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The Effects of the Straight Sections in Mark V.

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The straight sections in Mark V affect both the forced motion (equilibrium orbits) and the wave length of betatron oscillations. If we insert the straight sections along the spirals, keeping the scaling feature of the orbits, there are no serious difficulties. However, it is impractical to design such straight sections containing the cavities and injection equipment.

If ordinary radial straight sections are used, the scaling feature of the motion is lost and the shape of the orbits and the wave length of betatron oscillations vary with radius. However, this disturbance may be small when the lengths of the straight sections are very small.

The influences of the straight sections on the forced motion and the wave length of betatron oscillations are estimated roughly in the following articles.

A) Forced Motion

Two straight sections having length ℓ are inserted at

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the opposite sides of the circumference as in Fig. 1. We define s as the distance along the reference circle $r = r_1$ from a reference point at the edge of one straight section.

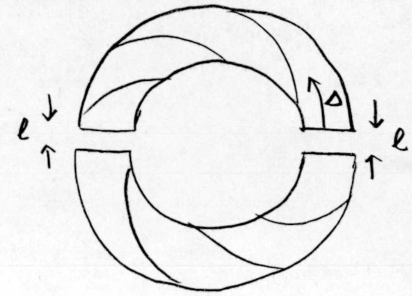

Fig. 1

In the absence of straight sections, we assume the equation for the forced motion, neglecting centrifugal force and higher order terms in the magnetic field, as follows.

$$\frac{d^2 x_f}{d\theta^2} = A \sin(\varphi(r_1) - N\theta) , \qquad \varphi(r_1) = \frac{\ln \frac{r_1}{r_0}}{w} \tag{1}$$

and x_f is

$$x_f = -\frac{A}{N^2} \sin(\varphi(r_1) - N\theta) \tag{2}$$

If the straight sections are inserted, equation (1) is modified to the form

$$\frac{d^2 x_f}{ds^2} = F(s) \tag{3}$$

where

$$\begin{cases} F(s) = \frac{A}{r_1^2} \sin\left(\varphi(r_1) - \frac{N}{r_1} s\right) & 0 < s < \pi r_1 \\ F(s) = 0 & \pi r_1 < s < \pi r_1 + \ell \\ F(s) = \frac{A}{r_1^2} \sin\left(\varphi(r_1) - \frac{N}{r_1}(s-\ell)\right) & \pi r_1 + \ell < s < L - \ell \\ F(s) = 0 & L - \ell < s < L \end{cases}$$

neglecting the fringing field.

Since the magnet structure still has the period L of S, F(s) may be expanded in a Fourier series.

$$F(s) = \sum \left(a_n \sin \frac{2\pi}{L} ns + b_n \cos \frac{2\pi}{L} ns \right)$$

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_0^L F(s) \sin \frac{2\pi}{L} ns \, ds \\ b_n = \dfrac{1}{L} \displaystyle\int_0^L F(s) \cos \frac{2\pi}{L} ns \, ds \end{cases}$$

The coefficients a_n and b_n are evaluated as follows:

$$\begin{aligned} a_n &= -\sin\varphi \left[\frac{\cos p_0 - 1 + \cos p_1 - \cos p_2}{p} + \frac{\cos g_0 - 1 + \cos g_1 - \cos g_2}{g} \right] \\ &\quad - \cos\varphi \left[\frac{\sin g_0 + \sin g_1 - \sin g_2}{g} - \frac{\sin p_0 + \sin p_1 - \sin p_2}{p} \right] \\ b_n &= \sin\varphi \left[\frac{\sin p_0 + \sin p_1 - \sin p_2}{p} + \frac{\sin g_0 + \sin g_1 - \sin g_2}{g} \right] \\ &\quad - \cos\varphi \left[\frac{1 - \cos p_0 - \cos p_1 + \cos g_2}{p} + \frac{\cos g_0 - 1 + \cos g_1 - \cos g_2}{g} \right] \end{aligned} \qquad (4)$$

where

$$p = 2\pi\left(n + \frac{N}{1-\delta}\right), \qquad g = 2\pi\left(n - \frac{N}{1-\delta}\right). \qquad \delta = \frac{2l}{L}$$

$$p_0 = \pi(n+N) - \pi n\delta, \quad g_0 = \pi(n-N) - \pi n\delta$$

$$p_1 = 2\pi(n+N) - \pi n\delta, \quad g_1 = 2\pi(n-N) - \pi n\delta$$

$$p_2 = \pi(n+N), \qquad g_2 = \pi(n-N)$$

Now x_f is given by

$$x_f = -A \frac{1}{(1-\delta)^2}\left(\sum \left(\frac{a_n}{n^2} \sin \frac{2\pi}{L} ns + \frac{b_n}{n^2} \cos \frac{2\pi}{L} ns \right) \right) \qquad (5)$$

The amplitude of the forced motion has the order of magnitude

$$\left[\sum \left\{ \left(\frac{a_n}{n^2}\right)^2 + \left(\frac{b_n}{n^2}\right)^2 \right\} \right]^{\frac{1}{2}} \qquad (6)$$

Assuming $\delta \ll 1$, the above a_n and b_n are expanded in terms of δ. The first terms are

$n \neq N$

$$N+n = \text{even} \quad \left\{ \begin{aligned} a_n &\sim \frac{2\cos\varphi \, Nn}{n^2 - N^2}\delta \\ b_n &\sim \frac{2\sin\varphi \, n^2}{n^2 - N^2}\delta \end{aligned} \right. \qquad (7)$$

$$N+n = \text{odd} \qquad a_n \sim b_n \sim 0$$

$n = N$

$$\begin{aligned} a_N &\sim -\cos\varphi - \frac{\pi}{2} N\delta \sin\varphi - \cos\varphi \frac{\delta}{2} \\ b_N &\sim \sin\varphi - \frac{\pi}{2} N\delta \cos\varphi - \sin\varphi \frac{\delta}{2} \end{aligned} \qquad (8)$$

The second terms of a_N and b_N represent a phase shift and can be eliminated by introducing aphase shift of the ridges between the two halves of the circumference, namely

$\varphi \rightarrow \varphi + \varphi_0$ at $\pi r_1 + \ell < s < L - \ell$ with $\varphi_0 \sim \delta$.

In this way we eliminate the second term of each of (8) and get for the amplitude estimate

$$\bar{A} \sim \left\{ \sum_{n \neq N} \frac{1}{n^4} (a_n^2 + b_n^2) + \frac{\delta^2}{4} \right\}^{1/2}$$

$$= 2\delta \left\{ \sum_{n \neq N} \frac{1}{n^2} \frac{n^2 + N^2}{n^2 - N^2} + \frac{1}{16} \right\}^{1/2} \qquad (9)$$

For example N = 5 for a model gives

$$\bar{A} \sim 0.7\ \delta$$

Since the amplitude of unperturbed motion is $\sim \frac{1}{N^2}$ the ratio of the increase of amplitude to the original amplitude is given by

$$\bar{A} N^2 \sim 17.5\ \delta \qquad (N = 5)$$

If we introduce a finite ($\gg \delta$) phase shift of the ridges between halves of the circumference, the N + n = odd terms do not vanish. With a phase shift of π , the N + n = even terms vanish instead. Also if the sign of N is changed in half of the circumference, i.e. the direction of ridges is changed, then p and g are interchanged and both N + n = even and odd terms exist.

In the actual magnet, the straight sections are not field free and the integrals $\int_{\pi r_1}^{\pi r_1 + \ell}$ and $\int_{L-\ell}^{L}$ have a value of the order of δ. It may be possible to modify the fringing field to such a shape that the above amplitude of the forced motion can be reduced.

B) Betatron wave length

Since the forced motion is deformed appreciably as treated in (A), the wave length of betatron oscillations will be affected greatly. However, this effect is not considered here.

There is another source of perturbation on σ from inserting the straight sections as reported in the previous reports, MURA TBE/DWK - 1, DWK - 11. The matrix method was used for obtaining the estimate of the effect of the position of the straight sections relative to the ridges in these reports. The same effects are examined by using Symon's smooth approximation assuming that the wave length of betatron oscillations is much larger than the length of a sector or of a straight section. It is not always necessary that the length of the straight sections is much smaller than the sector length.

In Laslett's linearlized equation for Mark V

$$\frac{d^2 v}{d\theta^2} + [A + B \cos N\theta + C \cos 2N\theta] v = 0$$

$$A = k + 1 - \frac{1}{2} \frac{(f/w)^2}{N^2 - (k+1)}, \quad B = \frac{f}{w}, \quad C = \frac{1}{2} \frac{(\frac{f}{w})^2}{N^2 - (k+1)}$$

θ is replaced by $t = N\theta$ for convenience, so that we have

$$\frac{d^2V}{dt^2} + (\mu_0 + \mu_1 \cos t + \mu_2 \cos 2t)V = 0$$

$$\mu_0 = \frac{A}{N^2}, \quad \mu_1 = \frac{B}{N^2}, \quad \mu_2 = \frac{C}{N^2} \tag{10}$$

If a straight section is inserted in a sector as in Fig. 2, the new sector (whose length is 2π in units of τ) consists of the focussing and defocussing part with period 2π of t and the field free part having the width $2\pi\Delta$ in units of τ.

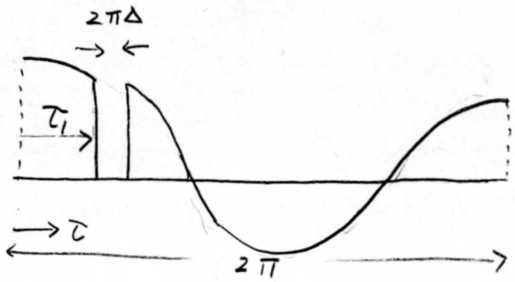

Fig.2

$$\tau = (1-\Delta)t$$

Equation (10) becomes

$$\frac{d^2V}{d\tau^2} + G(\tau)V = 0 \tag{11}$$

$$G(\tau) = \lambda_0 + \lambda_1 \cos\alpha\tau + \lambda_2 \cos 2\alpha\tau \qquad 0 < \tau < \tau_1$$

$$G(\tau) = 0 \qquad \tau_1 < \tau < \tau_1 + 2\pi\Delta$$

$$G(\tau) = \lambda_0 + \lambda_1 \cos\alpha(\tau - 2\pi\Delta) + \lambda_2 \cos 2\alpha(\tau - 2\pi\Delta)$$

$$\tau_1 + 2\pi\Delta < \tau < 2\pi$$

where

$$\lambda_0 = \alpha^2 \mu_0 \quad , \quad \lambda_1 = \alpha^2 \mu_1 \qquad \lambda_2 = \alpha^2 \mu_2$$

$$\alpha = \frac{1}{1-\Delta} \equiv 1 + \Delta'$$

This equation is written in the following form by expanding in Fourier series

$$\frac{d^2 V}{d\tau^2} + F(\tau) V = 0 \tag{12}$$

$$F(\tau) = a_0 + \sum a_j' \cos j'\tau + \sum b_j' \sin j'\tau$$

$$\begin{cases} a_0 = \lambda_0 (1-\Delta) \\ a_j' = \frac{1}{\pi} \int_0^{2\pi} G(\tau) \cos j'\tau \, d\tau \\ b_j' = \frac{1}{\pi} \int_0^{2\pi} G(\tau) \sin j'\tau \, d\tau \end{cases}$$

Applying the smooth approximation to equation (12),

$$\frac{d^2 V}{d\tau^2} + \left[a_0 + \frac{1}{2} \sum \left(\frac{a_j'^2}{j'^2} + \frac{b_j'^2}{j'^2} \right) \right] V = 0 \tag{13}$$

is obtained and σ_τ is given by

$$\left(\frac{\sigma_\tau}{2\pi}\right)^2 = \left[\, a_0 + \frac{1}{2}\left(\frac{a_j^2}{j^2} + \frac{b_j^2}{j^2}\right) \right] \tag{14}$$

The coefficients a_j and b_j are given by

$$\pi a_j = -\frac{\lambda_0}{j}\sin 2\pi j\Delta + \lambda_1\left[\frac{-1}{\alpha+j}\sin\pi j\Delta\cos\{(\alpha+j)\tau_1+\pi j\Delta\} + \frac{1}{\alpha-j}\sin\pi j\Delta\cos\{(\alpha-j)\tau_1-\pi j\Delta\}\right]$$
$$+\lambda_2\left[\frac{-1}{2\alpha+j}\sin\pi j\Delta\cos\{(2\alpha+j)\tau_1+\pi j\Delta\} + \frac{1}{2\alpha-j}\sin\pi j\Delta\cos\{(2\alpha-j)\tau_1-\pi j\Delta\}\right]$$

(15)

$$\pi b_j = \frac{\lambda_0}{j}(1-\cos 2\pi j\Delta) + \lambda_1\left[\frac{-1}{\alpha+j}\sin\pi j\Delta\sin\{(\alpha+j)\tau_1+\pi j\Delta\} - \frac{1}{\alpha-j}\sin\pi j\Delta\sin\{(\alpha-j)\tau_1-\pi j\Delta\}\right]$$
$$+\lambda_2\left[\frac{-1}{2\alpha+j}\sin\pi j\Delta\sin\{(2\alpha+j)\tau_1+\pi j\Delta\} - \frac{1}{2\alpha-j}\sin\pi j\Delta\sin\{(\alpha-j)\tau_1-\pi j\Delta\}\right]$$

In the case $\Delta \ll 1$, all a_j and b_j except a_1 and a_2 are of the order Δ and $\Delta \sim \Delta'$. Hence if only first order terms in Δ are kept in the expression for σ, only a_1 and a_2 need be considered.

$$a_1 \doteq \lambda_1 + 2\Delta\left[-\lambda_0 - \frac{\lambda_2}{2}\cos\tau_1 - \frac{\lambda_1}{4}\cos 2\tau_1 - \frac{\lambda_2}{6}\cos 3\tau_1\right]$$

$$a_2 \doteq \lambda_2 + 2\Delta\left[-\lambda_0 - \lambda_1\cos\tau_1 - \frac{\lambda_1}{3}\cos 3\tau_1 - \frac{\lambda_2}{4}\cos 4\tau_1\right] \tag{16}$$

Then σ_τ is given by

$$\left(\frac{\sigma_\tau}{2\pi}\right)^2 \doteq \lambda_0 + \frac{\lambda_1^2}{2} + \frac{\lambda_2^2}{8} + \frac{\Delta}{2}\left[-2\lambda_0 - 4\lambda_1\lambda_0 - \lambda_0\lambda_2 \right.$$
$$\left. + \lambda_1\lambda_2\cos\tau_1 - \lambda_1^2\cos 2\tau_1 - \lambda_1\lambda_2\cos 3\tau_1 - \frac{\lambda_2^2}{4}\cos 4\tau_1\right] \tag{17}$$

$$= \frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4} + \frac{\Delta}{2}\left[4\left(\frac{A}{N^2} + \frac{B^2}{N^4} + \frac{C^2}{4N^4}\right) - \frac{2A}{N^2} - \frac{2AB}{N^4} - \frac{AC}{N^4}\right.$$
$$\left. + \frac{BC}{N^4}\cos\tau_1 - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{BC}{N^4}\cos 3\tau_1 - \frac{C^2}{4N^4}\cos 4\tau_1\right]$$

or

$$\frac{\sigma_\tau}{2\pi} \doteq \left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2} + \frac{\Delta}{4\left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2}}\left[\frac{2A}{N^2} + \frac{4B^2}{N^4} + \frac{C^2}{N^4} - \frac{2AB}{N^4} - \frac{AC}{N^4}\right.$$
$$+ \frac{BC}{N^4}\cos\tau_1 - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{BC}{N^4}\cos 3\tau_1$$
$$\left. - \frac{C^2}{4N^4}\cos 4\tau_1\right] \tag{18}$$

If two straight sections are inserted in the circumference as in (A), ν_x becomes

$$\nu_x = (N-2)\left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2} + \left(\frac{\sigma_\tau}{2\pi}\right)_1 + \left(\frac{\sigma_\tau}{2\pi}\right)_2 \qquad (19)$$

N = even

$$\left(\frac{\sigma_\tau}{2\pi}\right)_1 = \left(\frac{\sigma_\tau}{2\pi}\right)_2$$

$$\nu_x = \nu_{x0} + \frac{\Delta}{2\left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2}}\left[\frac{2A}{N^2} + \frac{4B^2}{N^4} + \frac{C^2}{N^4} - \frac{2AB}{N^4} - \frac{AC}{N^4} + \frac{BC}{N^4}\cos\tau_1 - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{BC}{N^4}\cos 3\tau_1 - \frac{C^2}{4N^2}\cos 4\tau_1\right] \qquad (20a)$$

where $\nu_{x0} = N\left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2}$

N = odd

Since the τ_1's in each straight section differ by π, we get

$$\nu_x = \nu_{x0} + \frac{\Delta}{2\left[\frac{A}{N^2} + \frac{B^2}{2N^4} + \frac{C^2}{8N^4}\right]^{1/2}}\left[\frac{2A}{N^2} + \frac{4B^2}{N^4} + \frac{C^2}{N^4} - \frac{2AB}{N^4} - \frac{AC}{N^4} - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{C^2}{4N^4}\cos 4\tau_1\right] \qquad (20b)$$

They are written as follows, using $2\Delta \sim N\delta$

N = even

$$\nu_x - \nu_{x0} = \frac{N^2\delta}{4\nu_{x0}}\Big[\frac{2A}{N^2} + \frac{4B^2}{N^4} + \frac{C^2}{N^4} - \frac{2AB}{N^4} - \frac{AC}{N^4} + \frac{BC}{N^4}\cos\tau_1 - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{BC}{N^4}\cos 3\tau_1 - \frac{C^2}{4N^4}\cos 4\tau_1\Big] \quad (21a)$$

N = odd

$$\nu_x - \nu_{x0} = \frac{N^2\delta}{4\nu_{x0}}\Big[\frac{2A}{N^2} + \frac{4B^2}{N^4} + \frac{C^2}{N^4} - \frac{2AB}{N^4} - \frac{AC}{N^4} - \frac{B^2}{N^4}\cos 2\tau_1 - \frac{C^2}{4N^4}\cos 4\tau_1\Big] \quad (21b)$$

Example for a model

$N = 5$, $k = 0.6$, $\frac{\ell}{w} = 6.0$

$\nu_{x0} = 1.245$

$$\nu_x - \nu_{x0} = 5.020\left[0.2809 - 0.0576\cos 2\tau_1 - 0.000236\cos 4\tau_1\right]\delta \quad (22)$$

In this case the odd terms i.e. $\cos\tau_1$, $\cos 3\tau_1$, in σ_τ are much smaller than $\cos 2\tau_1$ term. Hence the part depending on τ_1 in the $\nu_x - \nu_{x0}$ can be reduced by inserting the straight sections in each quadrant of the circumference since N is odd and $\cos 2\tau_1$ term is eliminated by a phase shift $\pi/2$ of the ridges.

The above estimate is only the part due to the field free space in a sector and ν_x may be greatly changed by the deformation of the forced motion and the edge effect. It will be more effective to use Illiac in order to examine these effects.