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THE THEORETICAL PHYSICS OF THE ARGONAUT REACTOR

by
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THE THEORETICAL PHYSICS OF THE ARGONAUT REACTOR

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I. INTRODUCTION

The Argonaut is a versatile, low-power reactor designed and constructed at Argonne for use as a training facility and for the conduct of experiments in reactor physics.

The Argonaut assembly is comprised of a graphite cube with a central annulus formed by two concentric aluminum tanks. The inner tank contains a removable, graphite reflector. The outer tank, or core annulus, contains fuel assemblies, graphite, and water.

The fuel plates are made of 20% enriched U_3O_8 in an aluminum matrix and are clad with aluminum. A fuel assembly is composed of a cluster of 17 plates. Moderation is provided by graphite wedges between clusters and by water between fuel plates in each cluster.

The fuel loading is extremely flexible in principle. The annulus contains 24 graphite wedges outlining a like number of spaces of 3 x 6 in. cross section. Theoretically, each of these spaces could be filled with fuel assemblies or graphite blocks of equal cross section.

This report is a summary of the methods used (with some degree of success) to compute the critical properties of four typical core loadings shown in Fig. 1. Each loading requires a different type of solution of the critical problem.

The one-and-two-sided loadings can be approximated as slab arrangements. The 3-inch annular loading can be treated as a homogeneous annulus. The "two by six" loading can be regarded as an array of six equally spaced homogeneous cylinders imbedded in graphite. In treating each case the uranium content will be assumed to be pure U^{235}. (The addition of U^{238} in the amount four parts U^{238} to one part U^{235} will increase the critical loading by about 10%.)

In each instance, the composition will be established and group constants computed; then the critical problem will be solved for each case. Finally, a comparison with the results of Argonaut multiplication experiments will be made.

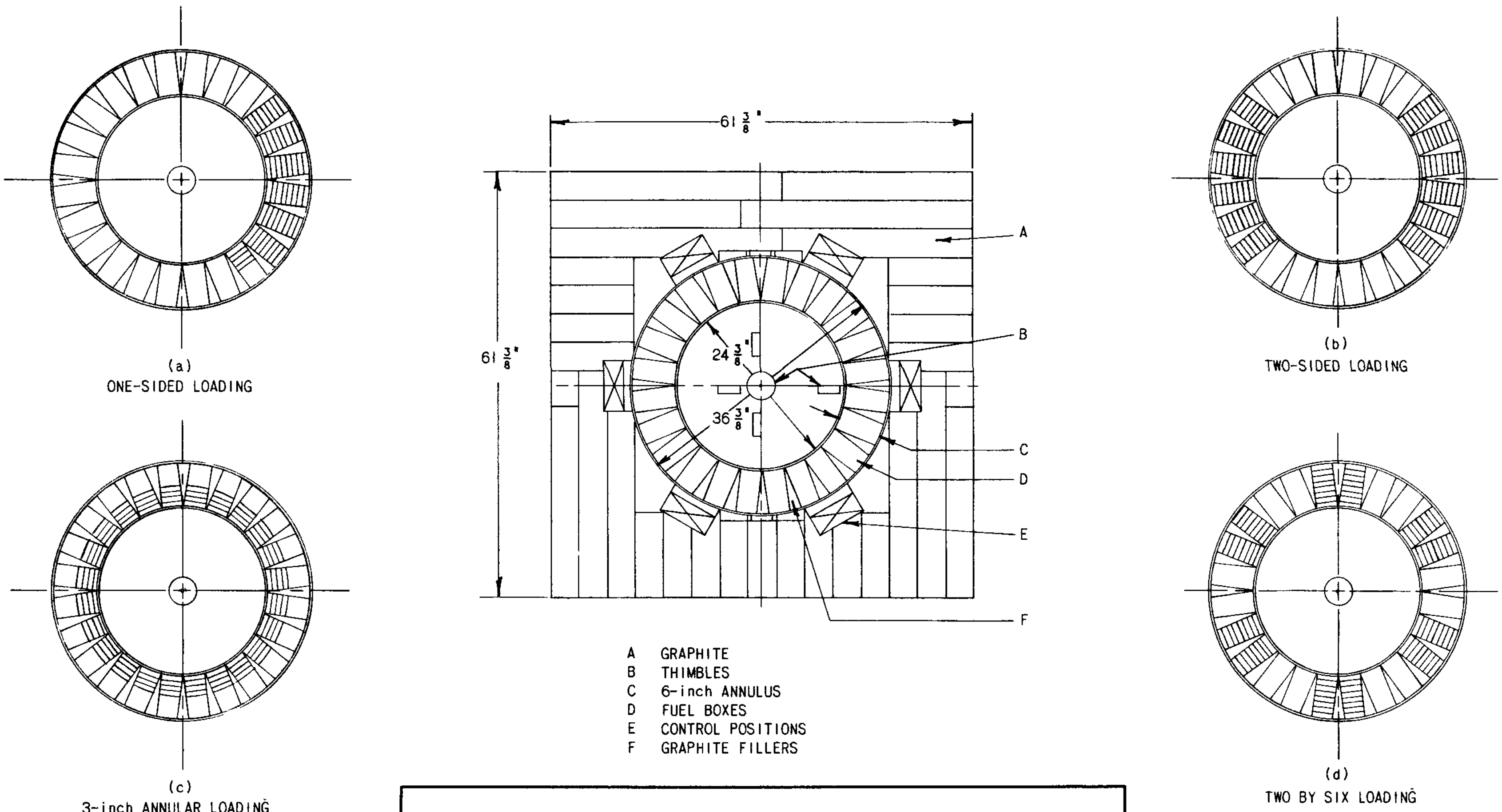

FIG. I
ARGONAUT LATTICE CONFIGURATIONS

II. COMPOSITION AND GROUP CONSTANTS

The internal and external reflectors are graphite, as are the fillers in the annulus. The fuel plates are aluminum and U^{235}. The moderator is water. One triangular, graphite filler piece is associated with each fuel box. The volume fractions of the various materials in the fuel region are listed in Table I.

Table I

COMPOSITION OF FUEL REGION

(Total volume of one fuel box: 8.34333 liters)

Material	Volume Fraction
Al	0.21952 (1 - X)
H_2O	0.52706 (1 - X)
C	0.25342
U^{235}	X (0.74658)

Two-group methods are employed throughout. The thermal group constants are taken from BNL-325[1] and are corrected for a Maxwellian distribution with a temperature of 25C. A non-(1/v) factor for the U^{235} cross section has been included. The thermal cross sections are listed in Table II.

Table II

THERMAL-GROUP CROSS SECTIONS

Material	Atom/cc	Σ_a (capture cross section, cm^{-1})	$\Sigma_{tr}(cm^{-1})$
H_2O	0.033326	0.01949	2.106
Al	0.060297	0.01229	0.0823
C	0.08024	0.00036	0.36913
U^{235}	0.04790	28.60836	

$\eta = 2.08$
$\alpha = 0.183$

[1] D. J. Hughes, J. A. Harvey, BNL-325 "Neutron Cross Sections," Brookhaven National Laboratory (July 1, 1955).

The fast-group cross sections are not so easily computed. An approximate method of some utility is to regard the age τ and fast diffusion coefficients D_f as universal functions of the water density and the excess scattering per water molecule. A reasonable estimate of the average scattering cross section for Al in the high-energy range is 4.28 b, and 3 b is a good estimate for C. Values of the age and D_f in Al-H_2O mixtures[2] provide calibration points.

In the following tabulation τ_R is the age reduced by the square of the water density. D_{fR} is the diffusion coefficient reduced by the water density, and S is the excess scattering per water molecule in barns.

S (barns)	τ_R (cm^2)	D_{fR} (cm)
3.87	22.9	0.804
5.81	20.96	0.709
7.75	19.95	0.637

From the volume fractions given in Table I and the concentrations given in the second column of Table II, it is found that the excess scattering per water molecule is 6.70 b. Linear interpolation yields: $\tau_R = 20.50\ cm^2$; $D_{fR} = 0.676$ cm. From the water density of 0.52706 g/cm^3 one quickly finds $\tau = 73.80\ cm^2$, $D_f = 1.28$ cm.

The age and fast diffusion coefficient can also be calculated by the methods of Deutsch.[3] Using these methods a value of $D_f = 1.25$ cm is obtained, in good agreement with the previous result; the age, however, is determined as 60 cm^2, in contrast to the 73.8 cm^2 found by the approximate method. A numerical integration scaled to the correct age in water yields: $\tau = 65\ cm^2$; $D_f = 1.27$ cm. The last set of values represents a compromise, but subsequent calculations will show that the age is slightly excessive. This favors the set: $\tau = 60\ cm^2$; $D_f = 1.25$ cm.

In the graphite reflector the fast diffusion coefficient is taken to be 1.1 cm and the age as 364 cm^2. Vertical reflector savings from the 1-foot water reflector top and bottom are taken as 15 cm. This yields an equivalent bare height of 75 cm. The macroscopic cross sections can now be computed and the relationship sought between loading and material buckling. Thus the following values are found.

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[2] "The Reactor Handbook," Vol. 1 (Physics), p. 486

[3] R. W. Deutsch, "Computing 3-Group Constants for Neutron Diffusion," Nucleonics, 15(1), pp. 47-51 (1957).

The moderator cross section:

$$\Sigma_{am} = 0.01297\,(1 - X) + 0.00009\ .$$

The U^{235} cross section:

$$\Sigma_{aU} = 28.60836X\ .$$

The total cross section:

$$\Sigma_a = 0.01306 + 28.59539X\ .$$

$$k = \eta f = \frac{(2.08)\,(\Sigma_{aU})}{\Sigma_a} = \frac{2.08094}{1 + \dfrac{0.45671}{(X)(10^3)}} \tag{1}$$

$$D_{th(core)} = \frac{1}{3\,(1.22160)} = 0.27286\ \text{cm}$$

The two-group critical condition is

$$(1 + \tau B^2)\,(1 + L^2B^2) = k_\infty\ , \tag{2}$$

which yields for the buckling

$$B^2 = -\frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{L^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{\tau} + \frac{1}{L^2}\right)^2 + 4\,\frac{k_\infty - 1}{L^2\tau}} \tag{3a}$$

and for the thermal diffusion area

$$L^2 = \frac{D_s}{0.01306 + 28.59539X}\ . \tag{3b}$$

Inverting the formulae,

$$L^2 = \frac{\eta - (1 + \tau B^2)}{\eta\,\dfrac{\Sigma_{am}}{D_s} + B^2\,(1 + \tau B^2)} \tag{4a}$$

$$\Sigma_{aU} = \frac{(1 + \tau B^2)\,(\Sigma_{am} + D_s B^2)}{\eta - (1 + \tau B^2)} \tag{4b}$$

The stage is now set for a series of critical calculations. In the succeeding sections three different techniques are developed, corresponding to the various geometries.

III. CRITICAL PROPERTIES OF AN ANNULUS

While straightforward solution of the two-group equations for the reflected annulus through matrix reduction is possible, such a technique is fraught with numerical problems arising from a need for excessive accuracy. A difference equation method, suitable for manual computation, was therefore developed. This technique by itself is not new, but the equations and boundary conditions used here differ from those previously employed and thus are displayed in full.

Consider first the core region $a \leq r \leq b$, where r is the radius. If the vertical buckling is neglected, for the moment, then the neutron diffusion equations with a source, Q, are

$$\frac{1}{r}\frac{d}{dr}(rJ) + \Sigma\phi = Q \quad ; \qquad J = -D\frac{d\phi}{dr} \quad , \tag{5}$$

where J is the current, D the diffusion coefficient, and ϕ the flux. Now let

$$\left.\begin{aligned} r_i &= r_0 + i\,\frac{(b-a)}{I} \quad ; \qquad r_0 = a \\ &\text{and} \\ \Delta r &= \frac{(b-a)}{I} \quad , \end{aligned}\right\} \tag{6}$$

where I is the number of equal intervals into which (a,b) is divided. Next assume that the flux may be represented by

$$\left.\begin{aligned} \phi(r) &= \phi_i + \frac{r - r_i}{\Delta r}(\phi_{i+1} - \phi_i) \qquad r_i \leq r \leq r_{i+1} \\ &\text{and that} \\ J(r) &= J_i + \frac{r - r_i}{\Delta r}(\phi_{i+1} - \phi_i) \qquad i = 0, 1, \ldots, I-1 . \end{aligned}\right\} \tag{7}$$

Substituting into Eq. (5) yields:

$$\left.\begin{aligned} \phi_{i+1} &= \phi_i - \frac{\Delta r}{2D}(J_{i+1} + J_i) \\ J_{i+1} &= J_i + \lambda_i(\alpha_i J_i - \beta_i \phi_i + q_i) \end{aligned}\right\} \tag{8}$$

The functions $\lambda_i, \alpha_i, \beta_i, q_i$ are functions of r_i and the material constants:

$$\lambda_i = \frac{\Delta r}{1 - \theta(2 + C_1)} = \frac{\Delta r}{C_3} \qquad C_1 = \frac{r_i}{r_{i+1}}$$

$$\alpha_i = S(2 + C_1) - C_2 \qquad C_2 = \frac{1}{r_{i+1}}$$

$$\beta_i = \Sigma_a^* \frac{1 + C_1}{2} \qquad \Sigma_a^* = \Sigma_a + DB_z^2$$

$$q_i = \gamma_i Q_{i+1} + \epsilon_i Q_i \qquad L^2 = \frac{D}{\Sigma^*}$$

$$\gamma_i = \frac{2 + C_1}{6} \qquad \theta = \frac{(\Delta r)^2}{12 L^2}$$

$$\epsilon_i = \frac{1 + 2C_1}{6} \qquad S = \frac{\Delta r}{6L^2}$$

Here B_z^2 is the buckling along the axis.

The use of Σ_a^* instead of Σ_a introduces the effect of the "equivalent bare height" of the reactor when it is assumed that the flux is the product $\Phi = \phi(r) \cos[B_z(Z)]$.

The boundary conditions can be expressed in the form $J = m\phi + d$. To see this, examine the case of the annulus with internal and external reflectors. In the interval $0 \leq r \leq a$, the fluxes have the analytical form:

$$\phi_f = AI_0(\mu_f r)$$

$$\phi_{th} = SAI_0(\mu_f r) + BI_0(\mu_{th} r) \qquad (9)$$

where the subscripts f and th refer to fast and thermal neutron groups, and μ_f and μ_{th} are material constants. Then in the interval $0 \leq r \leq a$,

$$J_f = -D_f \operatorname{grad} \phi_f$$

$$= -D_f \mu_f \frac{I_1(\mu_f r)}{I_0(\mu_f r)} \phi_f(\mu_f r) \quad ;$$

hence at r = a:

$$J_f = m_f \phi_f \quad , \qquad (10)$$

where

$$m_f = -\mu_f D_f \frac{I_1(\mu_f a)}{I_0(\mu_f a)} .$$

Similarly, at r = a:

$$J_{th} = m_{th}\phi_{th} + S\left(\frac{D_{th}}{D_f} m_f - m_{th}\right)\phi_f \tag{11}$$

where

$$m_{th} = -\mu_{th} D_{th} \frac{I_1(\mu_{th} a)}{I_0(\mu_{th} a)} .$$

The conditions at r = b are obtained in the same fashion. For an infinite reflector:

$$J_f = m_f' \phi_f \quad , \tag{12}$$

where

$$m_f' = \mu_f D_f \frac{K_1(\mu_f b)}{K_0(\mu_f b)} .$$

Also

$$J_{th} = m_{th}' \phi_{th} + S\left(\frac{D_{th}}{D_f} m_f' - m_{th}'\right)\phi_f \quad , \tag{13}$$

where

$$m_{th}' = \mu_{th} D_{th} \frac{K_1(\mu_s b)}{K_0(\mu_{th} b)} .$$

For a finite reflector of outer radius c, replace the function K_0 by

$$K_0(\mu r) - \frac{K_0(\mu c)}{I_0(\mu c)} I_0(\mu r)$$

and K_1 by

$$K_1(\mu r) + \frac{K_0(\mu c)}{I_0(\mu c)} I_1(\mu r) .$$

A complete description of the numerical procedure for solving Eqs. (8), together with Eqs. (10) and (12), is given in ANL-5687.[4] The example given here is that of a 3-in. thick annulus with an inner radius of 1 ft.

[4] B. I. Spinrad, C. N. Kelber, "A Two-Group Iteration Method for Annular Cylinders" ANL-5687 (February, 1957).

The group constants are listed in Section I. Tentatively, $\tau = 65\ cm^2$ and $D_f = 1.27$ cm. The core volume is 95.426 liters. The volume fraction of the core for 1 kg of U^{235} at 18.7 gm/cc is then

$$X/kg = \left(\frac{1}{18.7}\right)\left(\frac{1}{95.426}\right) = 5.603 \times 10^{-4}/kg \quad .$$

ARGONAUT ANNULAR CORE CONSTANTS

$X = 5.603 \times 10^{-4}$ M (M in kg)

$X = 2.2412 \times 10^{-3}$ at M = 4 kg

$\Sigma_{au} = 0.06412\ cm^{-1}$

$\Sigma_{am} = 0.01303\ cm^{-1}$

$\Sigma_a = 0.07715\ cm^{-1}$

$\Sigma_{af} = 0.01953\ cm^{-1}$

$k_\infty = \frac{2.08094}{1.20377} = 1.72868$

$D_{th} = 0.27286$ cm

$\tau = 65\ cm^2$

$D_f = 1.27$ cm

<u>Inner Reflector</u>

$\mu_f^2 = \frac{1}{\tau} + B_z^2 = 0.00448\ cm^{-2}$ $\qquad \mu_f = 0.06693\ cm^{-1}$ $\qquad D_f = 1.1$ cm

$\mu_{th}^2 = \frac{1}{L^2} + B_z^2 = 0.00215\ cm^{-2}$ $\qquad \mu_{th} = 0.04637\ cm^{-1}$ $\qquad D_{th} = 0.903$ cm

$$S_R = \left(\frac{D_f}{D_{th}}\right)\left(\frac{L^2}{\tau - L^2}\right) = \left(\frac{+1.1\ (2500)}{0.903}\right)\left(\frac{1}{\tau - L^2}\right) = \left(\frac{+1.1}{0.903}\right)\left(\frac{2500}{-2135}\right) = -1.42641$$

$I_0(2.0079) = 2.29219$

$I_1(2.0079) = 1.60240$

$I_0(1.3911) = 1.54554$

$I_1(1.3911) = 0.87793$

$K_0(2.50988) = 0.061622$

$K_1(2.50988) = 0.072988$

$K_0(1.73888) = 0.15757$

$K_1(1.73888) = 0.19848$

$I_0(2.50988) = 3.31482$

$I_1(2.50988) = 2.53937$

$\mu_f(30) = 2.0079$

$\mu_{th}(30) = 1.3911$

$\mu_f(37.5) = 2.50988$

$\mu_{th}(37.5) = 1.73888$

$m_f = -0.05942$; $m_f' = 0.08996$

$m_{th} = -0.02378$; $m_{th}' = 0.05846$

$d_{th} = +0.03565\ \phi_f$; $d_{th}' = -0.02194\ \phi_f$

$I_0(1.73888) = 1.91136$

$I_1(1.73888) = 1.24217$

Outer Reflector

$c = b + 30 \text{ cm} = 67.5 \text{ cm}$ $\mu_f c = 4.51778$ $\mu_{th} c = 3.12998$

$$\frac{K_0(\mu_f c)}{I_0(\mu_f c)} = \frac{0.0062753}{17.7570278} = 0.0003523 \qquad \frac{K_0(\mu_{th} c)}{I_0(\mu_{th} c)} = \frac{0.02993}{5.42596} = 0.00551$$

$$\phi_f = A(K_0 - 0.0003523\ I_0)$$

$$J_f = D_f \nabla\phi_f = -AD_f\mu_f\ (-K_1 - 0.0003523 I_1)$$

$$= AD_f\mu_f\ (K_1 + 0.0003523 I_1)$$

$$= D_f\mu_f\ \frac{K_1(\mu_f b) + 0.0003523\ I_1(\mu_f b)}{K_0(\mu_f b) - 0.0003523\ I_0(\mu_f b)}\ \phi_f$$

$$m_f' = \mu_f D_f\ \frac{K_1 + 0.0003523\ I_1}{K_0 - 0.0003523\ I_0}$$

m_{th}' similarly

$$m_f' = \mu_f D_f\ \frac{0.7388}{0.06046} = 0.08996$$

$$m_{th}' = +\mu_{th} D_{th}\ \frac{0.20532}{0.14704} = 0.05846$$

$$d_s' = -0.02194$$

The example is for a loading of 4 kg. The result is that the loading is subcritical by 1.9%. Previous calculations have shown that:

$$d \ln k = \tfrac{1}{4}\ d \ln M \quad ,$$

where M is the mass; therefore the predicted critical mass is about 4.32 kg. The critical mass determined experimentally by Lennox and Kelber[5] is 4.3 kg. This agreement is, in part, fortuitous, since the experiment cited used fuel enriched to 20% U^{235}. Also, some cracks in the reflector contributed to reactivity losses amounting to about 300 gm. Hence it appears more reasonable to use the fast constants: $\tau = 60\ cm^2$; $D_f = 1.25$ cm. For convenience, work sheets pertinent to this problem are included on the following pages.

[5] D. H. Lennox, C. N. Kelber, "Summary Report on the Hazards of the Argonaut Reactor" ANL-5647 (December, 1956), p. 11.

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Sheet No. 1

ANNULAR CYLINDRICAL CORE

Problem: Argonaut (4 kg)   Group: Slow

Geometry:

r_0 = 30 cm r_I = 37.5   No. of Intervals (I): 3 Δr: 2.5 cm

Group Parameters:

$\Sigma_a = 0.07715$ cm^{-1} B_z^2 : 0.00175 cm^{-2} D : 0.27286 cm

$\Sigma_a^* = 0.07763$ cm^{-1} L^2 : 3.51487 cm^2

Set-Up Parameters

η : 4.58110 θ : 0.14817   S : 0.11854

Formula	i = 0	1	2	3
$r_i = r_0 + i\,(\Delta r)$ (cm)	30	32.5	35	37.5
$C_1 = r_i/r_{i+1}$	0.92307	0.92857	0.93333	
$C_2 = 1/r_{i+1}$ (cm^{-1})	0.03076	0.02857	0.02667	
$C_3 = 1 - \theta\,(2 + C_1)$	0.56689	0.56607	0.56537	
$\lambda_i = \Delta r/C_3$ (cm)	4.41002	4.41641	4.42188	
$\alpha_i = S(2 + C_1) - C_2$ (cm^{-1})	0.31574	0.31858	0.32105	
$\beta_i = \Sigma_a^*\,(1 + C_1)/2$ (cm^{-1})	0.07465	0.07486	0.07505	
$\gamma_i = (2 + C_1)/6$	0.48719	0.48810	0.48890	
$\epsilon_i = (1 + 2C_1)/6$	0.47435	0.47619	0.47777	

Sheet No. 2

ANNULAR CYLINDRICAL CORE

Problem: Argonaut   Group: Fast

Geometry:

r_0 : 30 cm r_I : 37.5 cm   No. of Intervals (I): 3 Δr : 2.5 cm

Group Parameters:

Σ_a: 0.01953 cm^{-1} B_z^2 : 0.00175 cm^{-2} D : 1.27 cm

Σ_a^* ; 0.02175 cm^{-1} L^2 : 58.39080 cm^2

Set-Up Parameters:

η : 0.98425 θ : 0.00891   S : 0.00713

Formula	i = 0	1	2	3
$r_i = r_0 + i\,(\Delta r)$ (cm)	30	32.5	35	37.5
$C_1 = r_i/r_{i+1}$	0.92307	0.92857	0.93333	
$C_2 = 1/r_{i+1}$ (cm^{-1})	0.03076	0.02857	0.02667	
$C_3 = 1 - \theta\,(2 + C_1)$	0.97395	0.97391	0.97386	
$\lambda_i = r/C_3$ (cm)	2.56686	2.56697	2.56710	
$\alpha_i = S(2 + C_1) - C_2$ (cm^{-1})	-0.00992	-0.00769	-0.00576	
$\beta_i = \Sigma_a^*\,(1 + C_1)/2$	0.02092	0.02098	0.02103	
$\gamma_i = (2 + C_1)/6$	0.48719	0.48810	0.48890	
$\epsilon_i = (1 + 2C_1)/6$	0.47435	0.47619	0.47777	

Sheet No. 3

ANNULAR CYLINDRICAL CORE

Iteration Constants

Problem:

Homogeneous Conditions:

Fast: $J_0^H = -0.05942$ Thermal: $J_0^H = -0.02378$

Formulae:

$$J_{i+1} = J_i + \lambda_i(\alpha_i J_i - \beta_i \phi_i)$$

$$\phi_{i+1} = \phi_i - \eta(J_i + J_{i+1})$$

i	λ_i(cm)	α_i(cm^{-1})	β_i(cm^{-1})	ϕ_i^H	j_i^H
			Fast Group; $\eta = 0.98425$		
0	2.56686	-0.00992	0.02092	1.00000	-0.05942
1	2.56697	-0.00769	0.02098	1.16833	-0.11160
2	2.56710	-0.00576	0.02103	1.44777	-0.17231
3				1.86137	-0.24791
			Thermal Group; $\eta = 4.58110$		
0	4.41002	0.31574	0.07465	1.00000	-0.02378
1	4.41641	0.31858	0.07486	2.87752	-0.38606
2	4.42188	0.32105	0.07505	13.26118	-1.88057
3				62.88260	-8.9512

Sheet No. 4

ANNULAR CYLINDRICAL CORE

Iteration Sheet

Problem: Argonaut

i	γ_i	ϵ_i	Q_i	Q_{i+1}	q_i	ϕ_i^c	J_i^c	ϕ_i^c	J_i^c	ϕ_i
							Iteration No. 1			
		Fast					Fast Group; η = 0.98425			
0	0.48719	0.47435	0.07715	0.06172	0.06666	2.00000	-0.11884	1.44875	-0.08608	1.44871
1	0.48810	0.47619	0.04172	0.04629	0.05198	2.16825	-0.05210	1.52424	+0.00941	1.52416
2	0.48890	0.47777	0.04629	0.06172	0.05229	2.25340	-0.03441	1.45534	+0.06057	1.45528
3			0.06172			2.30825	-0.02132	1.28221	+0.11533	1.28213
		Thermal					Thermal Group; η = 4.58110			
0	0.48719	0.47435	0.02829	0.02977	0.02792	0.65000	0.03619	0.62500	0.03679	0.62158
1	0.48810	0.47619	0.02977	0.02842	0.02805	0.00386	-0.00430	0.42975	0.00583	0.41991
2	0.48890	0.47777	0.02842	0.02504	0.02582	0.76640	-0.05301	0.42205	-0.00415	0.37670
3			0.02504			2.23891	-0.26840	0.60571	-0.03594	0.39065
							Iteration No. 2			
		Fast					Fast Group; η = 0.98425			
0			0.07715	0.05212	0.06199	1.32428	-0.07869			
1			0.05212	0.04676	0.04764	1.39058	+0.01133			
2			0.04676	0.04849	0.4605	1.32184	+0.05851			
3			0.04849			1.16141	0.10449			
		Thermal					Thermal Group; η = 4.58110			
0			0.02586	0.02716	0.02550	0.56698	0.03373	0.56690		
1			0.02716	0.02582	0.02554	0.38259	0.00652	0.38236		
2			0.02582	0.02268	0.02342	0.34351	0.00201	0.34245		
3			0.02268			0.35973	-0.00555	0.35470		

Sheet No. 5

ANNULAR CYLINDRICAL CORE

Auxiliary Instructions and Constants

This sheet contains instructions for filling out the Iteration Sheets and provides space, if desired, for performing auxiliary computations.

1. γ_i and ϵ_i were computed on Sheet No. 1; they need not be computed for each iteration.

2. $Q_i^f = \Sigma_a^{th} \phi_i^{th}$, where ϕ_i^{th} is the most recently computed set; similarly,

$$Q_i^{th} = \Sigma_a^f \phi_i^f$$

3. $q_i = \gamma_i Q_{i+1} + \epsilon_i Q_i$

4. ϕ_0^c and J_0^c must satisfy the relations:

$$J_0^c = m \phi_0^c + b.$$

The constants m and b are characteristic of the problem and group and are listed in the accompanying box.

Group	m	b
Fast Thermal	-0.05942 -0.02378	$0.03565\phi_f$

5. Given ϕ_0^c and J_0^c, the iteration formulae are:

$$J_{i+1}^c = J_i^c + \gamma_i (\alpha_i J_i^c - \beta_i \phi_i^c + q_i)$$

$$\phi_{i+1}^c = \phi_i^c - \eta\left(J_i^c + J_{i+1}^c\right)$$

The constants γ_i, α_i, and β_i can be obtained by folding Sheet No. 3 and clipping to Sheet No. 1 or Sheet No. 2.

6. After ϕ_I^c and J_I^c have been computed, a constant "a" must be computed. The formula is:

$$a = \frac{J_I^c - m' \phi_I^c - b'}{m' \phi_I^H - J_I^H}$$

To aid this computation, the following box contains the fixed constants.

Group	ϕ_I^H	J_I^H	m'	b'
Fast	1.86137	-0.24791	0.08996	
Thermal	62.88260	-8.9512	0.05846	$-0.02194\phi_f$

Unless $\left| a\, \phi_0^H \right| < 0.01\ \phi_0^c$, the iteration is to be repeated, using as new starting values:

$$\phi_0^c + a\ \phi_0^H$$

and

$$J_0^c + aJ_0^H ,$$

with the "a" being that first computed. Space for this iteration is provided on Sheet No. 4. The following box is used for computation of "a."

COMPUTATION OF "a"

	Iteration 1			Iteration 1"					
Group	ϕ_I^c	J_I^c	a	ϕ_I^c	J_I^c	a	ϕ_I^c	J_I^c	a
Fast	2.30825	-0.02132	-0.55125	1.28221	0.11533	-0.0004	1.16141	0.6449	0.0002
Thermal	2.23881	-0.26840	-0.00557	0.60571	-0.03594	-9.00345	0.35973	-0.00055	-0.00008

7. Following the second determination of "a" (or the first, if $\left| a\phi_0^H \right| < 0.01\ \phi_0^c$), ϕ_i on Sheet No. 4 is computed from:

$$\phi_i = \phi_i^c + a\,\phi_i^H ,$$

where ϕ_i^c and "a" are the most recently computed values.

8. An estimate of k is obtained by completing the columns and boxes listed on Work Sheet No. 6. The superscripts "f" and "th" refer to fast and thermal groups, respectively.

Sheet No. 6

ANNULAR CYLINDRICAL CORE

COMPUTATION OF k

Iteration No. 1						Iteration No. 2				
i	$\phi_i^f\ r_i$	$\phi_{i_o}^{th} = \frac{Q_i^f}{\Sigma_a^{th}}$	ϕ_i^{th}	$\phi_i^f\ r_i\ \phi_{i_o}^{th}$	$\phi_i^f\ r_i\ \phi_i^{th}$	$\phi_i^f\ r_i$	$Q_i^f = \phi_{i_o}^{th}\ \Sigma_a^{th}$	ϕ_i^{th}	$\phi_i^f\ r_i\ Q_i^f$	$\phi_i^f\ r_i\ \phi_i^{th}$
0	43.46	1	0.62158	43.46	27.01387	39.7284	0.07715	0.56690	3.06505	22.52203
1	49.54	0.8	0.41991			45.19385	0.05212	0.38236		
2	50.93	0.6	0.37670			46.2644	0.04676	0.34245		
3	48.08	0.8	0.39065	38.464	18.78245	43.55288	0.04849	0.35470	2.11188	15.44821
			Sum A	152.114	85.78399			Sum A	9.69575	71.09379
			First term x $(1/2-\Delta r/6r_o)$: Last term x $(1/2+\Delta r/6r_I)$:							
			Sum B	40.60979	22.56533			Sum B	2.55027	18.70294
			Sum A minus Sum B	111.50421	63.21866			Sum A minus Sum B	7.14548	52.39085
				u	v				u	v

$k = \frac{u}{v} = 1.76378$

$k = \frac{u}{\Sigma_a^{th} v} = 1.76782$

Since $f = 0.83110$,

$\eta_1 = 2.1222$

$\rho_1 = -0.01989$

$\eta_2 = 2.12708$

$\rho_2 = -0.02213$

IV. CRITICAL PROPERTIES OF "Two by Six" LOADING

As shown in Fig. 1, the "two by six" loading consists of six groups of two fuel boxes each, separated by an equal amount of graphite. This configuration can be approximated by an array of six cylinders with centers equally spaced on the mean circumference of the annulus. A treatment of the problem of N cylinders has been made by Avery.[6] The work reported here is a condensation of the general treatment.

In this approach the fuel boxes are conceived as being a ring of cylindrical elements immersed in a block of graphite. Each fuel element has a flux within it, describable by an expansion about its center. In the reflector the flux is described as the sum (with undetermined coefficient) of fluxes about each fuel box. These are related to the expression of the flux about the origin, and the exterior boundary conditions and symmetry conditions are satisfied. Then the fluxes are determined at the fuel box face, and the problem solved.

The coordinate systems and appropriate addition theorems are given in Figs. 2 and 3.

Rigid Source Approximation

In this approximation the shape of the flux in each fuel box is assumed to be similar to that in a fuel box which is isolated in an infinite moderator bath. This is assumed for the asymptotic flux independently of the ordinary (J_0) flux. Since the ratio of asymptotic to ordinary flux will depend on the neighboring fuel boxes, the shape of the total flux cannot be fixed. Nevertheless, the situation is reminiscent of the potential of rigid, fixed bodies, and hence the term rigid source approximation is used.

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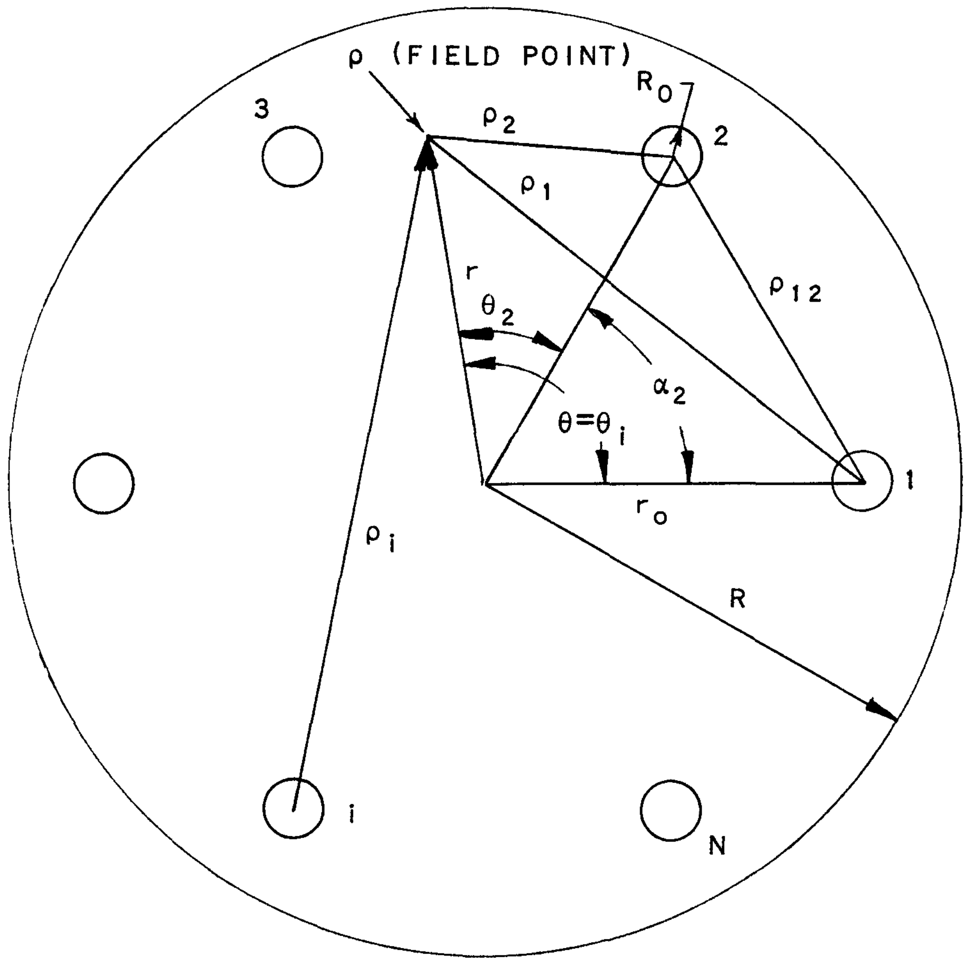
[6] R. Avery, "A Two-Group Diffusion Theory for a Ring of Cylindrical Rods." (To be published)



R = EXTRAPOLATED REFLECTOR RADIUS

r_o = RADIUS OF RING OF SOURCES

R_0 = RADIUS OF FUEL ELEMENT

r = DISTANCE OF FIELD POINT FROM ORIGIN

ρ_i = DISTANCE OF FIELD POINT FROM CENTER OF FUEL ROD i

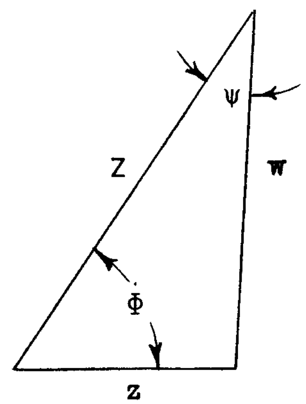
ρ_{ij} = CENTER TO CENTER DISTANCE, ROD i TO ROD j
$\rho_{ij} = 2r_o \sin(i-j)(\pi/N)$

θ_i = ANGULAR DISPLACEMENT OF FIELD POINT VECTOR RELATIVE TO BASE LINE FROM ORIGIN TO CENTER OF ROD i: $\theta_i \equiv \theta$

$$\alpha_i = \theta - \theta_i$$

FIG. 2
COORDINATE SYSTEM AND DEFINITIONS

$Z > z$



1. $$J_n(w) \begin{matrix}\cos\\ \sin\end{matrix} (n\psi) = \sum_{k=-\infty}^{\infty} J_{n+k}(Z)\, J_k(z) \begin{matrix}\cos\\ \sin\end{matrix} k\phi$$

2. $$Y_n(w) \begin{matrix}\cos\\ \sin\end{matrix} (n\psi) = \sum_{k=-\infty}^{\infty} Y_{n+k}(Z)\, J_k(z) \begin{matrix}\cos\\ \sin\end{matrix} k\phi$$

3. $$I_n(w) \begin{matrix}\cos\\ \sin\end{matrix} (n\psi) = \sum_{k=-\infty}^{\infty} (-)^k I_{n+k}(Z)\, I_k(z) \begin{matrix}\cos\\ \sin\end{matrix} k\phi$$

4. $$K_n(w) \begin{matrix}\cos\\ \sin\end{matrix} (n\psi) = \sum_{k=-\infty}^{\infty} K_{n+k}(Z)\, I_k(z) \begin{matrix}\cos\\ \sin\end{matrix} k\phi$$

FIG. 3
ADDITION THEOREMS FOR BESSEL FUNCTIONS

Under these assumptions,

$$\phi_c^f(\rho) = EJ_0(\ell\rho) + FI_0(m\rho)$$

and

$$\phi_c^{th}(\rho) = S_1\ EJ_0(\ell\rho) + S_2\ FI_0(m\rho)$$

are the fast and slow fluxes in the interior of any fuel box, where ρ is the radius vector from the center of the fuel box in question. Under the assumption of rigid sources, higher order terms containing angular dependence are dropped.

The reflector flux arises from the sources and hence can be described as a superposition of fluxes which are similar to that in a reflector about an isolated cylinder. Since the reflector is a moderating material there must be included, in addition, terms which are descriptive of the flux in the reflector as a whole. The only restriction on angular dependence of the reflector flux is that it be invariant under a rotation through $2\pi/N$ about the axis of the ring of fuel elements. (N is the number of fuel boxes.) Hence, one can write:

$$\phi_{ref}^f = \sum_{n=0}^{\infty} A_n I_{nN}(\mu_f r) \cos nN\theta + \sum_{i=1}^{N} K_0(\mu_f \rho_i) \ ;$$

$$\phi_{ref}^{th} = S_R\ \phi_{ref}^f + \sum_{n=0}^{\infty} B_n I_{nN}(\mu_{th} r) \cos nN\theta + C \sum_{i=1}^{N} K_0(\mu_{th}\rho_i).$$

The terms in A_n, B_n are characteristic of the reflector flux away from the sources. The source terms contain terms in K_0 only because higher angular dependence corresponds to non-rigid sources. We have used the boundary condition that the flux be finite at $r = 0$ to eliminate terms of the form $A_n\ K_n(\mu r)$. If $r > r_0$:

$$K_0(\mu\rho_i) = \sum_{n=0}^{\infty} \epsilon_n K_n(\mu r)\ I_n(\mu r_0) \cos n\theta_i \quad ,$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$, and $n \geq 1$. (For $r < r_0$, interchange the arguments I_n, K_n.) Hence, at $r = R(>r_0)$:

$$\phi_{ref}^f = \sum_{n=0}^{\infty} \left\{ A_n I_{nN}(\mu_f R) \cos nN\theta + \sum_{i=1}^{N} \epsilon_n K_n(\mu_f R)\ I_n(\mu_f r_0) \cos n\theta_i \right\}$$

with similar forms for ϕ_{ref}^{th}.

Now

$$\sum_{i=1}^{N} \cos n\theta_i = \cos n\theta \sum_{i=1}^{N} \cos n\alpha_i + \sin n\theta \sum_{i=1}^{N} \sin n\alpha_i \quad .$$

By symmetry, the sum over sin n $_i$ vanishes; also

$$\sum_{i=1} \cos n\alpha_i = 0$$

if n is not a multiple of N, and

$$\sum_{i=1} \cos n\alpha_i = N$$

if n is a multiple of N, including n = 0.

Therefore,

$$\phi^f_{ref} = \sum_{n=0}^{\infty} A_n I_{nN}(\mu_f r) \cos nN\theta + \cos nN\theta \; N \epsilon_n K_n(\mu_f r) I_n(\mu_f r_0) \quad .$$

At $r = R, \quad \phi^f_{ref} = 0 \quad ;$

thus

$$A_n = \frac{-N \epsilon_n K_n(\mu_f R) I_n(\mu_f r_0)}{I_{nN}(\mu_f R)} \quad .$$

Similarly,

$$B_n = \frac{-CN\epsilon_n K_{nN}(\mu_{th} R) I_{nN}(\mu_{th} r_0)}{I_{nN}(\mu_{th} R)}$$

Hence, as $R \rightarrow \infty$; $A_n, B_n \rightarrow 0$.

One can now match the fluxes and currents at the rod surfaces. Since the problem is symmetric with respect to the rods, choose rod No. 1. Write the equation for reflector flux about the origin of rod 1:

$$\phi^f_{ref}(\rho) = K_0(\mu_f\rho) + \sum_2^N K_0(\mu_f\rho_{i1}) + \sum_{n=0}^{\infty} A_n I_{nN}(\mu_f r) \cos nN\theta$$

$$= K_0(\mu_f\rho) + \sum_2^N K_0(\mu_f\rho_{i1})$$

$$- N \sum_{n=0}^{\infty} \frac{\epsilon_n K_{nN}(\mu_f R)\, I^2_{nN}(\mu_f r_0)}{I_{nN}(\mu_f R)} I_0(\mu_f\rho) \quad ,$$

where the formula:

$$I_n(\mu r) \cos n\theta = \sum_{k=-\infty}^{\infty} I_{n+k}(\mu r_0)\, I_k(\mu\rho) \cos k\phi$$

has been employed and only the angularly independent (rigid source) term has been kept. Then define

$$f_\nu(\mu_f) = \sum_{i=2}^{N} K_0(\mu_f\rho_{i1}) - N \sum_{n=0}^{\nu} \frac{\epsilon_n K_{nN}(\mu_f R)}{I_{nN}(\mu_f R)} I^2_{nN}(\mu_f r_0) \quad .$$

Let $f_\nu(\mu_{th})$ be similarly defined, and let $f(\mu_f) = \lim_{\nu \to \infty} f_\nu(\mu_f)$, etc.

Now, assume that R_0, the source radius, is small compared to the distance between sources so that the flux averaged over the surface of rod No. 1 may be represented by evaluating the fluxes as continued to the center of each rod. This assumption enables one to substitute

$$K_0(\mu_f\rho_{i1})$$

for

$$\int_0^{2\pi} K_0(\mu_f \left| \rho_{i1} + \rho \right|)\, (\rho d\theta / 2\pi)$$

in evaluating $f_\nu(\mu_f)$, etc.

With this restriction, the critical equations evolved are:

$$K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f RO) = EJ_0(\ell R_0) + FI_0(mR_0)$$

$$S_R \left\{K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f RO)\right\} + C\left\{K_0(\mu_{th} R_0) + f(\mu_{th})\, I_0(\mu_{th} R_0)\right\} = S_1\, EJ_0(\ell R_0) + S_2\, FI_0(mR_0)$$

$$-D_{fR}\mu_f \left[K_1(\mu_f R_0) - f(\mu_f)\, I_1(\mu_f R_0)\right] = -ED_{fc}\,\ell J_1(\ell R_0) + FD_{fc}\, mI_1(mRO)$$

$$-S_R\, D_{thR}\, \mu_f \left[K_1(\mu_f RO) - f(\mu_f)\, I_1(\mu_f R_0)\right] - CD_{thR}\mu_{th}\left[K_1(\mu_{th} R_0) = f(\mu_{th})\, I_1(\mu_{th} RO)\right] = -ES_1\, D_{thc}\,\ell J_1(\ell R_0) + F\, S_2 D_{thc}\, mI_1(mR_0)$$

This system of four equations in four unknowns must be solved for the critical bucklings. (The coefficient of ϕ_{fR} is arbitrarily chosen to be 1.)

Write the critical determinant:

$$D = \underbrace{\begin{vmatrix} K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f R_0) & 0 \\ S_R\left[K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f R_0)\right] & K_0(\mu_{th} R_0) + f(\mu_{th})\, I_0(\mu_{th} R_0) \\ -D_{fR}\mu_f\left[K_1(\mu_f R_0) - f(\mu_f)\, I_1(\mu_f R_0)\right] & 0 \\ -S_R D_{thR}\mu_f\left[K_1(\mu_f R_0) - f(\mu_f)\, I_1(\mu_f R_0)\right] & -D_{thR}\mu_{th}\left[K_1(\mu_{th} R_0) - f(\mu_{th})\, I_1(\mu_{th} R_0)\right] \end{vmatrix}}_{Y_R} \underbrace{\begin{vmatrix} 1 & 1 \\ +S_1 & +S_2 \\ -D_{fc}\,\ell(J_1/J_0)(\ell R_0) & D_{fc}\, m(I_1/I_0)(mR_0) \\ -S_1 D_{thc}\,\ell(J_1/J_0)(\ell r_0) & S_2\, D_{thc}\, m(I_1/I_0)(mR_0) \end{vmatrix}}_{Y_c}$$

Hence, an equivalent set of equations is $QY_c = 0$, where $QY_R \equiv 0$ defines Q.

$$Q = \begin{bmatrix} 1 & 0 & Q_1 & 0 \\ 0 & 1 & Q_2 & Q_3 \end{bmatrix}$$

where

$$Q_1 = \frac{K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f R_0)}{D_{fR}\mu_f\left[K_1(\mu_{th} R_0) - f(\mu_{th})\, I_1(\mu_f R_0)\right]}$$

$$Q_3 = \frac{K_0(\mu_{th} R_0) + f(\mu_{th})\, I_0(\mu_{th} R_0)}{D_{thR}\mu_{th}\left[K_1(\mu_{th} R_0) - f(\mu_{th})\, I_1(\mu_{th} R_0)\right]}$$

$$Q_2 = \frac{S_R}{D_{thR}}\left(D_{fR}\, Q_1 - D_{thR}\, Q_3\right) .$$

$f(\mu_f)$ may be approximated by $f_\nu(\mu_f)$ since

$$I_n(2X) = \frac{X^n}{n}\left\{1 + \frac{X^2}{n+1} \cdots\right\}$$

and

$$K_{n+1}(X) = \frac{2n}{X} K_n(X) + K_{n-1}(X).$$

Often it is found that $f_0(\mu)$ is adequate, since this corresponds to excluding Bessel functions of order $N >> 0$.

A numerical example of the "two by six" loading is given next.

$R = 75$ cm

$r_0 = 37.5$ cm

$R_0 = \sqrt{88.52555} = 9.40880$ cm (this preserves the volume of the fuel box)

$N = 6$

$\rho_{21} = 2r_0 \sin(\pi/6) = 37.5$ cm

$\rho_{31} = 2r_0 \sin(2\pi/6) = 64.95$ cm

$\rho_{41} = 2r_0 \sin(\pi/2) = 75$ cm

$\rho_{51} = \rho_{31} = 64.95$ cm

$\rho_{61} = \rho_{21} = 37.5$ cm

The graphite constants are:

$\mu_f = 0.06693\ \text{cm}^{-1}$ $D_f = 1.1$ cm

$\mu_{th} = 0.04637\ \text{cm}^{-1}$ $D_{th} = 0.903$ cm

$S = -1.42641$

Then the following Bessel functions are used:

X		$K_0(X)$	$I_0(X)$	$K_1(X)$	$I_1(X)$
$\mu_f R_0$ =	0.62973	0.74005	1.10162	1.219312	0.33073
$\mu_{th} R_0$ =	0.43629	1.03935	1.04816	1.96525	0.22338
$\mu_f \rho_{21}$ =	2.50988	0.06162	3.31482		
$\mu_f \rho_{31}$ =	4.34710	0.00758			
$\mu_f \rho_{41}$ =	5.01975	3.61024×10^{-3}			
$\mu_{th} \rho_{21}$ =	1.73888	0.15738	1.91137		
$\mu_{th} \rho_{31}$ =	3.01173	0.03427			
$\mu_{th} \rho_{41}$ =	3.47775	0.020100			
$\mu_f R$ =	5.01975	3.61204×10^{-3}	27.72000		
$\mu_{th} R$ =	3.47775	0.020100	7.24150		

$$f_0(\mu_f) = \sum_{i=2}^{6} K_0(\mu_f \rho_{i1}) - 6 \, \frac{K_0(\mu_f R)}{I_0(\mu_f R)} \; I_0^2(\mu_f r_0)$$

$$= 0.14201 - 6 \, \frac{(0.00361204)}{27.72} \; (3.31482)^2$$

$$= 0.13343$$

$$f_0(\mu_{th}) = 0.40340 - 6 \, \frac{(0.0201)}{7.2415} \; (1.91137)^2$$

$$= .34256$$

$$K_0(\mu_f R_0) + f_0(\mu_f) \, I_0(\mu_f R_0) = 0.88704$$

$$K_1(\mu_f R_0) - f_0(\mu_f) \, I_1(\mu_f R_0) = 1.17518$$

$$K_0(\mu_{th} R_0) + f_0(\mu_{th}) \, I_0(\mu_{th} R_0) = 1.29359$$

$$K_1(\mu_{th} R_0) - f_0(\mu_{th}) \, I_1(\mu_{th} R_0) = 1.88873 \quad .$$

Then

$Q_1 = 10.25235$

$Q_2 = 4.52886$

$Q_3 = 16.35669$

$$Q = \begin{bmatrix} 1 & 0 & 10.25235 & 0 \\ 0 & 1 & 4.52866 & 16.35669 \end{bmatrix} .$$

Equations (3a) and(3b) yield B^2 and L^2 for a given value of X.

$$\frac{X}{kg} = \left(\frac{1}{18.7}\right)\left(\frac{1}{100.11996}\right) = 5.3411 \times 10^{-4}/kg.$$

There is a slight difference in volume between the full 3-inch annular core and the "two by six" or "six by two" loadings. An estimated critical mass of 4.7 kg yields:

$$X = 2.51032 \times 10^{-3},$$

where

$k_\infty = 1.76062;$ $\Sigma_{au} = 0.07182\ cm^{-1};$

$\Sigma_a = 0.08484\ cm^{-1};$ $L^2 = 3.21617\ cm^2;$

$\frac{1}{L^2} = 0.31092\ cm^{-2};$ $\frac{1}{\tau} = \frac{1}{60} = 0.0166\ cm^{-2};$

$\frac{1}{L^2} + \frac{1}{\tau} = 0.32758\ cm^{-2};$ $\frac{1}{L^2\tau} = 0.00518\ cm^{-4};$

$$B^2 = \tfrac{1}{2}\left[- 0.32758 + (0.12306)^{\frac{1}{2}}\right] = 0.01161\ cm^{-2}$$

$$(B_z^2 = 0.00175\ cm^{-2})$$

The remainder of the calculation is carried out on the accompanying form sheets.

CORE - CYLINDER

Dimensions

Radius = R_0 = 9.4088 cm

Extrapolated bare height = H_z = 75 cm

$(\pi/H_z)^2 = B^2 = 0.00175\ cm^{-2}$

Constants

	$L^2 = \frac{1}{\kappa^2}$	$\kappa^2 = \frac{1}{L^2}$	$D = \frac{\lambda_t}{3} = N\sigma_a/\kappa^2$	$N\sigma_a = \kappa^2 D$
Fast	60 $cm^2 = \tau$	0.01666 cm^{-2}	1.25 cm	0 02083 cm^{-1}
Slow	3.21617 cm^2	0.31092 cm^{-2}	0.27286 cm	0.08484 cm^{-1}

$$\frac{N\sigma_{af}}{N\sigma_{ath}} = 0.24546$$

Functions of K

Test $\bar{\ell}^2 = 0.01161\ cm^{-2}$

$\kappa_f^2 + \kappa_{th}^2 + \bar{\ell}^2 = \bar{m}^2 = 0.33919\ cm^{-2}$

$\bar{m}^2 + B^2 = m^2 = 0\ 34094\ cm^{-2}$

m = 0.58390 cm^{-1}

$1 + L_{th}^2\ \ell^2 = 1.03734$

$1 + L_{th}^2\ m^2 = -0.09089$

$$\left(\frac{N\sigma_{af}}{N\sigma_{ath}}\right)\left(\frac{1}{1 + L_{th}^2\ \bar{\ell}^2}\right) = S_1 = 0.23662$$

$$\left(\frac{N_{af}}{N_{ath}}\right)\left(\frac{1}{1 - L_{th}^2\ \bar{m}^2}\right) = S_2 = -2.70062$$

Functions

mR_0	$I_0(mR_0)$	$I_1(mR_0)$	$I_1/I_0(mR_0)$	$mI_1/I_0(mR_0)$
5.49380	0 17459	0.15777	0.90368	0.52766

Matrix and Critical Relation

Y_{core} =

1	1
S_1 0.23662	S_2 -2.70062
D_f $-1.25 \frac{J_1}{J_0}(\ell r)$	$D_{th}\, m \frac{I_1}{I_0}(mR_0)$ 0.65958
$S_1 D_{th}$ $-0.06456 \frac{J_1}{J_0}(\ell R_0)$	$S_2 D_{th}\, m \frac{I_1}{I_0}(mR_0)$ -0.38883

$|QY|$ =

1 - 12.81544 y	7.76225
0.23662 - 6.71681 y	-6.06358

= 0

Where $y = \ell \frac{J_1}{J_0}(\ell R_0)$

$(129\ 97316 \frac{J_1}{J_0}(\ell R_0) = 7.91028$

$\ell R_0 \frac{J_1}{J_0}(\ell R_0) = 0.57262$

$\ell R_0 = 0\ 99815$; $\ell = 0.10608\ cm^{-1}$; $\ell^2 = 0.01125\ cm^{-2}$; $\bar{\ell}^2 = \ell^2 + B^2 = 0.01300\ cm^{-2}$

CORE - CYLINDER

Dimensions

Radius = R_0 = 9.4088 cm

Extrapolated bare height = H_z = 75 cm

$(\pi/H_z)^2 = B^2 = 0.00175\ \text{cm}^{-2}$

$X = 3.43 \times 10^{-3}$

Constants

	$L^2 = \frac{1}{\kappa_2}$	$\kappa^2 = \frac{1}{L^2}$	$D = \frac{\lambda_t}{3} = N\sigma_a/\kappa^2$	$N\sigma_a = \kappa^2 D$
Fast	60 cm² = τ	0.01666 cm⁻²	1.25 cm	0.02083 cm⁻¹
Slow	2.45356 cm²	0.40757 cm⁻²	0.27286 cm	0.11121 cm⁻¹

$\frac{N\sigma_{af}}{N\sigma_{a\,th}} = 0.18730$

Functions of K

Test $\bar{\ell}^2 = 0.01300\ \text{cm}^{-2}$

$1 + L_{th}^2\,\bar{\ell}^2 = 1.03190$

$\kappa_f^2 + \kappa_{th}^2 + \ell^2 = \bar{m}^2 = 0.43723\ \text{cm}^{-2}$

$1 - L_{th}^2\,\bar{m}^2 = -0.07277$

$\bar{m}^2 + B^2 = m^2 = 0.43898\ \text{cm}^{-2}$

$m = 0.66256\ \text{cm}^{-1}$

$$\left(\frac{N\sigma_{af}}{N\sigma_{a\,th}}\right)\left(\frac{1}{1 + L_{th}^2\,\ell^2}\right) = S_1 = 0.18150$$

$$\left(\frac{N_{af}}{N_{a\,th}}\right)\left(\frac{1}{1 - L_{th}^2\,m^2}\right) = S_2 = -2.57386$$

Functions

mR_0	$I_0(mR_0)$	$I_1(mR_0)$	$I_1/I_0(mR_0)$	$mI_1/I_0(mR_0)$
6.23389	0.16334	0.14960	0.91586	0.60681

Matrix and Critical Relation

Y_{core} =

1	1
S_1 0.18150	S_2 -2.57386
D_f $-1.25\,\ell\,\frac{J_1}{J_0}(\ell R_0)$	$D_f m \frac{I_1}{I_0}(mR_0)$ 0.75851
$S_1 D_{th}$	$S_2 D_{th} m \frac{I_1}{I_0}(mR_0)$
$-0.04952\ell\,\frac{J_1}{J_0}(\ell R_0)$	-0.42615

$|QY|$ =

1 - 12.81544 y	8.77651
0.18150 - 6.47081 y	-6.10923

= 0

$$(135.08360)\,\ell\,\frac{J_1}{J_0}(\ell R_0) = 7.70217$$

$$\ell R_0\,\frac{J_1}{J_0}(\ell R_0) = 0.53646$$

$\ell R_0 = 0.97026$; $\ell = 0.10312\ \text{cm}^{-1}$; $\ell^2 = 0.01063\ \text{cm}^{-2}$; $\bar{\ell}^2 = \bar{\ell}^2 + B = 0.01238\ \text{cm}^{-2}$

$$Q = \begin{pmatrix} 1 & 0 & \frac{Q_1}{10.25235} & 0 \\ 0 & 1 & \frac{Q_2}{4.52866} & \frac{Q_3}{16.35669} \end{pmatrix}$$

The critical buckling is $B^2 = 0.01257\ cm^{-2}$, whence $X = 3.095 \times 10^{-3}$ and the critical mass is $M_c = 5.795$ kg. This is slightly high, but is within 10% of the observed critical mass.

Another calculation of the same general type as the preceding has been made for a slightly different case with infinite reflector of better graphite, and fuel "rods" corresponding to two fuel boxes with no graphite included. The fast constants are somewhat better known in this case, but the idealization is not as good as before. The predicted critical mass is 4.7 kg; when corrected for a finite reflector and voids this is raised to about 5.1 kg. The perturbations calculated for this case are particularly revealing; the magnitude of these perturbations is not expected to change a great deal from case to case.

Let:

$R_0 = 8.66$ cm; $r_0 = 40$ cm; $N = 6$

Infinite Reflector:

$\rho_{21} = 2r_0 \sin(\pi/6) = 40$ cm

$\rho_{31} = 2r_0 \sin(2\pi/6) = 69.28$ cm

$\rho_{41} = 2r_0 \sin(\pi/2) = 80$ cm

$\rho_{51} = \rho_{31} = 69.28$ cm

$\rho_{61} = \rho_{21} = 40$ cm

In Graphite

$\mu_f = \sqrt{(1/\tau) + B^2} = 0.06701\ cm^{-1}$

$\mu_{th} = \sqrt{(1/L^2) + B_z^2} = 0.04626\ cm^{-1}$

$D_f = 1.1$ cm; $D_{th} = 0.9$ cm; $S_R = -1.44383$

$\mu_f R_0 = 0.58031$ $\mu_f \rho_{31} = 4.64245$

$\mu_{th} R_0 = 0.40061$ $\mu_{th} \rho_{31} = 3.20989$

$\mu_f \rho_{21} = 2.6804$ $\mu_f \rho_{41} = 5.3608$

$\mu_{th} \rho_{21} = 1.8504$ $\mu_{th} \rho_{41} = 3.70080$

	$K_0(X)$	$I_0(X)$	$K_1(X)$	$I_1(X)$
$X = \mu_f R_0$	0.80376	1.08598	1.36287	0.30254
$\mu_{th} R_0$	0.11319	1.04053	2.18035	0.20435
$\mu_f \rho_{21}$	0.05034			
$\mu_{th} \rho_{21}$	0.13703			
$\mu_f \rho_{31}$	0.00546			
$\mu_{th} \rho_{31}$	0.02744			
$\mu_f \rho_{41}$	0.00249			
$\mu_{th} \rho_{41}$	0.015617			

$$f(\mu_f) = \sum_{i=2}^{6} K_0(\mu_f \rho_{i1}) = 0.11409$$

$$f(\mu_{th}) = \sum_{i=2}^{6} K_0(\mu_{th} \rho_{i1}) = 0.34456$$

$$K_0(\mu_f R_0) + f(\mu_f)\, I_0(\mu_f R_0) = 0.96766$$

$$K_1(\mu_f R_0) - f(\mu_f)\, I_0(\mu_f R_0) = 1.32835$$

$$K_0(\mu_{th} R_0) + f(\mu_{th})\, I_0(\mu_{th} R_0) = 1.47172$$

$$K_1(\mu_{th} R_0) - f(\mu_{th})\, I_1(\mu_{th} R_0) = 2.10994$$

$Q_1 = 9.88264$

$Q_2 = 5.52201$

$Q_3 = 16.75336$

Core Constituents and Parameters

Material	Volume Fraction	Σ_a (cm^{-1})	Σ_{tr} (cm^{-1})
H_2O	0.71428 (1 - X)	0.01949	2.106
Al	0.28571 (1 - X)	0.01229	0.0823
U^{235}	X	28.2381	-

34

$\tau = 47.5 \text{ cm}^2$ $D_f = 1.193$ cm $D_{th} = 0.21817$ cm

$\Sigma_a^{mod} = 0.01743\ (1-X)$ $\Sigma_a^{U^{235}} = 28.2381\ (X)$

$\Sigma_a^{total} = \Sigma_a^{mod} + \Sigma_a^{U^{235}}$

$$\bar{\ell}^2 = \frac{k-1}{\tau + L^2} = \frac{(2.08)(\Sigma_a^{U^{235}}/\Sigma_a^{total})-1}{47.5 + (0.21817/\Sigma_a^{total})}$$

$$= \frac{30.51458X - 0.01743}{1.04610 + 1340.48183X}$$

Conversely:

$$X = \frac{0.01743 + 1.04610\ \bar{\ell}^2}{30.51458 - 1340.48183\ \bar{\ell}^2}$$

$X_{crit} = 0.00295$

therefore

$$\begin{aligned}\text{Critical mass} &= 18.7\ (6\pi R_0^2\ H)X \\ &= (1586.096)(X) \\ &= 4.68 \text{ kg}\end{aligned}$$

CORE - CYLINDER

Dimensions

Radius = R_0 = 8.66 cm

Extrapolated Bare Height = H_z = 75 cm

$(\pi/H_z)^2 = B^2 = 0.00175\ cm^{-2}$

X_1 = 0.00300

Constants

	$L^2 = \frac{1}{\kappa^2}$	$\kappa^2 = \frac{1}{L^2}$	$D = \frac{\lambda_t}{3} = \frac{N\sigma_a}{\kappa^2}$	$N\sigma_a = \kappa^2 D$
Fast	47.5 $cm^2 = \tau$	0.02105 cm^{-2}	1.193 cm	0.02511 cm^{-1}
Slow	2.13702 cm^2	0.96794 cm^{-2}	0.21817 cm	0.10209 cm^{-1}

$\frac{N\sigma_{af}}{N\sigma_{a\,th}} = 0.24595$

Functions of K

Test $\bar{\ell}^2 = 0.01429\ cm^{-2}$

$\kappa_f^2 + \kappa_{th}^2 + \bar{\ell}^2 = \bar{m}^2 = 0.050328\ cm^{-2}$

$\bar{m}^2 + B^2 = m^2 = 0.50503\ cm^{-2}$

$1 + L_{th}^2\,\bar{\ell}^2 = 1.03054$

$1 - L_{th}^2\,\bar{m}^2 = -0.07552$

$m = 0.71065\ cm^{-1}$

$$\left(\frac{N\sigma_{af}}{N\sigma_{a\,th}}\right)\left(\frac{1}{1 + L_{th}^2\bar{\ell}^2}\right) = S_1 = 0.23866$$

$$\left(\frac{N\sigma_{af}}{N\sigma_{a\,th}}\right)\left(\frac{1}{1 - L_{th}^2\bar{m}^2}\right) = S_2 = -3.25675$$

Functions

mR_0	$I_0(mR_0)$	$I_1(mR_0)$	$I_1/I_0(mR_0)$	$mI_1/I_0(mR_0)$
6.15432	0.16345	0.15042	0.92028	0.65400

Matrix and Critical Relation

Y_{core} =

1	1
S_1 0.23866	S_2 -3.25675
D_f $-1.193\,\frac{J_1}{J_0}(\ell R_0)$	$D_f m\,\frac{I_1}{I_0}(mR_0)$ 0.78022
$S_1 D_{th}$ $-0.4968\,\ell\,\frac{J_1}{J_0}(\ell R_0)$	$S_2 D_{th}\,m\,\frac{I_1}{I_0}(mR_0)$ -0.46469

QY =

1-11.78999 y	8.71063
0.23866 - 7.42006 y	-6.73341

= 0

$$(144.02023)\,\ell\,\frac{J_1}{J_0}(\ell R_0) = (8.81229)\,\ell\,\frac{J_1}{J_0}(\ell R_0) = 0.06118$$

$$\ell R_0\,\frac{J_1}{J_0}(\ell R_0) = 0.52982$$

$\ell R_0 = 0.965$; $\ell = 0.11143\ cm^{-1}$; $\ell^2 = 0.01242\ cm^{-2}$; $\bar{\ell}^2 = \ell^2 + B^2 = 0.01416\ cm^{-2}$

We can now determine the fluxes and their adjoints. Since this is a standard problem in algebra, we will not enter into the details here. For the critical system just studied we can now summarize the critical parameters:

$B^2 = 0.01419\ cm^{-2}\ (B_z^2 = 0.00175\ cm^{-2})$

$D_f = 1.193$ cm	$D_{th} = 0.21817$ cm
$S_1 = 0.24195$	$S_2 = 3.26653$
$L^2 = 2.16696\ cm^2$	$\tau = 47.5\ cm^2$
$S_1^\dagger = 0.59738$	$S_2^\dagger = -0.04426$

Then, in the rods we can write:

$$\phi_f = EJ_0(\ell r) + FI_0(mr)$$

$$\phi_{th} = S_1\,EJ_0(\ell r) + S_2\,FI_0(mr)$$

$$\phi_{th}^\dagger = E^\dagger J_0(\ell r) + F^\dagger I_0(mr)$$

$$\phi_f^\dagger = S_1^\dagger\ E^\dagger J_0(\ell r) + S_2^\dagger\ F^\dagger I_0(mr) \quad .$$

In the reflector immediately outside the rods we have:

$$\phi_f(r) = K_0(\mu_f r) + f(\mu_f)\,I_0(\mu_f r)$$

$$\phi_{th}(r) = S_R\phi_f + C\,[K_0(\mu_{th} r) + f(\mu_{th})\,I_0(\mu_{th} r)]$$

$$\phi_{th}^\dagger = K_0(\mu_{th} r) + f(\mu_{th})\,I_0(\mu_{th} r)$$

$$\phi_f^\dagger = S_R^\dagger\ \phi_{th}^\dagger + C^\dagger\,[K_0(\mu_f r) + f(\mu_f)\,I_0(\mu_f r)\ ]$$

Then:

$E = 1.23321$;	$F = -0.00048$;	$C = 1.16099$
$E^\dagger = 2.43286$;	$F^\dagger = -0.00595$;	$C^\dagger = 0.22576$

Perturbation calculations can now be made by means of the formula:

$$-\frac{\Delta\nu}{\nu} = \int dV \left[-\Delta D_f \nabla\phi_f \cdot \nabla\phi_f^\dagger - \Delta D_{th} \nabla\phi_{th} \cdot \nabla\phi_{th}^\dagger - \Delta\Sigma_f^a \phi_f^\dagger \phi_f \frac{-\Delta\Sigma_{th}^a \phi_{th} \phi_{th}^\dagger + \Delta\Sigma_f^a \phi_{th}^\dagger \phi_f + \nu\Delta\Sigma_{th}^{f\dagger} \phi_f \phi_{th}}{\int \nu\Sigma_{th}^f \phi_f^+ \phi_{th} \, dV} \right]$$

where superscripts a and f refer to absorption and fission, the subscripts f and th to fast and thermal. The integral is over the entire reactor volume and the curvature in the Z direction is introduced by assuming that the fluxes are proportional to cos (ZB_z).

The perturbation due to the introduction of a void is found by supposing that the material density is changed to the first order of infinitesimals; the resulting perturbation is proportional to the change in density and the proportionality constant is then assumed to be the perturbation due to the introduction of a void.

In the case of vertical displacements the equivalent bare reactor was introduced, the critical core height was found (and verified to be 60 cm), and the fluxes and adjoints determined. The reactivity change due to the uniform addition of fuel was found and the scale factor of 1.28355 was determined as being required to re-normalize this calculation to the same calculation based on the rods imbedded in the reflector. It was assumed this scale factor would apply to vertical displacements and the desired result was obtained in the usual way.

We now tabulate the various reactivity effects.

Change		
(1)	Uniform addition of fuel to core regions	$0.41906 \times 10^{-4} \Delta\Sigma_{th}^f (cm^2)$
(2)	Addition of fuel to internal thermal column	$1.105 \times 10^{-4} \Delta\Sigma_{th}^f (cm^2)$
(3)	Void at the center of the internal thermal column	-0.702×10^{-7}/cc
(4)	Void in the middle of the annular graphite	-2.862×10^{-7}/cc
(5)	Void in the graphite at the edge of the fuel box	-20.24991×10^{-7}/cc
(6)	Water replaces graphite at the center of internal thermal column	-1.62×10^{-6}/cc

Change		
(7)	Water replaces graphite at middle of annular graphite	-1.373×10^{-6}/cc
(8)	Water replaces graphite at edge of fuel box	-0.232×10^{-6}/cc
(9)	Displacement vertically of one fuel box 12 in. into top (or bottom) reflector	-0.00882
(10)	Uniform addition of void to the water in the fuel rods and reflectors	$-0.57\,(1 - \rho_{H_2O})$
(11)	Uniform temperature change	-1.065×10^{-4}/°C

Items (10) and (11) were found by direct computation.

The lifetime was found by the method of uniform addition of poison; e.g., write the time-dependent neutron balance equations:

$$D_f \Delta\phi_f - \Sigma_f^a\ \phi_f + k_\infty\ \Sigma_{th}\ \phi_{th} = \frac{1}{v_f}\ \frac{\partial\phi_f}{\partial t} \quad ;$$

$$D_{th}\ \Delta\phi_{th} - \Sigma_{th}^a\ \phi_{th} + \Sigma_f^a\ \phi_f = \frac{1}{v_{th}}\ \frac{\partial\phi_{th}}{\partial t} \quad .$$

Now assume

$$\phi_f,\ \phi_{th} \propto e^{\alpha t} \quad ,$$

where

$$\alpha = \frac{(\Delta k_{eff}/k_{eff})}{\ell} \quad ,$$

and, since $v_f >> v_{th}$, neglect the term in

$$\frac{1}{v_f}\ \frac{\partial\phi_f}{\partial t} \quad .$$

Then the equations can be rewritten:

$$D_f\ \Delta\phi_f - \Sigma_f^a\ \phi_f + k_\infty\ \Sigma_{th}^a\ \phi_{th} = 0$$

$$D_{th}\ \Delta\phi_{th} - [\Sigma_{th}^a + (\alpha/v_{th})\ ]\ \phi_{th} + \Sigma_f^a\ \phi_f = 0$$

$k_\infty \Sigma^a_{th} = \nu \Sigma^f_{th}$ and is independent of the poison present, so that we have perturbed the system through the uniform addition of thermal poison of the amount:

$$\Sigma^a_{th}(\text{poison}) = \frac{\alpha}{v_{th}} \quad .$$

If now we find the reactivity change $(\Delta k/k)$ induced by this perturbation, we have the following equation for the prompt neutron lifetime:

$$\Sigma^a_{th}(\text{poison}) = \frac{\alpha}{v_{th}} = \frac{(\Delta k/k)}{v_{th}} \quad ;$$

$$\ell = \frac{(\Delta k/k)}{v_{th}\, \Sigma^a_{th}(\text{poison})} \quad .$$

Actually, one should make the Maxwellian correction to $\Sigma^a_{th}(\text{poison})$ as one does to the other slow absorption cross sections. By taking various small values of α, successive estimates of ℓ can be obtained and the result extrapolated to $\alpha = 0$. For the critical system under study here, this yields a prompt neutron lifetime of 1.9×10^{-4} sec.

Particularly revealing is the worth of 1 cc of H_2O relative to 1 cc of void distributed in the interstitial graphite. As shown on page 37, if some of the graphite were replaced by water, introduction of a void into the water would yield a positive reactivity change. While the over-all void coefficient is still negative, it is possible for the temperature coefficient to be positive under these conditions. Other requirements disregarded, this possibility alone serves to require that as much as possible water be confined to the fuel-bearing region.

V. CRITICAL PROPERTIES OF ONE- AND TWO-SIDED LOADINGS

These loadings (Fig. 1) can be approximated by slabs. Because of the internal thermal column these become three-region problems.

The coordinate systems and designations used in these problems are:

I	II	III
Left Hand Reflector	Fuel Region	Right Hand Reflector
$0 \xrightarrow{\;x\;} x_1$	$0 \xrightarrow{\;y\;} y_1$	$0 \xrightarrow{\;z\;} z_1$

In Region I:

$$\left.\begin{aligned}
\phi_f &= A_1 \sinh \mu_f x \\
\phi_{th} &= SA_1 \sinh \mu_f x + A_2 \sinh \mu_{th} x \\
J_f &= D_f \frac{d\phi_f}{dx} = \mu_f D_f \operatorname{ctnh} \mu_f\ x\ A_1 \sinh \mu_f x \\
J_{th} &= S\, D_{th}\, \mu_f \operatorname{ctnh} \mu_f\, x\ A_1 \sinh \mu_f x \\
&\qquad + \mu_{th} D_{th} \operatorname{ctnh} \mu_{th}\, x\ A_2 \sinh \mu_{th} x
\end{aligned}\right\} \quad (14)$$

Thus

$$\begin{bmatrix} \phi_f \\ \phi_{th} \\ J_f \\ J_{th} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \\ \mu_f D_f \operatorname{ctnh} \mu_f x & 0 \\ S\,\mu_f D_{th} \operatorname{ctnh} \mu_{th} x & \mu_{th} D_{th} \operatorname{ctnh} \mu_{th} x \end{bmatrix} \begin{bmatrix} A_1 \sinh \mu_f x \\ A_2 \sinh \mu_s x \end{bmatrix} \equiv Y_I(x)\, V_I(x) \quad (15)$$

where Y_1 is a 2 x 4 matrix and Y_I is a 1 x 2 matrix.

In Region II:

$$\left.\begin{aligned}
\phi_f &= B_1 \sin \ell y + B_2 \cos \ell y + B_3 \sinh my + B_4 \cosh my \\
\phi_{th} &= S_1 B_1 \sin \ell y + S_1 B_2 \cos \ell y + S_2 B_3 \sinh my + S_2 B_4 \cosh my \\
J_f &= -D_f \ell B_1 \cos \ell y + \ell D_f B_2 \sin \ell y + mB_3 D_f \cosh my + mD_f B_4 \sinh my \\
J_{th} &= -S_1 D_{th} \ell B_1 \cos \ell y + S_1\, \ell D_{th}\, B_2 \sin \ell y + S_2 m B_3\, D_{th} \cosh my \\
&\qquad + S_2 m B_4\, D_{th} \sinh my
\end{aligned}\right\} \quad (16)$$

$$\begin{bmatrix} \phi_f \\ \phi_{th} \\ J_f \\ J_{th} \end{bmatrix} = \begin{bmatrix} \sin \ell y & \cosh \ell y & \sinh my & \cosh my \\ S_1 \sin \ell y & S_1 \cos \ell y & S_2 \sinh my & S_2 \cosh my \\ -D_f \ell \cos \ell y & \ell D_f \sin \ell y & mD_f \cosh my & mD_f \sinh my \\ -S_1 D_{th} \ell \cos y & S_1 \ell D_{th} \sin \ell y & S_2 m D_{th} \cosh my & S_2 m D_{th} \sinh my \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = Y_{II}(y) V_{II} \qquad (17)$$

where Y_{II} is a 4 x 4 matrix, and V_{II} is a 1 x 4 matrix.

In region III, there are two sets of conditions usable:

(a) One-sided loading. Region III is considered infinite in extent, and the flux vanishes at infinity.

(b) Two-sided loading. Region III is bounded by the median plane between the two slabs and the currents vanish there.

Case (a):

$$\left.\begin{aligned} \phi_f &= C_1 e^{-\mu_f z} \\ \phi_{th} &= C_2 e^{-\mu_{th} z} + S\, C_1 e^{-\mu_f z} \\ J_f &= -\mu_f D_f C_1 e^{-\mu_f z} \\ J_{th} &= -S \mu_f D_{th} C_1 e^{-\mu_f z} - \mu_{th} D_{th} C_2 e^{-\mu_{th} z} \end{aligned}\right\} \qquad (18)$$

$$\begin{bmatrix} \phi_f \\ \phi_{th} \\ J_f \\ J_{th} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \\ -\mu_f D_f & 0 \\ -S \mu_f D_{th} & -\mu_{th} D_{th} \end{bmatrix} \begin{bmatrix} C_1 e^{-\mu_f z} \\ C_2 e^{-\mu_{th} z} \end{bmatrix} = Y^a_{III} V^a_{III}(z) \qquad (19)$$

where Y^a_{III} is the 2 x 4 matrix, and V^a_{III} the 1 x 2 matrix.

Case (b):

$$\left.\begin{aligned} \phi_f &= C_1 \cosh \mu_f(z_1 - z) \\ \phi_{th} &= S\, C_1 \cosh \mu_f(z_1 - z) + C_2 \cosh \mu_{th}(z_1 - z) \\ J_f &= -\mu_f D_f C_1 \sinh \mu_f(z_1 - z) \\ J_{th} &= -S\mu_f D_{th} C_1 \sinh \mu_f(z_1 - z) - \mu_{th} D_{th} C_2 \sinh \mu_{th}(z_1 - z) \end{aligned}\right\} \quad (20)$$

or

$$\begin{bmatrix} \phi_f \\ \phi_{th} \\ J_f \\ J_{th} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S & 1 \\ -\mu_f D_f \tanh (z_1 - z) & 0 \\ -S\mu_f D_{th} \tanh (z_1 - z) & \mu_{th} D_{th} \tanh (z_1 - z) \end{bmatrix} \begin{bmatrix} C_1 \cosh {}_f(z_1 - z) \\ C_2 \cosh {}_{th}(z_1 - z) \end{bmatrix}$$

$$\equiv Y^b_{III}(z)\, V^b_{III}(z) \quad . \qquad (21)$$

The criticality condition in either case is of the form:

$$\left.\begin{aligned} Y_1(x_1)\, V_I(x_1) &= Y_{II}(o)\, V_{II}(o) \quad ; \\ Y_{III}(o)\, V_{III}(o) &= Y_{II}(y_1)\, V_{II}(y_1) \quad . \end{aligned}\right\} \quad (22)$$

Then write

$$V_{II} = Y^{-1}_{II}(o)\, Y_I(x_1)\, V_I(x_1)$$

and find

$$Y_{III}(o)\, V_{III}(o) = Y_{II}(y_1)\, Y^{-1}_{II}(o)\, Y_I(x_1)\, V_I(x_1) \quad .$$

Let:

$$Y = Y_{II}(y_1)\, Y^{-1}_{II}(o) \quad .$$

Let Q be such that:

$$QY_{III} \equiv 0 \quad .$$

The critical equation is then

$$\det Q\, Y\, Y_I(x_1) = 0 \quad . \tag{23}$$

Now

$$Y = Y_{II}(y_1)\, Y_{II}^{-1}(o) \; ;$$

hence

$$Y\, Y_{II}(o) = Y_{II}(y_1) \quad . \tag{24}$$

Let the elements of Y be y_{ij} , where i denotes row and j the column.

$$Y_{II}(o) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & S_1 & 0 & S_2 \\ -D_f \ell & 0 & mD_f & 0 \\ -S_1 D_{th} \ell & 0 & S_2 m D_{th} & 0 \end{bmatrix}$$

$$Y_{II}(y_1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ S_1 & S_1 & S_2 & S_2 \\ -D_f \ell \operatorname{ctn} \ell y_1 & \ell D_f \tan \ell y_1 & mD_f \operatorname{ctnh} my_1 & mD_f \tanh my_1 \\ -S_1 D_{th} \ell \operatorname{ctn} \ell y_1 & S_1 \ell D_{th} \tan y_1 & S_2 m D_{th} \operatorname{ctnh} my_1 & S_2 m D_{th} \tanh my_1 \end{bmatrix} \tag{25}$$

$$\times \begin{bmatrix} \sin \ell y & 0 & 0 & 0 \\ 0 & \cos \ell y & 0 & 0 \\ 0 & 0 & \sinh my_1 & 0 \\ 0 & 0 & 0 & \cosh my_1 \end{bmatrix} = Y_{II}^{*} D \quad .$$

For any row, i, one finds

$$\begin{aligned} y_{i1} + S_1\, y_{i2} &= a_{i2} \\ y_{i1} + S_2\, y_{i2} &= a_{i4} \end{aligned} \tag{26}$$

$$y_{i3}\,(-\ell D_f) + y_{i4}\,(-S_1\,\ell D_s) = a_{i1}$$

$$y_{i3}\,(mD_f) + y_{i4}\,(S_2 m D_s) = a_{i3}$$

where a_{ij} are the elements of $Y_{II}(y_1)$. Then

$$\left.\begin{aligned}
y_{i1} &= \frac{S_1 a_{i4} - S_2 a_{i2}}{S_1 - S_2} \\
y_{i2} &= \frac{a_{i2} - a_{i4}}{S_1 - S_2} \\
y_{i3} &= \frac{S_2}{S_1 - S_2}\;\frac{a_{i1}}{\ell D_f} + \frac{S_1}{S_1 - S_2}\;\frac{a_{i3}}{mD_f} \\
y_{i4} &= \frac{-1}{S_1 - S_2}\left[\frac{a_{i1}}{\ell D_{th}} + \frac{a_{i3}}{mD_f}\right]
\end{aligned}\right\} \quad (28)$$

If Eq. (23) is now multiplied through by $S_1 - S_2$ and if Y is redefined as

$$Y\,D^{-1} = Y^* \quad ,$$

where D is the diagonal matrix in Eq. (25), then the elements of Y^* are given by

$$y^*_{i1} = S_1\,a^*_{i4} - S_2\,a^*_{i2}\,, \quad \text{etc.},$$

where a^*_{i4} are the elements of $Y^*_{II}(y_1)$. This step makes numerical accuracy a little easier to attain.

$$Y_1(x_1) = \begin{bmatrix} 1 & 0 \\ S & 1 \\ \mu_f D_f \operatorname{ctnh} \mu_f x_1 & 0 \\ S\mu_f D_{th} \operatorname{ctnh} \mu_f x_1 & \mu_{th} D_{th} \operatorname{ctnh} \mu_{th} x_1 \end{bmatrix}$$

In both cases x_1 = 30 cm; $D_f D_{th}$ and S have all been given previously. The perpendicular buckling is now increased by the buckling in one lateral direction. Assuming a reflector saving of about 15 cm from the "infinite" graphite reflectors on either end, the equivalent bare length of a $6\frac{1}{2}$ box array is

$$60 \text{ cm} + 30 \text{ cm} = 90 \text{ cm}.$$

Hence the additional buckling is:

$$B_\ell^2 = \pi^2/(90)^2 = 0.00122 \text{ cm}^{-2} \quad ;$$

added to the vertical buckling $B_v^2 = .00175 \text{ cm}^{-2}$, this yields a total $B_z^2 = 0.00297 \text{ cm}^{-2}$. Then:

$$\mu_f^2 = 0.00570 \text{ cm}^{-2} \qquad \mu_f = 0.07549 \text{ cm}^{-1}$$

$$\mu_{th}^2 = 0.00337 \text{ cm}^{-2} \qquad \mu_{th} = 0.05805 \text{ cm}^{-1}$$

$$\mu_f x_1 = 2.2647 \qquad \mu_{th} x_1 = 1.7415$$

$$\text{ctnh}\,(2.2647) = 1.0237070 \qquad \text{ctnh}\,(1.7415) = 1.0633769$$

$$Y_1(x_1) = \begin{matrix} 1 & 0 \\ -1.42641000 & 1 \\ 0.08500761 & 0 \\ -0.09953991 & 0.05574131 \end{matrix}$$

Let the matrix Q for case (a) be denoted by Q_a:

$$Q_a = \begin{bmatrix} 1 & 0 & \dfrac{1}{\mu_f D_f} & 0 \\ 0 & 1 & \dfrac{S}{D_f}\left(\dfrac{1}{\mu_f} - \dfrac{1}{\mu_{th}}\right) & \dfrac{1}{\mu_{th} D_{th}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 12.04253422 & 0 \\ 0 & 1 & 5.16067523 & 19.07699761 \end{bmatrix} .$$

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For case (a) the critical loading will be in the neighborhood of 2 kg. The determinant will be examined for bucklings corresponding to 2 kg and 2.2 kg and the resulting function of buckling will be extrapolated or interpolated to find that buckling at which the determinant vanishes. This will yield a first estimate of the predicted critical mass.

The active volume is 54.231645 liters; hence the volume fraction per kg of U^{235} is

$$X = 9.86060 \times 10^{-4}/\text{kg}.$$

At 2 kg:	At 2.2 kg:
$X = 19.7212 \times 10^{-4}$	$X = 21.69332 \times 10^{-4}$
$\Sigma_{au} = 0.05641912\ \text{cm}^{-1}$	$0.06206097\ \text{cm}^{-1}$
$\Sigma_{am} = 0.01303442\ \text{cm}^{-1}$	$0.01303186\ \text{cm}^{-1}$
$L^2 = 3.92866935\ \text{cm}^2$	$3.63363576\ \text{cm}^2$
$B^2 = 0.01038971\ \text{cm}^{-2}$	$0.01089301\ \text{cm}^{-2}$
$k = 1.6896420$	1.71903181

Then from Eq. (28), the following elements y_{ij} are obtained for a loading of 2 kg:

$$Y = \begin{bmatrix} 3.11014699 & 0 & -25.77651937 & -49.41697474 \\ 0 & 3.11014699 & -8.77293518 & 7.12548526 \\ 1.25147827 & -0.29056933 & 1.36972917 & -3.27008372 \\ -0.05158443 & 0.43442164 & -0.12672376 & 2.90442658 \end{bmatrix}$$

$$Q_a Y = \begin{bmatrix} 18.18111688 & -3.49919110 & -9.28150897 & -88.79706984 \\ 5.47439673 & 9.89807366 & -4.12171678 & 45.65738446 \end{bmatrix}$$

$$DY_1 = A\,B$$

$$A = \begin{bmatrix} 0.96140553 & 0 & 0 & 0 \\ 0 & 0.27513524 & 0 & 0 \\ 0 & 0 & 1990.7455994 & 0 \\ 0 & 0 & 0 & 1440.7459348 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ -1.42641 & 1 \\ 0.08500761 & 0 \\ -0.09953991 & 0.05574131 \end{bmatrix} ;$$

therefore

$$DY_1 = \begin{bmatrix} 0.96140553 & 0 \\ -0.39245566 & 0.27513524 \\ 126.72472049 & 0 \\ -148.38871620 & 83.096131 \end{bmatrix} .$$

For a loading of 2.2 kg:

$$Y = \begin{bmatrix} 3.03728662 & 0 & -24.51333415 & -47.80130875 \\ 0 & 3.03728662 & -7.71465444 & 7.37453137 \\ 1.46835477 & -0.22773452 & 0.93201144 & -3.48124200 \\ -0.03675408 & 0.44497968 & -0.12264253 & 2.83448779 \end{bmatrix}$$

$$Q_a Y = \begin{bmatrix} 20.71999918 & -2.74250075 & -13.31239918 & -89.72428466 \\ 6.87654445 & 10.35089904 & -5.24449743 & 43.48248915 \end{bmatrix}$$

$$DY_1 = \begin{bmatrix} 0.97236816 & 0 & 0 & 0 \\ 0 & 0.23345268 & 0 & 0 \\ 0 & 0 & 1999.923897 & 0 \\ 0 & 0 & 0 & 1999.924147 \end{bmatrix} Y_1$$

$$= \begin{bmatrix} 0.97236816 & 0 \\ -0.33299924 & 0.23345268 \\ 170.00875066 & 0 \\ -199.07226960 & 111.47839185 \end{bmatrix} .$$

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Evaluating the determinants and extrapolating the determinant as a function of buckling to the point where it vanishes, one finds a critical mass of 1.6 kg, corresponding to a critical buckling of 0.00913 cm^{-2}. The observed critical mass is about 2 kg. Numerical accuracy is improved if, in evaluating

$$\Delta = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

one divides by cd :

$$\frac{\Delta}{cd} = \frac{a}{c} - \frac{b}{d}$$

and the quantities (a/c), (b/d) are of order of magnitude unity; thus the difficulty of computing the difference of two large but nearly equal numbers is avoided.

Case (b):

The volume of each slab is 6/6.5 times the volume of the slab in case (a). If the same concentrations are used as before only the matrix (Q_b) need be computed anew; there will be a slight increase in the buckling since the slab is slightly shorter, but this may be neglected for the moment and treated as a perturbation.

$$Y^b_{III}(o) = \begin{bmatrix} 1 & 0 \\ S & 1 \\ -\mu_f D_f \tanh \mu_f z_1 & 0 \\ -S\mu_f D_{th} \tanh \mu_f z_1 & -\mu_{th} D_{th} \tanh \mu_{th} z_1 \end{bmatrix} ;$$

therefore

$$Q_b = \begin{bmatrix} 1 & 0 & \dfrac{1}{+\mu_f D_f \tanh \mu_f z_1} & \\ 0 & 1 & \dfrac{-S}{D_f}\left(\dfrac{-1}{\mu_f \tanh \mu_f z_1} + \dfrac{1}{\mu_{th} \tanh \mu_{th} z_1}\right) & \dfrac{1}{+\mu_{th} D_{th} \tanh \mu_{th} z_1} \end{bmatrix}$$

$$\frac{1}{\tanh \mu_f z_1} = \operatorname{ctnh} \mu_f z_1 = 1.0237070$$

$$\frac{1}{\tanh \mu_{th} z_1} = \operatorname{ctnh} \mu_{th} z_1 = 1.0633769$$

$$Q_b = \begin{bmatrix} 1 & 0 & +12.32802657 & 0 \\ 0 & 1 & +6.16793088 & +20.2849752 \end{bmatrix} .$$

For $\ell^2 = 0.00741971\ cm^{-2}$:

$$Q_bY = \begin{bmatrix} 18.53840435 & -3.58214642 & -8.89046177 & -89.73065373 \\ 6.67264258 & 10.13016764 & -2.89512866 & 45.70802077 \end{bmatrix}$$

$$Q_bYDY_1 = \begin{bmatrix} 12205.8100748 & -7457.2557291 \\ -7146.9994285 & 3800.9468498 \end{bmatrix}$$

$$\frac{a}{b} - \frac{c}{d} = -1.707823 + 1.961946857$$

$$= 0.2541 \quad .$$

At 2.2 kg:

$$\ell^2 = 0.00792\ cm^2$$

$$\ell = 0.08901129\ cm$$

$$Q_bYDY_1 = \begin{bmatrix} 15866.8542495 & -10113.7691904 \\ -10113.7691915 & 4840.6490177 \end{bmatrix}$$

$$\frac{a}{b} - \frac{c}{d} = 0.52050$$

$$\ell_c = 0.08285\ cm^{-1}$$

$$\ell_c^2 = 0.00686\ cm^{-2}$$

$$B_c^2 = 0.00983\ cm^{-2}$$

The extrapolation yields a critical mass of 3.323 kg. The change in buckling due to the shorter length of the slabs is 0.00011 cm^{-2}; and perturbation theory indicates that this will increase the mass by about 6% or to about 3.52 kg. The observed critical mass for this configuration is 3.6 kg.

VI. SUMMARY

Three different methods of solving the two-group equations for the critical loading of Argonaut for as many different types of core configurations have been presented.

The critical mass is overestimated for the lattices with highest critical mass and is underestimated in the case of lowest critical mass. In the cases of highest interest, the two-sided loading and the 3-inch annular loading, the mass prediction is very good indeed. In any case, the difference between experiment and theoretical prediction is not more than 400 gm.

There is some question as to the proper value of the age; the value chosen for the 3-inch annular loading (Section III) yields an estimate of the critical mass which is known to be too high. Use of the lowest estimation of the age yields the results summarized in the preceding paragraph. The annular array is the only one in which a reasonably good idealization of the physical set up can be made; since the calculation has been coded for a large computing machine (AVIDAC) it is anticipated that more precise calculations will allow a better empirical determination of the approximate age. At present, the best value appears to be that predicted by the formula of Deutsch (See Section II).

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