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ATOMICS INTERNATIONAL

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ABSTRACT

One-group diffusion theory is employed to determine the flux for a large but finite cylindrical bare homogeneous reactor with a control rod inserted partially along the axis of the reactor. A critical equation is derived for the buckling.

I. INTRODUCTION

The flux for a large but finite cylindrical bare homogeneous thermal reactor with a control rod inserted along the axis at any position is to be obtained by use of diffusion theory. The flux is made to vanish along the extrapolated boundaries of the reactor and along the effective boundaries of the control rod. It is assumed that the rod is "black" to thermal neutrons and that the region in which the rod is not present is filled with core material. Calculations were made for the control rod inserted halfway into the reactor for various orders of approximation.

II. DIFFUSION EQUATION AND BOUNDARY CONDITIONS[1]

The one-group diffusion equation for the flux away from the control rod is

$$D\nabla^2\phi - \Sigma_a\phi + k\Sigma_a\phi = 0, \qquad \ldots(1)$$

where D is the diffusion coefficient, ∇^2 the Laplacian, Σ_a the absorption cross section of the core material, and k the multiplication constant. Equation (1) may be written as

$$\nabla^2\phi + B^2\phi = 0, \qquad \ldots(2)$$

where $B^2 = \frac{(k - 1)\Sigma_a}{D}$, and, because of the cylindrical symmetry,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad .$$

Figure 1 shows how the z- and r-axes were chosen, where H is the extrapolated height and R the extrapolated radius of the reactor, h is the length of insertion of the control rod and r_o its effective radius. We denote by V_1, the extrapolated volume of the reactor, and by S_1, its surface. We denote by V_2, the effective volume of the control rod, and by S_2, its surface.

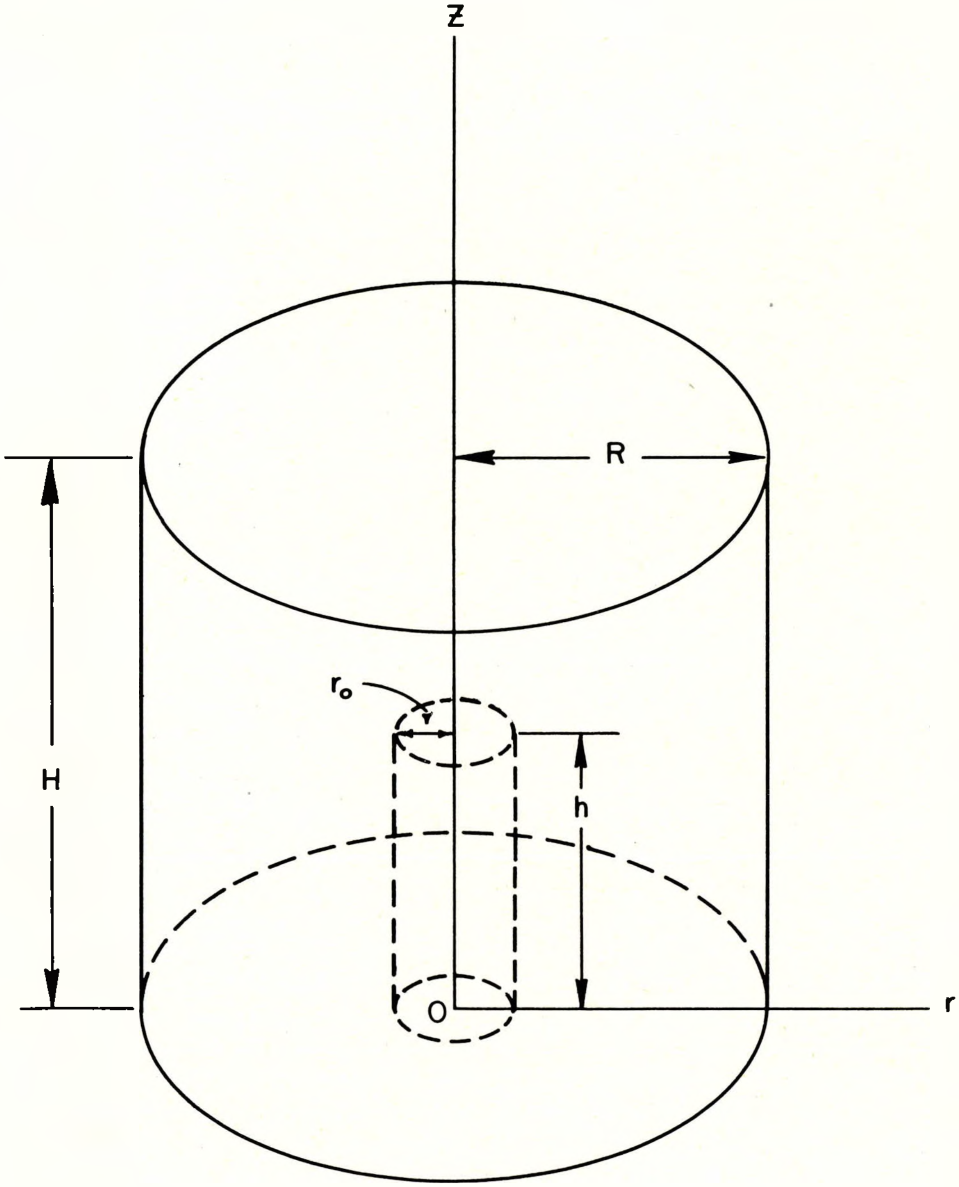

Fig. 1. Cylindrical Reactor with Central Control Rod

The boundary conditions are the following:

a) $0 \leq \phi < \infty$ in $V_1 - V_2$,

b) $\phi \Big|_{S_1} = 0$,

c) $\phi \Big|_{S_2} = 0$.

It has been assumed that elementary diffusion theory is applicable. Hence, if the reactor dimensions are large compared to λ_{tr}, the solutions obtained will be valid at distances greater than a few mean free paths away from the control rod.

III. REPRESENTATION OF THE CONTROL ROD ABSORPTION

By introducing a control rod in the reactor, we have introduced a strong extraneous sink, or absorber. We have assumed the control rod to be a "black" absorber. The control rod absorption strength will thus have the property of making the flux zero on the effective boundaries of the control rod. The control rod absorption S must be zero in the region $V_1 - V_2$. By applying the steady-state balance of neutrons equation, we find that S satisfies the Eq.:

$$\nabla^2\phi + B^2\phi = S \text{ in } V_2 \qquad \ldots (3)$$

$$= 0 \text{ in } V_1 - V_2 \ . \qquad \ldots (3a)$$

Furthermore, the proper choice of S makes ϕ satisfy the boundary condition II. c).

We assume that the sink S may be represented in the form:

$$S(r, z) = F(r)G(z) \text{ in } V_2 \qquad \ldots (4)$$

$$= 0 \text{ in } V_1 - V_2 \ . \qquad \ldots (4a)$$

This representation of $S(r, z)$ is not correct, but enables us to satisfy boundary conditions II. c) in the form:

$$\phi(r_o, z) = 0, \quad 0 \leq z \leq h, \qquad \ldots(5)$$

$$\phi(r, h) \cong 0, \quad 0 \leq r \leq r_o \quad . \qquad \ldots(5a)$$

Since the flux is being obtained by diffusion theory, satisfaction of Eq. (5a) does not give an inadequate physical description of the flux a few mean free paths away from the control rod.

For a control rod of radius $r_o \ll R$, we take $F(r)$ as:

$$F(r) = C, \quad 0 \leq r < r_o, \qquad \ldots(6)$$

$$= 0, \quad r_o < r \leq R; \qquad \ldots(6a)$$

where C is a constant proportional to the power level. We take $G(z)$ as:

$$G(z) = g(z), \quad 0 \leq z \leq h, \qquad \ldots(7)$$

$$= 0, \quad h < z \leq H \quad . \qquad \ldots(7a)$$

Equations (6), (6a), (7), and (7a) now satisfy Eq. (4) and (4a).

We expand $F(r)$ in a Fourier-Bessel series to get:

$$F(r) = \frac{2\,Cr_o}{R^2} \sum_{i=1}^{\infty} \frac{J_1(\xi_i r_o)}{\xi_i J_1^2(\xi_i R)} J_o(\xi_i r), \qquad \ldots(8)$$

where the ξ_i's are such that:

$$J_o(\xi_i R) = 0, \quad (i = 1, 2, 3, \ldots) \qquad \ldots(8a)$$

We expand G(z) in a Fourier-sine series to get:

$$G(z) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi z}{H} , \qquad \ldots (9)$$

where the b_n's are given by:

$$b_n = \int_0^h g(z) \sin \frac{n\pi z}{H} dz ,$$

and are to be determined later from the boundary condition in Eq. (5).

The expansions shown in Eq. (8) and (9) are so chosen so that S(r, z) satisfies boundary condition II. b), which leads ultimately to ϕ satisfying this boundary condition.

IV. SOLUTION OF DIFFUSION EQUATION BY USE OF HANKEL AND FOURIER TRANSFORMS[2]

In order to solve Eq. (3) subject to the boundary conditions in Section II, it is convenient to introduce the finite Hankel and finite Fourier transforms.

The finite Hankel transform of $\phi(r, z)$ is:

$$\bar{\phi}(\xi_i, z) = \int_0^R r\phi(r, z) J_o(\xi_i r)\, dr \quad . \qquad \ldots (10)$$

The inverse Hankel transform of $\bar{\phi}(\xi_i, z)$ is:

$$\phi(r, z) = \frac{2}{R^2} \sum_{i=1}^{\infty} \frac{\bar{\phi}(\xi_i, r) J_o(\xi_i r)}{J_1^2(\xi_i R)} , \qquad \ldots (11)$$

where the ξ_i's satisfy Eq. (8a).

One of the properties of the finite Hankel transforms which is of interest here is that

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi(r,z)\, J_o(\xi_i r)\, dr$$

$$= R\,\xi_i\,\phi(R,z) J_1(\xi_i R) - \xi_i^2\,\bar{\phi}(\xi_i, z) \quad . \qquad \ldots(12)$$

The finite Fourier transform of $\bar{\phi}(\xi_i, z)$ is

$$\bar{\bar{\phi}}(\xi_i, n) = \int_0^H \bar{\phi}(\xi_i, z) \sin\frac{n\pi z}{H}\, dz \quad . \qquad \ldots(13)$$

The inverse Fourier transform of $\bar{\bar{\phi}}(\xi_i, n)$ is

$$\bar{\phi}(\xi_i, z) = \frac{2}{H}\sum_{n=1}^{\infty} \bar{\bar{\phi}}(\xi_i, n) \sin\frac{n\pi z}{H} \quad . \qquad \ldots(14)$$

A property of the finite Fourier transform of interest here is that

$$\int_0^H \frac{\partial^2\bar{\phi}}{\partial z^2} \sin\frac{n\pi z}{H}\, dz = \frac{n\pi}{H}\left[(-1)^{n+1}\phi(r,H) + \phi(r,0)\right] - \frac{n^2\pi^2}{H^2}\bar{\bar{\phi}}(\xi_i, n) \quad . \qquad \ldots(15)$$

If we apply the double transform, resulting from the substitution of Eq. (10) in Eq. (13), we obtain:

$$\bar{\bar{\phi}}(\xi_i, n) = \int_0^R \int_0^H r\,\phi(r,z)\, J_o(\xi_i, r) \sin\frac{n\pi z}{H}\, dr dz \ , \qquad \ldots(16)$$

and then the substitution of Eq. (8) and (9) in Eq. (3), yields by use of Eq. (12), (13), (15), and (16):

$$R\xi_i\phi(R,z)J_1(\xi_i R) - \xi_i^2\bar{\bar{\phi}}(\xi_i, n) + \frac{n\pi}{H}\left[(-1)^{n+1}\phi(r,H) + \phi(r,0)\right]$$

$$-\frac{n^2\pi^2}{H^2}\bar{\bar{\phi}}(\xi_i, n) + B^2\bar{\bar{\phi}}(\xi_i, n) = Cr_o \frac{J_1(\xi_i r_o)}{\xi_i} b_n \ . \qquad \ldots(17)$$

We apply now the boundary condition II. b) to Eq. (17) to get

$$\bar{\bar{\phi}}(\xi_i, n) = \frac{Cr_o J_1(\xi_i r_o) b_n}{\xi_i\left[B^2 - \xi_i^2 - \frac{n^2\pi^2}{H^2}\right]} \ . \qquad \ldots(18)$$

Applying the inverse transforms in Eq. (11) and (14) to Eq. (18) we have

$$\phi(r,z) = \frac{4\,Cr_o}{R^2 H} \sum_{i,n=1}^{\infty} \frac{J_1(\xi_i r_o) b_n J_o(\xi_i r)\sin\frac{n\pi z}{H}}{\xi_i\left[B^2 - \xi_i^2 - \frac{n^2\pi^2}{H^2}\right] J_1^2(\xi_i R)} \ . \qquad \ldots(19)$$

In order to determine B^2 and the b_n's, it is necessary now to apply the boundary condition that the flux be zero on the effective radius of the control rod. We see at this point that our choice of $S(r,z)$ does not enable us to satisfy the boundary condition $\phi(r,h) \equiv 0$ for $0 \le r \le r_o$. This restriction of $S(r,z)$ thus causes a small error in the description of ϕ in a small region around the rod tip.

Applying the boundary condition in Eq. (5) to Eq. (19), we have:

$$\phi(r_o, z) = \frac{4\,Cr_o}{R^2 H} \sum_{i,n=1}^{\infty} \frac{a_i b_n}{C_{in}} \sin\frac{n\pi z}{H} = 0 \ , \quad 0 \le z \le h \ , \qquad \ldots(20)$$

where

$$\alpha_i = \frac{J_1(\xi_i r_o) J_o(\xi_i r_o)}{\xi_i J_1^2(\xi_i R)} ,$$

and

$$C_{in} = B^2 - \xi_i^2 - \frac{n^2 \pi^2}{H^2} .$$

The b_n's must have the property that make Eq. (7a) hold, while at the same time, Eq. (20) holds. These b_n's may be found to a greater and greater accuracy by solving a higher and higher order system of simultaneous linear homogeneous algebraic equations in the following way. Denote the M equally spaced points in the semi-open interval $0<z\leq h$ by $z_1, z_2, z_3, \ldots, z_M$. Denote the N equally spaced points in the open interval $h<z<H$, by $\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_N$.

We evaluate Eq. (20) at $z_1, z_2, \ldots, z_M$, and let n = 1, 2, ..., M+N in the summation to get an (M+N)xM system of linear homogeneous algebraic equations in the b_n's. We next evaluate Eq. (7a) at the points $\zeta_1, \zeta_2, \ldots, \zeta_N$, and let n = 1, 2, ..., M+N in the summation to get an (M+N)xN system of linear homogeneous algebraic equations in the b_n's. Eq. (20) and (7a) so evaluated must hold simultaneously. We thus get an (M+N)x(M+N) system of linear homogeneous algebraic equations for the b_n's.

The resulting (M+N)x(M+N) homogeneous system yields non-trivial b_n's, provided the determinant of these coefficients is equal to zero. The determinant thus derived is a function of B^2 and is zero for some B^2 such that $B_o^2<B^2<B_1^2$, where B_o^2 is the buckling for the rod completely withdrawn and B_1^2 is the buckling for the rod completely inserted. This determinantal equation may be regarded as a critical equation. It looks as follows:

$$
\begin{vmatrix}
\mu_{i1} \sin\frac{\pi z_1}{H} & \mu_{i2} \sin\frac{2\pi z_1}{H} & \cdots & \mu_{i(M+N)} \sin\frac{(M+N)\pi z_1}{H} \\
\mu_{i1} \sin\frac{\pi z_2}{H} & \mu_{i2} \sin\frac{2\pi z_2}{H} & \cdots & \mu_{i(M+N)} \sin\frac{(M+N)\pi z_2}{H} \\
\vdots & \vdots & & \vdots \\
\mu_{i1} \sin\frac{\pi z_M}{H} & \mu_{i2} \sin\frac{2\pi z_M}{H} & \cdots & \mu_{i(M+N)} \sin\frac{(M+N)\pi z_M}{H} \\
\sin\frac{\pi \zeta_1}{H} & \sin\frac{2\pi \zeta_1}{H} & \cdots & \sin\frac{(M+N)\pi \zeta_1}{H} \\
\sin\frac{\pi \zeta_2}{H} & \sin\frac{2\pi \zeta_2}{H} & \cdots & \sin\frac{(M+N)\pi \zeta_2}{H} \\
\vdots & \vdots & & \vdots \\
\sin\frac{\pi \zeta_N}{H} & \sin\frac{2\pi \zeta_N}{H} & \cdots & \sin\frac{(M+N)\pi \zeta_N}{H}
\end{vmatrix} = 0, \qquad \ldots(21)
$$

where

$$
\mu_{in} = \sum_{i=1}^{\infty} \frac{a_i}{C_{in}}, \quad (n = 1, 2, \ldots, M+N)
$$

Theoretically, one needs to evaluate an infinitely large determinant to determine B^2, but one can get a good approximation by evaluating a finite subdeterminant. The accuracy in B^2 obtained would be a function of the size of the determinant.

A procedure for examining the accuracy of the B^2 obtained will now be discussed. First, one would examine the effect on B^2 by increasing the order of the determinant resulting from the (M+N)x(M+N) homogeneous system. For a given accuracy desired for B^2 a certain order determinant would have to be evaluated.

A check on how satisfactory this B^2 is would be to see how well the boundary condition in Eq. (5) is satisfied by using the b_n's obtained from the homogeneous system of equations by use of this B^2. Satisfaction of this boundary condition is dependent on the number of b_n's and the B^2 obtained by these methods.

The homogeneous system can readily be solved for the b_n's by making b_1 arbitrary, striking out the Nth row, and then solving the resulting inhomogeneous system. The rest of the b_n's obtained will be constants times b_1, so b_1 would just be a normalizing constant for the solution of this eigenvalue problem. The b_n's are satisfactory if they are such that $\sum_n b_n^2 < \infty$.

V. NUMERICAL RESULTS

The IBM 704 was used to calculate the flux in various regions of the reactor with the control rod inserted halfway into the reactor. The dimensionless reactor and control rod parameters, $r = 0.02$, $R = 1$, and $H = 1$ were used. For this choice of dimensionless reactor parameters we have $B_o^2 = 8.251$ and $B_1^2 = 10.872$. Corresponding to the effective control rod radius, we also have[3] the physical control rod radius $r_p = 0.0690$.

The flux was calculated by use of Eq. (19) with the constant C chosen so that $\frac{4Cr_o}{R^2H} = 1$. In all calculations made the index i was run from 1 to 100 and the index n from 1 to M+N. The description of the flux is improved when the order of the linear algebraic homogeneous system of equations is increased and the resulting b_n's are used in the calculations. These fluxes calculated by use of Eq. (19) with the b_n's obtained from the solutions of the various (M+N)x(M+N) linear homogeneous algebraic systems are designated as $\phi_{M+N}(r, z)$. These results are shown in Fig. 2 to 4. See Fig. 1 for the location of the flux plotted in Fig. 2 to 4.

Table I shows how the buckling for the control rod inserted halfway into the reactor varied as the determinant in Eq. (21) was increased in size.

TABLE I

BUCKLING VS ORDER OF DETERMINANT

Buckling B^2	Order of Determinant
9.625	16 x 16
9.525	24 x 24
9.515	28 x 28
9.508	32 x 32

The method developed here is solely for the rod partially or completely inserted and excludes the case of complete withdrawal.

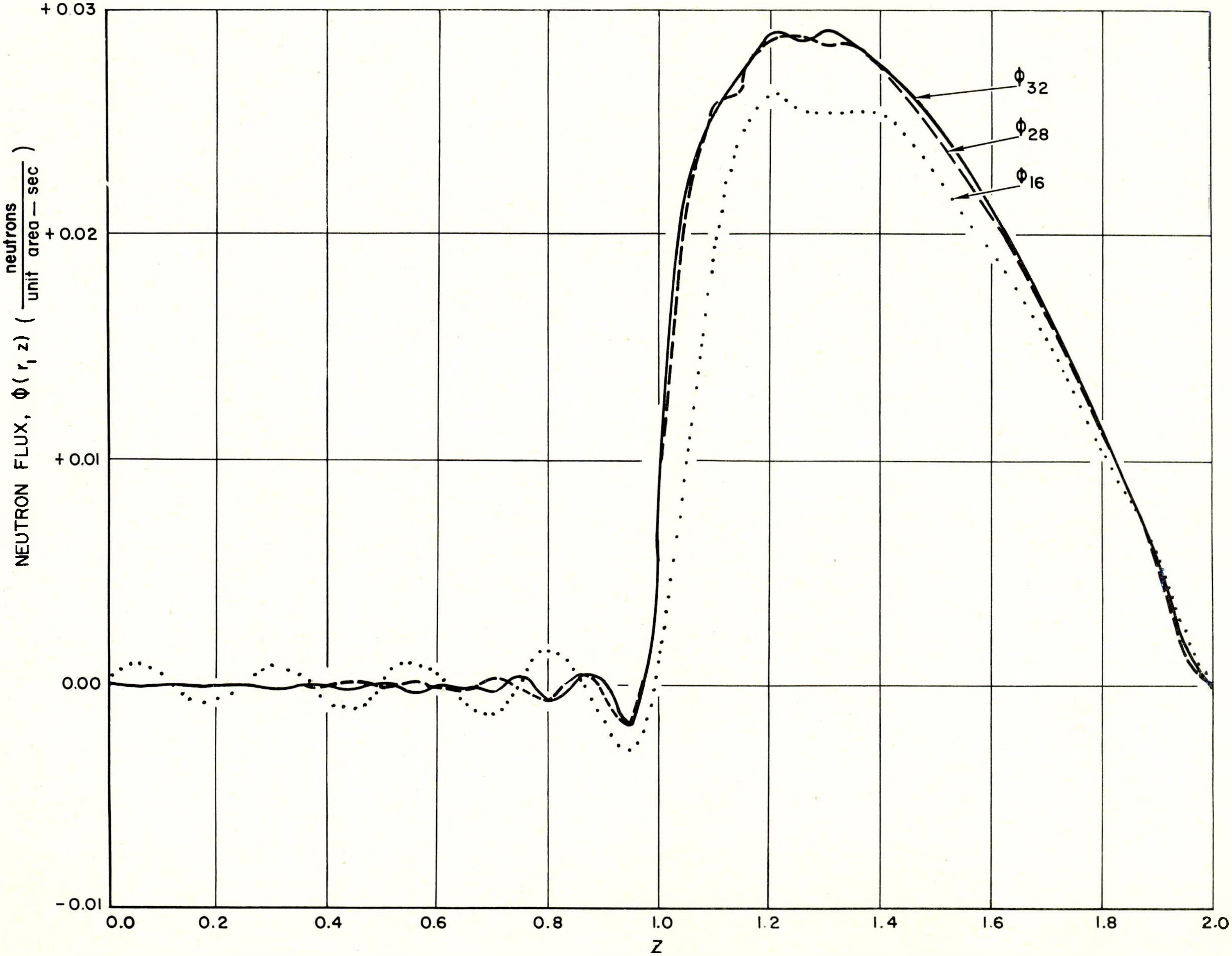

Fig. 2. Vertical Flux Plots for r_1 = 0.02R with the Control Rod Halfway Inserted

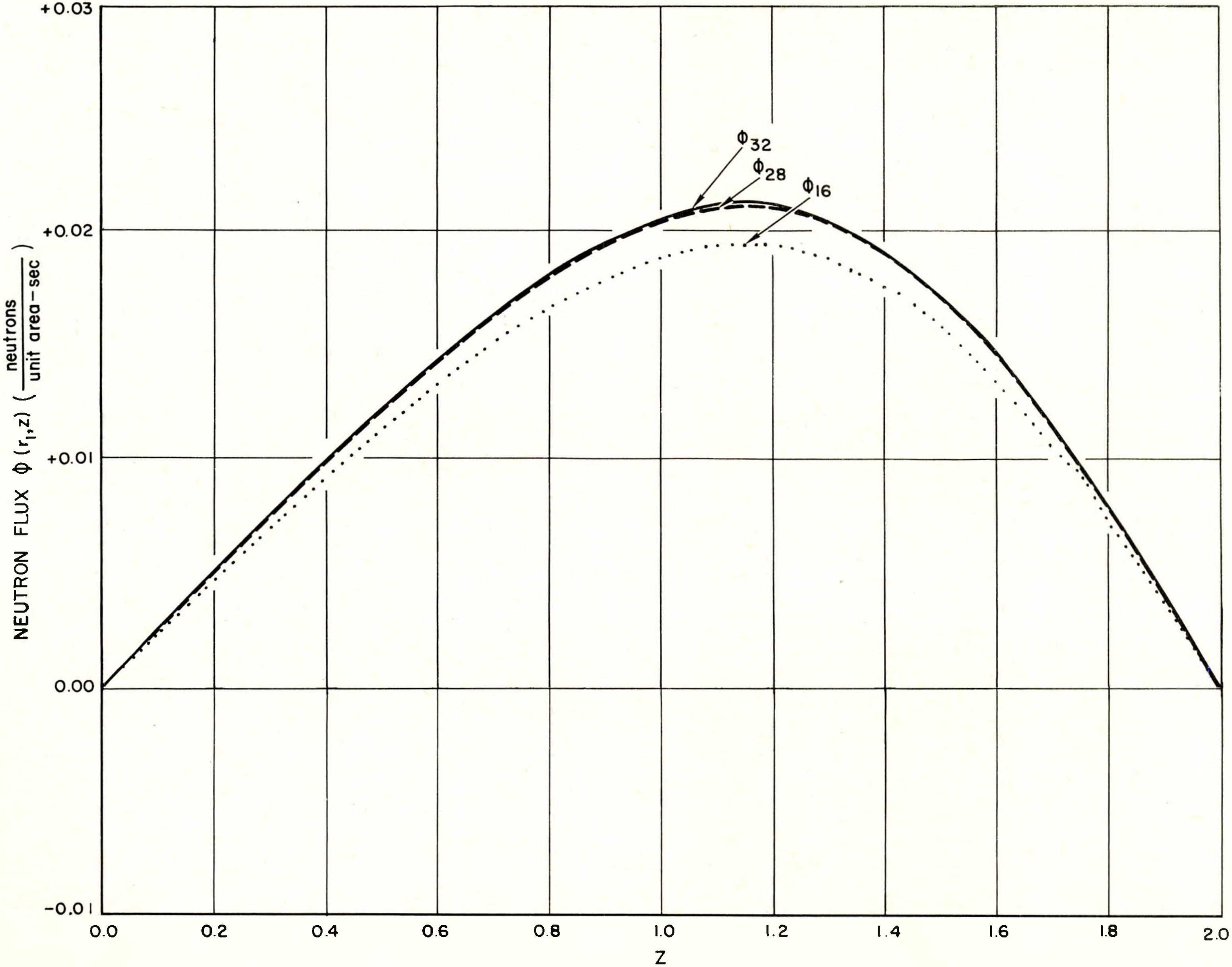

Fig. 3. Vertical Flux Plots for r_1 = 0.5R with the Control Rod Halfway Inserted

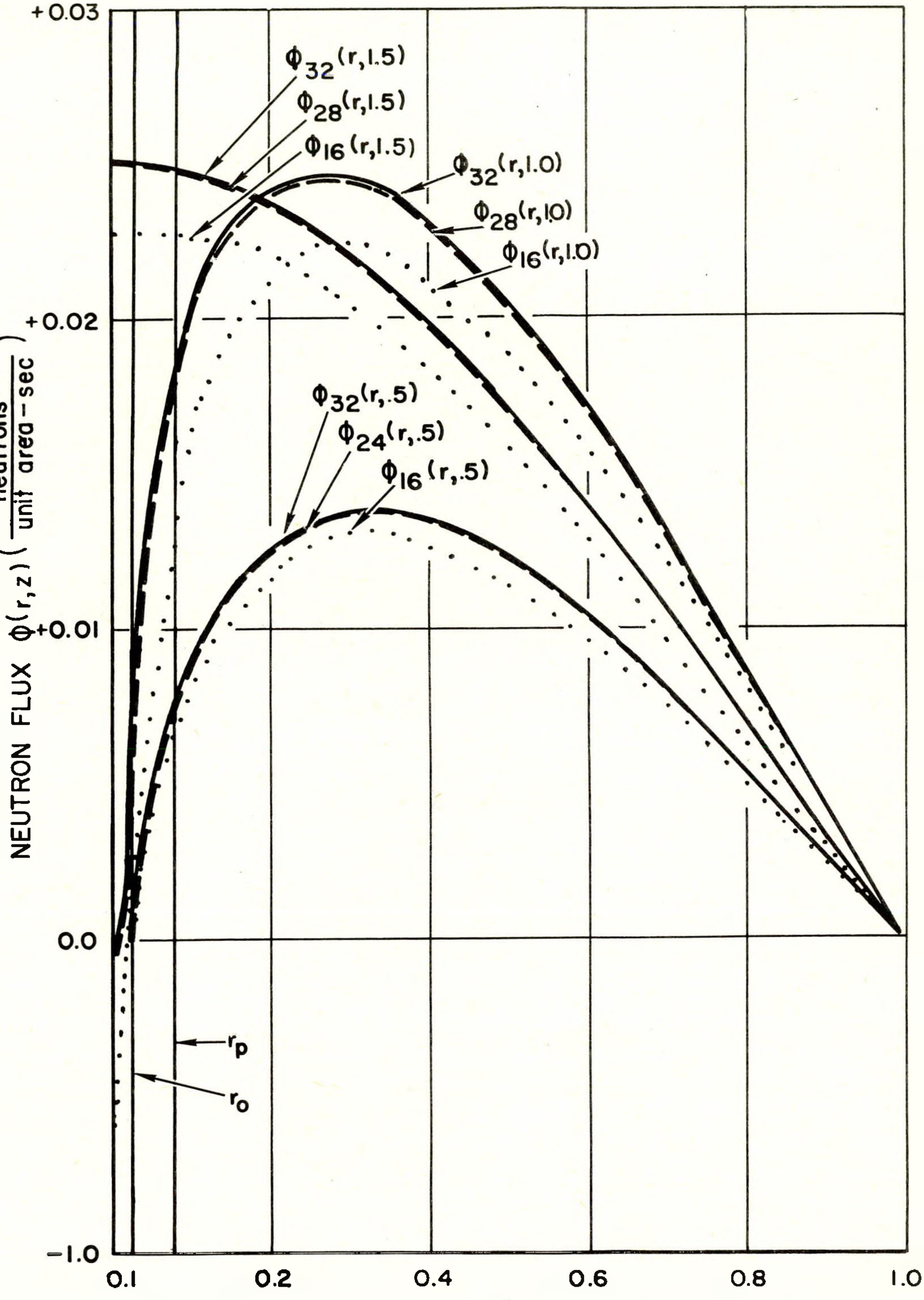

Fig. 4. Radial Flux Plots for $z = z_1 H$ with the Control Rod Halfway Inserted

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