-1-

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C-178-1

DISCLAIMER

NOTE ON THE NON-LINEAR KINETICS OF CIRCULATING-FUEL REACTORS

I. Introduction

The equations of motion for a somewhat idealized circulating-fuel reactor have been set up and discussed in several previous reports[(1)]. The assumptions made are, in essence, that the instantaneous power density is constant along the length of the fuel tubes, that the excess reactivity depends only upon the average fuel temperature, and that the fuel temperature at the inlet is kept constant. The effect of delayed neutrons is ignored.

It was suspected that its negative temperature coefficient makes this reactor inherently stable even without the damping provided by delayed neutrons. This was suggested in a crude way in reference (1) and subsequently proved in the linear approximation[(2)].

Specifically, it was proved that for the linear case no antidamped solutions exist, while from equation 4 of reference (2) it follows that the only undamped solutions are periodic with period equal to, or a submultiple of, Θ, the fuel transit time. It is the purpose of this note to prove the above statements in the non-linear case.

II. Analysis of the Motion

The equations of motion for the idealized system under consideration are:

$$\frac{1}{P}\frac{dP}{dt} = -\frac{\alpha}{\tau} T \tag{1}$$

$$\frac{dT}{dt} = \epsilon P - \frac{\epsilon}{\Theta}\int_{t-\Theta}^{t} P(\sigma)\,d\sigma \tag{2}$$

(1) Y-F10-98; ORNL-1227, p. 41.

(2) Y-F10-99.

where

- $P(t)$ = reactor power
- α = temperature coefficient of reactivity
- ϵ = reciprocal heat capacity
- Θ = fuel transit time
- τ = prompt generation time
- T = average fuel temperature

For convenience, we define T as the excess over the equilibrium temperature; i.e., for the steady state $T = 0$, $P = P_0$.

T can be eliminated from the equations of motion giving

$$-\frac{\alpha}{\tau}\frac{dT}{dt} = \frac{d^2}{dt^2}\log P = -\lambda\left\{P(t) - \frac{1}{\Theta}\int_{t-\Theta}^{t} P(\sigma)\,d\sigma.\right\} \qquad (3)$$

when $\lambda = \frac{\alpha\epsilon}{\tau}$. A simple quadrature puts (3) into a more convenient form. Note that

$$\frac{d}{dt}\int_0^{\Theta} P(t+\eta-\Theta)\,\eta\,d\eta = \int_0^{\Theta}\left[\frac{d}{d\eta}P(t+\eta-\Theta)\right]\eta\,d\eta$$

$$= \Theta P(t) - \int_0^{\Theta} P(t+\eta-\Theta)\,d\eta = \Theta P(t) - \int_{t-\Theta}^{t} P(\sigma)\,d\sigma.$$

We now integrate equation (3) to give

$$\frac{d}{dt}\log P = -\frac{\lambda}{\Theta}\int_0^{\Theta} P(t+\eta-\Theta)\,\eta\,d\eta + \text{const.}$$

Since the instantaneous power and temperature of the reactor are determined completely by the history of the system over the preceding fuel cycle, the integration constant must be independent of the initial conditions. To give the correct steady state solution the equation may be written

$$\frac{d}{dt}\log P = -\frac{\lambda}{\Theta}\int_0^{\Theta}\left[P(t+\eta-\Theta) - P_o\right]\eta\, d\eta. \qquad (4)$$

A. Boundedness of the Solutions

It is easy to show that for a given set of reactor constants P and T are bounded. Suppose that at $t = t_o$ $P = P_o$ and $\frac{dP}{dt} > 0$. So long as $\dot{P}$ remains positive $P > P_o$ and before t reaches $t_o + \Theta$ the integrand on the right hand side of (4) becomes positive thereby changing the sign of $\dot{P}$. Thus P must reach a maximum at $t = t_o + \Theta$ at the latest. But since $P(t) \geq 0$,

$$\frac{d}{dt}\log P \leq P_o \frac{\lambda\Theta}{2}$$

so that

$$P \leq P_o e^{P_o \lambda\Theta^2/2}.$$

This establishes an upper bound on P and proves that (4) has no antidamped solutions. A similar argument shows that P is bounded from below so that the reactor cannot shut itself off.

B. Asymptotic Solutions

We will now examine the solutions of (4) for large values of t, after the transients due to starting up have died away. The use of transform methods, while in general incapable of providing a complete solution of the non-linear problem, does give some insight into the properties of the solutions.

Differentiating equation (3) we get

$$\frac{d^3}{dt^3}\log P = -\lambda\left\{\frac{dP}{dt} - \frac{P(t) - P(t-\Theta)}{\Theta}\right\}. \qquad (5)$$

We take the Laplace transform of both sides and let

$$L\left\{P\right\} = \phi(s) \text{ and } L\left\{\log P\right\} = \psi(s)$$

for the time being considering them as independent. Note that

C-178-4

$$L\left\{P\ (t-\Theta)\right\} = \int_0^\infty e^{-st}\, P(t-\Theta)dt$$

$$= e^{-s\Theta}\, \phi(s) + \int_{-\Theta}^{0} e^{-st}\, P(t)dt.$$

Equation (5) then takes on the form

$$s^3\, \psi(s) + \lambda\left\{s - \frac{1-e^{-s\Theta}}{\Theta}\right\}\phi(s)$$

$$= \lambda\left\{A + \frac{e^{-s\Theta}}{\Theta}\int_{-\Theta}^{0} P(t)\, e^{-st} d\, t\right\} + s^2 B + s\, C + D$$

where A, B, C, D are constants determined by the initial conditions.

The time dependence of P and log P are given by the singularities of ϕ and ψ respectively. If $\phi(s)$ is regular ψ can be singular only at $s = 0$. For ϕ to be singular at a regular point of ψ we must have

$$1 - e^{-s\Theta} - s\Theta = 0.$$

Letting $s = u+iv$ we get

$$u\Theta = 1 - e^{-u\Theta}\cos v\Theta$$

$$v\Theta = e^{-u\Theta}\sin v\Theta$$

which can be satisfied only if $u < 0$ or $s = 0$. Thus all points at which ϕ is singular and ψ regular lie in the open left half plane or at $s = 0$.

Therefore if ϕ is singular on the imaginary axis, that point is also a pole of ψ. This statement means that for the asymptotic power the Fourier expansions of P and log P contain the same frequencies.

For sufficiently large t the transient part of ϕ can be neglected and we have

$$P = \sum_n a_n\, e^{i\omega_n t} \qquad \log P = \sum_n b_n e^{i\omega_n t}$$

where the a_n, b_n and ω_n are as yet undetermined.

Differentiating, we find

$$\sum_n i\omega_n b_n e^{i\omega_n t} = \frac{\sum i\omega_n a_n e^{i\omega_n t}}{\sum a_n e^{i\omega_n t}} .$$

Multiplying through and using the orthogonality of the circular functions, we get

$$\omega_r a_r = \sum_n \omega_n b_n a_{r-n} \tag{6}$$

The b_n can now be eliminated from the equations of motion. Equation (4) may be written

$$\sum i\omega_n b_n e^{i\omega_n t} = -\frac{\lambda}{\Theta} \int_0^\Theta \left[\sum_n a_n e^{i\omega_n(t+\eta-\Theta)} - P_o \right] \eta \, d\eta$$

which becomes, after some manipulation

$$i\omega_n b_n = -\frac{\lambda}{\Theta} \left[a_n \frac{1 - i\omega_n\Theta - e^{-i\omega_n\Theta}}{\omega n^2} - \frac{P_o\Theta^2}{2} \delta_{n,o} \right] . \tag{7}$$

Note that in the linear approximation $a_n = b_n$ ($\eta \neq 0$) and this reduces to equation (3) of reference 2. The last term in the bracket just removes the second order pole of ψ at the origin which would give divergent solutions for P. This term comes from the constant of integration and would not have appeared had we started with equation (3) or (5). Then a special argument would be needed to exclude the diverging solutions.

Passing to the limit of $\omega \to 0$ we find

$$a_o = P_o \tag{8}$$

so that, as we would expect, P_o is the average asymptotic power. Note that b_o is indeterminate, corresponding to the arbitrariness in units of P in the logarithm. With no loss of generality we may set $b_o = 0$.

Substituting from (7) into (6) we have

$$\omega_r a_r = \frac{i\lambda}{\theta} \sum_n a_{r-n} a_n \left(\frac{1 - i\omega_n\theta - e^{-i\omega_n\theta}}{\omega_n^2} - \frac{\theta^2}{2} \delta_{n,o} \right) . \tag{9}$$

This is of the form $\omega_r a_r = i \sum_n a_{r-n} a_n f_n$ where

the f_n have the property $f_n = f_{-n}^*$ (we order the ω's so that $\omega_{-n} = -\omega_n$). Furthermore, since P is real $a_{-r} = a_r^*$.

Consider the case $r = 0$.

$$\sum_n \left| a_n \right|^2 f_n = \frac{1}{2} \sum_n \left| a_n \right|^2 (f_n + f_{-n}) = \sum_n \left| a_n \right|^2 \mathrm{Re}(f_n) = 0.$$

$$\text{But } \mathrm{Re}(f_n) = \mathrm{Re} \frac{1 - i\omega_n - e^{-i\;{}_n\theta}}{\omega_n^2} \geq 0 \quad \text{for } n \neq 0$$

$$= 0 \qquad \text{for } n = 0.$$

and $\left| a_n \right|^2 \geq 0$. For the sum of a sequence of non-negative terms to be zero each term must separately vanish so that either $a_n = 0$ or $\mathrm{Re}(f_n) = 0$. In the latter case $e^{i\omega_n\theta} = 1$ or

$$\omega_n = \frac{2\pi n}{\theta} \tag{10}$$

This proves the desired result, namely that the asymptotic power is a periodic function of time and that the period is an integral submultiple of θ. Such a periodic variation of P makes the right hand side of (3) constant in time corresponding to the case of constant power extraction.

III. Numerical Estimates

While we have shown that the reactor equations have no antidamped solutions, this is not, from a practical standpoint, a guarantee of the safety of the reactor. The equations may admit oscillations of such large amplitude as to exceed the design limits of the system. For this reason it is desirable to obtain some estimate of the largest possible power and temperature excursions.

The method of section IIA provides an upper bound on the power which is much too crude. Somewhat improved estimates may be obtained by various analytical devices, but such bounds are still too high to be of much use. It was therefore decided to integrate numerically the equation of motion with initial conditions so chosen as to produce violent oscillations and see just how big the power and temperature surges can become. Two such integrations were carried out using the design constants of an ANP reactor.

$\alpha = 10^{-4}/^{o}C$

$\tau = 10^{-4}$ sec

$\epsilon = 2.67 \times 10^{-6}$ ^{o}C/watt sec

$P_o = 3 \times 10^{8}$ watts

$\theta = 1/8$ sec

One integration referred to the case in which the power is $4\ P_o$ for all times $t < 0$. At $t > 0$, Eq. (4) applies. The other case involved normal power P_o at all times up to 1/10 of a transit time before $t = 0$. For a period of 1/10 transit time just before $t = 0$, the power is $10\ P_o$. At $t > 0$, Eq. (4) is valid. The power and temperature as a function of time were calculated out through the first overswing. The results are shown in Figures 1 and 2. The transit time is taken as unit of time. The ordinate on the right side is P in units of P_o, and the ordinate on the left is the average temperature T in units of $\tau/\alpha\theta$. It can be seen that the extreme values of P and T were nearly exactly the same for the two cases, indicating that they are close to the most extreme values obtainable.

For the above reactor constants we find that P is bounded by

$$.013 < \frac{P}{P_o} < 2.1$$

while the bounds on T are

$$-50^{o}C < T < +26^{o}C.$$

The similar shape and limiting values of the curves in the two above cases is easy to understand. After the large initial disturbance, the reactor almost shuts itself off. The period of very low power is essentially the same in both cases, and it determines the future time behavior of the reactor.

For this reason, the following fictitious limiting case is plotted in Figure 3 in the same units as Figures 1 and 2. The power is zero for $t < 0$. At $t = 0$, the power is suddenly brought to P_o, and thereafter Eq. (4) takes over. Even in this case, the power and temperature oscillation is not much more violent than in Figures 1 and 2.

10

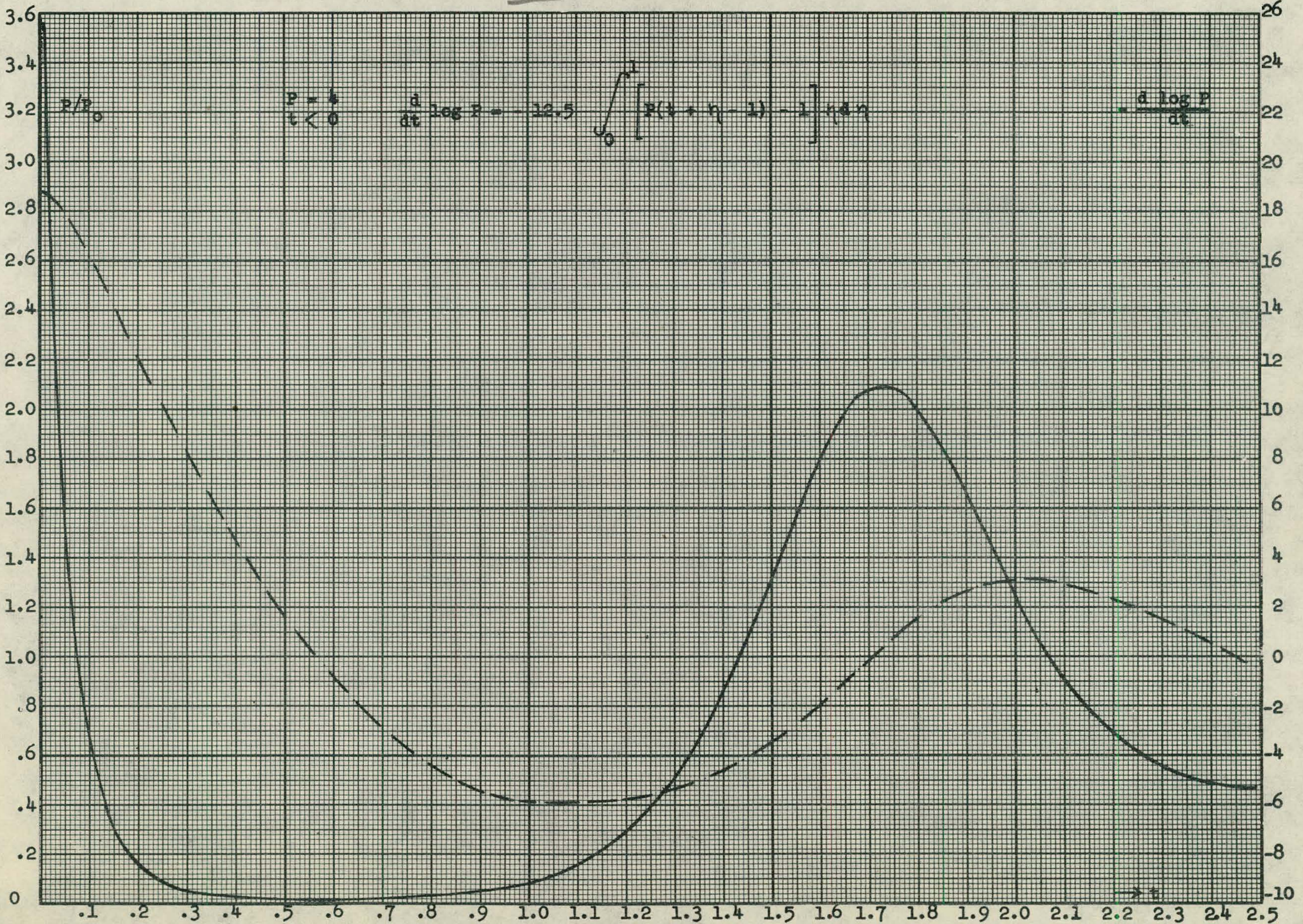

Figure 1

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11

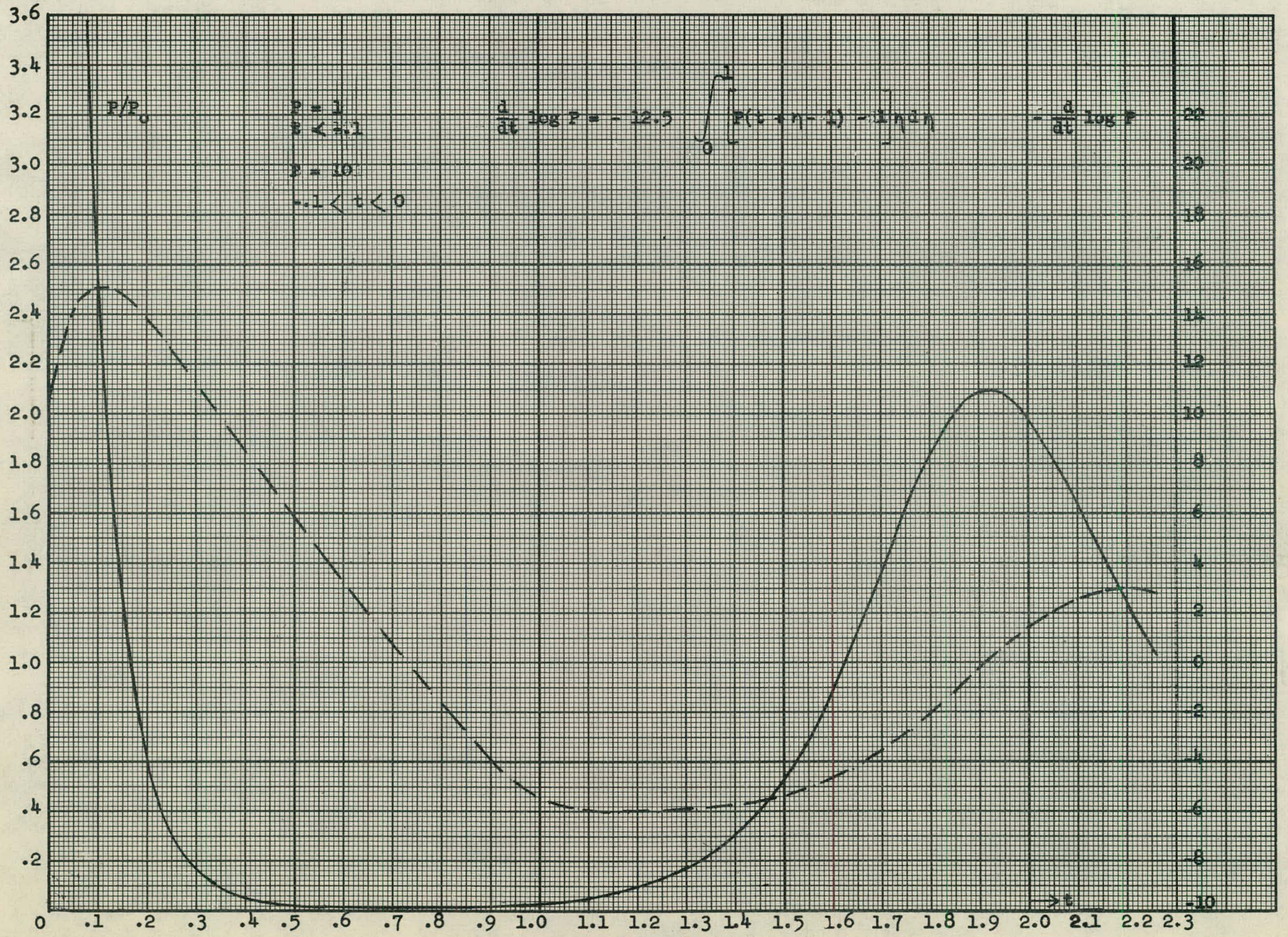
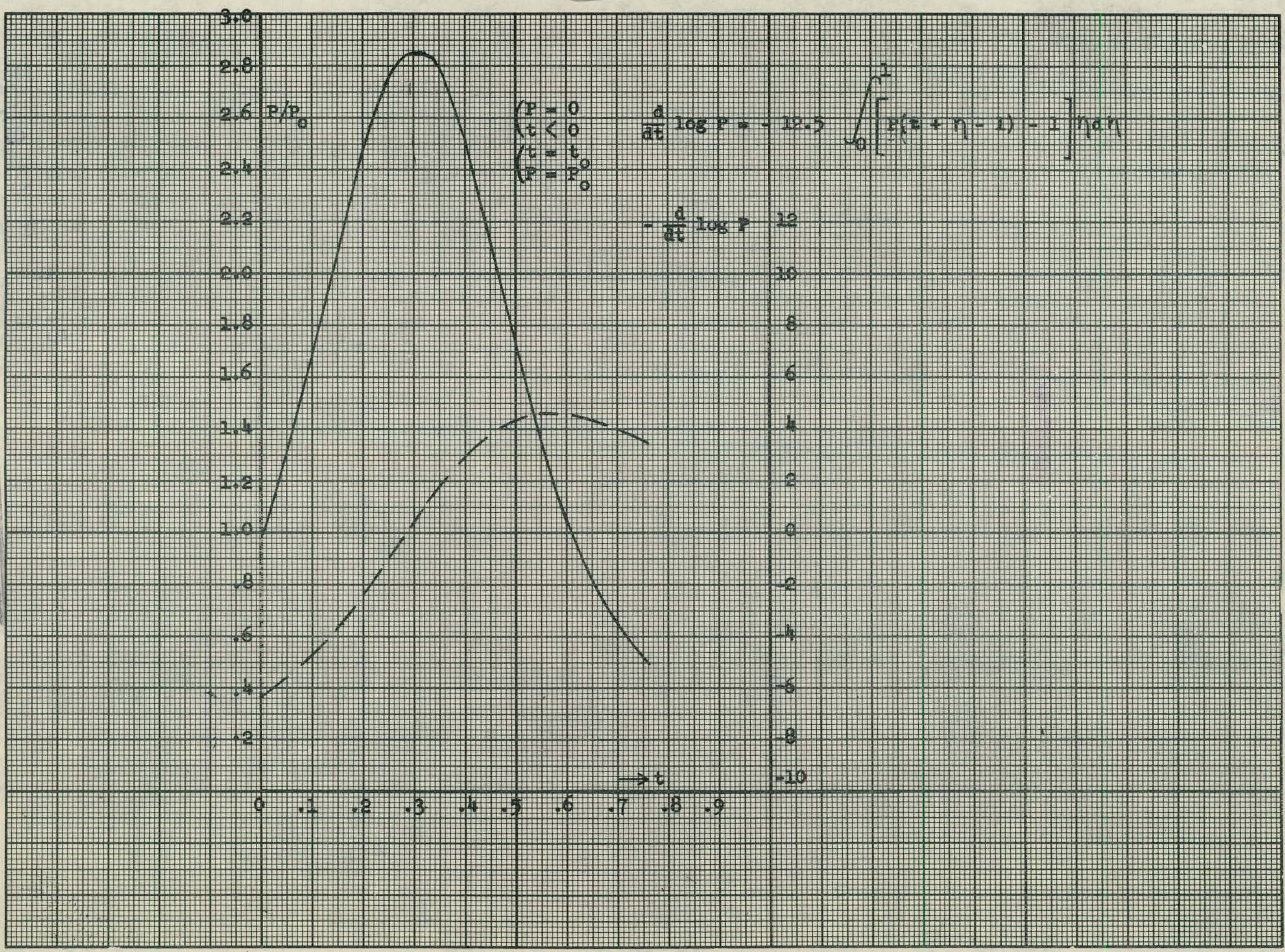


Figure 2

C-178-11

11

12



P/Po
(P = 0
(t < 0
(t = to
(P = Po
d/dt log P = - 12.5 ∫0^1 [P(t + η - 1) - 1] η dη
- d/dt log P
→ t
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
0 .1 .2 .3 .4 .5 .6 .7 .8 .9

Figure 3

1-12

5766

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