pi}\left[\frac{\overline{\nu}_i r_1^2}{8D_a - \overline{\nu}_i r_1^2} - \ln \frac{\overline{\nu}_i r_1^2}{8D_a - \overline{\nu}_i r_1^2}\right]$ (5-31)

The electron distribution is determined by

$$n = \frac{J_0/Ee(\mu_+ + \mu_-)}{(1 + br^2)^2 - J_0/E\sigma_1 T_-^{3/2}}$$

and the temperature distribution by substitution of this into the power balance equation (5-9). It is clear from the above that the electrons are not as tightly pinched as the current, but the equation determining the temperature is not simple.

7. Plasma Balance

The condition of plasma balance is obtained by equating the rate of generation of charged particles to their outward flow to the walls;

$$\nabla \Gamma_r = n\overline{\nu_i}$$

or

$$\frac{d}{dr}(2\pi r J_r) = 2\pi r n e \overline{\nu_i} = \frac{c\,\overline{\nu_i}}{T_+ + T_-}\frac{dI}{dr} \qquad (5\text{-}32)$$

In a substantially completely ionized discharge the ionization term will be small, so that we are looking for small departures from the Bennett relations and are therefore justified in making approximations in the ionization term. Treating $c\overline{\nu_i}(T_+ + T_-)$ as a constant, Equation (5-32) integrates to

$$2\pi r J_r = c\overline{\nu_i} I/(T_+ + T_-)$$

which with (5-20) and (5-25) gives

$$- r^2 \frac{dI}{d(r^2)} + I - \frac{\mu_0 I^2}{8\pi c} = \int_0^r \frac{\overline{\nu_i} I}{4D_a} dr^2 \qquad (5\text{-}33)$$

We now introduce the variable

$$s \equiv \overline{\nu_i} r^2/4D_a \qquad (5\text{-}34)$$

Treating $\overline{\nu_i}/4D_a$ as a constant and integrating the right hand side successively by parts we find

$$- s \frac{dI}{ds} + I - \frac{\mu_0 I^2}{8\pi c} = sI - \frac{s^2}{2} \frac{dI}{ds} + \ldots .$$

Neglecting higher terms the equation becomes

$$s(1 - s/2) \frac{dI}{ds} = (1 - s - \mu_0 I/8\pi c)\, I \qquad (5\text{-}35)$$

At the wall $dI/ds = 0$ and we obtain a corrected Bennett relation

$$\mu_0 I_1^2/8\pi = Ne(T_+ + T_-)_a\,(1 - s_1) \qquad (5\text{-}36)$$

where

$$s_1 = \overline{\nu_i}\, r_1^2/4D_a \qquad (5\text{-}37a)$$

Equation (5-36) is the pressure balance equation and shows that when there is a radial current a fraction s_1 of the kinetic pressure is supported by the walls, the remaining fraction $(1 - s_1)$ being held in by the magnetic field. At low currents $s_1 = 1$ and (5-37) comes very close to being the normal plasma balance equation

$$\overline{\nu_i} = D_a(2.4048/r_1^2) \qquad (5\text{-}37b)$$

where the number is the first root of a Bessel function. The approximations made in evaluating the integral in (5-33) have therefore introduced an error of at most $(1.2024)^2$.

Equation (5-35) is soluble and gives

$$I = \frac{\pi J_0 r^2 (1 - s/2)}{1 + br^2} \qquad \text{where} \quad b = (1 - s_1)\pi J_0/I_1 \qquad (5\text{-}38)$$

whence

$$J = J_0 \frac{1 - s(1 + br^2/2)}{(1 + br^2)^2} \qquad (5\text{-}39)$$

which differs insignificantly from (5-30) except that it vanishes at a radius $r_1^2 = 2I_1/\pi J_0 s_1$ (5-40)

The mean square radius of the current stream is

$$\overline{r^2} = \frac{\int_0^{J_0} r^2 dJ}{\int_0^{J_0} dJ} = \frac{\int_0^{r_1} J dr^2}{J_0} = \frac{I_1}{\pi J_0} = r_1^2 s_1/2$$

so that

$$s_1 = 2\overline{r^2}/r_1^2 \qquad (5\text{-}41)$$

can properly be named the pinch factor.

In a normal discharge the plasma balance equation (5-37b) determines the ionization frequency, and therefore the temperature in terms of the tube radius r_1. In a pinched discharge, on the other hand, the temperature is substantially determined by the pressure balance (5-36), and the plasma balance equation (5-37a) determines the mean square radius of the discharge. It is important to keep the two equations (5-36) and

(5-37a) distinct. Pressure balance takes place quickly and Equation (5-36) must therefore hold (except for pressure waves) even in a transient discharge. Plasma balance takes place much more slowly and the relation (5-37a) may not be required to hold in the transient state.

8. Total Power Balance

Knowing the distributions of J, n, and T throughout the tube one can, in principle, integrate Equation (5-8) over the cross-section to obtain the total power balance. It should be sufficiently accurate here to assume that both n and J vary as $(1 + br^2)^{-2}$, to use an average temperature in the ionization term, and to use the central temperature T_0 in the bremsstrahlung term. Or

$$EI_1 = Ne \left[2(T_+ + T_- + v_i)_a \overline{\nu_i} + \frac{2\beta Ne}{3\pi r_1^2 s_1} \right]$$

$$= \frac{Ne}{\pi r_1^2} \left[8\pi D_a (T_+ + T_- + v_i)_a s_1 + \frac{2\beta Ne}{3 s_1} \right] \qquad (5\text{-}42)$$

from which one sees that there is an optimum pinch given by

$$s_1^2 = \frac{\beta Ne}{12\pi D_a (T_+ + T_- + v_i)} = \frac{192\sqrt{T_0}}{(T_+ + T_-)_a^2} P_0^2 A\, i\, (1 - i) \qquad (5\text{-}43)$$

For pinches smaller than this, bremsstrahlung will be the dominant loss.

A similar integration of the arc current density equation (5-12) has not yet been made because it depends critically on how J/n and $J/T_-^{3/2}$ vary with the radius. However, the arc current equation must be of the form

$$EA = I_1 \left(\frac{\alpha}{Ne(\mu_+ + \mu_-)} + \frac{\gamma}{\sigma_1 T_-^{3/2}}\right) \tag{5-44}$$

where α and γ depend on the pinch s_1 and are both less than 1. These two terms can be referred to as the mobility and conductivity terms.

Equations (5-10, -36, -37, -42, and -44) contain the variables E, I_1, T_+, T_-, s_1, i. They determine the equilibrium values of these variables in terms of any one of them. However, as single-valued answers are desirable it is best to choose I_1 as the independent variable.

9. Numerical Evaluations

If a tube of A square meters cross-section is filled with gas at a pressure p_o mm of mercury, there will be

$$N_o = 3.5 \times 10^{22} p_o A \tag{5-45}$$

molecules of gas per meter length of tube. If i is the fraction of molecules doubly ionized (we neglect single ionization), so that i goes from zero to 1, the number N of electrons per meter will be given by

$$N_e = 11320\ p_o A_i \tag{5-46}$$

and the neutral gas pressure will be

$$p = p_o(1 - i) \tag{5-5}$$

Inserting these values in previous equations we find for the plasma balance equation:

$$\frac{\overline{\nu_i}}{4\pi p^2 D_a} = 5.22 \times 10^{8} \sqrt{\frac{v_i}{T_-}}\, e^{-v_i/T} \quad \text{or} \quad = 7.73 \times 10^{8} \frac{1 - 28.7\, T_-^{-5/4}}{(T_+ + T_-)T_-^{1/4}}$$

and $s_1 = \left(\frac{\overline{\nu_i}}{4\pi p^2 D_a}\right) p_o^2 A(1 - i)^2$ (5-47b)

Pressure balance equation:

$$I_1^2 = 2.264 \times 10^{11}(T_+ + T_-)_a p_o A i(1 - s_1) \text{ (amps)}^2 \tag{5-48}$$

Power balance equation:

$$EI_1 = 2.93 \times 10^5 i \left[(T_+ + T_-)_a (T_+ + T_- + v_i)_a \frac{s_1}{1 - i} + 388\sqrt{T_o}\, p_o^2 A\, i/s_1\right] \frac{\text{watts}}{\text{meter}} \tag{5-49}$$

Arc current equation:

$$\frac{EA}{I_1} = \frac{\alpha}{3.52 \times 10^5} \frac{1 - i}{i} + \frac{\gamma}{3620\, T_-^{3/2}} \text{ ohm-meter} \tag{5-50}$$

Temperature ratio equation:

$$\frac{T_-}{T_- - T_+} = 1 + \frac{4.52 \times 10^7 p_o^2 A i(1 - i)}{(T_+ + T_-)_a T_-^{3/2} s_1} \frac{n}{\overline{n}} \tag{5-51}$$

where $\overline{n} = N/A$.

Multiplying (5-48) and (5-50) gives

$$EI_1 = 7.5 \times 10^5 (T_+ + T_-)_a p_o (1 - s_1) \left[\alpha(1 - i) + \frac{97.3\gamma i}{T_-^{3/2}}\right] \tag{5-52}$$

and equating this to (5-49) gives a relation between i and s_1 which, together with (5-47b), determines s_1 and i in terms of T. For currents below 20,000 amperes the bremsstrahlung and conductivity terms can be neglected and simpler relations result. Equations (5-49) and (5-52) yield

$$\frac{i}{(1 - i)^2} = \frac{2.56\, p_o}{T_+ + T_- + v_i} \frac{1 - s_1}{s_1} \tag{5-53}$$

whence

$$I_1 = 2.98 \times 10^5 \frac{i}{1 - i} \sqrt{(T_+ + T_-)(T_+ + T_- + v_i)\ A\ s_1} \qquad (5\text{-}54)$$

and

$$E = 0.845 \sqrt{(T_+ + T_-)(T_+ + T_- + v_i)\ s_1/A} \qquad (5\text{-}55)$$

where s_1 is proportional to p^2A and a function of the temperature according to (5-47b).

At very low currents s_1 is substantially equal to 1 and the temperature does not deviate much from $0.3\,v_i$ or 5 electron volts. Then the field E is nearly constant at

$$E = 8.65 \sqrt{s_1/A} \quad \text{volts/meter}$$

and the fractional ionization i goes up directly with the current. This is a normal glow discharge and holds up to $s_1 \approx 1/2$ or $I_1 \approx 5 \times 10^5 p_0 \sqrt{A}$.

Above this range s_1 can be neglected compared to 1 and we find

$$I_1 = 7.6 \times 10^5 \sqrt{\frac{T_+ + T_-}{T_+ + T_- + v_i} \frac{4\pi p^2 D_a}{\nu_i}}$$

$$EI_1 = 7.5 \times 10^5 (T_+ + T_-)\ p_0\ (1 - i)$$

The discharge now operates at roughly a constant power

$$EI = 37 \times 10^5 p_0 \quad \text{watts/meter}$$

When the ionization passes 50%, which occurs at about $I_1 = 7.5 \times 10^5 \sqrt{p_0 A}$ amperes, the factor $i(1 - s_1)$ is substantially constant and the temperature starts going up with the square of the current. The pinch factor s_1, which varies as $(1 - i)^2$, gets quite small and we can no longer neglect either the conductivity term or bremsstrahlung.

10. Limit of Validity

The statistical theory used in this discussion is valid only if the electrons suffer several collisions before being absorbed by the walls. With neutral gas in the tube this means that one must have

$$pr_1 > 2.4 \times 10^{-2} \sqrt{T_-} \qquad (5\text{-}56)$$

(p in mm Hg, r_1 in cm). Space-charge causes the electrons to make several transits before actually reaching the wall so that the limit is reduced to

$$pr_1 > 4.5 \times 10^{-3} \sqrt{T_-} \qquad (5\text{-}57)$$

As soon as the self-magnetic field is appreciable this will also lengthen the electron's path to the wall and further reduce the limit.

The effect of approaching this limit is to increase the losses above what has been calculated. Accordingly the field E, which normally decreases as p decreases, reverses and increases with decreasing p, producing a "Paschen minimum." The optimum condition for minimum power is close to the limit defined by (5-57).